Numerical Analysis of Partial Differential Equations

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Lecture Notes
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Spectral Element Methods

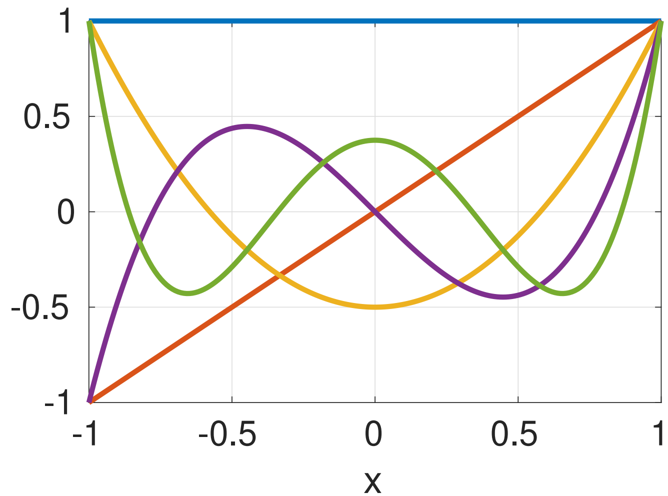
Legendre polynomials

The Legendre polynomials $\{L_k(x) \in \mathbb{P}_k,\ k = 0, 1, \ldots,\}$ are the eigenfunctions of the singular Sturm-Liouville problem

$$((1 - x^2)L_k'(x))' + k(k+1)L_k(x) = 0\ , \qquad -1 < x < 1.$$

They satisfy the recurrence relation

$$\boxed{\begin{aligned} & L_0(x) = 1, \quad L_1(x) = x, \ \text{and for } k \geq 1 \\ & L_{k+1}(x) = \frac{2k+1}{k+1} x L_k(x) - \frac{k}{k+1} L_{k-1}(x), \end{aligned}}$$



They are mutually orthogonal over the interval $(-1, 1)$ w.r.t the weight function $w(x) \equiv 1$, i.e.

$$\boxed{\int_{-1}^{1} L_k(x) L_m(x)\, dx = \begin{cases} \frac{2}{2k+1} & \text{if } k = m \\ 0 & \text{if } k \neq m \end{cases}}$$

Legendre projection

The **expansion** of any $u \in L^2(-1,1)$ in terms of the L_k's is

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k L_k(x),$$

and $\hat{u}_k = (k + \frac{1}{2}) \int_{-1}^{1} u(x) L_k(x)\, dx$ are named **modes** of the expansion.
The **truncated Legendre series** of u (L^2-projection of u over $\mathbb{P}_N$) is

$$P_N u = \sum_{k=0}^{N} \hat{u}_k L_k.$$

For any $u \in H^s(-1,1)$ with $s \in \mathbb{N}$, the **projection error** $(u - P_N u)$ satisfies the estimates

$$\|u - P_N u\|_{L^2(-1,1)} \leq C N^{-s} \|u\|_{H^s(-1,1)}, \quad \forall s \geq 0$$

$$\|u - P_N u\|_{L^2(-1,1)} \leq C N^{-s} |u|_{H^s(-1,1)}, \quad \forall s \leq N+1,$$

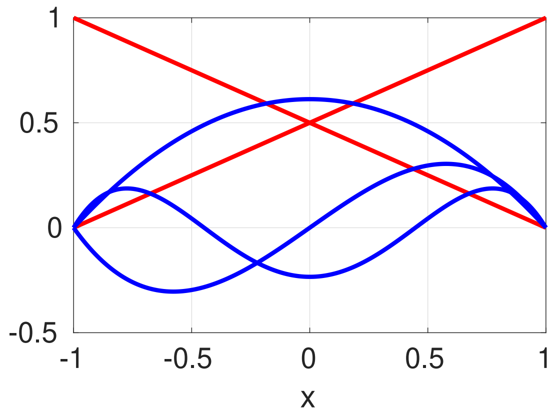
where $|u|_{H^s(-1,1)} = \|u^{(s)}\|_{L^2(-1,1)}$ is the **seminorm** in $H^s(-1,1)$.

The Legendre basis is adapted to the boundary

Remark: The Legendre basis $\{L_k\}$ is not suited to impose Dirichlet boundary conditions.

Boundary adapted modal basis (or **modified C^0 basis**)

$$\psi_0(x) = \frac{1}{2}(L_0(x) - L_1(x)) = \frac{1-x}{2}$$
$$\psi_N(x) = \frac{1}{2}(L_0(x) + L_1(x)) = \frac{1+x}{2}$$
$$\psi_{k-1}(x) = \frac{1}{\sqrt{2(2k-1)}}(L_{k-2}(x) - L_k(x))$$
for $k = 2, \ldots, N, \ -1 \leq x \leq 1$



$$u_N(x) = \sum_{k=0}^{N} \tilde{u}_k \psi_k(x) = \tilde{u}_0 \psi_0(x) + \sum_{k=1}^{N-1} \tilde{u}_k \psi_k(x) + \tilde{u}_N \psi_N(x), \ \text{for any } u_N \in \mathbb{P}_N$$

with $\tilde{u}_0 = u_N(-1)$ and $\tilde{u}_N = u_N(1)$.

If $u_N \in \mathbb{P}_N^0 = \{v_N \in \mathbb{P}_N : \ v_N(-1) = v_N(1) = 0\}$, then $u_N(x) = \sum_{k=1}^{N-1} \tilde{u}_k \psi_k(x)$.

Remark: Now we can easily impose Dirichlet b.c. but we lose the orthogonality property of the Legendre polynomials.

Diffusion-transport-reaction problem

Given: $\Omega = (-1, 1)$, $\mu, b, \sigma > 0$ const., $f : \Omega \to \mathbb{R}$,
look for $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -(\mu u')' + (bu)' + \sigma u = f & \text{in } \Omega, \\ u(-1) = 0, \quad u(1) = 0. \end{cases}$$

Set $V = H_0^1(\Omega)$. The **weak form** of the differential problem reads:
given $f \in L^2(\Omega)$,

$$\text{find } u \in V \text{ s.t.} \quad a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in V,$$

where

$$a(u, v) = \int_\Omega (\mu u' - bu)\, v'\, dx + \int_\Omega \sigma uv\, dx,$$
$$(f, v)_{L^2(\Omega)} = \int_\Omega f\, v\, dx.$$

Galerkin formulation

Set $V_N = \mathbb{P}_N^0$.

Spectral (Modal) Galerkin	$?u_N \in V_N :$	$a(u_N, v_N) = (f, v_N)_{L^2(\Omega)}$	$\forall v_N \in V_N$

Expand u_N w.r.t. the modal basis: $u_N(x) = \sum_{k=1}^{N-1} \tilde{u}_k \psi_k(x)$

and choose $v_N(x) = \psi_i(x)$ for any $i = 1, \ldots, N-1$.

The Spectral (Modal) Galerkin discretization of the weak problem reads:

look for $\tilde{\mathrm{u}} = [\tilde{u}_k]_{k=1}^{N-1}$,

$$\sum_{k=1}^{N-1} a(\psi_k, \psi_i)\tilde{u}_k = (f, \psi_i)_{L^2(\Omega)} \qquad \text{for any } i = 1, \ldots, N-1$$

A-priori error analysis for Spectral (Modal) Galerkin

Let $u_N \in V_N$ be the solution of

$$a(u_N, v_N) = (f, v_N)_{L^2(\Omega)} \qquad \forall v_N \in V_N,$$

If $u \in H^{s+1}(\Omega)$, with $s \geq 0$, thanks to Céa Lemma it holds:

$$\boxed{\|u - u_N\|_{H^1(\Omega)} \leq C(s) \left(\frac{1}{N}\right)^s \|u\|_{H^{s+1}(\Omega)}}$$

u_N converges with spectral accuracy with respect to N to the exact solution when the latter is smooth.

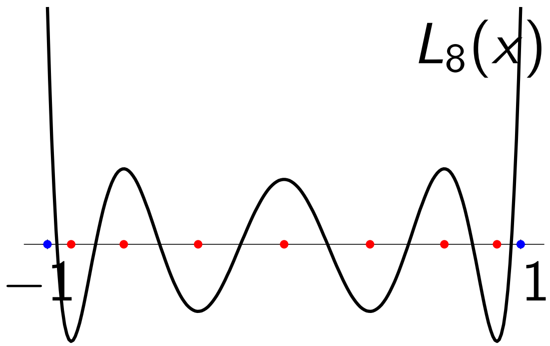
Remark. The stiffness matrix is full and difficult to compute (high order polynomials multiplied by variable coefficients to be integrated). Moreover, the mass matrix $M_{ij} = (\psi_j, \psi_i)_{L^2(-1,1)}$ is a full-band matrix and its inversion (e.g. for time-dependent problems) is quite expensive.

$\Rightarrow$ Lagrange **nodal** basis instead of the modal one.

Legendre-Gauss-Lobatto quadrature formulas

The Legendre-Gauss-Lobatto **(LGL) nodes and weigths** are

$$x_0 = -1,\ x_N = 1, \quad x_j \text{ zeros of } L'_N\ ,\ j = 1, \dots, N-1,$$
$$w_j = \frac{2}{N(N+1)} \frac{1}{[L_N(x_j)]^2}, \quad j = 0, \dots, N\ .$$



Interpolatory quadrature formulas

$$\boxed{\int_{-1}^{1} f(x)dx \simeq \sum_{j=0}^{N} f(x_j) w_j}$$

Degree of exactness $= 2N-1$, i.e.

$$\int_{-1}^{1} f(x)dx = \sum_{j=0}^{N} f(x_j) w_j \quad \forall f \in \mathbb{P}_{2N-1}. \qquad (1)$$

Discrete inner product and norm

Discrete inner product in $L^2(-1,1)$:

$$(u,v)_N = \sum_{j=0}^{N} u(x_j)v(x_j)w_j \tag{2}$$

Degree of exactness $= 2N-1$**,** i.e.:

$$(u,v)_{L^2(\Omega)} = (u,v)_N \qquad \text{only if } u\,v \in \mathbb{P}_{2N-1}. \tag{3}$$

Discrete norm in $L^2(-1,1)$:

$$\|u\|_N = (u,u)_N^{1/2}. \tag{4}$$

Norm equivalence: $\exists c_1,\ c_2 > 0$ s.t.

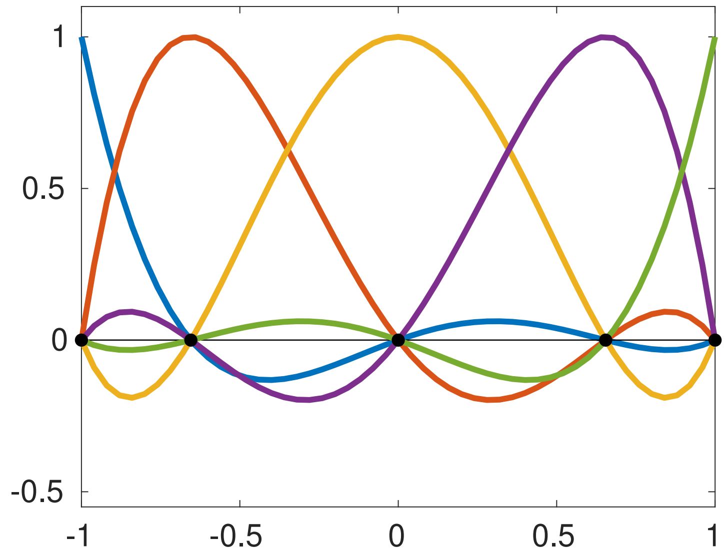
$$c_1\|v_N\|_{L^2(-1,1)} \le \|v_N\|_N \le c_2\|v_N\|_{L^2(-1,1)}, \qquad \forall v_N \in \mathbb{P}_N$$

Lagrange Interpolation at LGL nodes

$\{\varphi_0, \dots, \varphi_N\}$ = **characteristic Lagrange polynomials** in $\mathbb{P}_N$ w.r.t. the LGL nodes. It holds

$$\varphi_j(x) = \frac{1}{N(N+1)} \frac{(1-x^2)}{(x_j - x)} \frac{L'_N(x)}{L_N(x_j)}, \qquad \text{for } j = 0, \dots, N.$$

$\varphi_j(x_k) = \delta_{kj}$



$\varphi_j(x)$ when $N = 4$

$\{\varphi_j\}$ are orthogonal w.r.t. the discrete inner product $(\cdot,\cdot)_N$, i.e., the **mass matrix** M is diagonal:

$$M_{ij} = (\varphi_j, \varphi_i)_N = \delta_{ij} w_i, \qquad i, j = 0, \dots, N.$$

(w_i are the LGL weights)

Spectral Galerkin with Numerical Integration

Set $a_N(u_N, v_N) = (\mu u_N' - b u_N, v_N')_N + (\sigma u_N, v_N)_N$

Spectral GNI	$?u_N^{GNI} \in V_N :$	$a_N(u_N^{GNI}, v_N) = (f, v_N)_N$	$\forall v_N \in V_N$

Expand u_N^{GNI} w.r.t. the Lagrange basis: $u_N^{GNI}(x) = \sum_{i=0}^{N} u_N^{GNI}(x_i)\varphi_i(x)$
and choose $v_N(x) = \varphi_i(x)$ for any $i = 1, \ldots, N-1$.
The **Spectral GNI discretization** of the weak problem reads:

look for $\mathrm{u}^{GNI} = [u_N^{GNI}(x_j)]_{j=0}^{N}, u_N^{GNI}(x_0) = 0,\ u_N^{GNI}(x_N) = 0,\ :$

$$\sum_{j=0}^{N} a_N(\varphi_j, \varphi_i) u_N^{GNI}(x_j) = (f, \varphi_i)_N \qquad \text{for any } i = 1, \ldots, N-1$$

Characteristic Lagrange polynomials

$$\varphi_j \in \mathbb{P}_N : \; \varphi_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

$$(u, v)_N = \sum_{j=0}^{N} u(x_j) v(x_j) w_j$$

$$\begin{aligned} (\varphi_k, \varphi_m)_N &= \sum_{j=0}^{N} \underbrace{\varphi_k(x_j)}_{\delta_{kj}} \underbrace{\varphi_m(x_j)}_{\delta_{mj}} w_j \qquad 0 \leq k, m \leq N \\ &= \sum_{k=0}^{N} \delta_{km} w_k = \begin{cases} w_m & \text{if } k = m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$\Rightarrow \{\varphi_k\}$ is orthogonal under the $(\cdot, \cdot)_N$ discrete inner product.

GNI solution

GNI solution:

$$u_N(x) = \sum_{i=0}^{N} \alpha_i \varphi_i(x),$$

$\{\alpha_i\}$ unknown coefficients.
$x = x_j$ LGL node:

$$u_N(x_j) = \sum_{i=0}^{N} \alpha_i \underbrace{\varphi_i(x_j)}_{\delta_{ij}} = \alpha_j$$

$$\Downarrow$$

$$\boxed{u_N^{GNI}(x) = \sum_{j=0}^{N} u_N^{GNI}(x_j)\varphi_j(x)}$$

nodal expansion

$u_N^{GNI}(x_j)$ are the **nodal values**.

