Numerical Analysis of Partial Differential Equations

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Lecture Notes
A.Y. 2022-2023

Course information

Teachers

- Lecturer: Prof. A. Quarteroni (alfio.quarteroni@polimi.it)
- Teaching Assistant: Dr. F. Regazzoni (francesco.regazzoni@polimi.it)
- Projects Tutor: Dr. M. Bucelli (michele.bucelli@polimi.it)

Timetable

- Monday (11.15-13.15) room T.0.2
- Tuesday (9.15-11.15) room 3.1.6
- Tuesday (13.15-15.15) room T.2.3
- Thursday (10.15-12.15) room 2.1.3

Textbook

A. Quarteroni, *Numerical Models for Differential Problems*, 3rd edition, Springer, 2018

Office hours

Upon demand (please arrange an appointment by email).

Instructions and guidelines for the exam

The final exam consists of two parts: a written examination and a course project.

Written examination.

- The written exam is 2 hours long with open questions. It takes place in the computer room. The exam covers all the theoretical and practical subjects considered at the lectures and exercise/lab sessions.
- Part of the questions and problems are solved numerically using MATLAB; the exam includes the implementation and programming of numerical algorithms.
- The problems mainly focus on definitions, application of important lemmas and theorems, and important examples. Light calculations may be needed.
- The use of any form of course material is not allowed. The questions will be answered without books, notes, etc.
- The maximum grade achievable with the written examination is **28 points**.

Course project. As part of the assigned work for this course, students are required to complete a project on their own. Each student can choose among two types of project (Type A or Type B).

Instructions and guidelines for the exam

Type A Projects

A list of scientific publications on a topic related to the content of the course is provided. Students have to investigate the topic in detail, and prepare a report elaborating on the material provided.

- The projects can be carried out either individually or in teams of two people.
- The assessment of the project will be based on subject knowledge matters, oral delivery, quality of the report and - in case of team project - demonstration of teamwork and individual contributions.
- The assessment of the project will be based on the following steps
  - The last week of the course every student will present a **brief overview** of her/his project. This presentation lasts 5 minutes plus questions. Students should use slides.
  - One week before the final presentations, students have to submit their final **project report** to the lecturer. The report should be nearly 10 pages, no more than 20 pages long.
  - The **final project's presentation** lasts 10 minutes plus questions
- The maximum grade achievable is **3 points**.

Instructions and guidelines for the exam

Type B Projects

Students work on a project related to the course content. Each project has one or more tutors (MOX professors and researchers), who define the objectives of the work and follow the students as the project progresses. The projects require software implementations and numerical tests (in some cases, a starting code is provided).

- The projects are be carried out preferably in teams of three people.
- Students are allowed to choose a project that is related to their master thesis or to a project of another course.
- The assessment of the project will be based on subject knowledge matters, oral delivery, quality of the report and of the codes and - in case of team project - demonstration of teamwork and individual contributions.

Instructions and guidelines for the exam

Type B Projects

- The assessment of the project will be based on the following steps
  - The last week of the course every student will present a **brief overview** and plan of her/his project. This presentation lasts 5 minutes plus questions. Students should use slides.
  - One week before the final presentations, students have to submit their final **project report and the code** to the lecturer. The report should be nearly 20 pages, no more than 30 pages long.
  - The **final project's presentation** lasts 15 minutes plus questions
- The maximum grade achievable is **4 points**.

Projects: general rules

- The final exam mark is given by the sum of written exam grade and project grade. Both types of projects can achieve the *30 e lode* (merit) grade. The assignment of merit is at the discretion of the lecturer.
- The list of projects will be presented at the beginning of the course. Students will decide between Type A and Type B projects and express their preferences regarding the topic. In case of project proposed by the student, the topic has always to be agreed on by the lecturer.
- The scheduling of project final presentations will be made available on WeBeep. Typically, one session in correspondence of each written exam (the same day, or a few days apart) is scheduled. Please note that the project does not necessarily have to be discussed in the same session as the written examination.

