Collection of Problems that I think are Cool

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September 4, 2019

1 Math

Problem 1.1

Let $\mathbf{X} = \mathbb{R}^{n\times n}$ random matrix. Show that probability that $\det(\mathbf{X}) = 0$ is 0. That is, almost all $n \times n$ random matrices are invertible.

Problem 1.2 When middle school algebra surpase college calculus

For $n \in \mathbb{R}$, prove that the following optimization problem

$$\begin{aligned} \max \quad & xy \\ \text{s.t.} \quad & x + y = n. \end{aligned}$$

has optimal point $x = y = \frac{n}{2}$. Then show that, with the additional constraint that $x, y, n \in \mathbb{Z}$, the solution is achieved by $x = \lceil n/2 \rceil, y = \lfloor n/2 \rfloor$.

Proof. Since $y = n - x$, the problem is equivalent to maximizing $x(n-x)$ with no constraint. Completing the square, we have

$$x(n-x) = -(x^2 - nx) = -\left(x - \frac{n}{2}\right)^2 + \frac{n}{4}.$$

Since n is fixed, the objectice is maximized when $-\left(x - \frac{n}{2}\right)^2 = 0 \iff x = n/2 = y$. If we seek integer solutions, minimizing $x - n/2$ is achieved by rounding $n/2$ to the nearest integer. Thus y is just $n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$.

Note to self: the intuitive method of differentiation natural to all calculus students fails for the integer case, whereas completing the square method natural to middle school students is straightforward. $\square$

Problem 1.3

Continuing the previous problem, for $n \in \mathbb{R}$, prove that the following optimization problem

$$\begin{aligned} \max \quad & x_1 x_2 \cdots x_m \\ \text{s.t.} \quad & \sum_{i=1}^{m} x_i = n. \end{aligned}$$

has optimal point $x_i = \frac{n}{m}$. What would the solution look like with the additional constraint $x_i, n \in \mathbb{Z}$?

Problem 1.4

Given a line of length l, show that the maximum area it can enclose is achieved by a circle of radius $\frac{l}{2\pi}$.

Problem 1.5

Suppose matrix $\mathbf{M}$ is orthogonal and upper triangular, show that $\mathbf{M}$ is diagonal with ± 1 on the diagonal.

Proof. Write $M = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ where each $\mathbf{v}_i$ are column vectors with n terms. Then the upper triangularity of M implies that

$$\mathbf{v}_1 = \begin{bmatrix} a_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{...and so on.}$$

Since $\mathbf{M}$ is orthogonal, $\mathbf{v}_1^t \mathbf{v}_1 = 1 \iff a_{11} = \pm 1$. Furthermore, orthogonality implies that $\mathbf{v}_1^t \mathbf{v}_2 = 0 \iff a_{12} = 0 \iff v_2 = [0, a_{22}, 0, \ldots, 0]^t$. Again $a_{22} = \pm 1$ since $\mathbf{v}_2^t \mathbf{v}_2 = 1$. The result follows by induction on n. □

Problem 1.6 [Lange, 2010, Exercise 8.23]

Use the Gerschgorin circle theorem to estimate eigenvalues of the following matrix:

$$\begin{bmatrix} 4 & 0.2 & -0.1 & 0.1 \\ 0.2 & -1 & -0.1 & 0.05 \\ -0.1 & -0.1 & 3 & 0.1 \\ 0.1 & 0.05 & 0.1 & -3 \end{bmatrix}$$

Proof. Since the matrix is symmetric, checking along the rows or along the columns would yield the same invervals. The eigenvalues lie within the four disks as follows:

$$D(4,0.4) = [3.6,4.4], \quad D(-1,0.35) = [-1.35,-0.65]$$
$$D(3,0.3) = [2.7,3.3], \quad D(-3,0.25) = [-3.25,-2.75].$$

The actual eigenvalues are $4.0198, -3.00433, 2.99365$, and -1.00911, which indeed lies within our estimated invervals.

