Dirichlet BC in Firedrake

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Computational Fluid Dynamics
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The diffusion-reaction problem

Dirichlet boundary conditions

$$\begin{cases} -\Delta u + u = 0, & \text{in } \Omega := (0,1)^2, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

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Find $u = \check{u} + \mathcal{R}g \in V_\mathrm{D}$ with $\check{u} \in V_0$ s.t.

$$a(\check{u}, v) = F(v) - a(\mathcal{R}g, v) \quad \forall v \in V_0$$

The diffusion-reaction problem

Galerkin projection - Finite Element Method

mesh $\mathcal{T}_h$ of Ω

$X_h^p(\Omega) = \{u \in C^0(\overline{\Omega})\colon\, u|_K \in \mathbb{P}^p(K) \quad \forall K \in \mathcal{T}_h\}$

interpolator $\Pi_h^\partial : H^{1/2}(\Omega) \to X_h^p(\partial\Omega)$

<u>discrete</u> lifting operator $\mathcal{R}_h : H^{1/2}(\partial\Omega) \to V_h = X_h^p(\Omega)$

such that $\mathcal{R}_h g = \Pi_h^\partial g$ on $\partial\Omega$

discrete space $V_{0,h} = V_0 \cap X_h^p$

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(FEM): Find $u_h = \check{u}_h + \mathcal{R}_h g$ with $\check{u}_h \in V_{0,h}$ s.t.

$$a(\check{u}_h, v_h) = F(v_h) - a(\mathcal{R}_h g, v_h) \quad \forall v_h \in V_{0,h}$$

We then focus on finding $\check{u}_h \in V_{0,h}$!!

Algebraic formulation of FEM

$\{\varphi_i\}_{i=1}^{N_h}$ basis of the finite element space V_h.
Assume $\{\varphi_i\}_{i=1}^{N_h^\circ}$ with $N_h^\circ \leq N_h$ is a basis for $V_{0,h}$.
Then, (FEM) can be rewritten as:
Find $\check{u}_h = \sum_{j=1}^{N_h^\circ} u_j \varphi_j \in V_{0,h}$ such that

$$\sum_{j=1}^{N_h^\circ} u_j\, a(\varphi_j, \varphi_i) = F(\varphi_i) - a(\mathcal{R}_h g, \varphi_i)$$
$$\forall i = 1, \ldots, N_h^\circ$$

$$\Leftrightarrow \qquad A^\circ \check{\mathbf{U}} = \mathbf{b}^\circ$$

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... what about the remaining $N_h - N_h^\circ$ dofs?

Algebraic formulation of FEM

Discrete lifting

$$\mathcal{R}_h g \in V_h, \qquad \mathcal{R}_h g = \Pi^p_\partial g \text{ on } \partial\Omega$$

$\Rightarrow$ one can choose it in the form

$$\mathcal{R}_h g = \sum_{j=N_h^\circ+1}^{N_h} g_j \varphi_j$$

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$$u_h = \check{u}_h + \mathcal{R}_h g \quad \leftrightarrow \quad \mathbf{U} = \begin{bmatrix} \check{\mathbf{U}} \\ \mathbf{G} \end{bmatrix}$$

$$\text{(FEM) } A\mathbf{U} = \mathbf{b} \quad \leftrightarrow \quad \begin{bmatrix} A^\circ & 0 \\ 0 & I \end{bmatrix} \mathbf{U} = \begin{bmatrix} \mathbf{b}^\circ \\ \mathbf{G} \end{bmatrix}$$

Algebraic formulation of FEM

Firedrake code

```
bc = DirichletBC(V, g, [1,2,3,4])

A = assemble(a, bcs=list(bc))
rhs = assemble(L)

solver = LinearSolver(A,
            solver_parameters={...})
# NB: bc and g are stored in A

u_h = Function(V)
solver.solve(u_h, rhs)
```