Algebraic form of Spectral GNI

$$A^{GNI}\mathrm{u}^{GNI} = \mathrm{f}^{GNI}$$

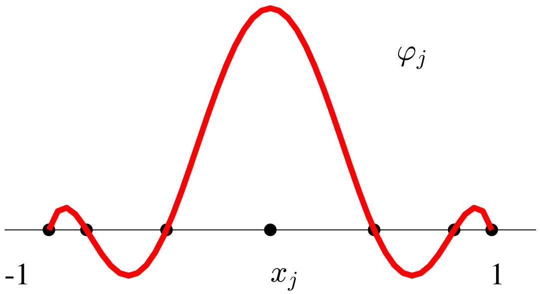
with $A_{ij}^{GNI} = a_N(\varphi_j, \varphi_i)$, for $i = 1, \ldots, N-1$, $j = 0, \ldots, N$, and $f_i^{GNI} = (f, \varphi_i)_N$ for $i = 1, \ldots, N-1$:

$$A^{GNI} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & a_N(\varphi_j, \varphi_i) & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \qquad \mathrm{f}^{GNI} = \begin{bmatrix} 0 \\ \vdots \\ f_i^{GNI} \\ \vdots \\ 0 \end{bmatrix}$$

$$a(u,v) = \int_{-1}^{1} \mu u' v' - \int_{-1}^{1} buv' + \int_{-1}^{1} \sigma uv, \qquad (f,v) = \int_{-1}^{1} fv$$

$$a_N(u,v) = (\mu u', v')_N - (bu, v')_N + (\sigma u, v)_N, \qquad (f,v)_N = (f,v)_N$$

$$A_{ij}^{GNI} = a_N(\varphi_j, \varphi_i) = \underbrace{(\mu \varphi_j', \varphi_i')_N}_{A} - \underbrace{(b\varphi_j, \varphi_i')_N}_{B} + \underbrace{(\sigma \varphi_j, \varphi_i)_N}_{C}$$



Assume: μ, b, $\sigma \in \mathbb{R}$.

$$C: \quad \sigma(\varphi_j, \varphi_i)_N = \sigma\delta_{ij}w_i = \begin{cases} \sigma w_i & i = j \\ 0 & i \neq j \end{cases} \quad \rightarrow \quad M = \sigma \underbrace{\begin{bmatrix} w_0 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_N \end{bmatrix}}_{\substack{\text{diagonal} \\ \text{weight matrix}}}$$

$$B: \quad -b(\varphi_j, \varphi_i')_N = -b\sum_{k=0}^{N} \underbrace{\varphi_j(x_k)}_{\delta_{jk}} \underbrace{\varphi_i'(x_k)}_{D_{ki}\neq 0} w_k \quad \rightarrow \quad \text{full matrix}$$

$$A: \quad \mu(\varphi_j', \varphi_i')_N = \mu\sum_{k=0}^{N} \underbrace{\varphi_j'(x_k)}_{D_{kj}} \underbrace{\varphi_i'(x_k)}_{D_{ki}} w_k \quad \rightarrow \quad \text{full matrix}$$

$D = (D_{ki})$ differentiation matrix (its entries can be computed off–line once and for all)

The case of non-constant coefficients μ, b, σ

Spectral Modal Galerkin method:

$$\underset{A}{\int_{-1}^{1} \mu\psi_k'\psi_m'} \quad - \underset{B}{\int_{-1}^{1} b\psi_k\psi_m'} \quad + \underset{C}{\int_{-1}^{1} \sigma \underbrace{\psi_k\psi_m}_{\mathbb{P}_{2N}}}$$

GNI method:
take for instance A:

$$(\mu\psi_k', \psi_m')_N = \sum_{j=0}^{N} \mu(x_j) \underbrace{\psi_k'(x_j)}_{D_{jk}} \underbrace{\psi_m'(x_j)}_{D_{jm}} w_j$$

I can compute $\mu(x_j)$!
Similarly for B and C.

Conclusion: A^{GNI} is still full as $A^{spectral}$.
However it is much easier to compute its coefficients, thanks to the nodal representation.

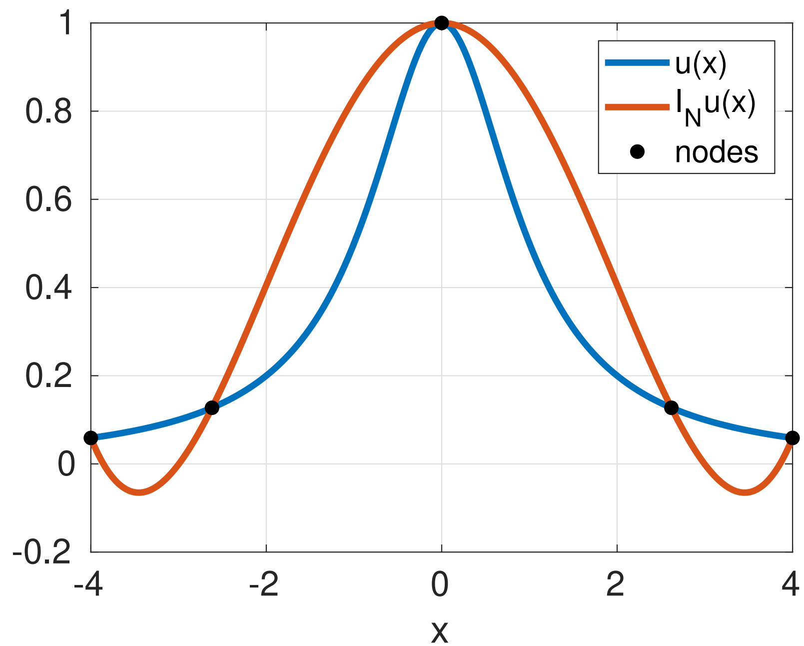
$$\boxed{u_N^{GNI}(x) = \sum_{k=0}^{N} u_N^{GNI}(x_k)\varphi_k(x)}$$ **nodal** expansion

$$\boxed{u_N^{spectral}(x) = \sum_{k=0}^{N} \tilde{u}_k \psi_k(x)}$$ **modal** expansion

Interpolation at LGL nodes

Global Lagrange polynomial of degree N that interpolates u at LGL nodes:

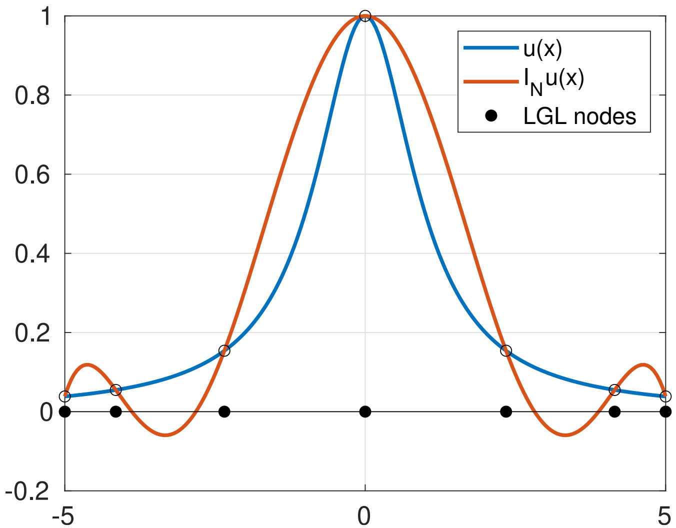
$$\boxed{I_N u(x) = \sum_{j=0}^{N} u(x_j)\varphi_j(x)}$$



Interpolation error:
For any $u \in H^{s+1}(-1,1)$ with $s \geq 0$, the **interpolation error** $(u - I_N u)$ satisfies the estimate

$$\|u - I_N u\|_{H^k(-1,1)} \leq C(s)\left(\frac{1}{N}\right)^{s+1-k} \|u\|_{H^{s+1}(-1,1)}, \text{ for } k = 0, 1.$$

Interpolation



LGL nodes not uniformly spaced$^{(*)}$

$$\boxed{I_N u(x_k) = u(x_k) \qquad 0 \le k \le N}$$

$I_N u \in \mathbb{P}_N$ = LGL interpolant of u

Interpolation error estimates (optimal):

$$\boxed{\|u - I_N u\|_{H^1(-1,1)} \le C \left(\frac{1}{N}\right)^s \|u\|_{H^{s+1}(-1,1)} \quad s \ge 1} \tag{5}$$

$$\boxed{\|u - I_N u\|_{L^2(-1,1)} \le C \left(\frac{1}{N}\right)^{s+1} \|u\|_{H^{s+1}(-1,1)} \quad s \ge 1}$$

$$\Rightarrow \|u - u_N^{GNI}\|_{H^1(-1,1)} ??$$

$(*)$ Runge phenomenon: if $\{x_j\}$ are uniformly spaced, then $\|u - I_N u\|_{C^0}$ could diverge as $N \nearrow \infty$

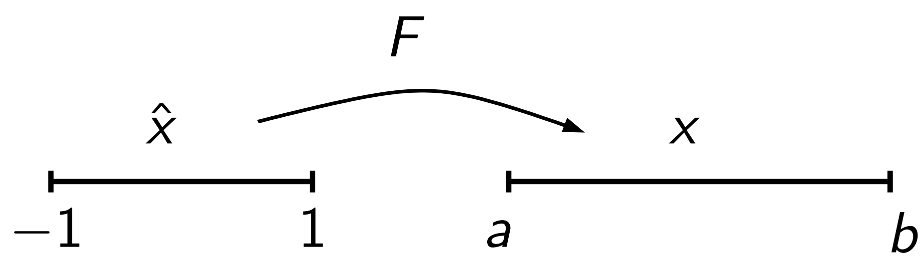
The quadrature error

Theorem. $\exists c > 0 : \ \forall f \in H^q(-1,1)$, with $q \geq 1$, $\forall \ v_N \in \mathbb{P}_N$ it holds:

$$\left| \int_{-1}^{1} f v_N \, dx - (f, v_N)_N \right| \leq c \left(\frac{1}{N} \right)^q \|f\|_{H^q(-1,1)} \|v_N\|_{L^2(-1,1)}.$$

Remark. Quadrature error behaves like the interpolation error

LGL nodes in $(a, b) \subset \mathbb{R}$:



Let $(\hat{\xi}_j, \hat{w}_j)$, for $j = 0, \ldots, N$, be the LGL nodes in $(-1, 1)$, then

$$\boxed{\int_a^b f(x)dx \simeq \sum_{j=0}^{N} f(\xi_j) w_j, \quad \xi_j = \frac{b-a}{2}\hat{\xi}_j + \frac{a+b}{2}, \quad w_j = \frac{b-a}{2}\hat{w}_j}$$

A-priori error analysis for Spectral GNI

Let $u_N^{GNI} \in V_N$ be the solution of

$$a_N(u_N^{GNI}, v_N) = (f, v_N)_N \qquad \forall v_N \in V_N.$$

If $u \in H^{s+1}(\Omega)$ and $f \in H^s(\Omega)$, with $s \geq 0$, then:

$$\boxed{\|u - u_N^{GNI}\|_{H^1(\Omega)} \leq C(s)\left(\frac{1}{N}\right)^s \left(\|u\|_{H^{s+1}(\Omega)} + \|f\|_{H^s(\Omega)}\right)}$$

u_N^{GNI} converges with spectral accuracy with respect to N to the exact solution u when the latter is smooth.

The proof is based on the Strang Lemma (a generalization of Céa Lemma). See later.

$$
\begin{array}{llll}
(WP) & ?u \in V: & a(u,v) = F(v), & \forall v \in V \\
(G) & ?u_N \in V_N: & a(u_N, v_N) = F(v_N), & \forall v_N \in V_N \\
(GNI) & ?u_N \in V_N: & a_N(u_N, v_N) = F_N(v_N) & \forall v_N \in V_N
\end{array}
$$

3 ingredients:

	V,	$a(\cdot,\cdot)$,	$F(\cdot)$	in (WP)
	V_N,	"	"	in (G)
	V_N,	$a_N(\cdot,\cdot)$	$F_N(\cdot)$	in (GNI).

- **(G) analysis**: Céa Lemma

$$\begin{aligned}\|u - u_N\|_{H^1(\Omega)} &\leq \underbrace{\inf_{v_N \in V_N} \|u - v_N\|_{H^1(\Omega)}}_{\text{distance of } V \text{ from } V_N} \\ &\leq \|u - I_N u\|_{H^1(\Omega)}\end{aligned}$$

- **(GNI) analysis**: I need something more than Céa Lemma. Phylosophically,

$$\begin{aligned}\|u - u_N\|_{H^1(\Omega)} &\leq \text{“distance” of } V \text{ from } V_N \\ &+ \text{“distance” of } a(\cdot,\cdot) \text{ from } a_N(\cdot,\cdot) \\ &+ \text{“distance” of } F(\cdot) \text{ from } F_N(\cdot).\end{aligned}$$

Strang Lemma

The Strang Lemma

Consider the problem

$$\text{find } u \in V: \quad a(u,v) = F(v) \quad \forall\, v \in V, \tag{6}$$

where V is a Hilbert space with norm $\|\cdot\|_V$, $F \in V'$ a linear and bounded functional on V and $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ a bilinear, continuous and coercive form on V.
Consider an approximation of (6) that can be formulated through the following generalized Galerkin problem

$$\text{find } u_h \in V_h: \quad a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h, \tag{7}$$

$\{V_h,\ h > 0\}$ being a family of finite-dimensional subspaces of V.
Let us suppose that the discrete bilinear form $a_h(\cdot,\cdot)$ is continuous on $V_h \times V_h$, and uniformly coercive on V_h, that is

$$\exists \alpha^* > 0 \text{ independent of h such that } a_h(v_h, v_h) \geq \alpha^* \|v_h\|_V^2 \qquad \forall\, v_h \in V_h.$$

Furthermore, let us suppose that F_h is a linear and bounded functional on V_h. (7);

The Strang Lemma (continued)

Then:

1. there exists a unique solution u_h to problem
2. such solution depends continuously on the data, i.e. we have

$$\|u_h\|_V \leq \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{F_h(v_h)}{\|v_h\|_V};$$

3. finally, the following a priori error estimate holds

$$\begin{aligned}\|u - u_h\|_V \leq \inf_{w_h \in V_h} \Bigg\{ & \left(1 + \frac{M}{\alpha^*}\right) \|u - w_h\|_V \\ & + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_V} \Bigg\} \\ & + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|F(v_h) - F_h(v_h)|}{\|v_h\|_V},\end{aligned} \tag{8}$$

M being the continuity constant of the bilinear form $a(\cdot, \cdot)$.

The Strang Lemma. Proof

The assumptions of the Lax-Milgram lemma for problem (7) are satisfied, so the solution of such problem exists and is unique. Moreover,

$$\|u_h\|_V \leq \frac{1}{\alpha^*}\|F_h\|_{V_h'},$$

$\|F_h\|_{V_h'} = \sup\limits_{v_h \in V_h \setminus \{0\}} \dfrac{F_h(v_h)}{\|v_h\|_V}$ being the norm of the dual space V_h' of V_h.

Let us now prove the error inequality (8). Let w_h be any function of the subspace V_h. Setting $\sigma_h = u_h - w_h \in V_h$, we have:

$$\begin{aligned}
\alpha^*\|\sigma_h\|_V^2 &\leq a_h(\sigma_h, \sigma_h) \quad \text{[by the coercivity of } a_h\text{]} \\
&= a_h(u_h, \sigma_h) - a_h(w_h, \sigma_h) \\
&= F_h(\sigma_h) - a_h(w_h, \sigma_h) \quad \text{[thanks to (7)]} \\
&= F_h(\sigma_h) - F(\sigma_h) + F(\sigma_h) - a_h(w_h, \sigma_h) \\
&= [F_h(\sigma_h) - F(\sigma_h)] + a(u, \sigma_h) - a_h(w_h, \sigma_h) \quad \text{[thanks to (6)]} \\
&= [F_h(\sigma_h) - F(\sigma_h)] + a(u - w_h, \sigma_h) + [a(w_h, \sigma_h) - a_h(w_h, \sigma_h)].
\end{aligned} \tag{9}$$

If $\sigma_h \neq 0$, (9) can be divided by $\alpha^* \|\sigma_h\|_V$, to give

$$
\begin{aligned}
\|\sigma_h\|_V \quad \leq \quad & \frac{1}{\alpha^*}\left\{ \frac{|a(u - w_h, \sigma_h)|}{\|\sigma_h\|_V} + \frac{|a(w_h, \sigma_h) - a_h(w_h, \sigma_h)|}{\|\sigma_h\|_V} \right. \\
& \qquad \left. + \ \frac{|F_h(\sigma_h) - F(\sigma_h)|}{\|\sigma_h\|_V} \right\} \\
\leq \quad & \frac{1}{\alpha^*}\left\{ M\|u - w_h\|_V + \sup_{v_h \in V_h \setminus \{0\}} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_V} \right. \\
& \qquad \left. + \sup_{v_h \in V_h \setminus \{0\}} \frac{|F_h(v_h) - F(v_h)|}{\|v_h\|_V} \right\} \quad \text{[by the continuity of } a\text{]}.
\end{aligned}
$$

If $\sigma_h = 0$ such inequality is still valid (it states that 0 is smaller than a sum of positive terms), although the proof breaks down.

