Exercise 1

Let us consider the Stokes problem defined in the rectangular domain $\Omega = (0,3) \times (0,1)$

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_{\mathrm{D1}} = \{0 \le x \le 3,\ y = 0\}, \\ \mathbf{u} = 1\mathbf{i} + 0\mathbf{j}, & \text{on } \Gamma_{\mathrm{D2}} = \{0 \le x \le 3,\ y = 1\}, \\ (\nabla \mathbf{u} - pI)\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\mathrm{N}} = \{x \in \{0,3\},\ 0 \le y \le 1\}. \end{cases}$$

1. Verify that $\mathbf{u} = y\mathbf{i}$, $p = 0$ is a solution of the problem.
2. Write the weak formulation of the differential problem.
3. Using the variational approach of Firedrake, write a solver for the problem, with $h = 0.1$ as mesh discretization parameter and $\mathbb{P}^1/\mathbb{P}^1$ finite elements.
4. Compute now the solution using the following pairs of finite element spaces

   $$\mathbb{P}^1/\mathbb{P}^0, \qquad \mathbb{P}^1/\mathbb{P}^1, \qquad \mathbb{P}^1_b/\mathbb{P}^1, \qquad \mathbb{P}^2/\mathbb{P}^1.$$

   Which of these pairs produce spurious pressure modes?
5. Use the results of the previous point to fill the following table.

	N_{dofs}	$\|\mathbf{u} - u_h\|_{L^2}$	$\|\nabla \mathbf{u} - \nabla u_h\|_{L^2}$	$\|p - p_h\|_{L^2}$
$\mathbb{P}^1/\mathbb{P}^0$				
$\mathbb{P}^1/\mathbb{P}^1$				
$\mathbb{P}^1_b/\mathbb{P}^1$				
$\mathbb{P}^2/\mathbb{P}^1$				

   What can you observe?

Exercise 2

Let us consider the Stokes problem defined in the unit square $\Omega := (0,1)^2$

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}, & \text{on } \partial\Omega. \end{cases}$$

1. Compute the functions $\mathbf{f}$ and $\mathbf{g}$ such that

   $$\mathbf{u}(x,y) = \begin{bmatrix} -\cos x \sin y \\ \sin x \cos y \end{bmatrix}, \qquad p(x,y) = -\frac{1}{4}(\cos(2x) + \cos(2y)) + \frac{\sin(2)}{4}.$$

2. Write the weak formulation of the differential problem.

3. Using the pairs of finite elements spaces $\mathbb{P}^2/\mathbb{P}^1$ and $\mathbb{P}^1_b/\mathbb{P}^1$ solve the problem with a uniform mesh with an increasing number of subdivisions n. For each pair of spaces fill the following table and verify numerically the convergence rates.

n	N_{dofs}	$\|\mathbf{u}-\mathbf{u}_h\|_{L^2}$	$\|\nabla\mathbf{u}-\nabla\mathbf{u}_h\|_{L^2}$	$\|p-p_h\|_{L^2}$
5				
10				
20				
40				

4. Let us introduce the following adjusted pressure:

$$\widehat{p}_h(x,y) = p_h(x,y) + \frac{1}{|\Omega|}\int_\Omega (p - p_h)d\mathbf{x}.$$

Repeat the convergence tests of the previous point, considering $\|p-\widehat{p}_h\|_{L^2}$ instead of $\|p-p_h\|_{L^2}$. What can you observe? Can you tell why?

Exercise 3

Let us consider the following Stokes problem over the domain $\Omega = (0,2)\times(-1,1)$:

$$\begin{cases} -\mu\Delta\mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega,\\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega,\\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_{\text{wall}} = \{0\le x\le 2,\ y\in\{-1,1\}\},\\ \mathbf{u} = (1-y^2)\mathbf{i} + 0\mathbf{j}, & \text{on } \Gamma_{\text{in}} = \{x=0,\ -1\le y\le 1\},\\ (\nabla\mathbf{u} - pI)\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{out}} = \{x=2,\ -1\le y\le 1\}, \end{cases}$$

with $\mu = 10^{-2}$.

1. Write the weak formulation of the differential problem, strongly enforcing Dirichlet boundary conditions.

2. Introduce the finite element method for the problem at hand, using $\mathbb{P}^2$ elements for velocity and $\mathbb{P}^1$ elements for pressure, and derive its algebraic formulation.

3. Assemble the algebraic problem in FreeFem++, using `LinearVariationalProblem`. Recall that the monolithic system has the following form:

$$\Sigma\mathbf{x} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \mathbf{P}\end{bmatrix} = \begin{bmatrix}\mathbf{F}\\ \mathbf{0}\end{bmatrix} = \mathbf{b}.$$

where A is the stiffness matrix, B and B^T correspond to the divergence and gradient operators, respectively, and $\mathbf{F}$ contains the boundary data.

4. Solve the algebraic problem assembled at the previous point using `LinearVariationalSolver`: use the iterative solver **gmres** on the system, with a block-Jacobi preconditioner (`'pc_type': 'bjacobi'`) combined with an inexact LU solver for the single blocks (`'sub_pc_type': 'ilu'`). Consider a relative tolerance of 10^{-5} and a maximum number of iterations 10000. Compare the cases with mesh size $h = 1/8, 1/16, 1/32$ in terms of number of iterations required by the GMRES solver: what can you observe?

5 (additional) Repeat the exercise for different values of μ: what can you observe in the solution?