Computation of Mean First Passage Times in a Markov Chain

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Introduction

Let $\boldsymbol{P} = (p_{ij})$ be the transition probability matrix of a finite state Markov chain. The first passage time T_{ij} is the number number of steps it takes to reach j starting from i. We will assume all $p_{ii} = 0$ and set $T_{ii} = 0$ in contrast to the standard first passage time formulation. Let $m_{ij} = \mathrm{E}(T_{ij})$ be the expected value of T_{ij} and collect the m_{ij} into a matrix $\boldsymbol{M}$. One can calculate the m_{ij} recursively via the system of equations

$$m_{ij} \quad = \quad 1 + \sum_k p_{ik} m_{kj}$$

for all $j \neq i$. It is natural to solve this equation iteratively via the recurrence

$$\begin{aligned} \boldsymbol{M}^{(n+1)} &= \boldsymbol{P}\boldsymbol{M}^{(n)} + \mathbf{1} \\ \mathrm{diag}(\boldsymbol{M}^{(n+1)}) &= \mathbf{0} \end{aligned}$$

starting from $\boldsymbol{M}^{(0)} = \mathbf{0}$. This recurrence has several advantages: (a) it exploits fast matrix times matrix multiplication, (b) it correctly maintains the diagonal entries, (c) it is monotonic in the sense that $\boldsymbol{M}^{(n+1)} \geq \boldsymbol{M}^{(n)}$ for all n, and (d) it converges to the minimal solution of the equations. Monotonicity follows by induction from the form of the updates and the obvious condition $\boldsymbol{M}^{(1)} \geq \boldsymbol{M}^{(0)}$. It is also clear by induction that if $\boldsymbol{M}$ solves the system of equations, then $\boldsymbol{M} \geq \boldsymbol{M}^{(n)}$ for all n. In view of this bound, the monotonic sequence $\boldsymbol{M}^{(n)}$ converges to a limit $\boldsymbol{M}^{(\infty)} \leq \boldsymbol{M}$.

Proposition 0.1. *The iterate* $m_{ij}^{(n)}$ *equals* $\mathrm{E}(T_{ij} 1_{\{T_{ij} \leq n\}})$ *and converges to* $\mathrm{E}(T_{ij})$ *whenever the latter is finite.*

Proof. This identification is clearly true for $n = 0$. Suppose $n > 0$ and T'_{kj} is a probabilistic replicate of T_{kj}. If X_1 denotes the state of the chain after one step starting at $X_0 = i$, then

$$\begin{aligned}
m_{ij}^{(n+1)} &= \mathrm{E}(T_{ij} 1_{\{T_{ij} \le n+1\}}) \\
&= \sum_k p_{ik} \, \mathrm{E}(T_{ij} 1_{\{T_{ij} \le n+1\}} \mid X_1 = k) \\
&= \sum_k p_{ik} \, \mathrm{E}\left[(T'_{kj} + 1) 1_{\{T'_{kj} + 1 \le n+1\}}\right] \\
&= 1 + \sum_k p_{ik} \, \mathrm{E}(T'_{kj} 1_{\{T'_{kj} \le n\}}) \\
&= 1 + \sum_k p_{ik} m_{kj}^{(n)}
\end{aligned}$$

proves our first claim. The monotone convergence theorem implies that $m_{ij}^{(n)}$ converges to $\mathrm{E}(T_{ij})$ given the later is finite. $\square$

References

[1] Lange, K (2010) *Applied Probability*, 2nd ed. Springer