We can now estimate the error between the solution u of (6) and the solution u_h of (7). Since $u - u_h = (u - w_h) - \sigma_h$, we obtain

$$\begin{aligned}
\|u - u_h\|_V &\leq \|u - w_h\|_V + \|\sigma_h\|_V \leq \|u - w_h\|_V \\
&\quad + \frac{1}{\alpha^*}\left\{ M\|u - w_h\|_V + \sup_{v_h \in V_h \setminus \{0\}} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_V} \right. \\
&\quad \left. + \sup_{v_h \in V_h \setminus \{0\}} \frac{|F_h(v_h) - F(v_h)|}{\|v_h\|_V} \right\} \\
&= \left(1 + \frac{M}{\alpha^*}\right)\|u - w_h\|_V + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|a(w_h, v_h) - a_h(w_h, v_h)|}{\|v_h\|_V} \\
&\quad + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|F_h(v_h) - F(v_h)|}{\|v_h\|_V}.
\end{aligned}$$

If the previous inequality holds $\forall w_h \in V_h$, it also holds when taking the infimum when w_h varies in V_h. Hence, we obtain (8). $\square$

By observing the right-hand side of inequality (8), we can recognize three different contributions to the approximation error $u - u_h$: the first is the best approximation error, the second is the error deriving from the approximation of the bilinear form $a(\cdot, \cdot)$ using the discrete bilinear form $a_h(\cdot, \cdot)$, and the third is the error arising from the approximation of the linear functional $F(\cdot)$ by the discrete linear functional $F_h(\cdot)$.

Remark If in the preceding proof we choose $w_h = u_h^*$, u_h^* being the solution to the Galerkin problem

$$u_h^* \in V_h : \; a(u_h^*, v_h) = F(v_h) \quad \forall\, v_h \in V_h,$$

then the term $a(u - w_h, \sigma_h)$ is null thanks to (6). It is therefore possible to obtain the following estimate, alternative to (8)

$$\begin{aligned} \|u - u_h\|_V \quad \leq \quad & \|u - u_h^*\|_V \\ & + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|a(u_h^*, v_h) - a_h(u_h^*, v_h)|}{\|v_h\|_V} \\ & + \frac{1}{\alpha^*} \sup_{v_h \in V_h \setminus \{0\}} \frac{|F(v_h) - F_h(v_h)|}{\|v_h\|_V}. \end{aligned}$$

The latter highlights the fact that the error due to the generalized Galerkin method can be bounded by the error of the Galerkin method plus the errors induced by the use of numerical integration for the computation of both $a(\cdot, \cdot)$ and $F(\cdot)$.

We now want to apply Strang's lemma to the GNI method, to verify its convergence. For simplicity we will only consider the one-dimensional case. Obviously, V_h will be replaced by V_N, u_h by u_N, v_h by v_N and w_h by w_N.
First of all, we begin by computing the error of the GLL numerical integration formula

$$E(g, v_N) = (g, v_N) - (g, v_N)_N,$$

g and v_N being a generic continuous function and a generic polynomial of $\mathbb{Q}_N$, respectively. By introducing the interpolation polynomial $I_N g$, we obtain

$$\begin{aligned}
E(g, v_N) &= (g, v_N) - (I_N g, v_N)_N \\
&= (g, v_N) - (I_{N-1}g, v_N) + \underbrace{(\overbrace{I_{N-1}g}^{\in\mathbb{Q}_{N-1}}, \overbrace{v_N}^{\in\mathbb{Q}_N})}_{\in\mathbb{Q}_{2N-1}} - (I_N g, v_N)_N \\
&= (g, v_N) - (I_{N-1}g, v_N) \\
&\quad + (I_{N-1}g, v_N)_N - (I_N g, v_N)_N \quad \text{[by (3)]} \\
&= (g - I_{N-1}g, v_N) + (I_{N-1}g - I_N g, v_N)_N.
\end{aligned} \tag{10}$$

The first summand of the right-hand side can be bounded from above using the Cauchy-Schwarz inequality as follows

$$|(g - I_{N-1}g, v_N)| \leq \|g - I_{N-1}g\|_{L^2(-1,1)} \|v_N\|_{L^2(-1,1)}. \tag{11}$$

To find an upper bound for the second summand, we must first introduce the two following lemmas, for the proof of which we refer to [Canuto, Hussaini, Quarteroni, Zang (2006)].

Lemma

The discrete scalar product $(\cdot,\cdot)_N$ defined in (2) is a scalar product on $\mathbb{Q}_N$ and, as such, it satisfies the Cauchy-Schwarz inequality

$$|(\varphi,\psi)_N| \leq \|\varphi\|_N \|\psi\|_N, \tag{12}$$

where the discrete norm $\|\cdot\|_N$ is given by

$$\|\varphi\|_N = \sqrt{(\varphi,\varphi)_N} \quad \forall\, \varphi \in \mathbb{Q}_N. \tag{13}$$

Lemma

The "continuous" norm of $L^2(-1,1)$ and the "discrete" norm $\|\cdot\|_N$ defined in (4) verify the inequalities

$$\|v_N\|_{L^2(-1,1)} \leq \|v_N\|_N \leq \sqrt{3}\|v_N\|_{L^2(-1,1)} \quad \forall\, v_N \in \mathbb{Q}_N, \tag{14}$$

hence they are uniformly equivalent on $\mathbb{Q}_N$.

By using first (13) and then (14) we obtain

$$
\begin{aligned}
|(I_{N-1}g - I_N g, v_N)_N| &\le \|I_{N-1}g - I_N g\|_N \, \|v_N\|_N \\
&\le 3\left[\|I_{N-1}g - g\|_{L^2(-1,1)} + \|I_N g - g\|_{L^2(-1,1)}\right] \|v_N\|_{L^2(-1,1)}.
\end{aligned}
$$

Using such inequality and (11), from (10) we can obtain the following upper bound

$$
|E(g, v_N)| \le \left[4\|I_{N-1}g - g\|_{L^2(-1,1)} + 3\|I_N g - g\|_{L^2(-1,1)}\right] \|v_N\|_{L^2(-1,1)}.
$$

Using the interpolation estimate

$$
\|f - I_N f\|_{H^k(-1,1)} \le C_s \left(\frac{1}{N}\right)^{s-k} \|f\|_{H^s(-1,1)}, \quad s \ge 1,\ k = 0, 1, \tag{15}
$$

valid for any $f \in H^s(-1,1)$, we have that

$$
|E(g, v_N)| \le C\left[\left(\frac{1}{N-1}\right)^s + \left(\frac{1}{N}\right)^s\right] \|g\|_{H^s(-1,1)} \, \|v_N\|_{L^2(-1,1)},
$$

provided that $g \in H^s(-1,1)$, for some $s \ge 1$.

Finally, as for each $N \geq 2$, $1/(N-1) \leq 2/N$, the Gauss-Legendre-Lobatto integration error results to be bound as

$$|E(g, v_N)| \leq C \left(\frac{1}{N}\right)^s \|g\|_{H^s(-1,1)} \|v_N\|_{L^2(-1,1)}, \tag{16}$$

for each $g \in H^s(-1,1)$ and for each polynomial $v_N \in \mathbb{Q}_N$.

At this point we are ready to evaluate the various contributions that intervene in (8). We anticipate that this analysis will be carried out in the case where suitable simplifying hypotheses are introduced on the differential problem under exam. We begin with the simplest term, i.e. the one associated with the functional F, supposing to consider a problem with homogeneous Dirichlet boundary conditions, in order to obtain $F(v_N) = (f, v_N)$ and $F_N(v_N) = (f, v_N)_N$. Provided that $f \in H^s(-1,1)$ for some $s \geq 1$, then,

$$\begin{aligned}
&\sup_{v_N \in V_N \setminus \{0\}} \frac{|F(v_N) - F_N(v_N)|}{\|v_N\|_V} = \sup_{v_N \in V_N \setminus \{0\}} \frac{|(f, v_N) - (f, v_N)_N|}{\|v_N\|_V} \\
&= \sup_{v_N \in V_N \setminus \{0\}} \frac{|E(f, v_N)|}{\|v_N\|_V} \leq \sup_{v_N \in V_N \setminus \{0\}} \frac{C\left(\frac{1}{N}\right)^s \|f\|_{H^s(-1,1)} \|v_N\|_{L^2(-1,1)}}{\|v_N\|_V} \\
&\leq C \left(\frac{1}{N}\right)^s \|f\|_{H^s(-1,1)},
\end{aligned} \tag{17}$$

having exploited relation (16) and having bounded the norm in $L^2(-1,1)$ by that in $H^s(-1,1)$.

As for the contribution arising from the approximation of the bilinear form,

$$\sup_{v_N \in V_N \setminus \{0\}} \frac{|a(w_N, v_N) - a_N(w_N, v_N)|}{\|v_N\|_V},$$

we cannot explicitly evaluate it without referring to a particular differential problem. We then choose, as an example, the one-dimensional diffusion-reaction problem

$$\begin{cases} Lu = -(\mu u')' + bu' + \sigma u = f \quad -1 < x < 1 \\ u(-1) = u(1) = 0, \end{cases} \tag{18}$$

supposing moreover that μ, b, and σ are constant. Incidentally, such problem satisfies homogeneous Dirichlet boundary conditions, in accordance with what was requested for deriving estimate (17). In such case, the associated bilinear form is

$$a(u, v) = (\mu u', v') + (bu', v) + (\sigma u, v),$$

while its GNI approximation is given by

$$a_N(u, v) = (\mu u', v')_N + (bu', v)_N + (\sigma u, v)_N.$$

We must then evaluate

$$a(w_N, v_N) - a_N(w_N, v_N) = (\mu w_N', v_N') - (\mu w_N', v_N')_N + (\sigma w_N, v_N) - (\sigma w_N, v_N)_N.$$

Since $w_N' v_N' \in \mathbb{Q}_{2N-2}$, if we suppose that μ is constant, the product $\mu w_N' v_N'$ is integrated exactly by the GLL integration formula, that is $(\mu w_N', v_N') - (\mu w_N', v_N')_N = 0$. Similarly, $(b w_n', v_N) - (b w_N', v_N)_N = 0$.
We now observe that

$$(\sigma w_N, v_N) - (\sigma w_N, v_N)_N = E(\sigma w_N, v_N) = E(\sigma(w_N - u), v_N) + E(\sigma u, v_N),$$

and therefore, using (16), we obtain

$$|E(\sigma(w_N - u), v_N)| \leq C\sigma \left(\frac{1}{N}\right) \|w_N - u\|_{H^1(-1,1)} \|v_N\|_{L^2(-1,1)}.$$

Setting $w_N = I_N u$ and using (5), we obtain

$$\|w_N - u\|_{H^1(-1,1)} = \|u - I_N u\|_{H^1(-1,1)} \leq C \left(\frac{1}{N}\right)^s \|u\|_{H^{s+1}(-1,1)}.$$

Then,

$$|E(\sigma u, v_N)| \leq C\sigma \left(\frac{1}{N}\right)^{s+1} \|u\|_{H^{s+1}(-1,1)} \|v_N\|_{L^2(-1,1)},$$

if we assume $u \in H^{s+1}(-1,1)$.

Hence,

$$\sup_{v_N \in V_N \setminus \{0\}} \frac{|a(w_N, v_N) - a_N(w_N, v_N)|}{\|v_N\|_V} \leq C^* \left(\frac{1}{N}\right)^s \|u\|_{H^s(-1,1)}. \tag{19}$$

We still need to estimate the first summand of (8). Having chosen $w_N = I_N u$ and exploiting (15) again, we obtain that

$$\|u - w_N\|_V = \|u - I_N u\|_{H^1(-1,1)} \leq C \left(\frac{1}{N}\right)^s \|u\|_{H^{s+1}(-1,1)}, \tag{20}$$

provided that $u \in H^{s+1}(-1,1)$, for a suitable $s \geq 1$. To conclude, thanks to (17), (19) and (20), from (8) applied to the GNI approximation of problem (18), and under the previous hypotheses, we find the following error estimate

$$\|u - u_N\|_{H^1(-1,1)} \leq C \left(\frac{1}{N}\right)^s \left(\|f\|_{H^s(-1,1)} + \|u\|_{H^{s+1}(-1,1)}\right).$$

The convergence analysis just carried out for the model problem (18) can be generalized (with a few technical difficulties) to the case of more complex differential problems and different boundary conditions.

Interpretation of GNI method as Collocation method

$$(P) \quad \begin{cases} Lu & = -(\mu u')' + (bu)' + \sigma u = f \quad -1 < x < 1 \\ & u(-1) = u(1) = 0. \end{cases}$$

$$(WP) \qquad ?u \in V = H_0^1(-1,1) : \qquad a(u,v) = F(v), \quad \forall v \in V$$

with

$$a(u,v) = \int_{-1}^{1} (\mu u' - bu)v' + \int_{-1}^{1} \sigma uv, \qquad F(v) = \int_{-1}^{1} fv.$$

$$(GNI) \quad \begin{cases} ?u_N \in V_N = \mathbb{P}_N^0 = \{v_n \in \mathbb{P}_N : \ v_N(\mp 1) = 0\} : \\ a_N(u_N, v_N) = F_N(v_N) \qquad \forall v_N \in V_N \end{cases}$$

with

$$a_N(u_N, v_N) = (\mu u_N' - bu_N, v_N')_N + (\sigma u_N, v_N)_N, \qquad F_N(v_N) = (f, v_N)_N.$$

Note that, thanks to the exactness of LGL quadrature formula:

$$
\begin{aligned}
a_N(u_N, v_N) &\overset{\text{(def. of } I_N)}{=} \quad (I_N \underbrace{(\mu u_N' - b u_N)}_{\in \mathbb{P}_N}, \underbrace{v_N'}_{\in \mathbb{P}_{N-1}})_N + (\sigma u_N, v_N)_N \\
&\overset{\text{(exactness)}}{=} \quad (I_N(\mu u_N' - b u_N), v_N') + (\sigma u_N, v_N)_N \\
&\overset{\text{(int. by parts)}}{=} \quad -(\underbrace{(I_N(\mu u_N' - b u_N))'}_{\in \mathbb{P}_{N-1}}, v_N'), \underbrace{v_N}_{\in \mathbb{P}_N}) + (\sigma u_N, v_N)_N \\
&\overset{\text{(exactness)}}{=} \quad (\underbrace{-(I_N(\mu u_N' - b u_N))' + \sigma u_N}_{= L_N u_N}, v_N)_N
\end{aligned}
$$

By comparison:

$$\begin{aligned} Lu &= -(\mu u' - bu)' + \sigma u \\ L_N u_N &= -\underbrace{I_N(\mu u_N' - bu_N)'}_{D\underbrace{(\mu u_N' - bu_N)}_{\Phi}} + \sigma u_N \end{aligned}$$

where $D\Phi$ is the pseudo–spectral derivative of Φ, that is

$$D\Phi := \frac{d}{dx} I_N \Phi \qquad \forall \Phi \in C^0(-1,1).$$

Obviously, if $\Phi \in \mathbb{P}_N$, then $D\Phi = \frac{d}{dx}\Phi$.