Note to self: From this it seems like the eigenvalue can be better estimated by $a_{ii} + \sum_{i \neq j} a_{ij}$, so the disk interval can be decreased by half. For instance, $D(4,0.4) = [4,4.4]$ since $0.2 - 0.1 + 0.1 = 0.1 =$ positive, so we can exclude the interval [3.6, 4].

□

2 Statistics

Problem 2.1

Consider a multiple linear regression where $n > p$ and $rank(\mathbf{X}) = p$. Let

$$\hat{\sigma}^2 = \frac{1}{n-p} \sum_{i=1}^{n} e_i^2$$

where $\mathbf{e} = (e_1, ..., e_n)^t = \mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}$ are the regression residuals and $\hat{\boldsymbol{\beta}}$ is the best linear unbiased estimator of $\boldsymbol{\beta}$. Show that $\hat{\sigma}^2$ is an unibased estimator of σ^2.

Proof. We have

$$\hat{\sigma^2} = \frac{1}{n-p} \sum_{i=1}^{n} e_i^2 = \frac{1}{n-p} (\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}}).$$

Also, $\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} = \mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} = \mathbf{H}\mathbf{y}$. Repeatedly applying cyclic permuation and linearity of trace operator, we have

$$\begin{aligned}
&\mathrm{E}\left((\mathbf{y} - \mathbf{H}\mathbf{y})^T(\mathbf{y} - \mathbf{H}\mathbf{y})\right) = \mathrm{E}(\mathbf{y}^T(\mathbf{I} - \mathbf{H})(\mathbf{I} - \mathbf{H})\mathbf{y}) = \mathrm{E}(\mathbf{y}^T(\mathbf{I} - \mathbf{H})\mathbf{y}) \\
&= \mathrm{tr}\left(\mathrm{E}(\mathbf{y}^T(\mathbf{I} - \mathbf{H})\mathbf{y})\right) = \mathrm{E}\left(\mathrm{tr}((\mathbf{X}\boldsymbol{\beta} + \varepsilon)^T(\mathbf{I} - \mathbf{H})(\mathbf{X}\boldsymbol{\beta} + \varepsilon))\right) \\
&= \mathrm{E}\left(\mathrm{tr}((\mathbf{X}\boldsymbol{\beta} + \varepsilon)^T(\mathbf{X}\boldsymbol{\beta} + \varepsilon - \mathbf{H}\mathbf{X}\boldsymbol{\beta} - \mathbf{H}\varepsilon))\right) = \mathrm{E}\left(\mathrm{tr}((\mathbf{X}\boldsymbol{\beta} + \varepsilon)^T(\mathbf{I} - \mathbf{H})\varepsilon)\right) \\
&= \mathrm{E}\left(\mathrm{tr}(\varepsilon^T(\mathbf{I} - \mathbf{H})\varepsilon)\right) = \mathrm{tr}\left((\mathbf{I} - \mathbf{H})\,\mathrm{E}(\varepsilon\varepsilon^T)\right) = \mathrm{tr}\left((\mathbf{I} - \mathbf{H})\,\mathrm{Var}(\varepsilon)\right) \\
&= \sigma^2\,\mathrm{tr}(\mathbf{I} - \mathbf{H}) = \sigma^2\left(\mathrm{tr}(\mathbf{I}_{n\times n}) - \mathrm{tr}(\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T)\right) = \sigma^2(n - p).
\end{aligned}$$

□

Problem 2.2

Show that sample mean and sample variance are 2 independent statistics.

3 Useful Tricks and Identities

Problem 3.1 [Dobson and Barnett, 2008, Chapter 3.4]

Let $\mathbf{X} \in \mathbb{R}^{n\times p}$, $\lambda_i \in \mathbb{R}$, and $\mathbf{x}_i^T \in \mathbb{R}^p$ be a row of $\mathbf{X}$. Show that

$$\sum_{i=1}^{n} \lambda_i \mathbf{x}_i \mathbf{x}_i^T = \mathbf{X}^T \begin{bmatrix} \lambda_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_n \end{bmatrix} \mathbf{X}$$

