Consider a stationary Navier-Stokes problem in the pipe Ω of Figure 1:

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 4y(1-y)\mathbf{i}, & \text{on } \Gamma_{\text{in}} := \{x=0,\, 0\le y\le 1\}, \\ (\nu\nabla\mathbf{u} - pI)\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{out}} := \{2\le x\le 3,\, y=3\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \Gamma_{\text{wall}} := \partial\Omega\backslash(\Gamma_{\text{in}}\cup\Gamma_{\text{out}}), \end{cases}$$

where the kinematic viscosity ν is equal to 10^{-2} and the boundary conditions are given by:

- no-slip boundary conditions on the lateral boundary Γ_{wall} of the pipe;
- a parabolic profile of velocity on the inlet boundary Γ_{in};
- homogeneous Neumann boundary conditions on the outlet boundary Γ_{out}.

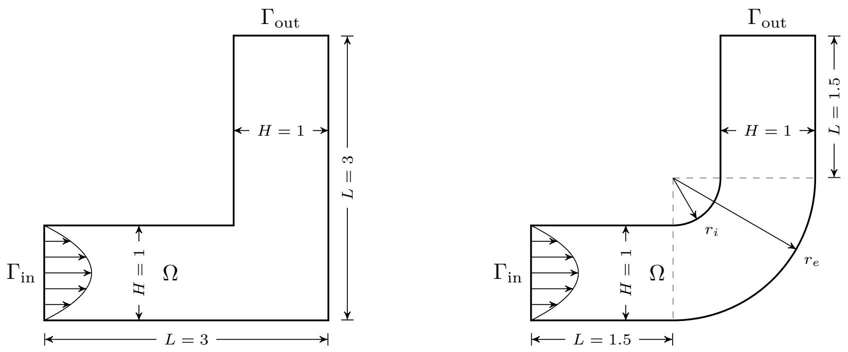

Figure 1: Domain Ω for Exercises 1 and 2 : miter joint (left: point 1 , `elbow1.msh`) and 90° circular elbow (right: point 2 , `elbow2.msh`).

Exercise 1

Let us take into account the following fixed-point iteration method: given $(\mathbf{u}_{k-1}, p_{k-1})$, the next iteration $(\mathbf{u}_k, p_k)$ is given by the solution of the linear problem

$$\begin{cases} (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u}_k - \nu\Delta\mathbf{u}_k + \nabla p_k = \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u}_k = 0, & \text{in } \Omega, \\ \mathbf{u}_k = 4y(1-y)\mathbf{i}, & \text{on } \Gamma_{\text{in}}, \\ (\nu\nabla\mathbf{u}_k - p_k I)\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{out}}, \\ \mathbf{u}_k = \mathbf{0}, & \text{on } \Gamma_{\text{wall}} = \partial\Omega\backslash(\Gamma_{\text{in}}\cup\Gamma_{\text{out}}). \end{cases}$$

1 Using the fixed-point iteration method defined above, solve the nonlinear problem over the geometry in Figure 1, left. As an initial condition $(\mathbf{u}_0, p_0)$, use the solution of the corresponding Stokes problem. Employ a stopping criterion based on the increment:

$$\frac{\|\mathbf{u}_k - \mathbf{u}_{k-1}\|_{H^1}}{\|\mathbf{u}_{k-1}\|_{H^1}} + \frac{\|p_k - p_{k-1}\|_{L^2}}{\|p_{k-1}\|_{L^2}} \le 10^{-3}.$$

2 Repeat the exercise using the smooth elbow geometry displayed in Figure 1, right. What differences can you observe?

Exercise 2

Let us take into account the Newton method, defined by the following iterative algorithm: given $(\mathbf{u}_{k-1}, p_{k-1})$, the next iteration $(\mathbf{u}_k, p_k)$ is given by the solution of the linear problem

$$\begin{cases} (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u}_k + (\mathbf{u}_k\cdot\nabla)\mathbf{u}_{k-1} - \nu\Delta\mathbf{u}_k + \nabla p_k = (\mathbf{u}_{k-1}\cdot\nabla)\mathbf{u}_{k-1}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u}_k = 0, & \text{in } \Omega, \\ \mathbf{u}_k = 4y(1-y)\mathbf{i}, & \text{on } \Gamma_{\text{in}}, \\ (\nu\nabla\mathbf{u}_k - p_k I)\mathbf{n} = \mathbf{0}, & \text{on } \Gamma_{\text{out}}, \\ \mathbf{u}_k = \mathbf{0}, & \text{on } \Gamma_{\text{wall}} = \partial\Omega\backslash(\Gamma_{\text{in}}\cup\Gamma_{\text{out}}). \end{cases}$$