To summarize:

$$(GNI) \Leftrightarrow (L_N u_N, v_N)_N = (f, v_N)_N \qquad \forall v_N \in V_N \qquad (*)$$

Take $v_N = \varphi_j$ (characteristic Lagrange basis function),

$$\varphi_j \in \mathbb{P}_N^0 : \quad \varphi_j(x_k) = \delta_{kj}, \qquad 1 \leq k, j \leq N-1.$$

Then

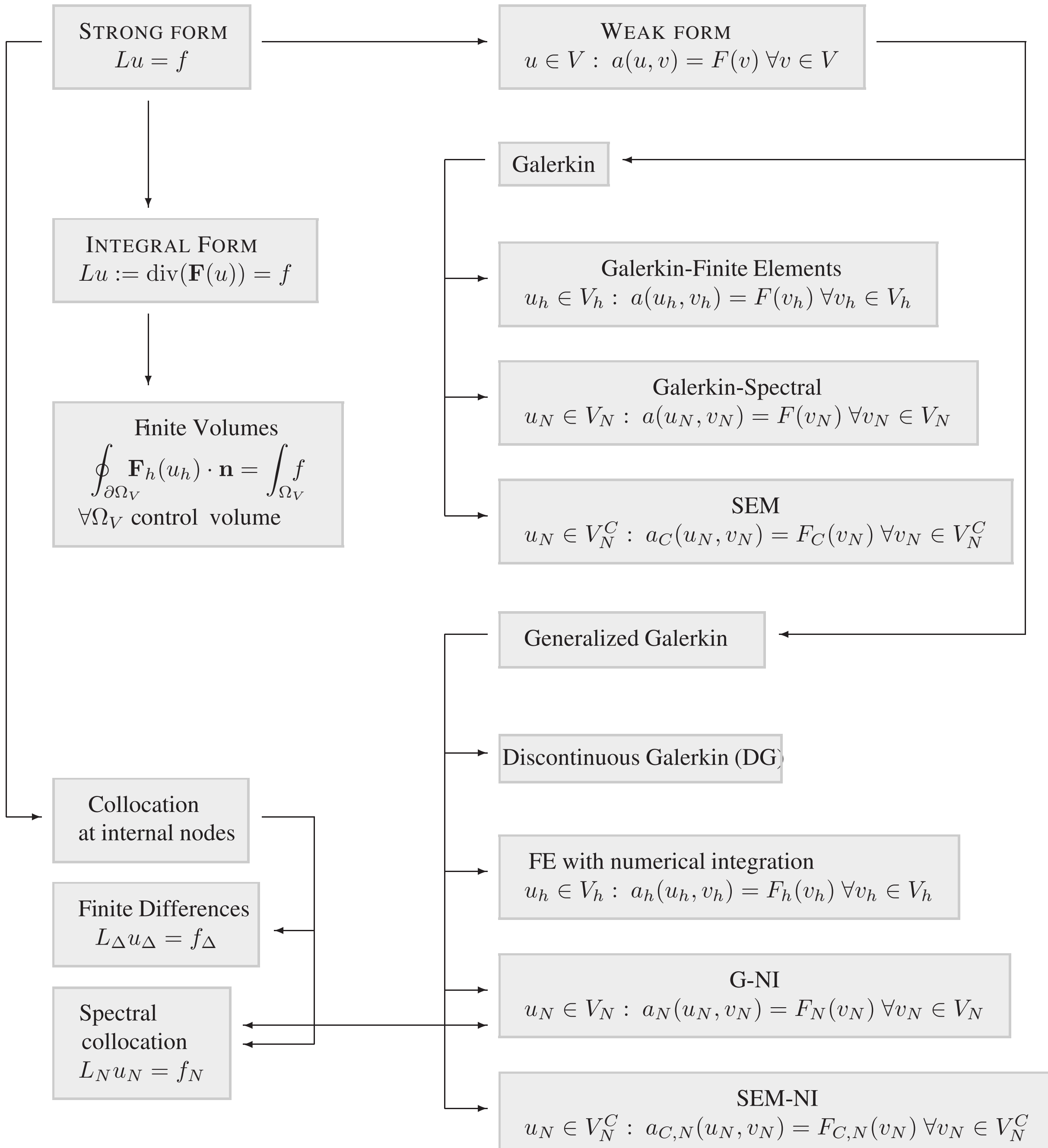
$$(*) \Leftrightarrow L_N u_N(x_j) w_j = f(x_j) w_j \qquad j = 1, \ldots, N-1,$$

whence

$$\begin{cases} L_N u_n(x_j) = f(x_j) & j = 1, \ldots, N-1 \\ u_N(x_0) = u_N(x_N) = 0. \end{cases}$$

Conclusion: $(GNI) \Leftrightarrow$ Collocation method, that is, the original equation $Lu = f$ is collocated at every internal LGL node, after replacing L by L_N. In turns, L_N is obtained from L by approximating every derivative by the corresponding pseudo–spectral derivative.

A global methodological picture

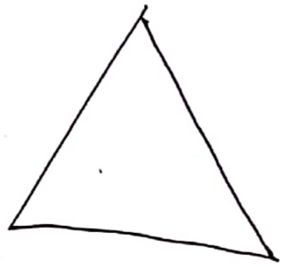


Polynomials on triangles & on quadrilaterals

2 different polynomial spaces:

FE case:

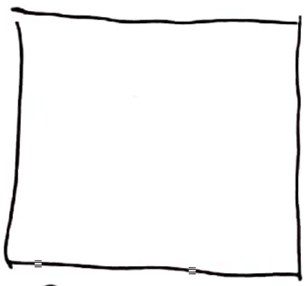
$d = 2$



TRIANGLE

$\mathbb{P}^{(r)}$: $v_h(x_1, x_2) = \sum_{k,\ell=0}^{r} a_{k\ell}\, x_1^k x_2^\ell$, $\boxed{k+\ell \le r}$

GNI case



QUADRILATERAL

$\mathbb{Q}_N = \mathbb{Q}^{(N)}$: $v_N(x_1, x_2) = \sum_{k,\ell=0}^{N} a_{k\ell}\, x_1^k x_2^\ell$

2D EXAMPLE

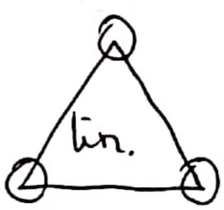
$\mathbb{P}^{(1)}$: $a_0 + a_1 x_1 + a_2 x_2$

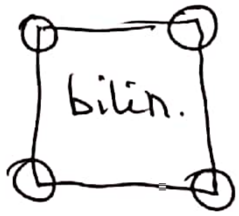
$\mathbb{Q}_1$: $a_0 + a_1 x_1 + a_2 x_2 + \boxed{a_3 x_1 x_2}$

$\mathbb{P}^{(2)}$: $a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1^2 + a_5 x_2^2$

$\mathbb{Q}_2$: $a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1 x_2 + a_4 x_1^2 + a_5 x_2^2 + \boxed{a_6 x_1^2 x_2^2 + a_7 x_1^2 x_2 + a_8 x_1 x_2^2}$

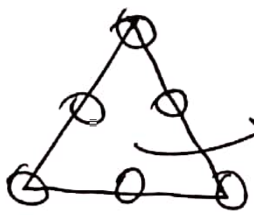
$r = 1$



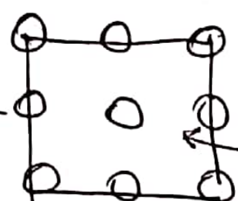


$N = 1$

$r = 2$



quadratic

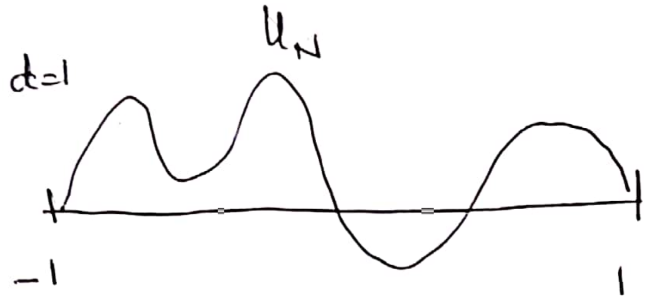


$N = 2$

biquadratic

o : DOF degrees of freedom

$$\begin{cases} Lu = f & \text{in } \Omega = (-1,1)^d \quad d=1,2,3 \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$



Reminder

GNI - 1D.

$$? \; u \in V : a(u,v) = F(v) \; \forall v \in V$$

$$\text{(GNI)} \quad \boxed{? \; u_N \in V_N : a_N(u_N, v_N) = F_N(v_N) \; \forall v_N \in V_N}$$

GNI

$$V_N = \{ v_N \in \mathbb{P}_N : v_N(\pm 1) = 0 \}$$

a_N, F_N : given by a and F provided we replace exact inner products by the LGL integration formula.

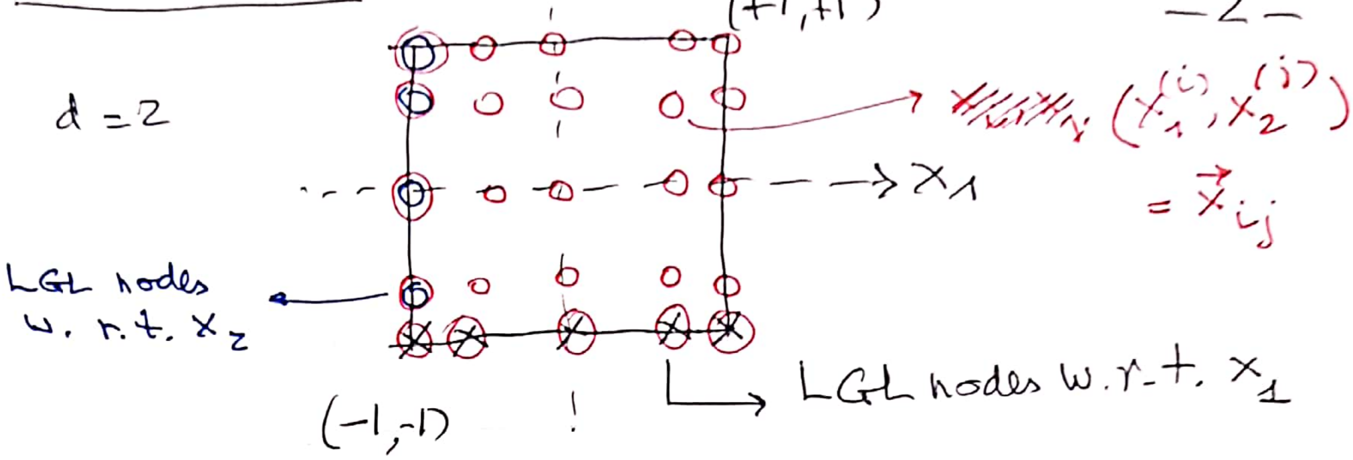
$$(u,v)_N = \sum_{j=0}^{N} u(x_j) v(x_j) w_j .$$

$\{x_j\}_{j=0}^{N}$ LGL nodes

$\{w_j\}_{j=0}^{N}$ LGL weights / coefficients.

MOVING to 2D

$d = 2$



I proceed in the same way, provided the following generalizations are adopted:

$$\mathbb{Q}_N = \{ q_N(x_1,..,x_3) = \sum_{k,l,m=0}^{N} a_{klm} x_1^k x_2^l x_3^m \quad a_{klm} \in \mathbb{R} \}$$

drop in 2D.

$$= \mathbb{P}_N(x_1) \dots \mathbb{P}_N(x_d) \quad \text{tensor product}$$

ex $d=2$

$$\boxed{\int_{-1}^{1}\int_{-1}^{1} u(\vec{x}) v(\vec{x}) d\vec{x} \simeq (u,v)_N = \sum_{i,j=0}^{N} u(\vec{x}_{ij}) v(\vec{x}_{ij}) w_{ij}} \quad (*)$$

$\vec{x}_{ij}$: LGL nodes in Ω^d — w_{ij}: weights in Ω^d.

$w_{ij} = w_i w_j$ — w_i: 1D LGL coeff.

(GN1) $? u_N \in V_N : a_N(u_N, v_N) = F_N(v_N) \; \forall v_N \in V_N$

$V_N = \{ v_N \in \mathbb{Q}_N : v_N|_{\partial\Omega} = 0 \}$

a_N, F_N: obtained from a and F by replacing scalar products with (*)

Strang Lemma yields:

(1) $\|u-u_N\|_{H^1(\Omega)} \le C_1 \left(\frac{1}{N}\right)^s \left(\|u\|_{H^{s+1}(\Omega)} + \|f\|_{H^s(\Omega)}\right)$

This is based on the following properties:

(2) $\|v - \Pi_N^{LGL} v\|_{H^k(\Omega)} \le C\left(\frac{1}{N}\right)^{s+1-k} \|v\|_{H^{s+1}(\Omega)}$

$k=0,1$

$\begin{cases} \Pi_N^{LGL} v \text{ is the interpolant of } v \text{ at the LGL nodes:} \\ \text{in 2D:} \quad \Pi_N^{LGL} v(\vec{x}_{ij}) = v(\vec{x}_{ij}) \quad i,j=0,..,N. \end{cases}$

Moreover:

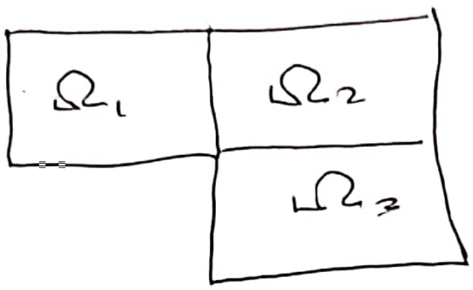
(4) $(u,v)_N = (u,v)$ provided $uv \in \mathbb{Q}_{2N-1}$

degree of exactness is $2N-1$

(5) $\begin{cases} |(u,v_N) - (u,v_N)_N| \le C_1\left(\frac{1}{N}\right)^q \|u\|_{H^q(\Omega)} \|v_N\|_{L^2(\Omega)} \\ \forall u \in H^q(\Omega), \forall v_N \in \mathbb{Q}_N. \end{cases}$

All these results hold thanks to the property that LGL in dimension $d \ge 2$ is obtained by d-tensor product of LGL in dimension 1.

What about more complex domains Ω?



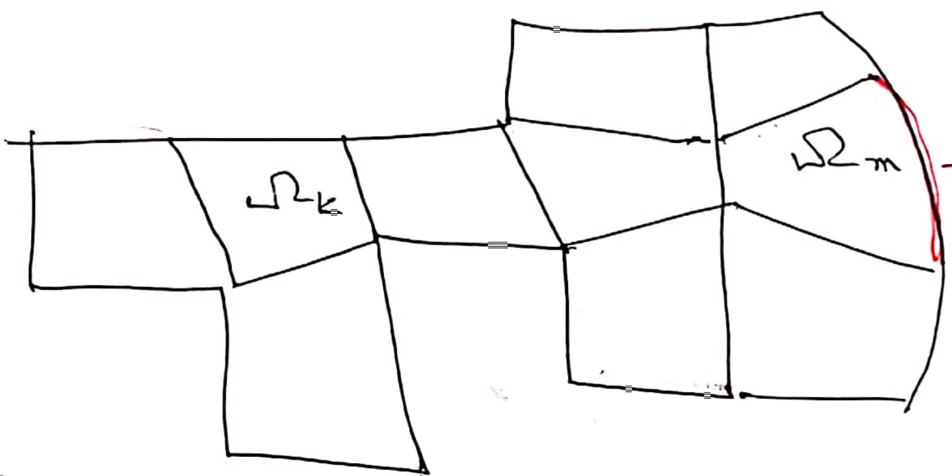
Ω_K "elements"

$\hat{\Omega} = (-1,1)^d$
Reference element
LGL nodes and weights

SEM : Spectral Element Method

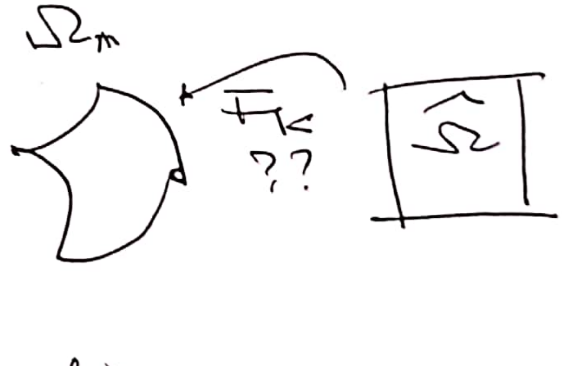
Spectral → Spectral method

Element → Finite Elements



→ Curved boundaries

Ω_K
quadrilateral
in 3D: parallelepipedal
→ polygons/polyhedra.



