Discontinuous Galerkin Methods*

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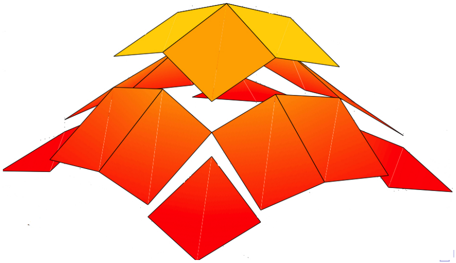
What are Discontinuous Galerkin methods?

Discontinuous Galerkin (DG) methods are a family of finite element methods for the approximation of partial differential equations

The idea

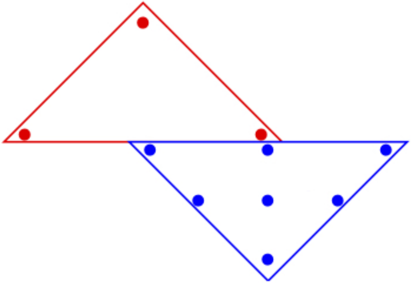
The discrete solution is seek in a discrete space made of polynomials that are completely discontinuous across mesh elements

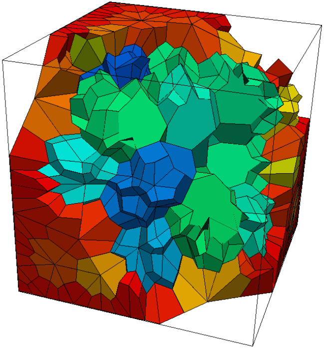
$$V_h \not\subseteq V$$



Features of DG Methods

- √ Wide range of PDE's treated within the same unified framework
- √ Weak approximation of boundary conditions
- √ Flexibility in mesh design
- √ Flexibility in polynomial degree distribution





- X Higher number of degrees of freedom
- X Larger algebraic linear systems to be solved (need of fast solvers)

Historical roots

- Introduced in the 70's for purely hyperbolic problems (Reed-Hill, Lesaint-Raviart)
- Extended in the mid 70's to second order elliptic PDEs (Douglas-Dupont) and to fourth order problems (Baker)
- Abandoned in 80's-90's due to a much larger number of degrees of freedom compared to their conforming cousins
- Great revival since the late 90's, application to a wide range of problems (linear/non-linear PDEs, time-dependent/stationary problems, spectrum approximation, multi-physics problems)
- DG formulation allows a modern revisitation of Finite Volume methods under a more rigorous mathematical setting and numerical analysis

Essential Bibliography: Books/Reviews

- B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, SIAM, 2008
- J.S. Hesthaven, T. Warburton, *Discontinuous Galerkin Methods. Algorithms, Analysis, and Applications*, Springer, 2008
- H. Fahs, *High-order discontinuous Galerkin methods for the Maxwell equations*, 2010, Editions Universitaires Europèennes
- D. A. Di Pietro, A. Ern, *Mathematical Aspects of Discontinuous Galerkin Methods*, Springer, 2012
- A. Cangiani, Z. Dong, E.H. Geourgolis, P. Houston. *hp–Version Discontinuous Galerkin Methods on Polygonal and Polyhedral Meshes*, Springer International Publishing, 2017
- P. F. Antonietti, A. Cangiani, J. Collis, Z. Dong, E. H. Georgoulis, S. Giani, P. Houston. *Review of discontinuous Galerkin finite element methods for partial differential equations on complicated domains*, Lect. Notes Comput. Sci. Eng., pp 279–308. Springer, 2016

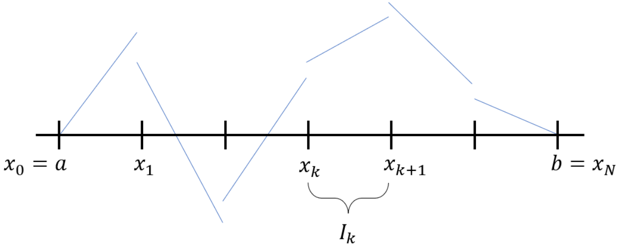
Essential Bibliography

- Elliptic/parabolic PDEs
  - Babuška & Zlámal (1973)
  - Douglas & Dupont (1976)
  - Baker (1977)
  - Wheeler (1978), Riviére & Wheeler (1999 $\longrightarrow$)
  - Arnold (1979, 1982)
  - Cockburn, Perugia & Schötzau (2000 $\longrightarrow$)
  - Arnold, Brezzi, Cockburn & Marini (SINUM, 2001/2002 $\longrightarrow$)
- Hyperbolic PDEs
  - Reed & Hill (1973, Los Alamos Technical report)
  - Lesaint & Raviart (1974, 1978)
  - Johnson, Nävert & Pitkäranta (1984), Johnson & Pitkäranta (1986)
  - Baumann (1997), Baumann & Oden (1997 $\longrightarrow$)
  - Cockburn & Shu (1989 $\longrightarrow$)
  - Houston & Süli (1999 $\longrightarrow$)

Warmup: 1D case

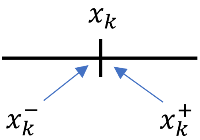
$$\begin{cases} -u'' = f & a < x < b \\ u(a) = 0, & u(b) = 0 \end{cases}$$

Aim: to use piecewise discontinuous polynomials.



$$\int_a^b -u''v = \int_a^b fv \implies -\sum_{k=0}^{N-1} \int_{I_k} u''v = \sum_{k=0}^{N-1} \int_{I_k} fv$$

Notation:



Assume both u and v to be discontinuous at the nodes $\{x_k\}$. Then:

$$-\sum_{k=0}^{N-1}\int_{I_k} u''v = \sum_{k=0}^{N-1}\left[\int_{I_k} u'v' - \left(u'v|_{x_{k+1}^-} - u'v|_{x_k^+}\right)\right] \tag{1}$$

$$
\begin{aligned}
\sum_{k=0}^{N-1}\left(u'v|_{x_{k+1}^-} - u'v|_{x_k^+}\right) &= u'(x_1^-)v(x_1^-) - u'(x_0^+)v(x_0^+) \\
&+ u'(x_2^-)v(x_2^-) - u'(x_1^+)v(x_1^+) \\
&+ \dots \\
&+ u'(x_N^-)v(x_N^-) - u'(x_{N-1}^+)v(x_{N-1}^+) \\
&= \sum_{k=0}^{N} [\![u'(x_k)v(x_k)]\!]
\end{aligned}
\tag{2}
$$

where we have defined the jump function:

$$
\begin{aligned}
&[\![\phi(x_0)]\!] := -\phi(x_0^+) \\
&[\![\phi(x_k)]\!] := \phi(x_k^-) - \phi(x_k^+) \qquad x_k\text{: interior node} \\
&[\![\phi(x_N)]\!] := \phi(x_N^-)
\end{aligned}
\tag{3}
$$

