Exercise 1

Let us consider the following diffusion-reaction problem defined in the unit square $\Omega = [0,1]^2$:

$$\begin{cases} -\Delta u + 3u = 2(y^2-1)e^{-x}, & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = -y^2, & \text{on } \Gamma_1 = \partial\Omega \cap \{x=0\}, \\ \nabla u \cdot \mathbf{n} = y^2, & \text{on } \Gamma_2 = \partial\Omega \cap \{x=1\}, \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Gamma_3 = \partial\Omega \cap \{y=0\}, \\ \nabla u \cdot \mathbf{n} = 1, & \text{on } \Gamma_4 = \partial\Omega \cap \{y=1\}. \end{cases}$$

1. Write the weak formulation of the differential problem.
2. Use the formulation of the previous point in order to obtain the finite elements formulation and the corresponding algebraic problem.
3. Solve the problem using the variational approach in Firedrake (use the `solve` command). Use the space $\mathbb{P}^1$ of continuous piecewise linear functions.
4. Extract the matrices and vectors associated to the finite element approximation of the problem and solve the problem algebraically.

Exercise 2

Let us consider the Laplace problem defined in the unit square $\Omega = [0,1]^2$

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega, \\ u = y(1-y), & \text{on } \Gamma_1 = \partial\Omega \cap \{x=0\}, \\ u = 0, & \text{on } \Gamma_2 = \partial\Omega \cap \{x=1\}, \\ \nabla u \cdot \mathbf{n} = 1, & \text{on } \Gamma_3 = \partial\Omega \cap \{y=0\}, \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \Gamma_4 = \partial\Omega \cap \{y=1\}. \end{cases}$$

1. Write the weak formulation of the differential problem, enforcing Dirichlet boundary conditions in the strong sense.
2. Derive the finite element formulation of the problem.
   From now on, consider **piecewise linear** finite elements.
3. Implement and solve the problem in Firedrake.
4. Extract the matrices and vectors associated to the finite element approximation of the problem. What is the structure of the stiffness matrix?