1 Using the Newton method defined above, solve the nonlinear problem, over the domain displayed in Figure 1, left. As an initial condition $(\mathbf{u}_0, p_0)$, use the solution of the corresponding Stokes problem. Employ a stopping criterion based on the increment:

$$\frac{\|\mathbf{u}_k - \mathbf{u}_{k-1}\|_{H^1}}{\|\mathbf{u}_{k-1}\|_{H^1}} + \frac{\|p_k - p_{k-1}\|_{L^2}}{\|p_{k-1}\|_{L^2}} \leq 10^{-3}.$$

2 Repeat the exercise using the smooth elbow geometry displayed in Figure 1, right. What differences can you observe?

Exercise 3

Let us consider the stationary Navier-Stokes problem defined in the domain Ω as in the Figure 2 (defined in `elbow3.edp`)

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{0}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 4y(1-y)\mathbf{i}, & \text{on } \Gamma_{\text{in}} := \{x = 0,\ 0 \leq y \leq 1\}, \\ (\nu\nabla\mathbf{u} - pI)\mathbf{n} = 0, & \text{on } \Gamma_{\text{out}} := \{x = 6,\ 2 \leq y \leq 3\}, \\ \mathbf{u} = \mathbf{0}, & \text{on } \partial\Omega\backslash(\Gamma_{\text{in}}\cup\Gamma_{\text{out}}), \end{cases}$$

where the kinematic viscosity ν is equal to 2×10^{-2} and the boundary conditions are:

- a no-slip boundary condition enforced on the lateral boundary of the pipe;
- a parabolic profile of velocity enforced on the inlet boundary Γ_{in};
- stress-free conditions on the outlet boundary Γ_{out}.

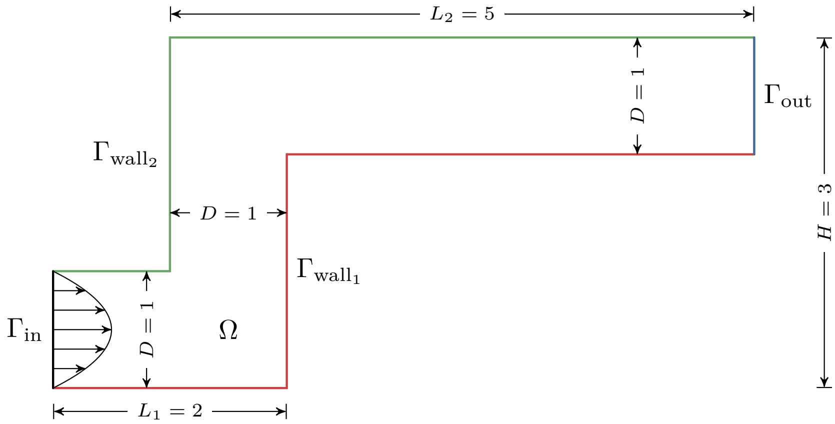

Figure 2: Domain Ω for Exercises 3 (`elbow3.edp`).

1. Modify the solver implemented in the previous exercises to account for the new geometry and the new value of ν. Then, solve the problem by the Newton method.

2. Compute and plot the stream function ψ given by the solution of the following elliptic boundary value problem

$$\begin{cases} -\Delta\psi = \omega, & \text{in } \Omega, \\ \nabla\psi \cdot \mathbf{n} = 0, & \text{on } \Gamma_{\text{in}} \cup \Gamma_{\text{out}}, \\ \psi = 2/3, & \text{on } \Gamma_{\text{wall}_1}, \\ \psi = 0, & \text{on } \Gamma_{\text{wall}_2}, \end{cases}$$

where ω is intensity of the vorticity field. Use $\mathbb{P}^1$ finite elements for ψ.
NB: in 3 dimensions, $\omega = |\boldsymbol{\omega}| = |\nabla \times \mathbf{u}|$. For 2-dimensional flows, the vorticity field is defined as orthogonal to the flow plane, with intensity $\omega = \partial_x u_y - \partial_y u_x$.