

Biomath 204 Homework 1

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Problem 1. Prove the Gauss-Markov theorem for β_0 in the following simple linear regression model:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i,$$

assuming $E(\epsilon_i) = 0$, $Var(\epsilon'_i) = \sigma^2$, $Cov(\epsilon_i, \epsilon_j) = 0$.

Proof. In class we derived that $b_0 = \bar{Y} - b_1 \bar{X}$. To show b_0 is unbiased, note:

$$E(\bar{Y}) = \frac{1}{n} E\left(\sum_i Y_i\right) = \frac{1}{n} \sum_i [\beta_0 + \beta_1 X_i] = \frac{1}{n} n \beta_0 + \frac{1}{n} \beta_1 \sum_i X_i = \beta_0 + \beta_1 \bar{X}$$

On the other hand, $E(\beta_1 X_i) = \beta_1 \bar{X}$, so

$$E(b_0) = E(\bar{Y} - \beta_1 X_i) = E(\bar{Y}) - E(\beta_1 X_i) = \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X} = \beta_0.$$

To show linearity in Y , recall in lecture we showed

$$b_1 = \sum_i k_i Y_i, \quad k_i = \frac{(X_i - \bar{X})}{\sum_i (X_i - \bar{X})^2}.$$

Using this, we have

$$\begin{aligned} b_0 &= \bar{Y} - b_1 \bar{X} \\ &= \bar{Y} - \left(\sum_i k_i Y_i\right) \bar{X} \\ &= \frac{1}{n} \sum_i Y_i - \frac{1}{n} \sum_i k_i Y_i \bar{X} n \\ &= \frac{1}{n} \sum_i [Y_i - k_i Y_i \bar{X} n] \\ &= \sum_i Y_i \left(\frac{1}{n} - k_i \bar{X}\right) \\ &= \sum_i Y_i c_i \end{aligned} \tag{1}$$

Finally, to show min variance, let \tilde{b}_0 be another estimator so that

$$\tilde{b}_0 = \sum_i Y_i r_i$$

For some r_i . We need to show that $V(\tilde{b}_0) \geq V(b_0)$.

$$V(\tilde{b}_0) = \sum V(Y_i) r_i^2 = \sigma^2 \sum r_i^2.$$

From here, following the strategy in lecture, define $d_i = r_i - c_i$, so that

$$\sigma^2 \sum r_i^2 = \sigma^2 \sum (d_i + c_i)^2 = \sigma^2 \sum [d_i^2 + 2d_i c_i + c_i^2]$$

Because $d_i^2 \geq 0$, if we could show $d_i c_i = 0$, then we are done, because $V(b_0) = \sigma^2 \sum c_i^2$. To show this, do a bunch of algebra:

$$\sum d_i c_i = \sum (r_i - c_i) c_i = \sum r_i c_i - \sum c_i^2 \quad (2)$$

Now because \tilde{b}_0 is another estimator, the r_i 's must satisfy the following properties that c_i 's from b_0 satisfy:

$$\begin{aligned} \sum c_i &= \sum \left(\frac{1}{n} - k_i \bar{X} \right) = 1 - \bar{X} \sum k_i = 1 - 0 = 1 \\ \sum c_i k_i &= \sum \left(\frac{1}{n} - k_i \bar{X} \right) k_i = \sum \frac{k_i}{n} - \bar{X} \sum k_i^2 = 0 - \frac{\bar{X}}{\sum (x - \bar{X})^2} \end{aligned}$$

By these two, we can evaluate the two terms from eq(2) as follows:

$$\begin{aligned} \sum r_i c_i &= \sum r_i \left(\frac{1}{n} - k_i \bar{X} \right) = \frac{1}{n} \sum r_i - \bar{X} \sum r_i k_i = 1 + \frac{\bar{X}}{\sum (X - \bar{X})^2} \\ \sum c_i^2 &= \sum \left(\frac{1}{n} - k_i \bar{X} \right)^2 = \sum \left(\frac{1}{n^2} - \frac{2k_i \bar{X}}{n} + \bar{X}^2 k_i^2 \right) = \frac{1}{n} + \frac{\bar{X}^2}{\sum (x - \bar{x})^2} \end{aligned}$$

Now putting everything together:

$$\sum r_i c_i - \sum c_i^2 = \frac{1}{n} + \frac{\bar{X}^2}{\sum (X - \bar{X})^2} - \frac{1}{n} - \frac{\bar{X}^2}{\sum (X - \bar{X})^2} = 0$$

Therefore $V(b_0) \leq V(\tilde{b}_0)$ and we have proven the gauss-markov theorem completely. \square

Problem 2. Given b_0, b_1 are least-square estimators for the above regression model, show that the point (\bar{X}, \bar{Y}) always falls on the line $Y_i = b_0 + b_1 X_i$.