Projects: general rules

- Projects assigned in the current academic year (2022-2023) must be discussed at the latest during the June 2024 exam session.
- Note that it is not mandatory to choose a project in case you know in advance that you will not have time to work on it during the designated period (i.e. so as to complete it by June 2024). In this case, you will be able to choose a project during the 2024 call or later.

Boundary value problems

Consider

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ + \text{ B.C.} & \text{on } \partial\Omega \end{cases} \tag{1}$$

- Ω: open bounded domain in $\mathbb{R}^d$, with $d = 2, 3$
- $\partial\Omega$: boundary of Ω
- f: given
- B.C. to be prescribed according to $\mathcal{L}$
- $\mathcal{L}$: 2nd order differential operator. Examples:
  1. $\mathcal{L}u = -\operatorname{div}(\mu\nabla u) + \mathbf{b}\cdot\nabla u + \sigma u$ (non-conservative form)
  2. $\mathcal{L}u = -\operatorname{div}(\mu\nabla u) + \operatorname{div}(\mathbf{b}u) + \sigma u$ (conservative form)

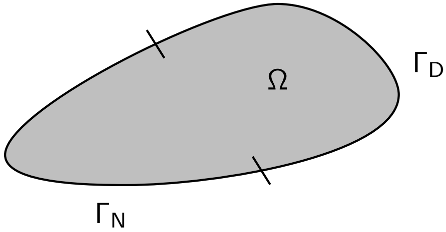
$\mu \in L^\infty(\Omega)$, $\mu(\mathbf{x}) \geq \mu_0 > 0$ $\quad$ $\sigma \in L^2(\Omega)$

$\mathbf{b} \in (L^\infty(\Omega))^d$ $\quad$ $f \in L^2(\Omega)$

Example

$$\begin{cases} \mathcal{L}u = -\operatorname{div}(\mu \nabla u) + \mathbf{b} \cdot \nabla u + \sigma\, u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_{\mathrm{D}} \\ \mu \nabla u \cdot \mathbf{n} = g & \text{on } \Gamma_{\mathrm{N}} \end{cases} \tag{2}$$

$$g \in L^2(\Gamma_{\mathrm{N}}), \qquad \partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}, \qquad \mathring{\Gamma}_{\mathrm{D}} \cap \mathring{\Gamma}_{\mathrm{N}} = \emptyset$$



Idea: v = suitable test function $\rightarrow$ multiply (2) by v: $(\mathcal{L}u)\, v = f\, v$
$\rightarrow$ integrate in Ω $\rightarrow$ apply integration by parts

$$\underbrace{\int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \mathbf{b} \cdot \nabla u\, v + \int_\Omega \sigma\, u\, v}_{=:a(u,v)}$$
$$= \int_\Omega f\, v + \underbrace{\int_{\Gamma_D} \mu \nabla u \cdot \mathbf{n}\, v}_{=0 \text{ if } v|_{\Gamma_D} = 0} + \int_{\Gamma_N} \underbrace{\mu \nabla u \cdot \mathbf{n}}_{=g}\, v \tag{3}$$

whence:

$$\begin{cases} \text{Find } u \in V = \{v \in H^1(\Omega), v|_{\Gamma_D} = 0\} =: H^1_{\Gamma_D}(\Omega) \\ a(u,v) = \langle F, v \rangle \qquad \forall\, v \in V \end{cases} \tag{4}$$

- $a\colon V \times V \rightarrow \mathbb{R}$ bilinear form
- $F\colon V \rightarrow \mathbb{R}$ linear form s.t. $\langle F, v\rangle \equiv F(v) = \int_\Omega f\, v + \int_{\Gamma_N} g\, v$

Theorem (Lax-Milgram)

Assume that:

i. V *Hilbert space with norm* $\|\cdot\|$ *and inner product* $(\cdot,\cdot)$
ii. $F \in V'$: $|F(v)| \leq \|F\|_{V'}\|v\| \, \forall\, v \in V$
iii. a *continuous:* $\exists M > 0$: $|a(u,v)| \leq M\|u\|\|v\| \, \forall\, u, v \in V$
iv. a *coercive:* $\exists \alpha > 0$: $a(v,v) \geq \alpha\|v\|^2 \quad \forall\, v \in V$

Then, there exists a unique solution u *of* (4).