Using (1) + (2) we obtain:

$$\sum_{k=0}^{N-1} \int_{I_k} u'v' - \sum_{k=0}^{N} [\![u'(x_k) v(x_k)]\!] = \sum_{k=0}^{N-1} \int_{I_k} fv \tag{4}$$

Define the average operator:

$$\begin{aligned} \{\!\{\phi(x_0)\}\!\} &:= \phi(x_0^+) \\ \{\!\{\phi(x_k)\}\!\} &:= \frac{1}{2}(\phi(x_k^-) + \phi(x_k^+)) \qquad x_k\text{: interior node} \\ \{\!\{\phi(x_N)\}\!\} &:= \phi(x_N^-) \end{aligned} \tag{5}$$

Magic formula

$$\sum_{k=0}^{N} \left[\!\left[u'(x_k)v(x_k) \right]\!\right] = \sum_{k=0}^{N} \left\{\!\left\{ u'(x_k) \right\}\!\right\} [\![v(x_k)]\!] + \sum_{k=1}^{N-1} \left[\!\left[u'(x_k) \right]\!\right] \{\!\{ v(x_k) \}\!\} \quad (6)$$

Remark

If u is the exact solution and $u \in \mathcal{C}^1([a,b])$, then $[\![u'(x_k)]\!] = 0$ for every interior node, and the second sum in (6) drops.

Proof of Magic formula (not for exam)

For $k = 0, N$ the l.h.s. of (6) yields (using (3) and (5)):

$$\left[\!\left[u'(x_0)v(x_0)\right]\!\right] = -u'(x_0^+)v(x_0^+)$$
$$\left[\!\left[u'(x_N)v(x_N)\right]\!\right] = u'(x_N^-)v(x_N^-)$$

whereas the r.h.s. of (6) yields

$$\{\!\{u'(x_0)\}\!\} [\![v(x_0)]\!] = -u'(x_0^+)v(x_0^+)$$
$$\{\!\{u'(x_N)\}\!\} [\![v(x_N)]\!] = u'(x_N^-)v(x_N^-)$$

We are left to check (6) limited to each interior point, that is:

$$\underbrace{\left[\!\left[u'(x_k)v(x_k)\right]\!\right]}_{(I)} \stackrel{?}{=} \underbrace{\{\!\{u'(x_k)\}\!\} [\![v(x_k)]\!] + \left[\!\left[u'(x_k)\right]\!\right] \{\!\{v(x_k)\}\!\}}_{(II)}$$

Thanks to (3):

$$(I) = u'(x_k^-)v(x_k^-) - u'(x_k^+)v(x_k^+)$$

Moreover:

$$\begin{aligned}
(II) &= \frac{1}{2}(u'(x_k^-) + u'(x_k^+))(v(x_k^-) - v(x_k^+)) \\
&+ \frac{1}{2}(u'(x_k^-) - u'(x_k^+))(v(x_k^-) + v(x_k^+)) \\
&= u'(x_k^-)v(x_k^-) - u'(x_k^+)v(x_k^+) \\
&+ \frac{1}{2}u'(x_k^-)\cancel{(-v(x_k^+) + v(x_k^+))} \\
&+ \frac{1}{2}u'(x_k^+)\cancel{(+v(x_k^-) - v(x_k^-))} = (I)
\end{aligned}$$

$\square$

We end up with the formulation (upon collecting (4) and (6)):

$$\underbrace{\sum_{k=0}^{N-1}\int_{I_k} u'v' - \sum_{k=0}^{N}\{\!\{u'(x_k)\}\!\}[\![v(x_k)]\!] - \sum_{k=1}^{N-1}[\![u'(x_k)]\!]\{\!\{v(x_k)\}\!\}}_{\mathcal{A}(u,v)} = \sum_{k=0}^{N-1}\int_{I_k} fv \qquad \forall v \in V \tag{7}$$

where

$$V = H^1_{\text{broken}}(\Omega) := \left\{v \in L^2(\Omega)\colon\ v|_{I_k} \in H^1(I_k)\,\forall k = 0,\dots,N-1\right\}$$

with the *broken norm*:

$$\|v\|_{H^1_{\text{broken}}(\Omega)} = \left(\sum_{k=0}^{N}\|v|_{I_k}\|^2_{H^1(I_k)}\right)^{1/2}$$

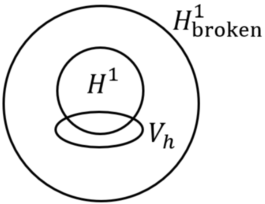
DG formulation

Let $V_h \subset V$

$$\text{Find } u_h \in V_h : \mathcal{A}(u_h, v_h) = \sum_{k=0}^{N-1} \int_{I_k} f v_h \qquad \forall v_h \in V_h \tag{8}$$

Remark

V_h is not a subspace of $H^1(\Omega)$ (see figure below).



Formulation (8) is not well-posed. Modify it like this: in (7)

- drop 3rd term because $[\![u'(x_k)]\!] = 0$
- add the *symmetrization term* ($=0$ if u is the exact solution)

$$-\sum_{k=0}^{N} \theta \left\{\!\left\{ v'(x_k) \right\}\!\right\} [\![u(x_k)]\!]$$

  with

  - $\theta = 1$ (SIP) Symmetric Interior Penalty
  - $\theta = -1$ (NIP) Non-symmetric Interior Penalty
  - $\theta = 0$ (IIP) Incomplete Interior Penalty

- add the *stabilization term* ($=0$ if u is the exact solution)

$$+\sum_{k=0}^{N} \gamma [\![u(x_k)]\!] [\![v(x_k)]\!]$$

New bilinear form

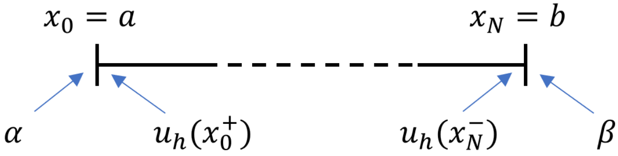
$$
\begin{aligned}
\mathcal{A}^*(u_h, v_h) = & \underbrace{\sum_{k=0}^{N-1} \int_{I_k} u_h' v_h'}_{(i)} \underbrace{- \sum_{k=0}^{N} \{\!\{ u_h'(x_k) \}\!\} [\![v_h(x_k)]\!]}_{(ii)} \\
& \underbrace{- \sum_{k=0}^{N} \theta \{\!\{ v_h'(x_k) \}\!\} [\![u_h(x_k)]\!]}_{(iii)} + \underbrace{\sum_{k=0}^{N} \gamma [\![u_h(x_k)]\!] [\![v_h(x_k)]\!]}_{(iv)}
\end{aligned}
\tag{9}
$$

Neumann BC:
Impose (N) BC through $\{\!\{ u'(x_k) \}\!\}$ in (*ii*). In this case, we have $\sum_{k=1}^{N-1}$ in (*ii*) and, consequently, we write $\sum_{k=1}^{N-1}$ in (*iii*) for symmetry.