Let us start from a 1D example

Spectral (Element) Methods - S(E)M

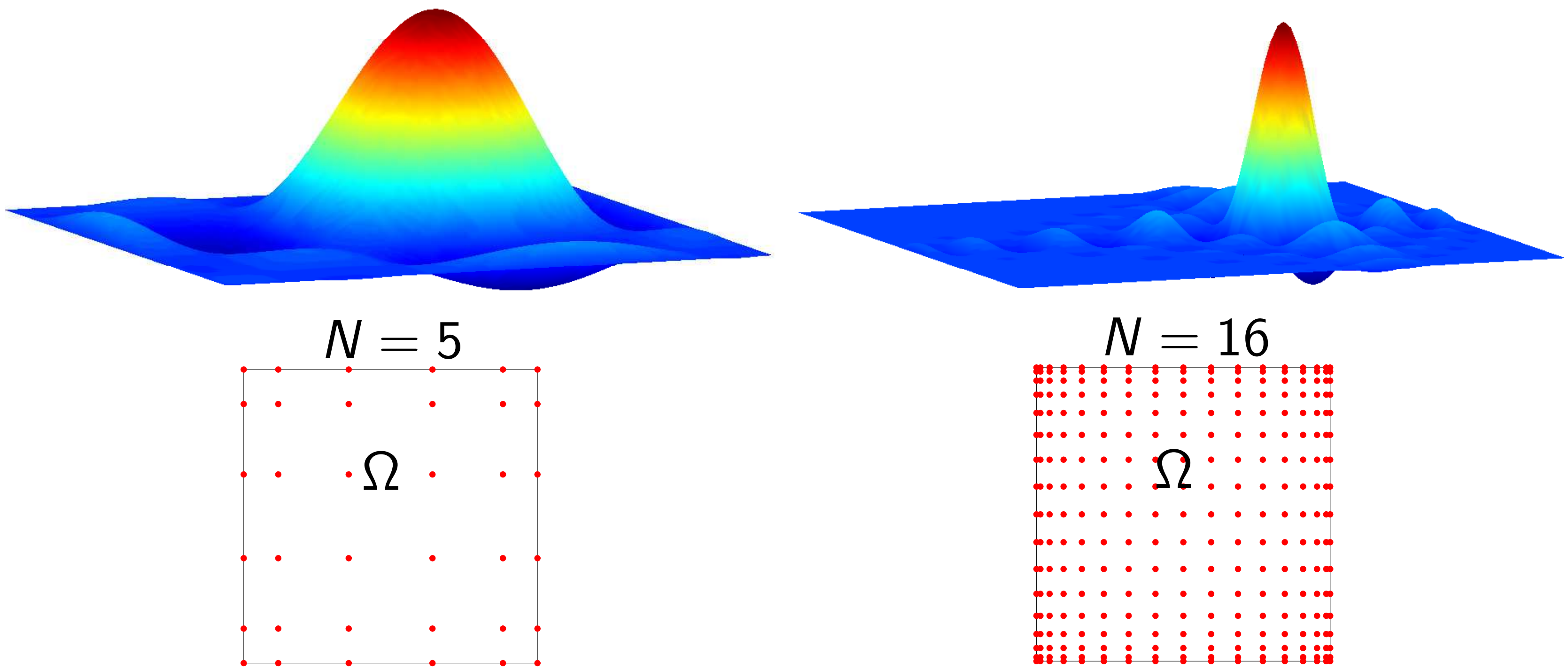
- S(E)M are **high-order** numerical methods to solve boundary value problems
- they are alternative to low-order FEM, but you can combine them with FEM (e.g., through a MORTAR approach)
- historically they were designed on quadrilaterals (**quads**)
- they have been extended to **simplices** more recently
- in SEM, **continuity** at interface elements is imposed (otherwise one speaks about DG-SEM)
- SEM are also known as either spectral/hp or $hp-$FEM

Nomenclature

Gottlieb & Orszag (1977),
Canuto, Hussaini, Quarteroni, & Zang (1988),
Bernardi & Maday (1992)

Spectral Methods: one quad element Ω and global support of the polynomial basis functions on Ω.

One parameter: $N =$ polynomial degree ($\nearrow$)

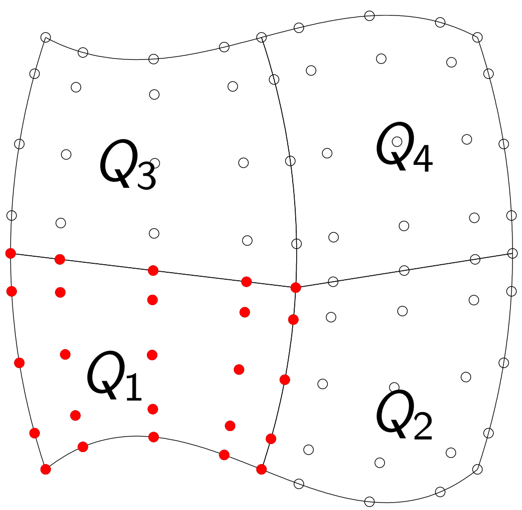


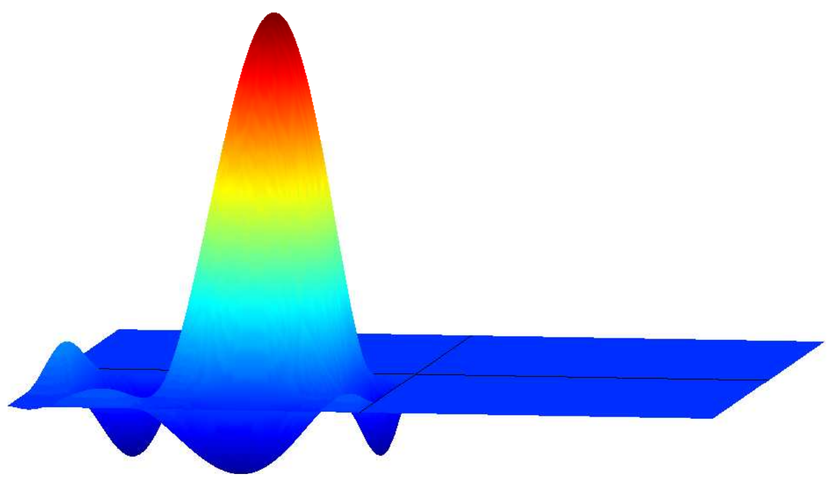
Nomenclature

Patera (1984)

Spectral Element Methods: conformal partition of quads in Ω, global C^0 basis functions (local polynomials) with local support.

Two parameters: N = pol. degree ($\nearrow$)
h = mesh size (=elements diameter) ($\searrow 0$)



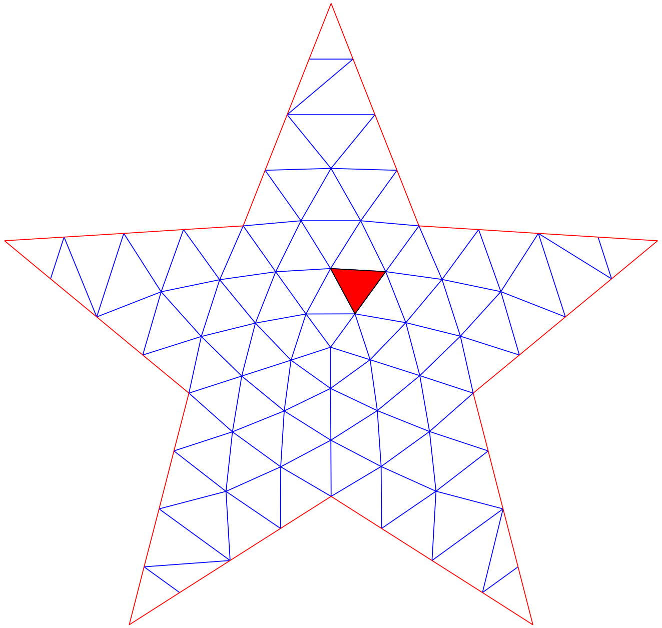


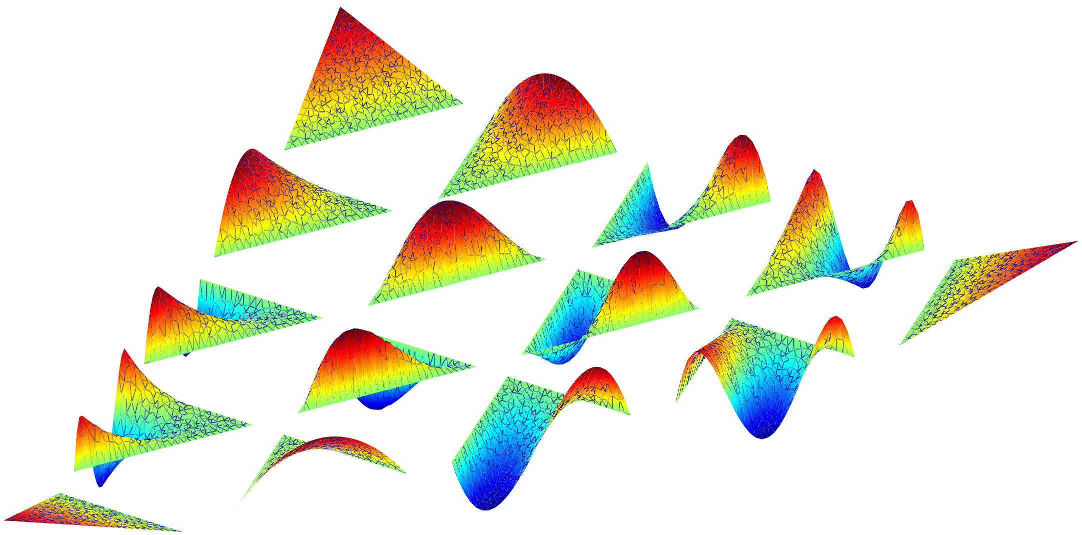
Nomenclature

Patera (1984) for SEM on quads,
Dubiner (1991), Sherwin & Karniadakis (1995) for SEM on simplices

spectral/*hp* conformal partition of quads/simplices in Ω,
global C^0 basis functions (local polynomials) with local support.

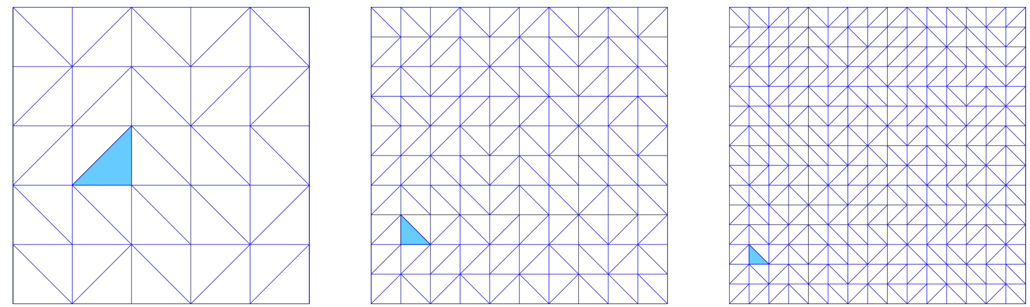
Two parameters: $p(= N)$ = pol. degree ($\nearrow$)
h = mesh size (=elements diameter) ($\searrow 0$)





Nomenclature

h-FEM: fixed low degree
refinement in h
(simplices and quads)
One parameter:
h =mesh size
(the same on quads)



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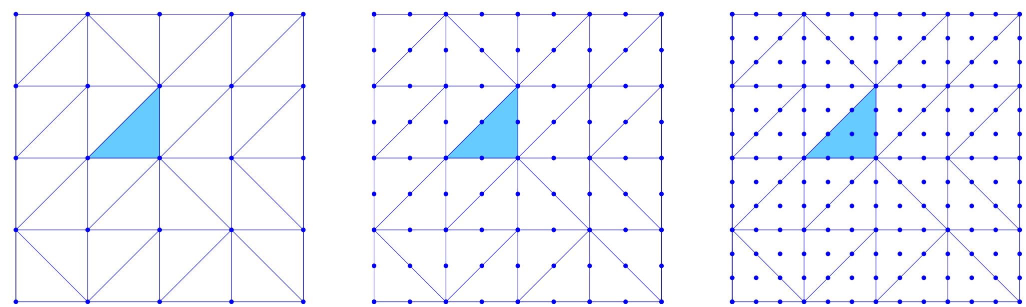
p-FEM: fixed h
refinement in p
(simplices and quads)
One parameter:
p =pol. degree
(the same on quads)



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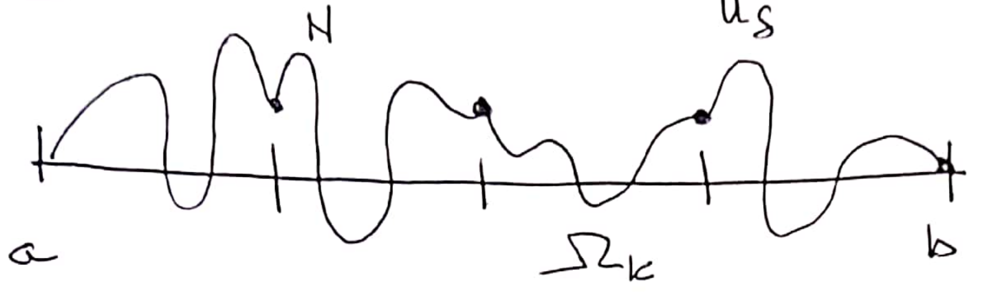
hp-FEM: refinement in both h and p (simplices and quads)

Two parameters: $p =$ pol. degree ($\nearrow$)
$h =$ mesh size (=elements diameter) ($\searrow 0$)

The borderline between spectral/*hp* and *hp*-FEM seems invisible

1D   SEM – NI



$$\bar{\Omega} = [a,b] = \bigcup_{k=1}^{M} \bar{\Omega}_k \qquad \delta = (N, h)$$

local polynomial degree   size of every element

Ω_k = elements = e_k

$$\boxed{\begin{array}{l} ?\ u_\delta \in V_\delta = \{ v_\delta \in C^0(\bar{\Omega}),\ v_\delta|_{\Omega_k} \in \mathbb{P}_N,\ v_\delta(a,b) = 0 \}. \\ a_\delta(u_\delta, v_\delta) = F_\delta(v_\delta) \quad \forall v_\delta \in V_\delta \end{array}}$$

SEM(–NI)

$$a_\delta(u_\delta, v_\delta) = \sum_{k=1}^{M} a_\delta^k(u_\delta, v_\delta)$$

F_δ defined similarly.

$$a_\delta^k = \left\{ \begin{array}{l} \text{First restrict } a(\cdot,\cdot) \text{ to } \Omega_k \\ \text{then replace integrals by LGL numerical} \\ \text{integration on } \Omega_k. \end{array} \right.$$

