Exercise 1

Let us consider the following Navier-Stokes problem in the unit square $\Omega = (0,1)^2$:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \quad \forall t \in (0,T),\\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \quad \forall t \in (0,T),\\ \mathbf{u} = \mathbf{u}_{ex}, & \text{on } \partial\Omega, \quad \forall t \in (0,T),\\ \mathbf{u} = \mathbf{0}, & \text{in } \Omega, \quad \text{for } t = 0, \end{cases}$$

where the forcing term $\mathbf{f}$ is given by

$$\mathbf{f}(\mathbf{x},t) = \begin{bmatrix} -2\cos x \sin y \left(\cos(2t) + \dfrac{1}{\mathrm{Re}}\sin(2t)\right) \\ 2\sin x\cos y\left(\cos(2t) + \dfrac{1}{\mathrm{Re}}\sin(2t)\right)\end{bmatrix},$$

and the exact solution $(\mathbf{u}_{\mathrm{ex}}, p_{\mathrm{ex}})$ is defined by

$$\mathbf{u}_{\mathrm{ex}}(\mathbf{x},t) = \begin{bmatrix} -\cos x \sin y \sin(2t) \\ \sin x \cos y \sin(2t)\end{bmatrix}, \qquad p_{\mathrm{ex}}(\mathbf{x},t) = -\frac{\cos(2x)+\cos(2y)}{4}\sin^2(2t).$$

1. Introduce the backward Euler method to discretize the problem in time, with a semi-implicit treatment of the nonlinear term.
   Introducing also suitable finite element spaces, derive the fully discrete problem associated to a timestep advancement.

2. Rewrite the fully discrete problem introduced at point 1 using the penalty method to impose Dirichlet boundary conditions and derive its algebraic formulation.

3. For the problem obtained at point 2, write a resolution algorithm based of the pressure matrix method, preconditioned with the Cahouet-Chabard preconditioner

$$P_{\mathrm{CC}}^{-1} = \frac{1}{\mathrm{Re}}M_p^{-1} + \frac{1}{\Delta t}K_p^{-1}, \qquad \text{where } (K_p)_{lk} = \int_\Omega \nabla\psi_k \cdot \nabla\psi_l \, d\mathbf{x}.$$

   Starting from the given template file, implement the solver described at the previous point and solve the problem with $\mathrm{Re} = 10, T = 0.3, h = 0.1, \Delta t = 0.1$.

Exercise 2

1. Consider different timesteps $\Delta t = 1, 1/2, 1/4, 1/8$. For each value of Δt, compute the error for the velocity and the pressure using the norm of the spaces $L^\infty(0,1,H^1(\Omega))$ and $L^\infty(0,1,L^2(\Omega))$, respectively. What can you conclude on the convergence rate of the method?

2. Repeat Exercise 1 using the following time-advancing schemes:
   - two-step BDF semi-implicit scheme;

- two-step BDF explicit scheme.

In both cases, use a step of the backward Euler method to initialize the solver, and a second-order linear extrapolation

$$\mathbf{u}^{n+1} \simeq 2\mathbf{u}^n - \mathbf{u}^{n-1}$$

to approximate the advection field.