Non-homogeneous Dirichlet BC:
Impose (D) BC as follows. In (*iii*) and (*iv*) replace $[\![u_h(x_0)]\!]$ and $[\![u_h(x_N)]\!]$ with the following definitions:

$$
\begin{aligned}
[\![u_h(x_0)]\!] &:= \alpha - u_h(x_0^+) \qquad && \text{if } u(a) = \alpha \\
[\![u_h(x_N)]\!] &:= u_h(x_N^-) - \beta \qquad && \text{if } u(b) = \beta
\end{aligned}
$$

Special case: $\alpha = \beta = 0$ if homogeneous Dirichlet conditions.



Do not change the definition of $[\![v_h(x_0)]\!]$ and $[\![v_h(x_N)]\!]$.

Now in (9), split sums as follows

$$
\begin{aligned}
&\mathcal{A}^*(u_h, v_h) \\
&= \sum_{k=0}^{N-1} \int_{I_k} u_h' v_h' \\
&- \sum_{k=1}^{N-1} \{\!\{ u_h'(x_k) \}\!\} [\![v_h(x_k)]\!] + u_h'(x_0^+) v_h(x_0^+) - u_h'(x_N^-) v_h(x_N^-) \\
&- \sum_{k=1}^{N-1} \theta \{\!\{ v_h'(x_k) \}\!\} [\![u_h(x_k)]\!] - \left(\theta v_h'(x_0^+)(\alpha - u_h(x_0^+)) + \theta v_h'(x_N^-)(u_h(x_N^-) - \beta) \right) \\
&+ \sum_{k=1}^{N-1} \gamma [\![u_h(x_k)]\!] [\![v_h(x_k)]\!] + \gamma(\alpha - u_h(x_0^+))(-v_h(x_0^+)) + \gamma(u_h(x_N^-) - \beta) v_h(x_N^-)
\end{aligned}
\tag{10}
$$

Now move terms including α and β to the r.h.s.

On the l.h.s. it remains:

$$
\begin{aligned}
&\widetilde{\mathcal{A}}(u_h, v_h) \\
&= \sum_{k=0}^{N-1} \int_{I_k} u_h' v_h' \\
&- \sum_{k=1}^{N-1} \{\!\{ u_h'(x_k) \}\!\} [\![v_h(x_k)]\!] + u_h'(x_0^+) v_h(x_0^+) - u_h'(x_N^-) v_h(x_N^-) \\
&- \sum_{k=1}^{N-1} \theta \{\!\{ v_h'(x_k) \}\!\} [\![u_h(x_k)]\!] + \left(\theta u_h(x_0^+) v_h'(x_0^+) - \theta u_h(x_N^-) v_h'(x_N^-) \right) \\
&+ \sum_{k=1}^{N-1} \gamma [\![u_h(x_k)]\!] [\![v_h(x_k)]\!] + \gamma u_h(x_0^+) v_h(x_0^+) + \gamma u_h(x_N^-) v_h(x_N^-)
\end{aligned}
\tag{11}
$$

On the r.h.s. we have:

$$
\mathcal{F}(v_h) = \sum_{k=0}^{N-1} \int_{I_k} f v_h + \theta(\alpha v_h'(x_0^+) - \beta v_h'(x_N^-)) + \gamma(\alpha v_h(x_0^+) + \beta v_h(x_N^-)) \tag{12}
$$

Remark

Note that for $\theta = 1$, $\widetilde{\mathcal{A}}(u_h, v_h) = \widetilde{\mathcal{A}}(v_h, u_h) \rightarrow$ *symmetry!*

(DG) for non-homogeneous Dirichlet conditions:

$$\text{Find } u_h \in V_h \colon \widetilde{\mathcal{A}}(u_h, v_h) = \mathcal{F}(v_h) \qquad \forall v_h \in V_h \tag{13}$$

with $\mathcal{F}$ depending on f, α and β.

Note that in (11), if we define $[\![u_h(x_0)]\!]$ and $[\![u_h(x_N)]\!]$ as $[\![v_h(x_0)]\!]$ and $[\![v_h(x_N)]\!]$:

$$
\begin{aligned}
&-\sum_{k=1}^{N-1} \{\!\{u_h'(x_k)\}\!\} [\![v_h(x_k)]\!] + u_h'(x_0^+)v_h(x_0^+) - u_h'(x_N^-)v_h(x_N^-) \\
&\qquad = -\sum_{k=0}^{N} \{\!\{u_h'(x_k)\}\!\} [\![v_h(x_k)]\!] \\
&-\sum_{k=1}^{N-1} \theta \{\!\{v_h'(x_k)\}\!\} [\![u_h(x_k)]\!] + \left(\theta u_h(x_0^+)v_h'(x_0^+) - \theta u_h(x_N^-)v_h'(x_N^-)\right) \\
&\qquad = -\sum_{k=0}^{N} \theta \{\!\{v_h'(x_k)\}\!\} [\![u_h(x_k)]\!] \\
&+\sum_{k=1}^{N-1} \gamma [\![u_h(x_k)]\!] [\![v_h(x_k)]\!] + \gamma u_h(x_0^+)v_h(x_0^+) + \gamma u_h(x_N^-)v_h(x_N^-) \\
&\qquad = +\sum_{k=0}^{N} \gamma [\![u_h(x_k)]\!] [\![v_h(x_k)]\!]
\end{aligned}
$$

Multidimensional case

$$\text{Model problem} \quad \longrightarrow \quad \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Let us introduce a finite element triangulation $\mathcal{T}_h$ (for simplicity, triangles in 2D, tetrahedra in 3D). However, this time the conformity constraint that we we have assumed in the standard approach with continuous elements may be violated.
- Take the equation $-\Delta u = f$, multiply it by a (elementwise smooth) test function v and integrate over an element $\mathcal{K} \in \mathcal{T}_h$

$$\int_{\mathcal{K}} -\Delta u v = \int_{\mathcal{K}} f v$$

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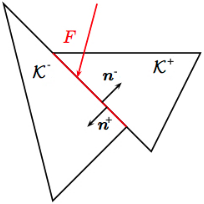

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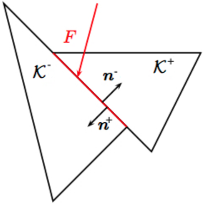

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Multidimensional case (cont'd)

- Integrate by parts and sum over all the elements $\mathcal{K} \in \mathcal{T}_h$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \nabla u \cdot \mathbf{n}_{\mathcal{K}} v = \int_{\Omega} fv$$