Ex: $\left\{ \begin{array}{l} Lu \equiv -(\mu u')' + bu' + \sigma u = f \quad \bar{a} < x < \bar{b} \\ u(\bar{a}) = u(\bar{b}) = 0 \end{array} \right.$

$$?\ u \in V = H_0^1(\bar{a},\bar{b}):\ \underbrace{\int_{\bar{a}}^{\bar{b}} \mu u'v' + \int_{\bar{a}}^{\bar{b}} bu'v + \int_{\bar{a}}^{\bar{b}} \sigma uv}_{a(u,v)} = \int_{\bar{a}}^{\bar{b}} fv \quad \forall v \in V.$$

$$a(u,v) = \sum_{k=1}^{M} \underbrace{a^k(u,v)}_{\text{restriction of } a \text{ to } \Omega_k}$$

$$a_\delta(u,v) = \sum_{k=1}^{M} a_\delta^k(u_\delta, v)$$

thus:

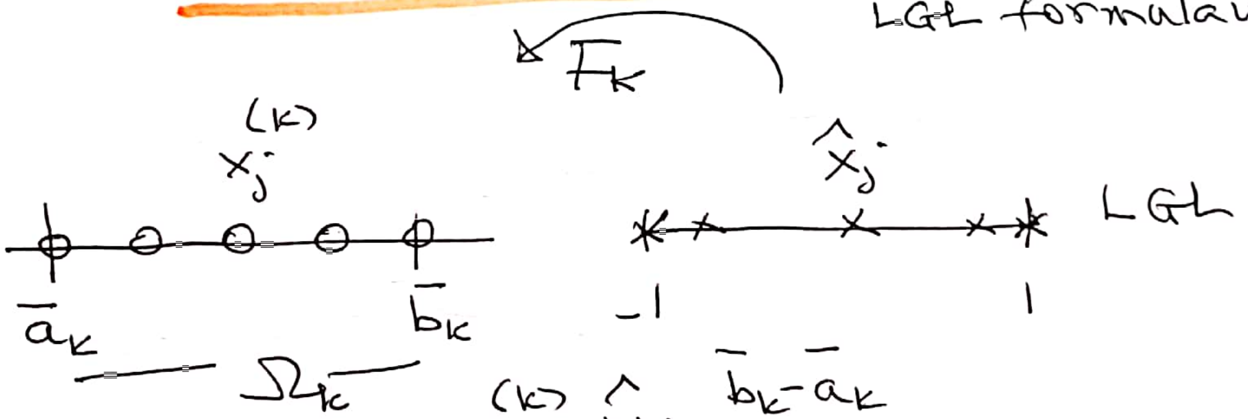
$$a^k(u,v) = \int_{\Omega_k} \mu u' v' + \int_{\Omega_k} b u' v + \int_{\Omega_k} \sigma u v$$

$$a_\delta^k(u,v) = (\mu u', v')_{\delta,\Omega_k} + (b u', v)_{\delta,\Omega_k} + (\sigma u, v)_{\delta,\Omega_k}$$

<u>Now</u>

$$\boxed{(\Phi, \Psi)_{\delta,\Omega_k} = \sum_{j=0}^{N} \Phi(x_j^{(k)})\, \Psi(x_j^{(k)})\, w_j^{(k)}}$$

LGL formula in Ω_k.



$$w_j^{(k)} = \hat{w}_j \cdot \frac{\bar{b}_k - \bar{a}_k}{2}$$

Similarly:

$$\boxed{\begin{aligned} F_\delta(v) &= \sum_{k=1}^{M} (f, v)_{\delta,\Omega_k} \\ &= \sum_{k=1}^{M} \sum_{j=0}^{N} f(x_j^{(k)})\, v(x_j^{(k)})\, w_j^{(k)} \end{aligned}}$$

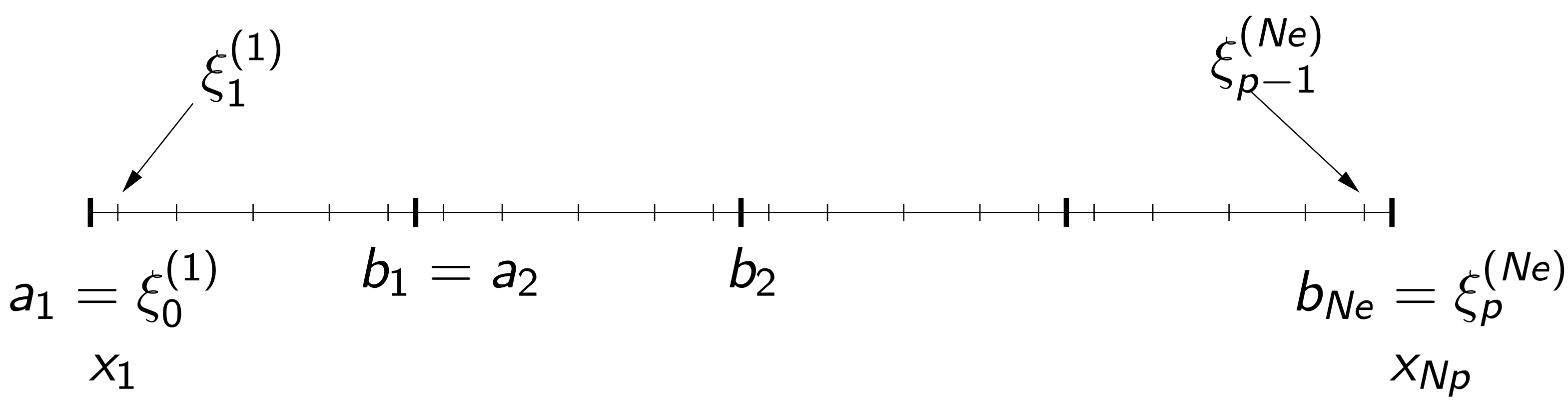
1D Spectral Elements

Let $p \geq 1$ integer and $\mathbb{P}_p$ the space of polynomials of degree $\leq p$.[1]

$$\Omega = \bigcup_{k=1}^{Ne} I_k$$

with I_k disjoint elements s.t. $I_k = F_k((-1,1))$, and

$$F_k : \xi \mapsto x = \frac{b_k - a_k}{2}\xi + \frac{b_k + a_k}{2}.$$



[1] From now on p plays the role of N (the polynomial degree)

Piece-wise polynomials

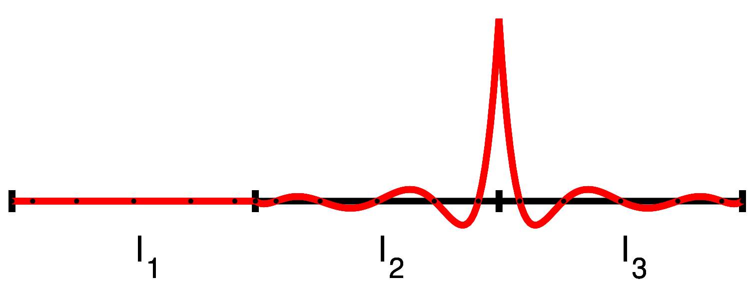
$Np = p \cdot Ne + 1$ is the total number of nodes in $\Omega \subset \mathbb{R}$.
Nodal Lagrange basis functions $\{\varphi_i\}_{i=1}^{Np}$ w.r.t. the LGL nodes x_i (unique ordering in Ω).
Lagrange functions φ_i are globally continuous in $\overline{\Omega}$, and locally polynomials of degree p.

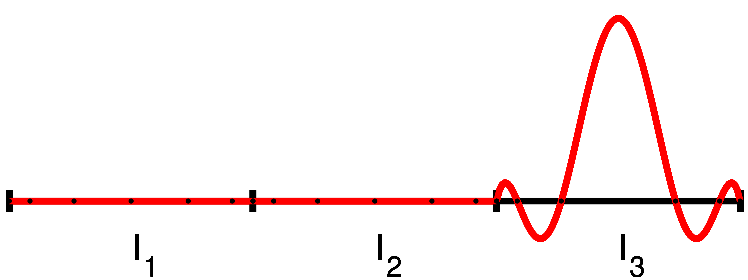
Set: $X_\delta = \{v \in C^0(\overline{\Omega}) : \; v|_{I_k} \in \mathbb{P}_p, \;\; \forall I_k\}$

$h_k = \mathrm{meas}(I_k)$,
mesh size $h = \max_k h_k$,
polynomial degree p

$$\delta = (h, p)$$

$$v_\delta(x) = \sum_{i=1}^{Np} v_\delta(x_i)\varphi_i(x) \quad \forall v_\delta \in X_\delta$$





Composite LGL quadrature formulas

Let $(\hat{\xi}_j, \hat{w}_j)$ (for $j = 0, \ldots, p$) be the LGL nodes and weigths in $\widehat{\Omega} = (-1, 1)$.

Local:
LGL quadrature

$$\int_{I_k} u(x)v(x)dx \simeq (u, v)_{\delta, I_k} = \sum_{j=0}^{p} u(\xi_j)v(\xi_j)w_j$$

with $\xi_j = \frac{b_k - a_k}{2}\hat{\xi}_j + \frac{b_k + a_k}{2}$ and $w_j = \frac{b_k - a_k}{2}\hat{w}_j$.

Global: composite
LGL quadrature

$$\int_{\Omega} u(x)v(x)dx \simeq (u, v)_{\delta, \Omega} = \sum_{k=1}^{Ne} (u, v)_{\delta, I_k}$$

Quadrature error: $\exists c > 0 : \ \forall f \in H^r(\Omega), \ r \geq 1, \ p \geq 1 : \forall v_\delta \in X_\delta$ (recall that $\delta = (h, p)$)

$$\left| \int_{\Omega} f v_\delta \, dx - (f, v_\delta)_{\delta, \Omega} \right| \leq c \ h^{\min(p, r)} \left(\frac{1}{p} \right)^r \|f\|_{H^r(\Omega)} \|v_\delta\|_{L^2(\Omega)}$$

Interpolation error: $\exists C > 0 : \ \forall v \in H^{s+1}(\Omega), \ s \geq 1 :$

$$\|v - \Pi_\delta^{LGL} v\|_{H^k(\Omega)} \leq C h^{\min(p+1, s+1) - k} \left(\frac{1}{p} \right)^{s+1-k} \|v\|_{H^{s+1}(\Omega)}$$

Diffusion-transport-reaction problem (reminder)

Given: $\Omega = (\bar{a}, \bar{b})$, $\mu, b, \sigma > 0$ const., $f : \Omega \to \mathbb{R}$,
look for $u : \Omega \to \mathbb{R}$ s.t.

$$\begin{cases} -(\mu u')' - (bu)' + \sigma u = f & \text{in } \Omega, \\ u(\bar{a}) = 0, \quad u(\bar{b}) = 0. \end{cases}$$

Set $V = H_0^1(\Omega)$. The **weak form** of the differential problem reads:
given $f \in L^2(\Omega)$,

$$\text{find } u \in V \text{ s.t.} \quad a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in V,$$

where

$$a(u, v) = \int_\Omega (\mu u' - bu)\, v'\, dx + \int_\Omega \sigma u v\, dx,$$
$$(f, v)_{L^2(\Omega)} = \int_\Omega f\, v\, dx.$$

SEM with Galerkin Numerical Integration (SEM-GNI)

Set $a_\delta(\varphi_j, \varphi_i) = (\nu\varphi_j' - b\varphi_j, \varphi_i')_{\delta,\Omega} + (\sigma\varphi_j, \varphi_i)_{\delta,\Omega}$

$$SEM - GNI \quad ?u_\delta^{GNI} \in V_\delta : \quad a_\delta(u_\delta^{GNI}, v_\delta) = (f, v_\delta)_{\delta,\Omega} \qquad \forall v_\delta \in V_\delta$$

Expand u_δ^{GNI} w.r.t. the Lagrange basis: $u_\delta^{GNI}(x) = \sum_{i=1}^{Np} u_\delta^{GNI}(x_i)\varphi_i(x)$

and choose $v_\delta(x) = \varphi_i(x)$ for any $i = 1, \ldots, Np$.
The **SEM-GNI discretization** of the weak problem reads:

look for $\mathrm{u}^{GNI} = [u_\delta^{GNI}(x_j)]_{j=1}^{Np}$, $u_\delta^{GNI}(x_1) = 0$, $u_\delta^{GNI}(x_{Np}) = 0$, *and*

$$\sum_{j=1}^{Np} a_\delta(\varphi_j, \varphi_i) u_\delta^{GNI}(x_j) = (f, \varphi_i)_{\delta,\Omega} \qquad \text{for any } i = 1, \ldots, Np$$

or equivalently $A^{GNI}\mathrm{u}^{GNI} = \mathrm{f}^{GNI}$
with $A_{ij}^{GNI} = a_\delta(\varphi_j, \varphi_i)$ and $f_i^{GNI} = (f, \varphi_i)_{\delta,\Omega}$.

Error analysis for SEM-GNI

Let $u_\delta^{GNI} \in V_\delta$ be the SEM-GNI solution of

$$a_\delta(u_\delta^{GNI}, v_\delta) = (f, v_\delta)_{\delta,\Omega} \qquad \forall v_\delta \in V_\delta.$$

By applying the **Strang Lemma**[2] it holds:

$$\begin{aligned}\|u - u_\delta^{GNI}\|_V \leq \quad & \|u - u_\delta\|_V \\ & + \frac{1}{\mu^*} \sup_{v_\delta \in V_\delta \setminus \{0\}} \frac{|a(u_\delta, v_\delta) - a_\delta(u_\delta, v_\delta)|}{\|v_\delta\|_V} \\ & + \frac{1}{\mu^*} \sup_{v_\delta \in V_\delta \setminus \{0\}} \frac{|(f, v_\delta)_{L^2(\Omega)} - (f, v_\delta)_{\delta,\Omega}|}{\|v_\delta\|_V}\end{aligned}$$

where μ^* is the coercivity constant of a_δ, i.e. $a_\delta(v_\delta, v_\delta) \geq \mu^* \|v_\delta\|_V^2$ and u_δ is the SEM-Galerkin solution.

[2] where u_δ is the interpolant of u upon the subspace V_δ (u_δ = best fit of u)

Bounding the three errors

For any $u \in H^{s+1}(\Omega)$ and $f \in H^r(\Omega)$

$$\|u - u_\delta\|_V + \sup_{v_\delta \in V_\delta \setminus \{0\}} \frac{|a(u_\delta, v_\delta) - a_\delta(u_\delta, v_\delta)|}{\|v_\delta\|_V} \leq c h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)}$$

and

$$\sup_{v_\delta \in V_\delta \setminus \{0\}} \frac{|(f, v_\delta)_{L^2(\Omega)} - (f, v_\delta)_{\delta,\Omega}|}{\|v_\delta\|_V} \leq c h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)}$$

thus

$$\boxed{\begin{aligned}\|u - u_\delta^{GNI}\|_{H^1(\Omega)} \leq C\Big(& h^{\min(p,s)} \left(\tfrac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} \\ & + h^{\min(p,r)} \left(\tfrac{1}{p}\right)^r \|f\|_{H^r(\Omega)}\Big)\end{aligned}}$$

i.e., u_δ^{GNI} converges with spectral accuracy with respect to p and algebraic accuracy with respect to h to the exact solution.

