Exercise 1

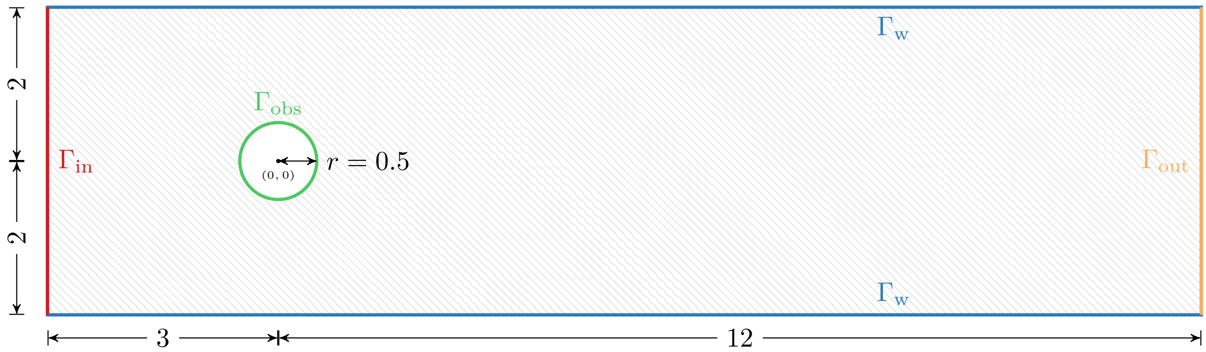
Let us consider the Navier-Stokes problem defined in the domain Ω as in the Figure below

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \dfrac{1}{\mathrm{Re}}\,\mathrm{div}\,(\nabla\mathbf{u}+\nabla\mathbf{u}^T) + \nabla p = \mathbf{0}, & \text{in } \Omega, t\in(0,T),\\ \mathrm{div}\,\mathbf{u} = 0, & \text{in } \Omega, t\in(0,T),\\ \mathbf{u} = \frac{3}{8}(2-y)(2+y)\mathbf{i}, & \text{on } \Gamma_{\mathrm{in}} = \{x=-3,\ -2\le y\le 2\}, t\in(0,T),\\ \left[\dfrac{1}{\mathrm{Re}}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right) - pI\right]\mathbf{n} = h_{\mathrm{out}}\mathbf{n}, & \text{on } \Gamma_{\mathrm{out}} = \{x=12,\ -2\le y\le 2\}, t\in(0,T),\\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega\backslash(\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}}), t\in(0,T),\\ \mathbf{u}|_{t=0} = \mathbf{u}_{\mathrm{Stokes}}, & \text{in } \Omega, t=0, \end{cases}$$

where the Reynolds number Re is equal to 200, boundary conditions are given by:

- no-slip conditions on the lateral boundary of the pipe and on the boundary of the obstacle;
- a parabolic velocity profile on the inlet boundary Γ_{in};
- non-homogeneous Neumann boundary conditions at the outlet boundary Γ_{out}, with an outward-directed normal stress $h_{\mathrm{N}} = 0.5$;

and $\mathbf{u}_{\mathrm{Stokes}}$ is the solution of the corresponding steady Stokes problem.



1 Discretize the problem using the stable pair of spaces $\mathbb{P}^1_b/\mathbb{P}^1$ and solve the problem using the incremental Chorin-Temam projection scheme coupled with the backward Euler semi-implicit scheme.

Given the solution $\mathbf{u}^n, p^n$ at a certain time instant t^n, the scheme to compute $\mathbf{u}^{n+1}, p^{n+1}$ is described by the following steps:

1) Solve the predictor step to obtain a temporary velocity $\tilde{\mathbf{u}}$:

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}-\mathbf{u}^n}{\Delta t} + (\mathbf{u}^n\cdot\nabla)\tilde{\mathbf{u}} - \dfrac{1}{\mathrm{Re}}\,\mathrm{div}\,(\nabla\tilde{\mathbf{u}}+\nabla\tilde{\mathbf{u}}^T) = -\nabla p^n, & \text{in } \Omega,\\ \tilde{\mathbf{u}} = (1-y^2)\mathbf{i}, & \text{on } \Gamma_{\mathrm{in}},\\ \dfrac{1}{\mathrm{Re}}\left(\nabla\tilde{\mathbf{u}}+\nabla\tilde{\mathbf{u}}^T\right)\mathbf{n} = h_{\mathrm{N}}\mathbf{n}, & \text{on } \Gamma_{\mathrm{out}},\\ \tilde{\mathbf{u}} = \mathbf{0}, & \text{on } \partial\Omega\backslash(\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}}). \end{cases}$$

2) Solve the Poisson problem for the pressure increment δp:

$$\begin{cases} -\Delta \delta p = -\dfrac{1}{\Delta t} \operatorname{div} \tilde{\mathbf{u}}, & \text{in } \Omega, \\ \delta p = 0, & \text{on } \Gamma_{\text{out}}, \\ \nabla \delta p \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \backslash \Gamma_{\text{out}}. \end{cases}$$

3) Projection step

$$\mathbf{u}^{n+1} = \tilde{\mathbf{u}} - \Delta t \, \nabla \delta p.$$

4) Pressure update

$$p^{n+1} = p^n + \delta p.$$

2 Compute the drag and lift forces exerted by the fluid on the obstacle at the final time $T = 50$, defined as follows:

$$F_{\text{D}} = \int_{\Gamma_{\text{obs}}} \left[p\mathbf{n} - \frac{1}{\text{Re}} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)\mathbf{n} \right] \cdot \mathbf{i} \, ds, \, F_{\text{L}} = \int_{\Gamma_{\text{obs}}} \left[p\mathbf{n} - \frac{1}{\text{Re}} (\nabla \mathbf{u} + \nabla \mathbf{u}^T)\mathbf{n} \right] \cdot \mathbf{j} \, ds,$$

where Γ_{obs} is the boundary of the obstacle.
NB: Use `FacetNormal(mesh)` to access the normal vector $\mathbf{n}$ in Firedrake.

3 Consider now free-slip conditions on the wall boundaries

$$\Gamma_{\text{w}} = \{-3 \leq x \leq 4, y = -2, 2\} = \partial\Omega \setminus (\Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_{\text{obs}}).$$

Rewrite the steps of the projection scheme in this case, modify the implementation and run the simulation. What can you observe?