- To deal with the boxed term we need some further notation

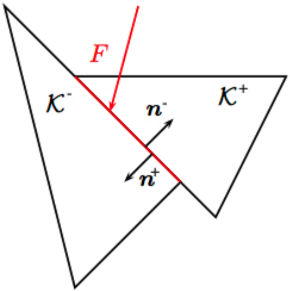
Multidimensional case (cont'd)

- Integrate by parts and sum over all the elements $\mathcal{K} \in \mathcal{T}_h$

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \boxed{\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\partial\mathcal{K}} \nabla u \cdot \mathbf{n}_{\mathcal{K}} v} = \int_{\Omega} fv$$

- To deal with the boxed term we need some further notation

Trace Operators



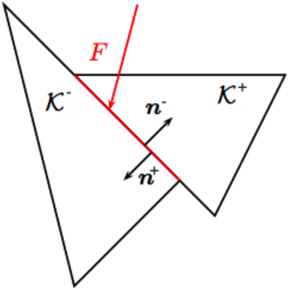
- for any $F \in \mathcal{F}_h^I$(= set of interior faces) shared by $\mathcal{K}^{\pm}$

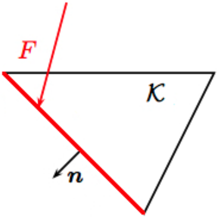
$$\{\!\{v\}\!\} = (v^+ + v^-)/2 \qquad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-$$
$$\{\!\{\boldsymbol{\tau}\}\!\} = (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)/2 \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\tau}^- \cdot \boldsymbol{n}^-$$

- for any $F \in \mathcal{F}_h^B$(= set of boundary faces)

$$\{\!\{v\}\!\} = v \qquad [\![v]\!] = v\boldsymbol{n} \qquad \{\!\{\boldsymbol{\tau}\}\!\} = \boldsymbol{\tau} \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau} \cdot \boldsymbol{n}$$

Trace Operators





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$$\{\!\{v\}\!\} = (v^+ + v^-)/2 \qquad [\![v]\!] = v^+\boldsymbol{n}^+ + v^-\boldsymbol{n}^-$$
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- for any $F \in \mathcal{F}_h^B$(= set of boundary faces)

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Magic Formula (Arnold, 82)

$\forall \boldsymbol{\tau}$ vector function

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}} \boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}} v = \sum_{F\in\mathcal{F}_h}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}_h^I}\int_F [\![\boldsymbol{\tau}]\!]\,\{\!\{v\}\!\}$$

Magic Formula (Arnold, 82)

$$\underbrace{\sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{\tau}\cdot\mathbf{n}_K v}_{(A)} = \underbrace{\sum_{F\in\mathcal{F}_h}\int_F \{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}_h^I}\int_F [\![\boldsymbol{\tau}]\!]\,\{\!\{v\}\!\}}_{(B)}$$

Proof (not for exam)

Observe that

$$(A) = \sum_{F\in\mathcal{F}_h^I}\int_F(\boldsymbol{\tau}^+\cdot\mathbf{n}^+v^+ + \boldsymbol{\tau}^-\cdot\mathbf{n}^-v^-) + \sum_{F\in\mathcal{F}_h^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v$$

$$(B) = \sum_{F\in\mathcal{F}_h^I}\int_F(\{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!] + [\![\boldsymbol{\tau}]\!]\,\{\!\{v\}\!\}) + \sum_{F\in\mathcal{F}_h^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v$$

Magic Formula (Arnold, 82)

$$\underbrace{\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}}v}_{(A)} = \underbrace{\sum_{F\in\mathcal{F}_h}\int_F \{\boldsymbol{\tau}\}\cdot[\![v]\!] + \sum_{F\in\mathcal{F}_h^I}\int_F [\![\boldsymbol{\tau}]\!]\,\{v\}}_{(B)}$$

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Observe that

$$(A) = \sum_{F\in\mathcal{F}_h^I}\int_F(\boldsymbol{\tau}^+\cdot\mathbf{n}^+v^+ + \boldsymbol{\tau}^-\cdot\mathbf{n}^-v^-) + \boxed{\sum_{F\in\mathcal{F}_h^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v}$$

$$(B) = \sum_{F\in\mathcal{F}_h^I}\int_F(\{\boldsymbol{\tau}\}\cdot[\![v]\!] + [\![\boldsymbol{\tau}]\!]\,\{v\}) + \boxed{\sum_{F\in\mathcal{F}_h^B}\int_F\boldsymbol{\tau}\cdot\mathbf{n}v}$$

Proof (not for exam)

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}_h^I$

$$\int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-) = \int_F (\{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![v]\!] + [\![\boldsymbol{\tau}]\!] \{\!\{v\}\!\})$$

Magic Formula: proof (cont'd) (Arnold, 82)

Proof (not for exam)

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}_h^I$

$$\int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-) = \underbrace{\int_F (\{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![v]\!] + [\![\boldsymbol{\tau}]\!] \{\!\{v\}\!\})}_{(C)}$$

Using the definition of the jump and average operators and that $\mathbf{n}^+ = -\mathbf{n}^-$ we have

$$(C) = \frac{1}{2} \int_F (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)(v^+ - v^-) \cdot \mathbf{n}^+ + (v^+ + v^-)(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-) \cdot \mathbf{n}^+$$

$$= \frac{1}{2} \int_F (2\boldsymbol{\tau}^+ v^+ + \boldsymbol{\tau}^- v^+ - \boldsymbol{\tau}^+ v^- - 2\boldsymbol{\tau}^- v^- + v^- \boldsymbol{\tau}^+ - v^+ \boldsymbol{\tau}^-) \cdot \mathbf{n}^+$$

Magic Formula: proof (cont'd) (Arnold, 82)

Proof (not for exam)

Therefore, it is enough to show that on each internal face $F \in \mathcal{F}_h^I$

$$\int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-) = \underbrace{\int_F (\{\!\{\boldsymbol{\tau}\}\!\} \cdot [\![v]\!] + [\![\boldsymbol{\tau}]\!] \{\!\{v\}\!\})}_{(C)}$$

$$(C) = \frac{1}{2} \int_F (\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-)(v^+ - v^-) \cdot \mathbf{n}^+ + (v^+ + v^-)(\boldsymbol{\tau}^+ - \boldsymbol{\tau}^-) \cdot \mathbf{n}^+$$

$$= \frac{1}{2} \int_F (2\boldsymbol{\tau}^+ v^+ \cancel{+ \boldsymbol{\tau}^- v^+} \cancel{- \boldsymbol{\tau}^+ v^-} - 2\boldsymbol{\tau}^- v^- \cancel{+ v^- \boldsymbol{\tau}^+} \cancel{- v^+ \boldsymbol{\tau}^-}) \cdot \mathbf{n}^+$$

$$= \int_F (\boldsymbol{\tau}^+ \cdot \mathbf{n}^+ v^+ + \boldsymbol{\tau}^- \cdot \mathbf{n}^- v^-)$$