Spectral Element Methods on quads

Strong points (of the most used form nowadays)

1. nodal (Lagrange) basis
2. the interpolation nodes are the Legendre Gauss Lobatto (LGL) nodes
3. when Numerical Integration is used (SEM-GNI) then the **quadrature nodes** are exactly the **interpolation nodes** and Lagrange basis is orthogonal w.r.t. the discrete L^2 inner product (induced by quadrature) $\Longrightarrow$ diagonal mass matrices (in $\mathbb{R}^d$, $d \geq 1$)
4. tensorial structure of the basis functions in $\mathbb{R}^d$ (with $d \geq 2$) $\Longrightarrow$ high computational efficiency

A diffusion-reaction problem

Given $\mu(x) \geq \mu > 0$ and $\sigma(x) \geq 0$, both in $L^\infty(\Omega)$; $f \in L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d$ and $d = 2, 3$,

look for $u : \Omega \to \mathbb{R}$:

strong form	$\begin{cases} -\nabla \cdot (\mu \nabla u) + \sigma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$

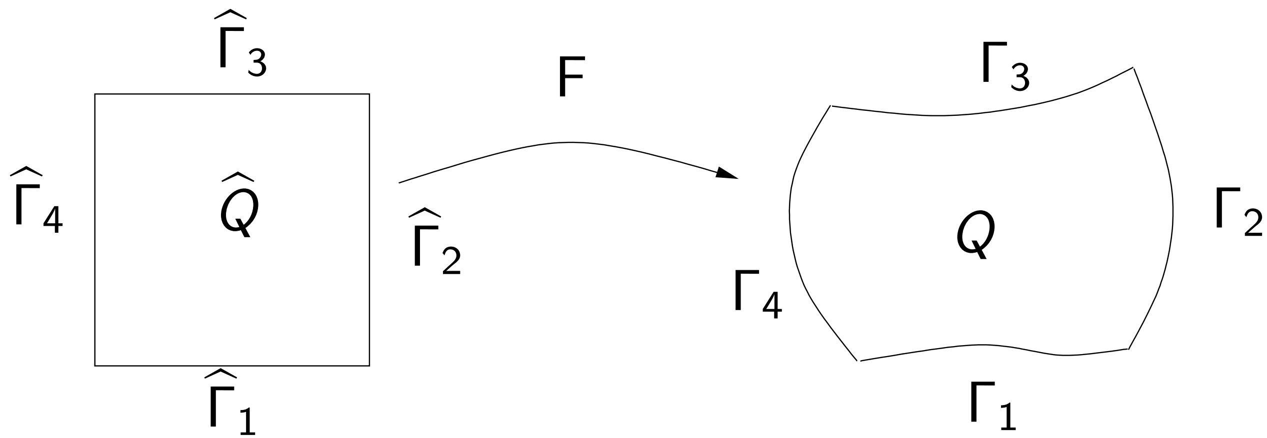
By setting $V = H_0^1(\Omega)$, $\quad a(u, v) = \int_\Omega \mu \nabla u \cdot \nabla v d\Omega + \int_\Omega \sigma u v d\Omega$,

$(f, v)_{L^2(\Omega)} = \int_\Omega f v d\Omega$

weak form	$?u \in V : \quad a(u, v) = (f, v)_{L^2(\Omega)} \qquad \forall v \in V$

The computational domain $Q \subset \mathbb{R}^d$, $(d \geq 2)$

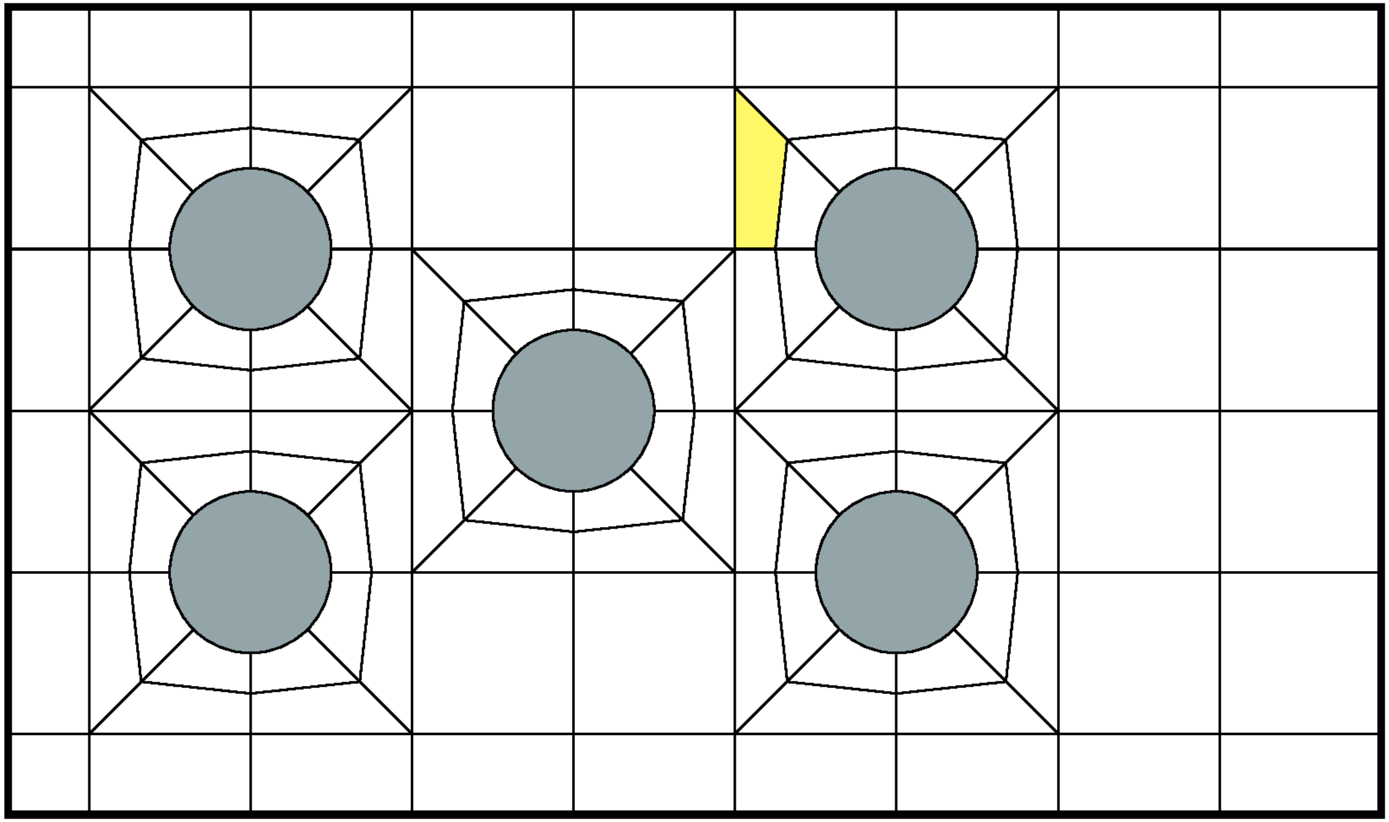
Reference domain: $\widehat{Q} = (-1,1)^d$.
Lipschitz bounded domain $Q \in \mathbb{R}^d$: $\exists \mathrm{F} : \widehat{Q} \rightarrow Q$ bijective and differentiable



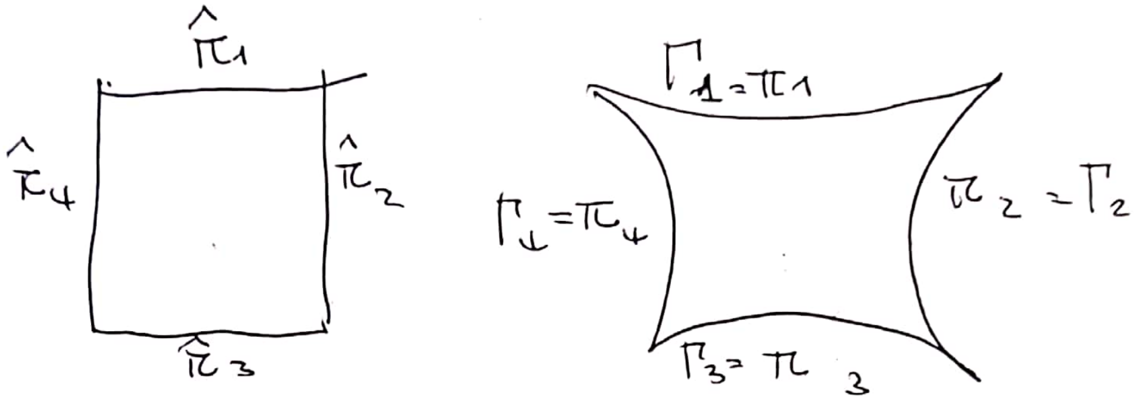
Γ_j **arcs** or **edges** of Q.

An example of computational domain

Partition in quadrilaterals
SEM (or *hp*−fem) on quads. $\mathcal{T} = \{Q_k\}_{k=1}^{Ne}$ is a conforming partition of Ω: $\Omega = \cup_{k=1}^{Ne} Q_k$ and $\exists \mathrm{F}_k : \widehat{Q} \to Q_k$ bijective and differentiable (for any $k = 1, \ldots, Ne$)

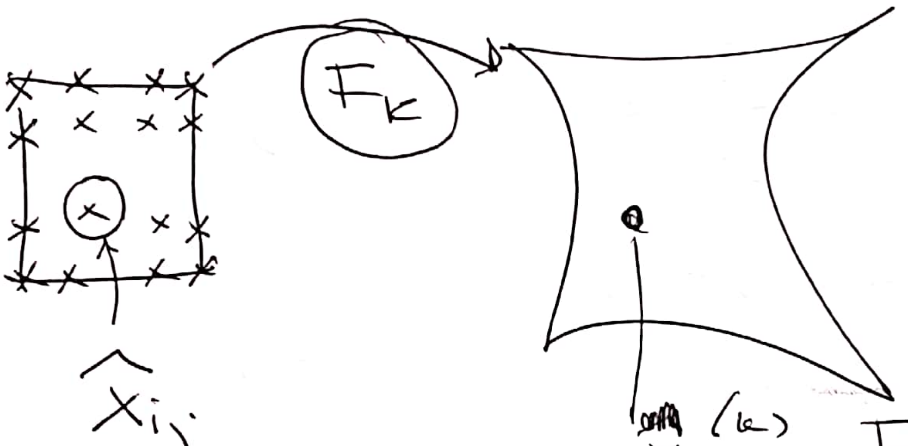


$$\hat{\Omega} \xrightarrow{F_K} \Omega_K$$



$(-1,1)^2$

$$\pi_e^{(K)} : [-1,1] \longrightarrow \Gamma_e$$



$$x_{ij}^{(K)} = F_K(\hat{x}_{ij})$$

$\forall i,j \quad \forall K.$

$$\boxed{\text{Gordon \& Hall map}}$$

Mapping a Square into a Curved Quadrilateral.

How to design mappings F_k

Conformal mappings preserve orthogonality, the divergence and the gradient (Milne-Thomson 1966, Israeli 1981, Trefethen 1980, Gordon-Hall 1973)

The simplest ones are linear blending mappings

In $\mathbb{R}^2$, given the maps $\boldsymbol{\pi}_\ell^{(k)} : [-1,1] \to \Gamma_\ell$ (arcs in $\mathbb{R}^2$) for $\ell = 1, \dots, 4$, $\mathsf{F}_k : \widehat{Q} \to Q_k$ is defined as

$$\boxed{\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = \mathsf{F}_k\left(\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}\right) = & \frac{1-\hat{y}}{2}\boldsymbol{\pi}_1^{(k)}(\hat{x}) + \frac{1+\hat{y}}{2}\boldsymbol{\pi}_3^{(k)}(\hat{x}) \\ & + \frac{1-\hat{x}}{2}\left[\boldsymbol{\pi}_4^{(k)}(\hat{y}) - \frac{1+\hat{y}}{2}\boldsymbol{\pi}_4^{(k)}(1) - \frac{1-\hat{y}}{2}\boldsymbol{\pi}_4^{(k)}(-1)\right] \\ & + \frac{1+\hat{x}}{2}\left[\boldsymbol{\pi}_2^{(k)}(\hat{y}) - \frac{1+\hat{y}}{2}\boldsymbol{\pi}_2^{(k)}(1) - \frac{1-\hat{y}}{2}\boldsymbol{\pi}_2^{(k)}(-1)\right]\end{aligned}}$$

Similar construction in 3D, now $\boldsymbol{\pi}_\ell : [-1,1]^2 \to \Sigma_\ell$ (faces in $\mathbb{R}^3$) for $\ell = 1, \dots, 6$.

Finite dimensional spaces

Let $p \geq 1$ integer and $\mathbb{Q}_p$ the space of polynomials of degree $\leq p$ w.r.t. each variable $x_1, \ldots, x_d$.
Set

$$X_\delta = \{v \in C^0(\overline{\Omega}) : \ v|_{Q_k} = \hat{v} \circ \mathsf{F}_k^{-1}, \ \ \text{with } \hat{v} \in \mathbb{Q}_p(\widehat{Q}), \ \forall Q_k \in \mathcal{T}\}$$

mesh size $h = \max_k h_k, \ h_k = \mathrm{diam}(Q_k)$, polynomial degree p

$$\Rightarrow \delta = (h, p)$$

SM & SEM RECAP

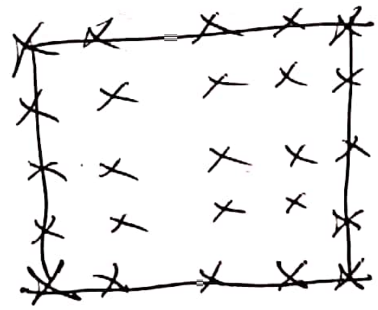
$$(1) \quad \begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$(2) \quad ?\, u \in V : a(u,v) = F(v) \quad \forall v \in V$$

$$\boxed{\begin{array}{l} \text{(SM)} \quad ?\, u_N \in V_N : a(u_N, v_N) = F(v_N) \quad \forall v_N \in V_N \\ \qquad \Omega = (-1,1)^d \qquad d = 1,2,3 \end{array}}$$

$$V_N = \begin{cases} \mathbb{P}_N^0 & 1D \\ \mathbb{Q}_N^0 & 2D, 3D \end{cases}$$

$$\boxed{\text{(GNI)} \quad ?\, u_N \in V_N : a_N(u_N, v_N) = F_N(v_N) \quad \forall v_N \in V_N}$$



Ω

LGL nodes & weights

a_N & F_N are obtained from a & F by replacing exact scalar products (Φ, Ψ) by discrete scalar products $(\Phi, \Psi)_N$ (LGL)

For both cases (SM) & (GNI) we can prove:

$$(3) \quad \|u - u_N\|_{H^1(\Omega)} \leq C \left(\frac{1}{N}\right)^s \|u\|_{H^{s+1}(\Omega)}, \quad s \geq 1,$$

(3) follows from these results:

(a) $$\| v - \Pi_N^{LGL} v \|_{H^1(\Omega)} \le C\left(\frac{1}{N}\right)^s \|v\|_{H^{s+1}(\Omega)}$$

(b) $$\left\{\begin{array}{l} |(f, v_N) - (f, v_N)_N| \le C\left(\frac{1}{N}\right)^q \|f\|_{H^q(\Omega)} \|v_N\|_{L^2(\Omega)} \\ \forall f \in H^q(\Omega),\ q \ge 1,\ \forall v_N \in V_N \end{array}\right.$$

(c) Strang Lemma (generalization of Céa lemma)

Need to estimate

$$(G) \qquad \begin{array}{l} V \longrightarrow V_N \\ a \longrightarrow a \\ F \longrightarrow F \end{array} \qquad \left.\begin{array}{l} d(V, V_N) \\ - \\ - \end{array}\right\} \text{Céa}$$

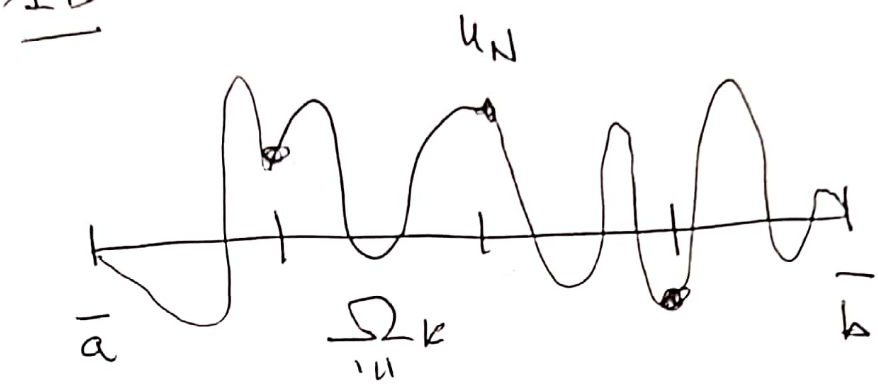
$$(GNI) \qquad \begin{array}{l} V \longrightarrow V_N \\ a \longrightarrow a_N \\ F \longrightarrow F_N \end{array} \qquad \left.\begin{array}{l} d(V, V_N) \\ d(a, a_N) \\ d(F, F_N) \end{array}\right\} \text{Strang}$$

Limitations: Geometrical simplicity

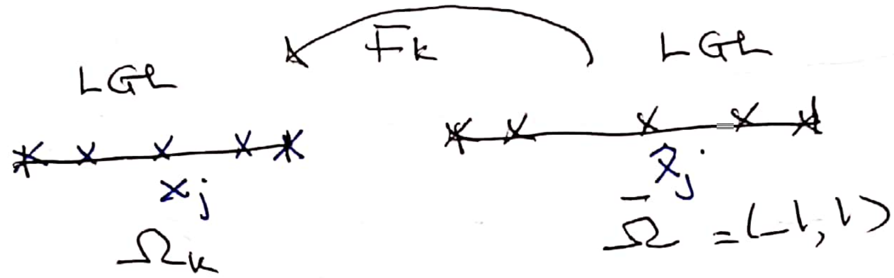
Ω must be a cube / parallelepipedal region

$\rightsquigarrow$ generalization: SEM (& SEM-NI).