Remark: V' = dual space of V = { linear and bounded (i.e. continuous,) maps from V to $\mathbb{R}$ } with norm:

$$\|F\|_{V'} = \sup_{v \in V \setminus \{0\}} \frac{|F(v)|}{\|v\|_V}$$

Moreover:

$$\alpha\|u\|^2 \leq a(u,u) = F(u) \leq \|F\|_{V'}\|u\|$$

Hence:

$$\|u\| \leq \frac{\|F\|_{V'}}{\alpha} \quad \rightarrow \quad \text{stability / continuous dependence on data}$$

Questions:

- How to prove assumptions of Lax-Milgram Lemma?

  $\rightarrow$ case by case
- what if the assumptions are not satisfied (in particular, iv)?

  $\rightarrow$ try Nečas Theorem

Nečas theorem

To formulate Nečas Theorem, consider a slightly more general problem than (4):

Problem

$$\begin{cases} \text{Find } u \in V \\ a(u, w) = \langle F, w \rangle \qquad \forall\, w \in W \end{cases} \tag{5}$$

$a\colon V \times W \to \mathbb{R}$ bilinear form
$F\colon W \to \mathbb{R}$ linear and continuous (that is $F \in W'$)

Theorem (Nečas)

Assume that $F \in W'$. Consider the following conditions.

- **A** *a continuous:* $\exists M > 0 \colon |a(u,w)| \leq M \|u\|_V \|w\|_W \; \forall \, u \in V, w \in W$
- **B** *inf-sup condition:* $\exists \alpha > 0 \colon \forall v \in V \quad \sup_{w \in W \setminus \{0\}} \dfrac{a(v,w)}{\|w\|_W} \geq \alpha \|v\|_V$
- **C** $\forall \, w \in W, w \neq 0, \exists \, v \in V \colon a(v,w) \neq 0$

Conditions A – B – C are necessary and sufficient for the existence and uniqueness of a solution of (5), for any $F \in W'$.
Moreover (continuous dependence on data):

$$\|u\|_V \leq \frac{1}{\alpha} \|F\|_{W'} \tag{6}$$

Proof of Thm. 3: see PDE2 course by Prof. S. Salsa.

N.B. **B** can be restated as: $\exists \alpha > 0 \colon \inf_{v \in V} \sup_{w \in W \setminus \{0\}} a(v,w) \geq \alpha \|v\|_V \|w\|_W$

Remark

If $W = V$, then Thm. 3 provides necessary and sufficient conditions for existence and uniqueness of solutions of problem. 4 (as a special case).

Note that in this case:

A $\iff$ iii

Whereas:

B yields:

$$\exists \alpha > 0 \colon \forall v \in V \sup_{w \in V \setminus \{0\}} \frac{a(v,w)}{\|w\|_V} \geq \alpha \|v\|_V \tag{7}$$

C yields:

$$\forall\, w \in V, w \neq 0, \exists\, v \in V \colon a(v,w) \neq 0 \tag{8}$$

(7) and (8) are more general / weaker conditions than coercivity (i.e. iv).
Indeed, by taking $w = v$, iv $\implies$ (7), (8)

Approximation

Galerkin method (for problem (4))

$$\text{Find } u_h \in V_h : a(u_h, v_h) = \langle F, v_h \rangle \quad \forall\, v_h \in V_h \tag{G}$$

Petrov-Galerkin method (for problem (5))

$$\text{Find } u_h \in V_h : a(u_h, w_h) = \langle F, w_h \rangle \quad \forall\, w_h \in W_h \tag{PG}$$

where:

- $\{V_h, h > 0\}$ finite dimensional subspaces of V
- $\{W_h, h > 0\}$ finite dimensional subspaces of W
- $\dim V_h = \dim W_h = N_h < +\infty$

Existence and uniqueness

Corollary of Lax-Milgram Lemma (being V_h a closed subspace of V)