Multidimensional case(cont'd)

Thanks to

Magic formula

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\boldsymbol{\tau}\cdot\mathbf{n}_{\mathcal{K}}v=\sum_{F\in\mathcal{F}_h}\int_F\{\!\{\boldsymbol{\tau}\}\!\}\cdot[\![v]\!]+\sum_{F\in\mathcal{F}_h^I}\int_F[\![\boldsymbol{\tau}]\!]\{\!\{v\}\!\}$$

we obtain

$$-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\nabla u\cdot\mathbf{n}_{\mathcal{K}}v=-\sum_{F\in\mathcal{F}_h}\int_F\{\!\{\nabla u\}\!\}\cdot[\![v]\!]-\sum_{F\in\mathcal{F}_h^I}\int_F[\![\nabla u]\!]\{\!\{v\}\!\}.$$

Then,

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v-\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\partial\mathcal{K}}\nabla u\cdot\mathbf{n}_{\mathcal{K}}v=\int_{\Omega}fv$$

Multidimensional case(cont'd)

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v-\sum_{F\in\mathcal{F}_h}\int_F\{\!\{\nabla u\}\!\}\cdot[\![v]\!]-\sum_{F\in\mathcal{F}_h^I}\int_F[\![\nabla u]\!]\,\{\!\{v\}\!\}=\int_\Omega fv$$

Multidimensional case(cont'd)

$$\sum_{\mathcal{K}\in\mathcal{T}_h}\int_{\mathcal{K}}\nabla u\cdot\nabla v-\sum_{F\in\mathcal{F}_h}\int_F\{\!\{\nabla u\}\!\}\cdot[\![v]\!]+\cancel{\sum_{F\in\mathcal{F}_h^I}\int_F[\![\nabla u]\!]\,\{\!\{v\}\!\}}=\int_\Omega fv$$

If we assume that $u\in H^2(\Omega)$, then $[\![\nabla u]\!]=0\ \forall\, F\in\mathcal{F}_h^I$. This condition on the jump of the gradient needs to be intended in the sense of the traces. Note that this regularity assumption on u is fulfilled if, e.g. f is in L^2 and the computational domain is a convex polygon, thanks to the property of elliptic regularity.

Multidimensional case (cont'd)

- Use that $[\![u]\!] = 0 \ \forall\, F \in \mathcal{F}_h$ (since $u \in H^2(\Omega) \cap H^1_0(\Omega)$) to add a symmetry term

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] = \int_\Omega fv$$

where ∇_h is the elementwise gradient (v is only piecewise smooth).

- We also add a stabilization term that controls the jumps (note: this is consistent as $[\![u]\!] = 0$ for exact solution)

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}_h} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\}$$
$$+ \sum_{F\in\mathcal{F}_h} \int_F \gamma\, [\![u]\!] \cdot [\![v]\!] = \int_\Omega fv$$

where γ is a stabilization function (that might depend on the discretization parameters) [Douglas-Dupont, Wheeler, Arnold], see later ......

Multidimensional case (cont'd)

- Use that $[\![u]\!] = 0 \ \forall F \in \mathcal{F}_h$ (since $u \in H^2(\Omega) \cap H_0^1(\Omega)$) to add a symmetry term

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h v\}\!\} \cdot [\![u]\!] = \int_\Omega fv$$

  where ∇_h is the elementwise gradient (v is only piecewise smooth).

- We also add a stabilization term that controls the jumps (note: this is consistent as $[\![u]\!] = 0$ for exact solution)

$$\sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla u \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla u\}\!\} \cdot [\![v]\!] - \sum_{F\in\mathcal{F}_h} \int_F [\![u]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h} \int_F \gamma\, [\![u]\!] \cdot [\![v]\!] = \int_\Omega fv$$

  where γ is a stabilization function (that might depend on the discretization parameters) [Douglas-Dupont, Wheeler, Arnold], see later ......

Multidimensional case (cont'd)

- For $p_{\mathcal{K}} \geq 1$, define the DG discrete space

$$V_h^p = \left\{ v_h \in L^2(\Omega) : v_h|_{\mathcal{K}} \in \mathbb{P}^{p_{\mathcal{K}}}(\mathcal{K}) \quad \forall \mathcal{K} \in \mathcal{T}_h \right\} \boxed{\not\subseteq H_0^1(\Omega)}$$

- Discretize $u \rightsquigarrow u_h$, $v \rightsquigarrow v_h$

Multidimensional case (cont'd)

- For $p_{\mathcal{K}} \geq 1$, define the DG discrete space

$$V_h^p = \left\{ v_h \in L^2(\Omega) : v_h|_{\mathcal{K}} \in \mathbb{P}^{p_{\mathcal{K}}}(\mathcal{K}) \quad \forall \mathcal{K} \in \mathcal{T}_h \right\} \boxed{\not\subseteq H_0^1(\Omega)}$$

- Discretize $u \rightsquigarrow u_h$, $v \rightsquigarrow v_h$

$$\text{Find } u_h \in V_h^p \quad \text{s.t.} \quad \mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

$$\mathcal{A}(w, v) = \sum_{\mathcal{K} \in \mathcal{T}_h} \int_{\mathcal{K}} \nabla w \cdot \nabla v - \sum_{F \in \mathcal{F}_h} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!] - \sum_{F \in \mathcal{F}_h} \int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F \in \mathcal{F}_h} \int_F \gamma\, [\![w]\!] \cdot [\![v]\!]$$

The class of Interior Penalty DG methods

$$\text{Find } u_h \in V_h^p \quad \text{s.t.} \quad \mathcal{A}(u_h, v_h) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

Note: $\mathcal{A}$ depends on the triangulation (whence $\mathcal{A} = \mathcal{A}_h$) and it actually differs from the bilinear form associated to the original weak problem in infinite dimension:

$$\mathcal{A}(w, v) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla w \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!]$$
$$- \theta \sum_{F\in\mathcal{F}_h} \int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h} \int_F \gamma\, [\![w]\!] \cdot [\![v]\!]$$

Nonetheless, we use this simplified notation (without the subindex h) for ease of reading.

$$\mathcal{A}(w,v) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla w \cdot \nabla v - \sum_{F\in\mathcal{F}_h} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!]$$
$$- \theta \sum_{F\in\mathcal{F}_h} \int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h} \int_F \gamma\, [\![w]\!] \cdot [\![v]\!]$$

- $\theta = 1$: Symmetric Interior Penalty (SIP). [Wheeler, 78],[Arnold, 82]
- $\theta = -1$: Non-symmetric Interior Penalty (NIP). [Riviére, Wheeler & Girault, 99]
- $\theta = 0$: Incomplete Interior Penalty (IIP). [Dawson, Sun, Wheeler, 04]

