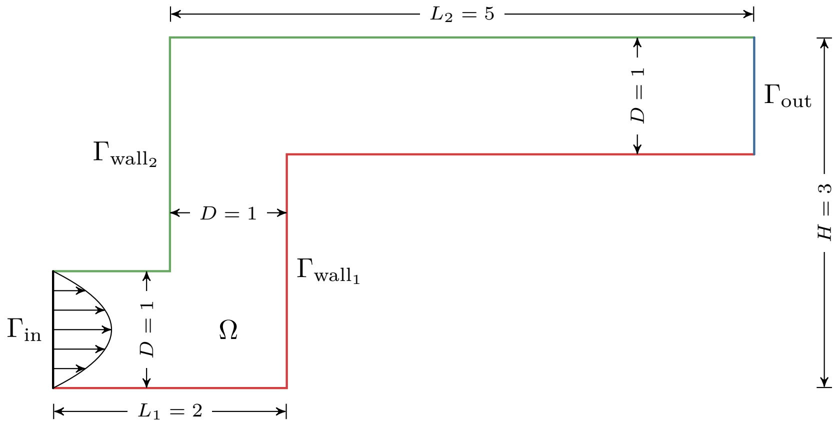

Figure 1: Domain Ω for Exercise 1.

Exercise 1

Let us consider the stationary Navier-Stokes problem defined in the domain Ω as in Figure 1 (defined in `elbow3.msh` and `elbow3bis.msh`):

$$\begin{cases} (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \end{cases}$$

where the kinematic viscosity ν is equal to $2 \cdot 10^{-2}$ and the boundary conditions are:

- a parabolic profile of velocity $\mathbf{u}_{\text{in}} = 4y(1-y)\mathbf{i}$ on the inlet boundary Γ_{in};
- stress-free conditions on the outlet boundary Γ_{out}.
- one of the following on the lateral walls $\Gamma_{\text{wall}_1} \cup \Gamma_{\text{wall}_2}$:
  - (i) no-slip conditions: $\mathbf{u} = \mathbf{0}$;
  - (ii) free-slip conditions: $\mathbf{u} \cdot \mathbf{n} = 0, \quad (\nu \nabla \mathbf{u} - pI)\mathbf{n} \cdot \mathbf{t} = 0$, where $\mathbf{t} \perp \mathbf{n}$.

1. Derive the weak formulations of the <u>nonlinear</u> problems corresponding to both case (i) and (ii), using a strong imposition of Dirichlet conditions.

2. Consider the no-slip case (i).
Starting from the provided template, implement the problem in Firedrake, using its residual-based <u>nonlinear solver</u> and export to Paraview the velocity $\mathbf{u}$, pressure p and stream function ψ (defined as in Lab 6).

3. Modify the code to solve case (ii) and compare the results in Paraview: what can you observe?

4. Compute the following quantities and compare cases (i) and (ii) in terms of them:
   - drag force exerted by the fluid onto the lower downstream wall $\Gamma = \{x \in (2, 6), y = 2\}$

$$F_{\text{D}} = -\int_{\Gamma} (\nu \nabla \mathbf{u} - pI)\mathbf{n} \cdot \mathbf{i} \, ds;$$

   - total pressure jump

$$\Delta p = p_{\text{in}} - p_{\text{out}} = \frac{1}{|\Gamma_{\text{in}}|} \int_{\Gamma_{\text{in}}} p \, ds - \frac{1}{|\Gamma_{\text{out}}|} \int_{\Gamma_{\text{out}}} p \, ds.$$

Exercise 2 - homework

Consider the following time-dependent, nonlinear Navier-Stokes problem:

$$
\begin{cases}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{\mathrm{Re}}\Delta\mathbf{u} + \nabla p = \mathbf{0} & \text{in } \Omega, t \in (0,T),\\
\operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, t \in (0,T),\\
\mathbf{u} = 1\mathbf{i} & \text{on } \Gamma_{\text{in}} = \{\mathbf{x} \in \partial\Omega, x = -3\}, t \in (0,T),\\
(\nu\nabla\mathbf{u} - pI)\mathbf{n} = \mathbf{0} & \text{on } \Gamma_{\text{out}} = \{\mathbf{x} \in \partial\Omega, x = 12\}, t \in (0,T),\\
\mathbf{u}\cdot\mathbf{n} = 0, \quad (\nu\nabla\mathbf{u} - pI)\mathbf{n}\cdot\mathbf{t} = 0 & \text{on } \Gamma_{\text{wall}} = \{\mathbf{x} \in \partial\Omega, y = -2, 2\}, t \in (0,T),\\
\mathbf{u} = \mathbf{0} & \text{on } \Gamma_{\text{cyl}} = \{\mathbf{x} \in \partial\Omega, (2x)^2 + (2y)^2 = 1\}, t \in (0,T),\\
\mathbf{u} = \mathbf{0} & \text{in } \Omega, t = 0,
\end{cases}
\tag{1}
$$

where the domain Ω is provided in `cylinder-ns.msh`, $T = 10$, $\mathrm{Re} = 10$.

Starting from the code implemented in Exercise 1, and using the hints provided in the template, solve problem (1) by the Implicit Euler method with an implicit treatment of the advection term, using $\Delta t = 0.2$ and the Firedrake's nonlinear solver for each time step.

Warning: running the solution of the problem may take several seconds for each timestep.