Proof. This is a definition problem. By definition we have

$$\mathbf{X}^T\mathbf{X} = \begin{bmatrix} | & & | \\ \mathbf{x}_1 & — & \mathbf{x}_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & \mathbf{x}_1^T & — \\ & | & \\ — & \mathbf{x}_n^T & — \end{bmatrix} \equiv \begin{bmatrix} c_{11} & \cdots & c_{ij} \\ \vdots & & \vdots \\ & & c_{nn} \end{bmatrix}$$

Therefore $c_{11} = x_{11}x_{11} + x_{21}x_{21} + \ldots + x_{n1}x_{n1}$. Similarly,

$$\sum_{i=1}^{n} \mathbf{x}_i \mathbf{x}_i^T = \begin{bmatrix} d_{11} & \cdots & d_{ij} \\ \vdots & & \vdots \\ & & d_{nn} \end{bmatrix} \iff d_{11} = \left(\mathbf{x}_1\mathbf{x}_1^T\right)_{11} + \left(\mathbf{x}_2\mathbf{x}_2^T\right)_{11} \ldots + \left(\mathbf{x}_n\mathbf{x}_n^T\right)_{11} = c_{11}.$$

Therefore the entries match up judiciously. □

Problem 3.2 Exact 2nd order Taylor's expansion

Suppose $f \in C^2(\mathbb{R})$. Show that there exists $y \in (x_0, x)$ such that:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(y)(x - x_0)^2.$$

This motivates the quadratic upper bound principle, which is used ubiquitously in MM algorithms.

Proof. Applying fundamental theorem of calculus twice, we have

$$\begin{aligned} f(x) &= f(x_0) + \int_{x_0}^{x} f'(x_1)dx_1 \\ &= f(x_0) + \int_{x_0}^{x} \left(f'(x_0) + \int_{x_0}^{x_1} f''(x_2)dx_2 \right) dx_1 \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} \int_{x_0}^{x_1} f''(x_2)dx_2dx_1. \end{aligned}$$

By mean value theorem, there exists $y \in (x_0, x_1)$ such that $\int_{x_0}^{x_1} f''(x_2)dx_2 = f''(y)(x_1 - x_0)$. Thus

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^{x} f''(y)(x_1 - x_0)dx_1 \\ &= f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(y)(x - x_0). \end{aligned}$$

□

Problem 3.3 Clever use of Cauchy-Schwarz [Lange, 2016, Exercise 1.4.18]

Prove the majorization

$$\begin{aligned} (x+y-z)^2 \le -(x_n + y_n - z_n)^2 + 2(x_n + y_n - z_n)(x+y-z) \\ +3[(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2] \end{aligned}$$

which separtes the variables x, y, and z. In examples 1.3.6 and 1.3.7 this would facilitate penalizing parameter curvature rather than changes in parameter values.

Proof. First, move the first two terms on the right to the left:

$$(x+y-z)^2 - 2(x_n + y_n - z_n)(x+y-z) + (x_n + y_n - z_n)^2 \le 3[(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2]$$

The left can be factored cleanly as

$$\big((x+y-z) - (x_n + y_n - z_n)\big)^2 \le 3[(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2]$$

$$\iff (a+b+c)^2 \le 3a^2 + 3b^2 + 3c^2$$

where $a = x - x_n, b = y - y_n, c = z - z_n$. Now define $v = (1,1,1), u = (a,b,c)$. By Cauchy-Schwarz, we obtain the desired result:

$$(a+b+c)^2 \le 3(a^2 + b^2 + c^2).$$

□

4 Real worl Application Problems

Problem 4.1

Suppose we have a huge number of samples and small number of covariate (large n small p) problem and we wish to fit a linear regression. Furthremore, every day millions of new sample points are generated, i.e. $\hat{\boldsymbol{\beta}}$ must be updated continuously whenever new data arrives. How would one obtain $\hat{\boldsymbol{\beta}}$ without saving larger and larger matrices?