On the imposition of (Dirichlet) boundary conditions

- The above formulation applies to the case of homogeneous Dirichlet boundary conditions (that are enforced weakly).
- In the case of non-homogeneous Dirichlet boundary conditions of the form

$$u = g_D \qquad \text{on } \partial\Omega,$$

the r.h.s has to be modified as

$$\int_\Omega fv - \theta \sum_{F \in \mathcal{F}_h^B} \int_F g_D \nabla_h v \cdot \mathbf{n} + \sum_{F \in \mathcal{F}_h^B} \int_F \gamma g_D v$$

On the imposition of (Neumann) boundary conditions

- In the case of Neumann boundary conditions of the form

$$\nabla u \cdot \mathbf{n} = g_N \qquad \text{on } \partial\Omega,$$

the bilinear form has to be modified as

$$\mathcal{A}(w,v) = \sum_{\mathcal{K}\in\mathcal{T}_h} \int_{\mathcal{K}} \nabla w \cdot \nabla v - \sum_{F\in\mathcal{F}_h^I} \int_F \{\!\{\nabla_h w\}\!\} \cdot [\![v]\!]$$
$$- \theta \sum_{F\in\mathcal{F}_h^I} \int_F [\![w]\!] \cdot \{\!\{\nabla_h v\}\!\} + \sum_{F\in\mathcal{F}_h^I} \int_F \gamma\, [\![w]\!] \cdot [\![v]\!]$$

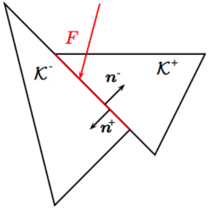
and the r.h.s has to be modified as

$$\int_\Omega fv + \sum_{F\in\mathcal{F}_h^B} \int_F g_N v$$

See Section 2.4 of [B. Rivière. *Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation*, SIAM, 2008] for more details.

The stabilization function γ

$$\sum_{F \in \mathcal{F}_h} \int_F \gamma \, [\![w]\!] \cdot [\![v]\!] \qquad \gamma = \alpha \frac{\mathtt{p}^2}{\mathtt{h}}$$



$$\mathtt{p} = \begin{cases} \max\{p_{\mathcal{K}^+}, p_{\mathcal{K}^-}\} & \text{if } F \in \mathcal{F}_h^I \\ p_{\mathcal{K}} & \text{if } F \in \mathcal{F}_h^B \end{cases} \qquad \mathtt{h} = \begin{cases} \min\{h_{\mathcal{K}^+}, h_{\mathcal{K}^-}\} & \text{if } F \in \mathcal{F}_h^I \\ h_{\mathcal{K}} & \text{if } F \in \mathcal{F}_h^B \end{cases}$$

Assumptions

$$h_F \approx h_{\mathcal{K}^+} \approx h_{\mathcal{K}^-}, \quad p_{\mathcal{K}^+} \approx p_{\mathcal{K}^-} \quad \Longrightarrow \gamma = \mathcal{O}\left(\frac{p^2}{h}\right)$$

A little bit of theory: notation

- For an integer $s \geq 1$, define the broken Sobolev space

$$H^s(\mathcal{T}_h) = \left\{ v \in L^2(\Omega) \, : \, v|_{\mathcal{K}} \in H^s(\mathcal{K}) \;\; \forall \mathcal{K} \in \mathcal{T}_h \right\}$$
$$\|v\|^2_{H^s(\mathcal{T}_h)} = \sum_{\mathcal{K} \in \mathcal{T}_h} \|v\|^2_{H^s(\mathcal{K})}$$

- Define also

$$\|v\|^2_{L^2(\mathcal{F}_h)} = \sum_{F \in \mathcal{F}_h} \|v\|^2_{L^2(F)}$$

A little bit of theory: notation (cont'd)

- Define the following norms

$$\|v\|_{\mathrm{DG}}^2 = \|\nabla_h v\|_{L^2(\Omega)}^2 + \left\|\gamma^{1/2} [\![v]\!]\right\|_{L^2(\mathcal{F}_h)}^2 \qquad \forall\, v \in H^2(\mathcal{T}_h)$$

$$|\!|\!|v|\!|\!|_{\mathrm{DG}}^2 = \|v\|_{\mathrm{DG}}^2 + \left\|\gamma^{-1/2} \{\!\{\nabla_h v\}\!\}\right\|_{L^2(\mathcal{F}_h)}^2 \qquad \forall\, v \in H^2(\mathcal{T}_h)$$

where $\nabla_h v$ is the element-wise gradient, i.e.,

$$(\nabla_h v)|_{\mathcal{K}} = \nabla(v|_{\mathcal{K}}) \qquad \forall \mathcal{K} \in \mathcal{T}_h.$$

Notice that $V_h^p \subset H^2(\mathcal{T}_h)$. It can be shown that

$$\|v\|_{\mathrm{DG}} \underset{(trivial)}{\leq} |\!|\!|v|\!|\!|_{\mathrm{DG}} \not\lesssim \|v\|_{\mathrm{DG}} \qquad \forall\, v \in H^2(\mathcal{T}_h)$$

$$\|v_h\|_{\mathrm{DG}} \underset{(trivial)}{\leq} |\!|\!|v_h|\!|\!|_{\mathrm{DG}} \underset{\text{(see next slide)}}{\lesssim} \|v_h\|_{\mathrm{DG}} \qquad \forall\, v_h \in V_h^p$$

A little bit of theory: $|||v_h|||_{\mathrm{DG}} \lesssim \|v_h\|_{\mathrm{DG}} \ \forall v_h \in V_h^p$ (not for exam)

Trace/inverse estimate for polynomial spaces

$$\|\nabla\eta\|_{L^2(F)} \leq C_{\mathrm{I}} \frac{p}{h^{1/2}} \|\nabla\eta\|_{L^2(\mathcal{K})} \qquad \forall F \subset \partial\mathcal{K} \qquad \forall \eta \in \mathbb{P}^p(\mathcal{K})$$

Using the above trace-inverse estimate

$$\begin{aligned}
\left\|\gamma^{-1/2}\ \{\!\{\nabla_h v_h\}\!\}\right\|^2_{L^2(\mathcal{F}_h)} &\leq \frac{h}{p^2\alpha} \sum_{F\in\mathcal{F}_h} \|\{\!\{\nabla_h v_h\}\!\}\|^2_{L^2(F)} \\
&\lesssim \frac{h}{\alpha p^2} \frac{p^2}{h} C_{\mathrm{I}} \sum_{\mathcal{K}\in\mathcal{T}_h} \|\nabla_h v_h\|^2_{L^2(\mathcal{K})} \\
&\lesssim \frac{1}{\alpha} C_{\mathrm{I}} \|\nabla_h v_h\|^2_{L^2(\Omega)}
\end{aligned}$$

Key properties

1. Continuity on $H^2(\mathcal{T}_h) \times V_h^p$:

$$|\mathcal{A}(v, w_h)| \lesssim |||v|||_{\mathrm{DG}} \|w_h\|_{\mathrm{DG}} \qquad \forall\, v \in H^2(\mathcal{T}_h), \quad \forall\, w_h \in V_h^p$$

Remark: $|\mathcal{A}(v, w_h)| \not\lesssim \|v\|_{\mathrm{DG}} \|w_h\|_{\mathrm{DG}}$

2. Coercivity on $V_h^p \times V_h^p$:

$$\mathcal{A}(v_h, v_h) \gtrsim \|v_h\|_{\mathrm{DG}}^2 \qquad \forall\, v_h \in V_h^p.$$

Remark: For SIP and IIP the penalty parameter α has to be chosen large enough. For NIP $\alpha = 0$ is fine, provided that $p \geq 2$ (however the analysis is more technical).

