Exercise 1

Let us consider the stationary Navier-Stokes problem defined in the unit square $\Omega := [0,1]^2$

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 1\mathbf{i} + 0\mathbf{j}, & \text{on } \Gamma^{\mathrm{up}} = \{0 \le x \le 1,\, y = 1\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega\backslash\Gamma^{\mathrm{up}}. \end{cases} \tag{1}$$

1. Using the stable pair of spaces $\mathbb{P}^2/\mathbb{P}^1$, solve the problem using the Newton method. Each nonlinear iteration should be solved using the `UMFPACK` solver applied to the monolithic system, and the stop criterion should be based on the relative increment. Solve the problem with $\mathrm{Re} = 30$. Does the solver converge to a proper solution?

2. Repeat the previous point using the mean-null projection of the pressure, defined as

$$\bar{p} = p - \int_\Omega p\, d\mathbf{x}.$$

   Do you notice any difference?

3. Consider now different values of the Reynolds number: $\mathrm{Re} = 3, 30, 300, 3000$. Comment on the differences in the results.

4. If the pair of spaces $\mathbb{P}^1/\mathbb{P}^1$ is considered, then the SUPG-stabilized version of the linearized problem can be written as follows:

$$\begin{aligned} a(\mathbf{u}_k,\mathbf{v}) + c(\mathbf{u}_k,\mathbf{u}_{k-1},\mathbf{v}) + c(\mathbf{u}_{k-1},\mathbf{u}_k,\mathbf{v}) + b(\mathbf{v},p) - b(\mathbf{u},q) + \\ + \sum_{K\in\mathcal{T}_h} \delta_K((\mathbf{u}_k\cdot\nabla)\mathbf{u}_{k-1} + (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u}_k + \nabla p, (\mathbf{v}\cdot\nabla)\mathbf{u}_{k-1} + (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{v} + \nabla q)_K + \\ + \sum_{K\in\mathcal{T}_h} \delta_K(\operatorname{div}\mathbf{u}_k, \operatorname{div}\mathbf{v})_K = \\ = c(\mathbf{u}_{k-1},\mathbf{u}_{k-1},\mathbf{v}) + \sum_{K\in\mathcal{T}_h} \delta_K((\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u}_{k-1}, (\mathbf{v}\cdot\nabla)\mathbf{u}_{k-1} + (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{v} + \nabla q)_K, \end{aligned}$$

   where $(\cdot,\cdot)_K$ denotes the L^2 inner product over the element K. The stabilization parameter should depend on the local Reynolds number Re_K: [1]

$$\delta_K = \begin{cases} \delta\frac{h_K}{\overline{u}_K}, & \text{for } \mathrm{Re}_K \ge 1, \\ \delta\, h_K^2\, \mathrm{Re}, & \text{for } \mathrm{Re}_K < 1, \end{cases}$$

   where δ is a given scalar and

$$\mathrm{Re}_K = \frac{\overline{u}_K h_k}{\nu}, \qquad \overline{u}_K = \frac{1}{|K|}\int_K |\mathbf{u}_k|\, d\mathbf{x},$$

   or, for brevity, $\delta_K = \delta\frac{h_K}{\overline{u}_K}\min\{1, \mathrm{Re}_K\}$. Implement this stabilization, with $\delta = 1$ and compare the results with the ones obtained in the previous points.
   Beware that $|\mathbf{u}|$ can reach very small values, thus the computation of δ_K should be carefully dealt with.

[1] Notice that, for the problem at hand, $\nu = \frac{1}{\mathrm{Re}}$.

Exercise 2

For the same problem of the previous exercise:

1. Using the stable pair of spaces $\mathbb{P}^2/\mathbb{P}^1$, solve the problem using the fixed-point iteration method. Use the skew-symmetric version of the trilinear form associated to advection. Each nonlinear iteration should be solved using the `UMFPACK` solver applied to the monolithic system, and the stop criterion should be based on the residual. Try to solve the problem with different values of Reynolds number ($\mathrm{Re} = 3, 30, 300, 3000$). Does the solver converge to a proper solution?

2. Introduce the SUPG stabilization for the solution of the linearized problem.

   The stabilization parameter should depend on the local Reynolds number Re_K: [2]

   $$\delta_K = \begin{cases} \delta \frac{h_K}{\overline{u}_K}, & \text{for } \mathrm{Re}_K \geq 1, \\ \delta\, h_K^2 \,\mathrm{Re}, & \text{for } \mathrm{Re}_K < 1, \end{cases}$$

   where δ is a given scalar and

   $$\mathrm{Re}_K = \frac{\overline{u}_K h_k}{\nu}, \qquad\qquad \overline{u}_K = \frac{1}{|K|}\int_K |\mathbf{u}_k|\, d\mathbf{x},$$

   or, for brevity, $\delta_K = \delta \frac{h_K}{\overline{u}_K} \min\{1, \mathrm{Re}_K\}$.

   Compare the solutions of the previous point with the solutions obtained with the SUPG stabilization, with $\delta = 0.25$.

3. Repeat the exercise using the pair of spaces $\mathbb{P}^1/\mathbb{P}^1$ and compare the SUPG stabilization with the Brezzi-Pitkäranta stabilization.

Exercise 3

We want to compare different choices for the diffusion term and the boundary conditions in the following problem: $\Omega := [0, 3] \times [0, 1]$

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{\mathrm{Re}}\operatorname{div} G(\mathbf{u}) + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 6y(1-y)\mathbf{i}, & \text{on } \Gamma_{\text{left}} = \{x = 0,\, 0 \leq y \leq 1\}, \\ \frac{1}{\mathrm{Re}} G(\mathbf{u})\mathbf{n} - p\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{right}} = \{x = 3,\, 0 \leq y \leq 1\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_{\text{bottom}} = \{0 \leq x \leq 3,\, y = 0\}, \\ \star, & \text{on } \Gamma_{\text{top}} = \{0 \leq x \leq 3,\, y = 1\}, \end{cases} \tag{2}$$

where the possible choices for the operator G and the boundary condition $\star$ are:

[2] Notice that, for the problem at hand, $\nu = \frac{1}{\mathrm{Re}}$.

$$G(\mathbf{u}) = \nabla \mathbf{u} \quad \text{vs.} \qquad G(\mathbf{u}) = \varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$$

$$\begin{array}{ccc} \star : \text{ no-slip} & \text{vs.} & \star : \text{ free slip} \\ \mathbf{u} = \mathbf{0} & & \mathbf{u} \cdot \mathbf{n} = 0, \quad [\frac{1}{\mathrm{Re}} G(\mathbf{u}) - pI]\mathbf{n} \cdot \mathbf{t} = 0 \end{array}$$

1. For each combined choice of G and $\star$, derive the weak form of the problem.

2. Modify the code of the previous lab session (`lab4ex1.edp`) to solve problem (2) and compare the results outcoming from the different choices. How would you model the flow between two infinite, rough planes?