Proof. Let $\mathbf{y}_i$ and $\mathbf{X}_i$ denote the samples and corresponding data of day i. Then up to day n, the concatenated full design matrix $\mathbf{X}$ and full sample vector $\mathbf{y}$ is

$$[\mathbf{X}\mathbf{y}] = \begin{bmatrix} [\mathbf{X}_1\mathbf{y}_1] \\ \vdots \\ [\mathbf{X}_n\mathbf{y}_n] \end{bmatrix}.$$

Of course we do not want to store this entire matrix because it gets bigger each day. Fortunately, the gram matrix of $[\mathbf{X}\mathbf{y}]$ is a small $p \times p$ matrix and can be readily computed:

$$\begin{aligned} [\mathbf{X}\mathbf{y}]^t[\mathbf{X}\mathbf{y}] &= [\mathbf{X}_1\mathbf{y}_1]^t[\mathbf{X}_1\mathbf{y}_1] + \ldots + [\mathbf{X}_n\mathbf{y}_n]^t[\mathbf{X}_n\mathbf{y}_n] \\ &= \begin{bmatrix} \mathbf{X}_1^t\mathbf{X}_1 & \mathbf{X}_1^t\mathbf{y} \\ \mathbf{y}_1^t\mathbf{X}_1 & \mathbf{y}_1^t\mathbf{y}_1 \end{bmatrix} + \ldots + \begin{bmatrix} \mathbf{X}_n^t\mathbf{X}_1 & \mathbf{X}_n^t\mathbf{y} \\ \mathbf{y}_n^t\mathbf{X}_1 & \mathbf{y}_n^t\mathbf{y}_1 \end{bmatrix}. \end{aligned}$$

By property of the sweep operator, we know that sweeping on this full gram matrix have the property:

$$\begin{aligned} \text{sweep}\left([\mathbf{X}\mathbf{y}]^t[\mathbf{X}\mathbf{y}]\right) &= \begin{bmatrix} -(\mathbf{X}^t\mathbf{X})^{-1} & (\mathbf{X}^t\mathbf{X})\mathbf{X}^t\mathbf{y} \\ \mathbf{y}^t\mathbf{X}(\mathbf{X}^t\mathbf{X}) & \mathbf{y}^t\mathbf{y} - \mathbf{y}^t\mathbf{X}(\mathbf{X}^t\mathbf{X})^y\mathbf{X}^t\mathbf{y} \end{bmatrix} \\ &= \begin{bmatrix} -\sigma^{-2}\text{Cov}(\hat{\beta}) & \hat{\beta} \\ \hat{\beta}^t & ||\mathbf{y} - \hat{\mathbf{y}}||_2^2 \end{bmatrix} \end{aligned}$$

Therefore, we store the *sum* of all preceeding days of data in the form of a gram matrix. When new data arrives, we add the new data's gram matrix to the previous sum and sweep until the 2nd to last entry. Then the fitted model $\hat{\beta}$ will be on the top right column. Since $n \gg p$, the gram matrix is small and thus easy to store. □

Problem 4.2 Modeling count data [Dobson and Barnett, 2008, 3.5.b]

To model count data, one can choose among Poisson, Negative Binomial, and Binomial distributions. Given a set of observations y_i and assuming a common rate parameter, how would one decide which of these distribution are more appropriate?

Proof. The 3 different models under consideration are:

$$
\begin{aligned}
y_i &\sim \text{Poisson}(\lambda_i) \\
y_i &\sim \text{NegBin}(r, p) \\
y_i &\sim \text{Binomial}(n, p).
\end{aligned}
$$

The simplest way is to use the relationship between mean and variance of $\mathbf{y}$. For Poisson, $\mathrm{E}(by) = \mathrm{Var}(by)$. For negative binomial, $\mathrm{Var}(\mathbf{y}) > \mathrm{E}(\mathbf{y})$. And for Binomial, $\mathrm{E}(\mathbf{y}) > \mathrm{Var}(\mathbf{y})$.

$\square$

References

[Dobson and Barnett, 2008] Dobson, A. J. and Barnett, A. G. (2008). *An introduction to generalized linear models*. Chapman and Hall/CRC.

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[Lange, 2016] Lange, K. (2016). *MM optimization algorithms*, volume 147. SIAM.