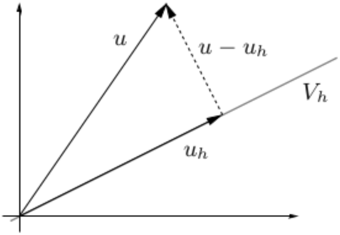
Stability

Uniform bound w.r.t. h:

$$\boxed{\|u_h\| \leq \frac{\|F\|_{V'}}{\alpha}}$$

Consistency = Galerkin orthogonality

$$\boxed{a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h}$$



Convergence (Céa Lemma)

$$
\begin{aligned}
\alpha\|u-u_h\|^2 &\leq a(u-u_h, u-u_h) \\
&= a(u-u_h, u-v_h) + \underbrace{a(u-u_h, v_h-u_h)}_{=0 \text{ (Galerkin orthogonality)}} \\
&\leq M\|u-u_h\|\|u-v_h\| \quad \forall\, v_h \in V_h
\end{aligned}
$$

Hence:

$$
\|u-u_h\| \leq \frac{M}{\alpha}\|u-v_h\| \quad \forall\, v_h \in V_h
$$

Finally:

$$
\boxed{\|u-u_h\| \leq \frac{M}{\alpha} \inf_{v_h\in V_h} \|u-v_h\|}
$$

Analysis of (G)

Assumption: *space saturation*

$$\forall\, v \in V \qquad \lim_{h\to 0} \inf_{v_h \in V_h} \|v - v_h\| = 0$$

Then:

$$\boxed{\lim_{h\to 0} \|u - u_h\| = 0}$$

convergence

Rate of convergence (ex: Finite Elements)

$\mathcal{T}_h = \bigcup K$: triangulation of Ω (of Ω_h actually)
$V_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}) \colon v_h|_K \in \mathbb{P}^r(K)\, \forall\, K \in \mathcal{T}_h, v_h|_{\Gamma_\mathrm{D}} = 0\}$, for $r \geq 1$

$$\inf_{v_h \in V_h} \|u - v_h\| \leq \|u - \overline{u}_h\| \text{ (suitable choice of } \overline{u}_h)$$

Example: $\overline{u}_h = \Pi_h^r u$ Finite Element interpolant

$$\boxed{\|u - \overline{u}_h\| \leq C\, h^r\, |u|_{H^{r+1}(\Omega)}}$$

provided $u \in V \cap H^{r+1}(\Omega)$.

Examples of polynomial spaces

1D ($\mathbb{P}^r$):

$$p(x) = \sum_{k=0}^{r} a_k \, x^k$$

2D ($\mathbb{P}^r$):

$$p(x_1, x_2) = \sum_{\substack{k=0,\dots,r \\ m=0,\dots,r \\ k+m\leq r}} a_{km} \, x_1^k x_2^m$$

3D tetrahedra ($\mathbb{P}^r$):

$$p(x_1, x_2, x_3) = \sum_{\substack{k=0,\dots,r \\ m=0,\dots,r \\ n=0,\dots,r \\ k+m+n\leq r}} a_{kmn} \, x_1^k x_2^m x_3^n$$

3D hexahedra ($\mathbb{Q}^r$):

$$p(x_1, x_2, x_3) = \sum_{\substack{k=0,\dots,r \\ m=0,\dots,r \\ n=0,\dots,r}} a_{kmn} \, x_1^k x_2^m x_3^n$$

Basis functions

Having defined a basis $\{\phi_j(\mathbf{x})\}_{j=1}^{N_h}$ for the space V_h, each function $v_h \in V_h$ can be expanded as a linear combination of elements of the basis, suitably weighted by the coefficients $\{v_j\}_{j=1}^{N_h}$:

$$v_h(\mathbf{x}) = \sum_{j=1}^{N_h} v_j \, \phi_j(\mathbf{x})$$

We will use the notation $\mathbf{v} = (v_1, \ldots, v_{N_h})^T$ to denote a vector $\mathbf{v} \in \mathbb{R}^{N_h}$ collecting all the basis coefficients (also called *degrees of freedom*).
A basis is called **lagrangian** if it satisfies the following property:

$$\phi_i(\mathbf{x}_j) = \delta_{ij} \qquad \forall\, 1 \leq i, j \leq N_h$$

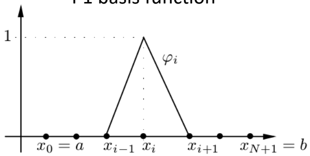
for a suitable collection of points $\{\mathbf{x}_j\}_{j=1}^{N_h}$ called *nodes*. When the basis is lagrangian, the following property holds:

$$v_h(\mathbf{x}_j) = v_j \qquad \forall\, 1 \leq j \leq N_h$$

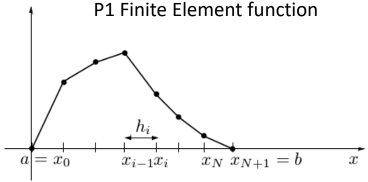
Examples of triangulations and FE functions

1D

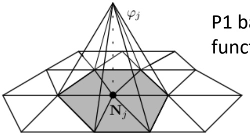
P1 basis function



P1 Finite Element function

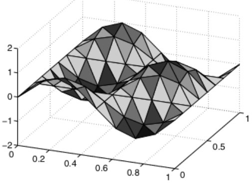


2D

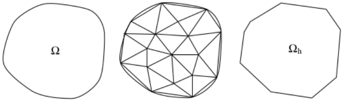


P1 basis function

P1 Finite Element function



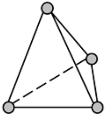
triangulation

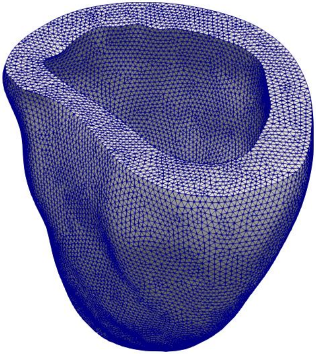


Examples of triangulations and FE functions

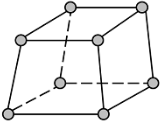
3D

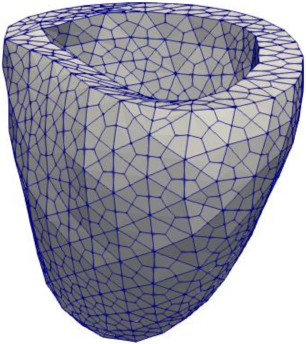
tetrahedra





hexahedra





Condition i

V Hilbert space, as it is a closed subspace of the Hilbert space $H^1(\Omega)$.

$(u, v) = \int_\Omega u\, v + \int_\Omega \nabla u \cdot \nabla v$ scalar product

$\|v\| = \left(\int_\Omega v^2 + \int_\Omega |\nabla v|^2\right)^{1/2}$ norm

Condition ii $(F \in V')$

$$
\begin{aligned}
|\langle F, v\rangle| &= \left| \int_\Omega f\, v + \int_{\Gamma_N} g\, v \right| \\
&\leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)} \|v\|_{L^2(\Gamma_N)} && \text{(Cauchy-Schwarz inequality)} \\
&\leq \|f\|_{L^2(\Omega)} \|v\| + \|g\|_{L^2(\Gamma_N)} C_{\text{trace}} \|v\| && \text{(definition of } \|\cdot\| \text{ and trace ineq.)} \\
&= \left(\|f\|_{L^2(\Omega)} + C_{\text{trace}} \|g\|_{L^2(\Gamma_N)}\right) \|v\|
\end{aligned}
$$

$$
\|F\|_{V'} = \sup_{v \neq 0} \frac{|\langle F, v\rangle|}{\|v\|} \leq \|f\|_{L^2(\Omega)} + C_{\text{trace}} \|g\|_{L^2(\Gamma_N)} < +\infty
$$

Condition iii (a continuous)

$$
\begin{aligned}
|a(u,v)| &\leq \|\mu\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + \|\mathbf{b}\|_{L^\infty}\|\nabla u\|_{L^2}\|v\|_{L^2} \\
&\quad + \|\sigma\|_{L^2}\|u\|_{L^4}\|v\|_{L^4} \\
&\leq \underbrace{\left(\|\mu\|_{L^\infty} + \|\mathbf{b}\|_{L^\infty} + \|\sigma\|_{L^2}\right)}_{M}\|u\|\|v\| \qquad \forall u.v \in V
\end{aligned}
$$

