Exercise 1

Let us consider the Stokes problem defined in the unit square $\Omega := [0,1]^2$

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = 0, & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 1\mathbf{i} + 0\mathbf{j}, & \text{on } \Gamma^{\mathrm{up}} = \{0 \le x \le 1,\, y = 1\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega \backslash \Gamma^{\mathrm{up}}. \end{cases}$$

1. Assemble the matrices defining the algebraic form of the FEM apprixmation of the problem, using the stable pair of spaces $\mathbb{P}^2/\mathbb{P}^1$. Use the penalty method for the imposition of Dirichlet BCs.

2. Performing a block Gaussian elimination the previous system can be reduced to the equation
$$S\mathbf{P} = \chi, \tag{1}$$
where $S = BA^{-1}B^T$ is the Schur complement and $\chi = BA^{-1}\mathbf{f}$. This system can be preconditioned by the pressure mass matrix M_p.
Write a function that represens the matrix S and the preconditioner M_p (see template code).

3. Solve problem (1) by the precondiioned conjugate gradient method, with M_p as preconditioner. When the **`LinearCG`** function returns, **`x`** contains the result of last iteration. The **`precon`** argument is the preconditioner, its default value is the identity operator and can be defined as a function with the same signature of the residual function.

   Use **`LinearCG`** to solve the pressure equation, then compute the velocity.

4. Repeat now the previous step using, as preconditioner:
   - nothing (no preconditioner);
   - the pressure mass matrix M_p;
   - the lumped pressure mass matrix: $(\tilde{M_p})_{ij} = \begin{cases} \sum_{k=0}^{N_p} (M_p)_{ik} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$
   - the diagonal of the pressure mass matrix: $(D_p)_{ij} = \begin{cases} (M_p)_{ij} & \text{if } i = j, \\ 0 & \text{otherwise;} \end{cases}$

   How many iterations are needed in order to reach convergence?

Exercise 2

Let us now consider the unsteady Stokes problem defined in the unit square $\Omega := [0,1]^2$

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + \nabla p = 0, & \text{in } \Omega \ \forall t \in (0,1), \\ \operatorname{div} \mathbf{u} = 0, & \text{in } \Omega \ \forall t \in (0,1), \\ \mathbf{u}(t,\mathbf{x}) = t\mathbf{i} + 0\mathbf{j}, & \text{on } \Gamma^{\text{up}} = \{0 \leq x \leq 1,\ y = 1\} \ \ \forall t \in (0,1), \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega \backslash \Gamma^{\text{up}} \ \forall t \in (0,1), \\ \mathbf{u} = \mathbf{0}, & \text{in } \Omega \text{ for } t = 0. \end{cases}$$

Solve the problem using the building blocks created in Exercise 1.