Proof. Intuitively, if we have a line that we know best estimates a set of data, then that line should be placed so that the sum of squared error is minimized. As in any refression, X_i are the inputs (observed) and Y_i are the outputs (predicted). Here we are asked to prove that if \bar{X} was the input, then \bar{Y} must be the output.

The formal proof has already been given in lecture, though. Let

$$Q = \sum_i \epsilon_i^2 = \sum_i [Y_i - b_0 - b_1 X_i]^2$$

$$\frac{\partial Q}{\partial b_0} = -2 \sum_i [Y_i - b_0 - b_1 X_i]$$

Setting the above expression equal to zero (i.e. finding the minimum or maximum), we have

$$\begin{aligned} \sum_i Y_i - nb_0 - b_1 \sum_i X_i &= 0 \iff \bar{Y} - b_0 - b_1 \bar{X} = 0 \\ \Rightarrow \bar{Y} &= b_0 + b_1 \bar{X} \end{aligned}$$

Thus the point (\bar{X}, \bar{Y}) is on the regression line defined by b_0 and b_1 (i.e. input \bar{X} spits out \bar{Y}). Because $\frac{\partial^2 Q}{\partial b_0^2} = 2$, the function is concave upwards, so this point is indeed a minimum. \square

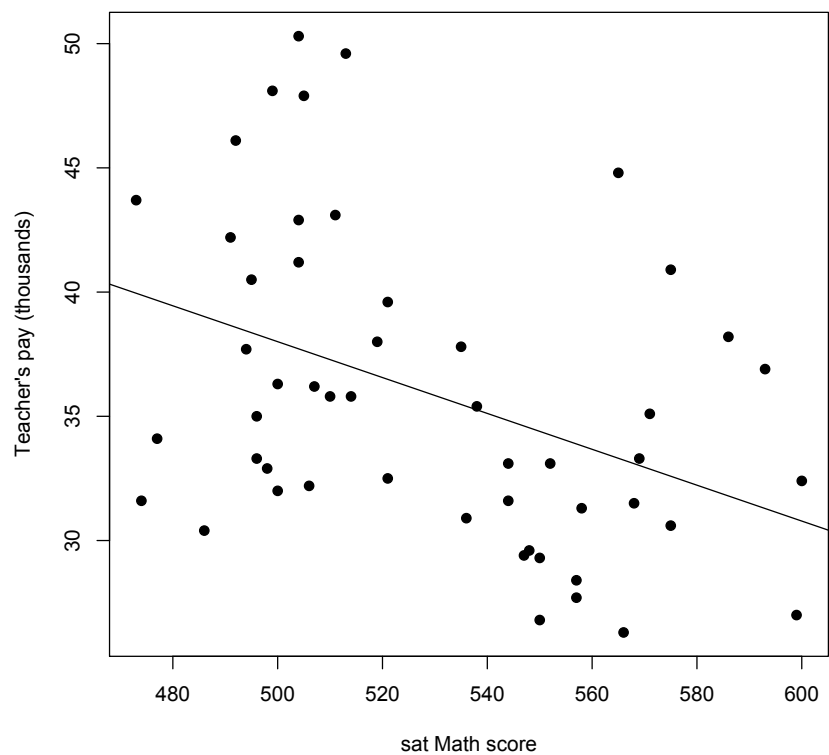
Problem 4. Using methods described in section 3.1, examine the quantitative variables of "States.txt". Characterize the distribution of the variables in terms of symmetry or skewness; non-normality or apparent normality, number of modes, and presence/absence of unusual values.

Proof. For the States.txt dataset, I compared the following: "teachers pay vs student performance (math SAT)" and "percent of state population without high school education vs student performance (verbal SAT)." I really should have used a histogram to illustrate these graphs, but according to the textbook I should divide the 50 states into $2\sqrt{50} \approx 14$ bins so that the graph doesn't appear too overwhelming. However I'm new to R and had a hard time figuring out how to do that since the data were given in terms of the 50 separate states, so I just plotted everything with scatter plot in the hope that it's more illustrative than 50 bars (see next page).

For the top graph, interestingly, the higher the teacher's salary the lower their student's SAT scores (i.e. these two variables have negative covariance). Intuitively we expect higher salary to reflect a higher qualification and hence teaching ability, but apparently that isn't the case. On the other hand, the bottom graph shows that while a state's population without high school education could vary considerably (from 13 to 35), that does not have any effect on student's verbal SAT score. This is another rather strange phenomenon since we would expect a more educated state to treat SAT more seriously and hence be more successful at it.

The mode for both graph's X axis is around 500. Contrary to popular beliefs, the mean is definitely not around 500. From eye inspection both graphs are obviously not normal. Finally, if we ignore the part that some states have as much as 35 percent of the population without a high school diploma, there is little unusual values. \square

Teacher pay VS student performance



Student performance vs % of people without high school