3. Strong-consistency

$$\mathcal{A}(u, v_h) = \int_\Omega f v_h \;\forall\, v_h \in V_h^p \quad \Longrightarrow \mathcal{A}(u - u_h, v_h) = 0 \;\forall\, v_h \in V_h^p$$

(Galerkin orthogonality)

4. Approximation. Let $\Pi_h^p u \in V_h^p$ be a suitable approximation of u. Then

$$|||u - \Pi_h^p u|||_{\mathrm{DG}} \lesssim ??$$

Abstract error estimate (not for exam)

$$\|u - u_h\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

For the analysis, suppose, for simplicity, that

- the exact solution u is (at least) $u \in H^2(\mathcal{T}_h)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

Note: the final result holds in the more general case, but the analysis is more technical.

Abstract error estimate (not for exam)

$$\|u - u_h\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

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- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

Note: the final result holds in the more general case, but the analysis is more technical.

$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p \times V_h^p) \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Galerkin othogonality)} \\
&\lesssim \left|\!\left|\!\left|\Pi_h^p u - u\right|\!\right|\!\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Continuity on } H^2(\mathcal{T}_h) \times V_h^p)
\end{aligned}$$

Abstract error estimate (not for exam)

$$\|u - u_h\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

For the analysis, suppose, for simplicity, that

- the exact solution u is (at least) $u \in H^2(\mathcal{T}_h)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

Note: the final result holds in the more general case, but the analysis is more technical.

$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p \times V_h^p\text{)} \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Galerkin othogonality)} \\
&\lesssim \left|\left|\left|\Pi_h^p u - u\right|\right|\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Continuity on } H^2(\mathcal{T}_h) \times V_h^p\text{)}
\end{aligned}$$

Abstract error estimate (not for exam)

$$\left\|u - u_h\right\|_{\mathrm{DG}} \leq \left\|u - \Pi_h^p u\right\|_{\mathrm{DG}} + \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} \quad \text{(triangle inequality)}$$

For the analysis, suppose, for simplicity, that

- the exact solution u is (at least) $u \in H^2(\mathcal{T}_h)$
- the mesh $\mathcal{T}_h$ is quasi-uniform (h=mesh-size)
- $p_{\mathcal{K}} = p$ for any $\mathcal{K} \in \mathcal{T}_h$

Note: the final result holds in the more general case, but the analysis is more technical.

$$\begin{aligned}
\left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}}^2 &\lesssim \mathcal{A}(\Pi_h^p u - u_h, \Pi_h^p u - u_h) && \text{(Coercivity on } V_h^p \times V_h^p\text{)} \\
&\lesssim \mathcal{A}(\Pi_h^p u - u, \Pi_h^p u - u_h) && \text{(Galerkin othogonality)} \\
&\lesssim \left|\left|\left|\Pi_h^p u - u\right|\right|\right|_{\mathrm{DG}} \left\|\Pi_h^p u - u_h\right\|_{\mathrm{DG}} && \text{(Continuity on } H^2(\mathcal{T}_h) \times V_h^p\text{)}
\end{aligned}$$

$$\left\|u - u_h\right\|_{\mathrm{DG}} \lesssim \left|\left|\left|u - \Pi_h^p u\right|\right|\right|_{\mathrm{DG}}$$

Approximation bound for $|||u - \Pi_h^p u|||_{\mathrm{DG}}$ (not for exam)

Set $e_\pi = u - \Pi_h^p u$ and recall that

$$|||e_\pi|||_{\mathrm{DG}}^2 = \underbrace{\|\nabla_h e_\pi\|_{L^2(\Omega)}^2}_{(I)} + \underbrace{\sum_{F\in\mathcal{F}_h} \left\|\gamma^{1/2} \, [\![e_\pi]\!]\right\|_{0,F}^2}_{(II)} + \underbrace{\sum_{F\in\mathcal{F}_h} \left\|\gamma^{-1/2} \, \{\!\{\nabla_h e_\pi\}\!\}\right\|_{0,F}^2}_{(III)}$$

We make use of the following approximation result

An approximation result (not for exam) [Babuska, Suri, 1987]

Let $\mathcal{K} \in \mathcal{T}_h$ and let $v \in H^{s+1}(\mathcal{K})$, $s \geq 0$. Then, there exists a sequence of operators $\Pi_h^p v : H^{s+1}(\mathcal{K}) \longrightarrow \mathbb{P}^p(\mathcal{K})$, $p = 1, 2, \dots$ such that

$$\left\| v - \Pi_h^p v \right\|_{H^m(\mathcal{K})} \lesssim \frac{h^{\min(p,s)+1-m}}{p^{s+1-m}} \|v\|_{H^{s+1}(\mathcal{K})} \quad 0 \leq m \leq s+1$$

The hidden constant is independent of v, h and p, but depends on the shape-regularity of $\mathcal{K}$ an on s.

For the proof, see [Babuska, Suri, 1987], for $d = 2$. The case $d = 3$ is analogous.

Approximation bound for $\|u - \Pi_h^p u\|_{\mathrm{DG}}$ (cont'd) (not for exam)

Then, for the term (I) we get

$$(I) = \left\|\nabla_h(u - \Pi_h^p u)\right\|^2_{L^2(\Omega)} \lesssim \frac{h^{2\min(p,s)}}{p^{2s}} \|v\|^2_{H^{s+1}(\mathcal{T}_h)}$$

To estimate (II), we make use of the multiplicative trace inequality

Multiplicative trace inequality

$$\|\eta\|^2_{L^2(\partial\mathcal{K})} \lesssim \|\eta\|_{L^2(\mathcal{K})}\|\nabla\eta\|_{L^2(\mathcal{K})} + h^{-1}\|\eta\|^2_{L^2(\mathcal{K})} \quad \forall\,\eta \in H^1(\mathcal{K})$$

where the hidden constant depends on the shape regularity of $\mathcal{K}$ and on the dimension d.

