Random Graph theory

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Most materials from this note is taken from [1, 2]

1 Erdos-Renyi Graph Model

- We use $G(n, p)$ to denote an undirected (Erdos-Renyi) graph with n nodes.
- An edge is formed between 2 nodes with probability $p \in (0, 1)$ **independently** of other edges.
- A graph is **connected** when there is a path between every pair of vertices.

When $p = p(n)$ is a function of n, we may be interested in the behavior of $G(n, p(n))$ as $n \to \infty$.

1.1 Warm-up

Q1. What is the probability that a vertex is isolated in $G(n, p)$? Ans: A given node i cannot form an edge with each of the remaining $n-1$ nodes. Thus the probability is $(1-p)^{n-1}$.

Q2. What is the expected number of edges in $G(n, p)$? The total number of edges in a graph is $\binom{n}{2}$, and each of these edges form with probability p. So we expect $p\binom{n}{2}$ edges overall.

2 Sharp Threshold for Connectivity

Theorem 2.1 Erdos-Renyi 1961

Consider a graph $G \sim G(n, p(n))$ where $p(n) = \lambda \frac{\ln(n)}{n}$. Then as $n \to \infty$,

$$\begin{cases} P(G \text{ connected}) \to 0 & \text{if } \lambda < 1 \\ P(G \text{ connected}) \to 1 & \text{if } \lambda > 1 \end{cases}$$

References

[1] Acemoglu, D. and Ozdaglar, A. (2009). Lecture 3: Erdos-Renyi graphs and Branching Processes.

[2] Ramchandran, K. (2009). Random Graphs.