N.B. Sobolev embedding: $\|v\|_{L^4} \leq \|v\| \quad \forall\, v \in H^1$

Condition iv (a coercive)

This condition holds under the assumptions:

$$\boxed{\sigma - \frac{1}{2}\operatorname{div}\mathbf{b} \geq 0 \text{ in } \Omega, \qquad \mathbf{b}\cdot\mathbf{n} \geq 0 \text{ on } \Gamma_N} \tag{9}$$

$$\begin{aligned} a(v,v) &= \int_\Omega \mu\,|\nabla v|^2 + \underbrace{\int_\Omega \mathbf{b}\cdot\frac{1}{2}\nabla\left(v^2\right)}_{\int_\Omega -\frac{1}{2}\operatorname{div}\mathbf{b}(v^2)+\frac{1}{2}\int_{\partial\Omega}\mathbf{b}\cdot\mathbf{n}\,v^2} + \int_\Omega \sigma\,v^2 \\ &= \int_\Omega \mu\,|\nabla v|^2 + \int_\Omega \left(\sigma - \frac{1}{2}\operatorname{div}\mathbf{b}\right) v^2 + \frac{1}{2}\int_{\Gamma_N} \mathbf{b}\cdot\mathbf{n}\,v^2 \\ &\geq \mu_0 \|\nabla v\|^2_{L^2} \end{aligned}$$

Poincaré inequalilty

If Γ_D is a set of positive measure (in 1D, it is sufficient that Γ_D contains a single point), then:

$$\exists\, C_\mathrm{P} > 0 \colon \|v\|_{L^2(\Omega)} \leq C_\mathrm{P} \|\nabla v\|_{L^2(\Omega)} \qquad \forall\, v \in H^1_{\Gamma_\mathrm{D}}(\Omega)$$

Then: $\|v\|^2 = \|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)} \leq \left(1 + C_\mathrm{P}^2\right) \|\nabla v\|^2_{L^2(\Omega)}$

Whence: $\|\nabla v\|^2_{L^2(\Omega)} \geq \left(1 + C_\mathrm{P}^2\right)^{-1} \|v\|^2$

In conclusion (coercivity):

$$a(v, v) \geq \frac{\mu_0}{1 + C_\mathrm{P}^2} \|v\|^2$$

Remark: what if (9) is not satisfied?
Try with Nečas Theorem.

The following assumptions are equivalent to those of Nečas Theorem:

- a Weak coercivity (Gårding inequality)
  $\exists\, \alpha, \lambda > 0\colon a(v,v) \geq \alpha \|v\|^2 - \lambda \|v\|^2_{L^2(\Omega)} \quad \forall v \in V$
- b Uniqueness condition:
  $(a(u,v) = 0 \,\forall\, v \in V) \implies u = 0$
  Typically proven by maximum principle (see PDE2 course by Prof. S. Salsa)

Similar result in finite dimension. Consider problem (G).

Condition a holds in the subspace as well:
$\exists\, \alpha, \lambda > 0\colon a(v_h,v_h) \geq \alpha \|v_h\|^2 - \lambda \|v_h\|^2_{L^2(\Omega)} \quad \forall v_h \in V_h$

Condition a does not imply:
$(a(u_h,v_h) = 0 \,\forall\, v_h \in V_h) \implies u_h = 0.$
the latter property needs to be proven!

Conditioning of the stiffness matrix

Reminder: if A is spd, then $K_2(A) = \dfrac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$

Proposition

If $a(\cdot,\cdot)$ is symmetric and coercive, then A is spd

Proof.