Approximation bound for $\|\|u - \Pi_h^p u\|\|_{\mathrm{DG}}$ (cont'd) (not for exam)

$$
\begin{aligned}
(II) = \sum_{F \in \mathcal{F}_h} \left\| \gamma^{1/2} \, [\![e_\pi]\!] \right\|_{0,F}^2 &\lesssim \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \|e_\pi\|_{L^2(\partial \mathcal{K})}^2 \\
&\lesssim \frac{p^2}{h} \sum_{\mathcal{K} \in \mathcal{T}_h} \|e_\pi\|_{L^2(\mathcal{K})} \|\nabla e_\pi\|_{L^2(\mathcal{K})} + h^{-1} \|e_\pi\|_{L^2(\mathcal{K})}^2 \\
&\lesssim \frac{h^{2\min(p,s)}}{p^{2s-1}} \|u\|_{H^{s+1}(\mathcal{T}_h)}^2
\end{aligned}
$$

Analogously, we have

$$
\begin{aligned}
(III) = \sum_{F \in \mathcal{F}_h} \left\| \gamma^{-1/2} \, \{\!\{ \nabla_h e_\pi \}\!\} \right\|_{0,F}^2 &\lesssim \frac{h}{p^2} \sum_{\mathcal{K} \in \mathcal{T}_h} \|\nabla e_\pi\|_{L^2(\partial \mathcal{K})}^2 \\
&\lesssim \frac{h^{2\min(p,s)}}{p^{2s+1}} \|u\|_{H^{s+1}(\mathcal{T}_h)}^2
\end{aligned}
$$

Approximation bound for $|||u - \Pi_h^p u|||_{\mathrm{DG}}$ (cont'd) (not for exam)

If the exact solution u is sufficiently regular then

$$|||u - \Pi_h^p u|||_{\mathrm{DG}} \lesssim \frac{h^{\min(p,s)}}{p^{s-1/2}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

In particular, if $p \geq s$, the estimate becomes

$$|||u - \Pi_h^p u|||_{\mathrm{DG}} \lesssim \left(\frac{h}{p}\right)^s p^{1/2} \|u\|_{H^{s+1}(\mathcal{T}_h)}.$$

Energy-norm error estimate

Remark: recall the abstract error estimate: $\|u - u_h\|_{\mathrm{DG}} \lesssim |||u - \Pi_h^p u|||_{\mathrm{DG}}$

If u is sufficiently regular then

$$\|u - u_h\|_{\mathrm{DG}} \lesssim \frac{h^{\min(p,s)}}{p^{s-1/2}} \|u\|_{H^{s+1}(\mathcal{T}_h)}$$

For the SIP and IIP methods the above estimate holds provided that the penalty constant α is chosen sufficiently large.

- The bound is optimal in h and suboptimal in p by a factor $p^{1/2}$. See, for example, [Houston, Schwab, Suli, 2001], [Riviere, Wheeler, Girault, 1999], [Perugia, Schotzau, 2001].
- Optimal error estimates with respect to p can be shown using the projector of [Georgoulis & Suli, 2005] provided the solution belongs to a suitable augmented space, or whenever a continuous interpolant can be built; cf. [Stamm & Wihler, 2010].

L^2-norm error estimates by duality argument

An estimate for the L^2-error can be obtained by using a duality argument.

Elliptic regularity

Assume that Ω is such that the following ellitpic regularity result holds: for any $g \in L^2(\Omega)$, the solution z of the problem

$$-\Delta z = g \quad \text{in } \Omega \qquad\qquad z = 0 \quad \text{on } \partial\Omega$$

satisfies $z \in H^2(\Omega)$ and

$$\|z\|_{H^2(\Omega)} \lesssim \|g\|_{L^2(\Omega)}$$

This property was already used in the finite element Galerkin method for continuous piecewise polynomials (Chapter 1 of this course). A sufficient condition for this property to be true is that Ω is a convex polygonal/polyhedral domain.

L^2–norm error estimates by duality argument (cont'd)

If the exact solution $u \in H^s(\Omega)$, $s \geq 2$ and if u_h is the solution obtained with the SIP method ($\theta = 1$), it holds

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p,s)+1}}{p^{s+1/2}} \|u\|_{H^{s+1}(\Omega)}$$

The essential ingredient in the duality argument for proving L^2 estimates is the following adjoint consistency property

$$\mathcal{A}(v_h, z) = \int_\Omega f v_h \quad \forall\, v_h \in V_h^p$$

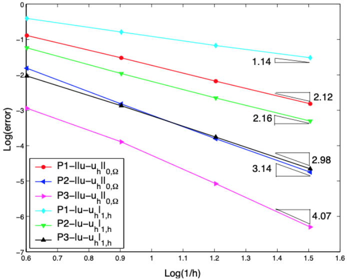
L^2–norm error estimates by duality argument (cont'd)

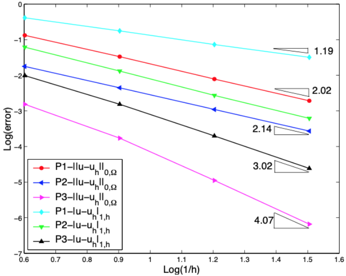
For NIP and IIP formulations it only holds

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \frac{h^{\min(p,s)}}{p^{s-1/2}} \|u\|_{H^{s+1}(\Omega)}$$

since the corresponding bilinear form is non-symmetric and does not satisfy the adjoint consistency property

$p = 1, 2, 3,\ \alpha = 10$, unstructured triangular meshes





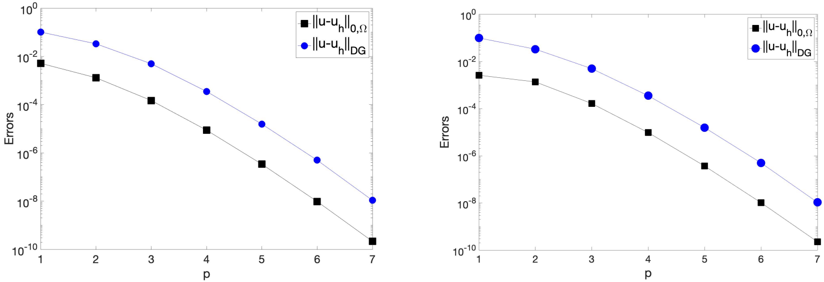
Computed errors in the L^2 and H^1 norms versus $1/h$ (loglog scale).

Remark. For NIP formulation we numerically observe that

$$\|u - u_h\|_{L^2(\Omega)} \lesssim \begin{cases} h^{p+1} & \text{if } p \text{ is odd} \\ h^{p} & \text{if } p \text{ is even} \end{cases}$$

Numerical results: SIP (left) and NIP (right) methods

$h = 1/2$, $\alpha = 10$, unstructured triangular meshes



Computed errors in the L^2 and H^1 norms versus p (semilog scale).