Orthogonal polynomials for the sum of a normal and a uniform random variable

Jithin Jith

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1 Sum of a normal and a uniform random variable

Let X and Y be two independent random variables. Let us define their sum

$$Z = X + Y \tag{1}$$

as another random variable. Let us denote the probability density of X by $f_X(x)$ and that of Y by $f_Y(y)$. Then, the probability density of Z is given by [Papoulis, 1991]

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(z-y, y)\, dy \\ &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy. \end{aligned} \tag{2}$$

Now, let X be normally distributed, i.e.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2} \right), \tag{3}$$

where μ and σ are the mean and standard deviation of X, respectively. Let Y be uniformly distributed, i.e.

$$f_Y(y) = \begin{cases} \frac{1}{b-a} & a \leq y \leq b \\ 0 & \text{elsewhere} \end{cases}, \tag{4}$$

where a and b are the lower and upper limits of the uniform distribution. Then, then $f_Z(z)$, given by Eq. 2, can be written as

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy \\ &= \frac{1}{b-a} \int_a^b f_X(z-y)\, dy \\ &= \frac{1}{b-a} \left[F_X(z-a) - F_X(z-b) \right]. \end{aligned} \tag{5}$$

Here $F_X(\cdot)$ is the cumulative distribution of X which is given by [Papoulis, 1991]

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)\, dt \\ &= \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2}\sigma} \right) \right], \end{aligned} \tag{6}$$

where

$$\text{erf}(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} \exp\left(-t^2\right) \, dt \tag{7}$$

is the error function. Now, Eq. 5 can be rewritten as

$$f_Z(z) = \frac{1}{2(b-a)} \left[\text{erf}\left(\frac{z-a-\mu}{\sqrt{2}\sigma} \right) - \text{erf}\left(\frac{z-b-\mu}{\sqrt{2}\sigma} \right) \right]. \tag{8}$$

2 Orthogonal polynomials for $f_Z(z)$

Polynomials orthogonal with respect to $f_Z(z)$ should obey the orthogonality condition

$$\int_{\mathcal{R}} \psi_i(z) \psi_j(z) f_Z(z) \, dz = \delta_{ij}, \tag{9}$$

where ψ_i are the orthogonal polynomials and δ_{ij} is the Kronecker delta. Orthogonal polynomials up to order n can be constructed using the three-term recurrence relation [Gautschi, 1982] given below:

$$p_{i+1}(z) = (x - \alpha_i) p_i(z) - \beta_i p_{i-1}(z), \quad i = 0, 1, \ldots, n \tag{10}$$

$$p_{-1}(z) = 0, \qquad p_0(z) = 1, \tag{11}$$

where

$$\alpha_i = \frac{\int_{\mathcal{R}} z p_i^2(z) f_Z(z) \, dz}{\int_{\mathcal{R}} p_i^2(z) f_Z(z) \, dz} \qquad \beta_i = \frac{\int_{\mathcal{R}} p_i^2(z) f_Z(z) \, dz}{\int_{\mathcal{R}} p_{i-1}^2(z) f_Z(z) \, dz}, \tag{12}$$

and $\beta_0 = 1$. These polynomials p_i are then normalised as

$$\psi_i(z) = \frac{p_i(z)}{\sqrt{\beta_0 \beta_1 \ldots \beta_i}} \tag{13}$$

to obtain the orthogonal polynomials ψ_i.

2.1 Gauss quadrature

Using Gauss quadrature, an integral is expressed as

$$\int_{\mathcal{R}} g(z) f_Z(z) \, dz \approx \sum_{i=1}^{n+1} g(z_j) w_j, \tag{14}$$

where z_j are the quadrature points and w_j are the weights. This approximation is exact if $g(z)$ has a degree $\leq 2n+1$. The quadrature points and weights can be determined from the eigenproblem of the tridiagonal matrix [Golub and Welsch, 1969]

$$\mathbf{J} = \begin{bmatrix} \alpha_0 & \beta_0 & & & \\ \sqrt{\beta_1} & \alpha_1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \alpha_{n-1} & \sqrt{\beta_n} \\ & & & \sqrt{\beta_n} & \alpha_n \end{bmatrix}. \tag{15}$$

which is called the Jacobi matrix of the polynomials ψ_i. Let the eigenproblem of $\mathbf{J}$ be written as

$$\mathbf{JU} = \mathbf{US}, \tag{16}$$

where $\mathbf{S}$ is a matrix whose diagonal contains the eigenvalues and $\mathbf{U}$ is a matrix whose columns contain the corresponding eigenvectors. Then the quadrature points are given by S_{ii} and the corresponding weights are given by U_{1i}.

References

Athanasios Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, 1991. ISBN 978-0-07-048477-1. Google-Books-ID: 4IwQAQAAIAAJ.

W. Gautschi. On Generating Orthogonal Polynomials. *SIAM Journal on Scientific and Statistical Computing*, 3(3):289–317, September 1982. ISSN 0196-5204.

Gene H. Golub and John H. Welsch. Calculation of Gauss Quadrature Rules. *Mathematics of Computation*, 23(106):221, April 1969. ISSN 00255718.