Symmetry: $A_{ij} = a(\phi_j, \phi_i) = a(\phi_i, \phi_j) = A_{ji}$

For any $\mathbf{v} \in \mathbb{R}^{N_h}$:

$$\begin{aligned} \mathbf{v}^T A \mathbf{v} &= \sum_{i,j} A_{ij} v_i v_j = \sum_{i,j} a(\phi_j, \phi_i) v_i v_j \\ &= a\Big(\sum_j v_j \phi_j, \sum_i v_i \phi_i\Big) = a(v_h, v_h) \geq \alpha \|v_h\|^2 > 0 \end{aligned}$$

if ($v_h \neq 0 \iff \mathbf{v} \neq \mathbf{0}$). Hence, A is pos. def. $\square$

A-norm

Definition

If A is spd, we define the *A-norm* of $\mathbf{v}$ as

$$\begin{aligned}\|\mathbf{v}\|_A &:= (A\mathbf{v}, \mathbf{v})^{1/2} \\ &= \left(\sum_{i,j} a_{ij}\, v_i\, v_j\right)^{1/2}\end{aligned}$$

Conditioning of the stiffness matrix

We can prove that $\exists C_1, C_2 > 0 : \forall \lambda_h$ eigenvalue of A:

$$\alpha\, C_1\, h^d \leq \lambda_h \leq M\, C_2\, h^{d-2} \qquad d = 1, 2, 3$$

whence:

$$\frac{\lambda_{\max}(A)}{\lambda_{\min}(A)} \leq \frac{M\, C_2}{\alpha\, C_1} h^{-2}$$

This is indeed an asymptotic estimate (also a lower bound, with a different constant). Then:

$$\boxed{K_2(A) = \mathcal{O}(h^{-2})}$$

Implication. If we use the conjugate gradient method to solve $A\mathbf{u} = \mathbf{f}$, then:

$$\|\mathbf{u}^{(k)} - \mathbf{u}\|_A \leq 2 \left(\frac{\sqrt{K_2(A)} - 1}{\sqrt{K_2(A)} + 1} \right)^k \|\mathbf{u}^{(0)} - \mathbf{u}\|_A$$

Same with gradient method, with $\sqrt{K_2(A)}$ replaced by $K_2(A)$.
$\implies$ need for preconditioners!

Remark. How many iterations to reduce the relative error by a factor 10 with the gradient method?

$$\left(\frac{K_2(A) - 1}{K_2(A) + 1} \right)^k < 10^{-1} \iff k > \frac{\log(10)}{\log\left(\frac{K_2(A)-1}{K_2(A)+1}\right)}$$

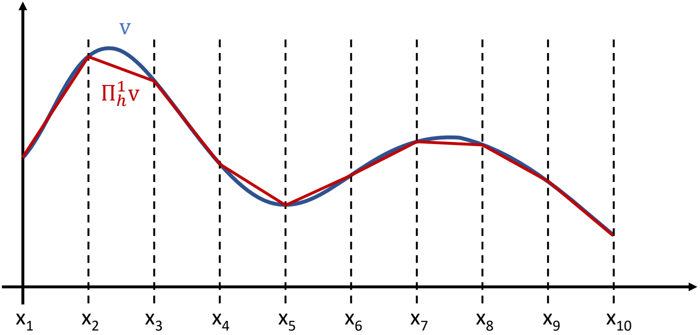
With the conjugate gradient method:

$$k > \frac{\log(10)}{\log\left(\frac{\sqrt{K_2(A)}-1}{\sqrt{K_2(A)}+1}\right)}$$

Interpolation Error Estimate

Use continuous FE of degree $r \geq 1$. Assume $v \in H^{r+1}(\Omega)$ let $\Pi_h^r v$ be the finite element interpolant of v at the FE nodes, that is:

$$\Pi_h^r v \in X_h^r, \qquad \Pi_h^r(\mathbf{x}_i) = v(\mathbf{x}_i) \quad \forall \text{ node } \mathbf{x}_i \text{ of } \mathcal{T}_h$$



Interpolation Error Estimate

Then for $m = 0, 1$, $\exists C = C(r, m, \hat{k})$ such that:

$$|v - \Pi_h^r v|_{H^m(\Omega)} \leq C \left(\sum_{K \in \mathcal{T}_h} h_K^{2(r+1-m)} |v|^2_{H^{r+1}(K)} \right)^{1/2} \tag{10}$$

$h_K = \mathrm{diam}(K)$
since $h_K \leq h \quad \forall K$, this yields:

$$\boxed{|v - \Pi_h^r v|_{H^m(\Omega)} \leq C\, h^{r+1-m} |v|_{H^{r+1}(K)} \qquad \forall\, v \in H^{r+1}(\Omega), m = 0, 1} \tag{11}$$

Finite Element error estimate

Recall that:

$$
\begin{aligned}
\|u - u_h\| = \|u - u_h\|_{H^1(\Omega)} \\
&\leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \\
&\leq \frac{M}{\alpha} \|u - \Pi_h^r u\|_{H^1(\Omega)}
\end{aligned}
$$

Using (10):

$$
\|u - u_h\| \leq C \frac{M}{\alpha} \left(\sum_{K \in \mathcal{T}_h} h_K^{2r} |u|^2_{H^{r+1}(\Omega)} \right)^{1/2} \tag{12}
$$

Using (11):

$$
\|u - u_h\| \leq C \frac{M}{\alpha} h^r |u|_{H^{r+1}(\Omega)} \tag{13}
$$

Error estimate for $\|u - u_h\|_{L^2(\Omega)}$

Adjoint form

Consider a bilinear form $a\colon V \times V \to \mathbb{R}$. The *adjoint form* a^* is defined as:

$$
\begin{aligned}
&a^*\colon V \times V \to \mathbb{R} \\
&a^*(v, w) = a(w, v) \quad \forall\, v, w \in V
\end{aligned}
$$

Remark: If a is symmetric, then $a^* = a$

$\forall\, g \in L^2(\Omega)$, consider the adjoint problem:

$$
\begin{cases}
\text{Find } \phi = \phi(g) \in V \\
a^*(\phi, v) = (g, v) = \int_\Omega g\, v \qquad \forall\, v \in V
\end{cases}
\tag{14}
$$

Assume that $\phi \in H^2(\Omega) \cap V$ (elliptic regularity)

Example

Consider $\mathcal{L} = -\Delta$. Then the solution of

$$\begin{cases} -\Delta\phi = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $\phi \in H^2(\Omega)$. Moreover:

$$\exists\, C_1 > 0: \quad \|\phi(g)\|_{H^2(\Omega)} \leq C_1 \|g\|_{L^2(\Omega)} \tag{15}$$

Take now $g = e_h = u - u_h$ in (14). Then:

$$\begin{aligned} \|e_h\|^2_{L^2(\Omega)} &= (e_h, e_h) = a^*(\phi, e_h) = a(e_h, \phi) \\ &= a(e_h, \phi - \phi_h) \qquad \text{(Galerkin orthogonality, for } \phi_h \in V_h) \\ &\leq M \|e_h\|_{H^1(\Omega)} \|\phi - \phi_h\|_{H^1(\Omega)} \end{aligned}$$

Take $\phi_h = \Pi^1_h \phi$. Then:

$$\begin{aligned} \|e_h\|^2_{L^2(\Omega)} &\leq M \|e_h\|_{H^1(\Omega)} \|\phi - \Pi^1_h \phi\|_{H^1(\Omega)} \\ &\leq M \|e_h\|_{H^1(\Omega)}\, C_2\, h |\phi|_{H^2(\Omega)} \qquad \text{(for (11) with } m = r = 1) \\ &\leq M \|e_h\|_{H^1(\Omega)}\, C_2\, h\, C_1 \|e_h\|_{L^2(\Omega)} \qquad \text{(for (15))} \end{aligned}$$

Whence:

$$\begin{aligned} \|e_h\|_{L^2(\Omega)} &\leq M\, C_1\, C_2\, h \|e_h\|_{H^1(\Omega)} \\ &\leq M\, C_1\, C_2\, h\, C_3 h^r |u|_{H^{r+1}(\Omega)} \qquad \text{(for (13))} \end{aligned}$$

Conclusion:

$$\boxed{\|e_h\|_{L^2(\Omega)} \leq \overline{C} h^{r+1} |u|_{H^{r+1}(\Omega)}} \tag{16}$$