(SEM) 1D



$\Omega_k = I_k = e_k$



$$w_j = \frac{|\Omega_k|}{2} \hat{w}_j$$

(SEM). ? $u_\delta \in V_\delta$: $a(u_\delta, v_\delta) = F(v_\delta) \quad \forall v_\delta \in V_\delta$

$\delta = (h, N)$

$h_k = |\Omega_k|$

$h = \max_k h_k$

$$V_\delta = \{ v_\delta \in C^0(\bar{\Omega}) : v_\delta|_{\Omega_k} \in \mathbb{Q}_N \ \forall k, \ v_\delta|_{\partial\Omega} = 0 \}$$

SEM-NI

? $u_\delta \in V_\delta$: $a_\delta(u_\delta, v_\delta) = F_\delta(v_\delta) \quad \forall v_\delta \in V_\delta$

a_δ & F_δ are obtained by replacing on every Ω_k the exact integrals by the LGL numerical integration on Ω_k

Analysis [N → p]

(a) $\|v - \Pi_\delta^{LGL} v\|_{H^1} \leq C h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|v\|_{H^{SH}}$

$\|v - \Pi_\delta^{LGL} v\|_{L^2} \leq C h^{\min(p,s)+1} \left(\frac{1}{p}\right)^{SH} \|v\|_{H^{SH}}$

$(s \geq 1)$

spectral accuracy contribution (cyan)

geometrical accuracy contribution (orange)

(b) $|(f, v_\delta) - (f, v_\delta)_\delta| \leq C h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r} \cdot \|v_\delta\|_{L^2}$

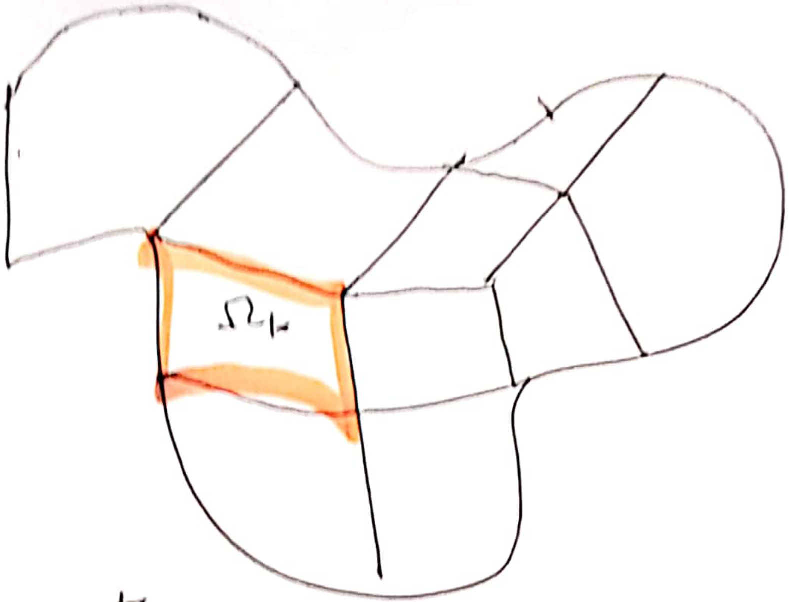
$(r \geq 1)$

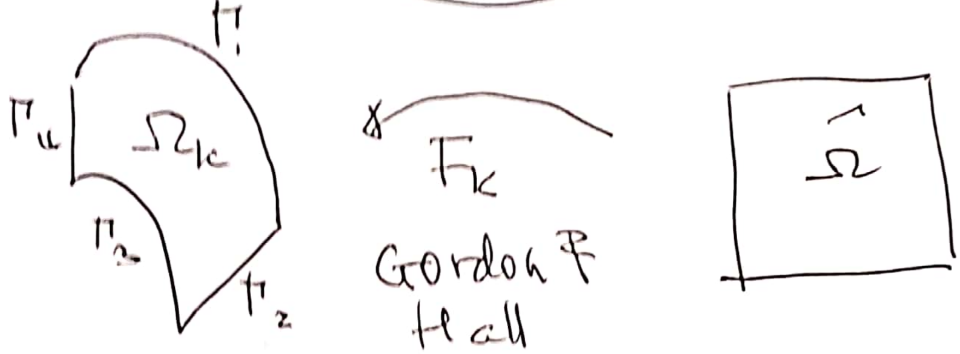
(c) Strang Lemma

$$\Rightarrow \boxed{\|u - u_\delta^{GNI}\|_{H^1} \leq C\left(h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|u\|_{H^{SH}} + h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r} \right)}$$

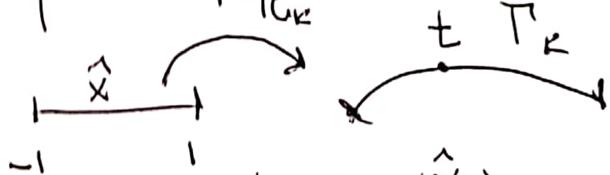
(error estimate)

Valid for any space dimension !!!





It only requires parametrization of



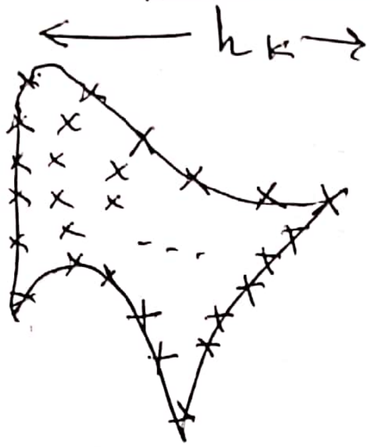
$t = \pi_k(\hat{x})$.

→ By F_k we recover LGL nodes & weights on every Ω_k from $\hat{\Omega}$.

Using independent h_k and p_k on Ω_k:

$$\boxed{\|u-u_\delta^{GNI}\|_{H^1} \le c_0 \left(\sum_{k=1}^{\#elem} h_k^{2\min(p_k,s_k)} \left(\frac{1}{p_k}\right)^{2s_k} \|u\|^2_{H^{s_k}(\Omega_k)} + \sum_{k=1}^{\#elem} h_k^{2\min(p_k,r_k)} \left(\frac{1}{p_k}\right)^{2r_k} \|f\|^2_{H^{r_k}(\Omega_k)} \right)^{1/2}}$$

Design Principle



Ω_K

solution $u|_{\Omega_K}$

and $f|_{\Omega_K}$

are "very smooth"

(s_K, r_K "large")

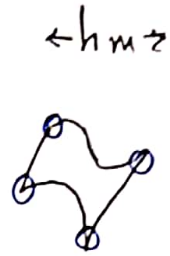
Then use large h_K

and large p_K

→ exploiting the spectral accuracy

↓

"SM" regime



Ω_M

solution $u|_{\Omega_M}$

and $f|_{\Omega_M}$

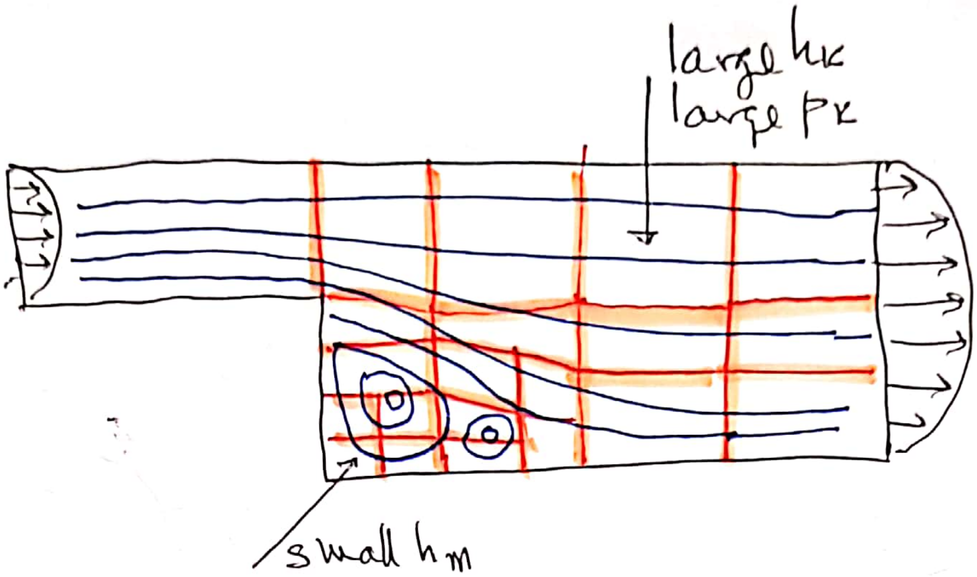
are functions with low regularity, then use

small h_K

and small p_K

↓

"FE" regime



flow in a channel with a step.

"philosophy":

$$\underbrace{\text{Global error}}_{E} \simeq \Big(\sum_K \big(\underbrace{\text{local errors}}_{E_K}\big)^2 \Big)^{1/2}$$

Best way is to ensure that local errors are "equilibrated": $E_K \sim E_M \ \forall K, M$

Now:

$$E_K \simeq E_K^{(1)} \, E_K^{(2)}(U_K, f_K)$$

$E_K^{(1)} \leftarrow h_K$, $\quad E_K^{(2)} \leftarrow p_K$, $\quad U_K \leftarrow u|_{\Omega_K}$, $\quad f_K \leftarrow f|_{\Omega_K}$

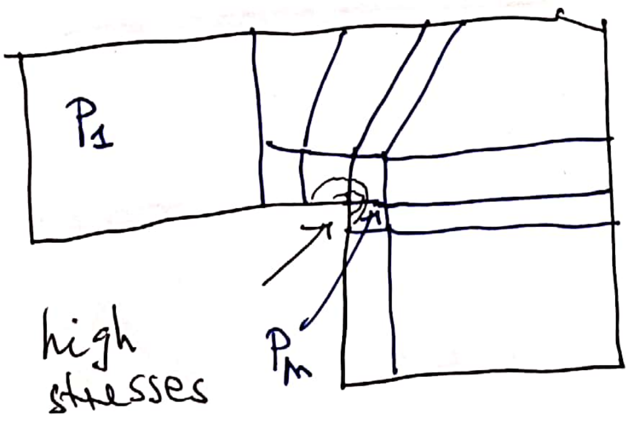


plate / structural mechanics

$p_M \ll p_1$

$h_1 \gg h_M$.

Physical problem

$$\underset{\substack{\uparrow \\ \text{unknown}}}{Lu} = \underset{\nwarrow \text{known}}{f}$$

2 ways of assessing the quality of u_δ

1 - Validation

Physical measures : I compare physical measure with u_δ

2 - Verification

$\underbrace{\|u - u_\delta\|_{H^1}}_{??} \leq ?$ .. $\underbrace{\|u\|_{H^{s+1}}}_{??}$

error estimates.

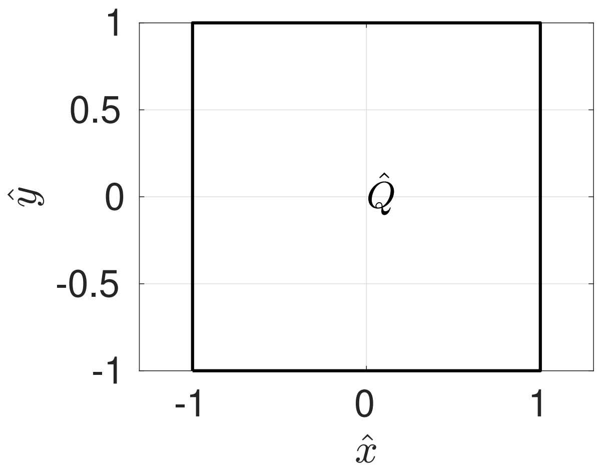
Benchmark math. problem: (Toy problems)

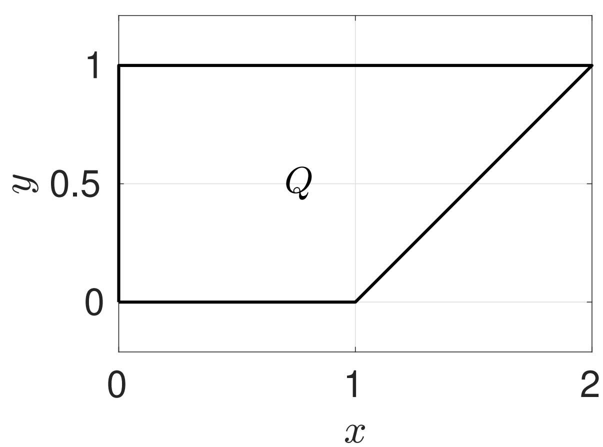
ⓐ I give u (my choice) [I can decide its regularity]

ⓑ I compute $f := Lu$
I compute $g := u|_{\partial\Omega}$

ⓒ Now I solve $\begin{cases} Lu = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$

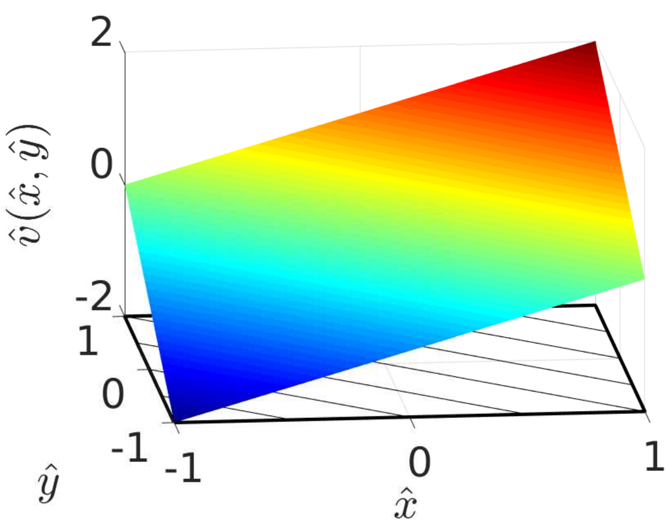
numerically $\longrightarrow u_\delta$.

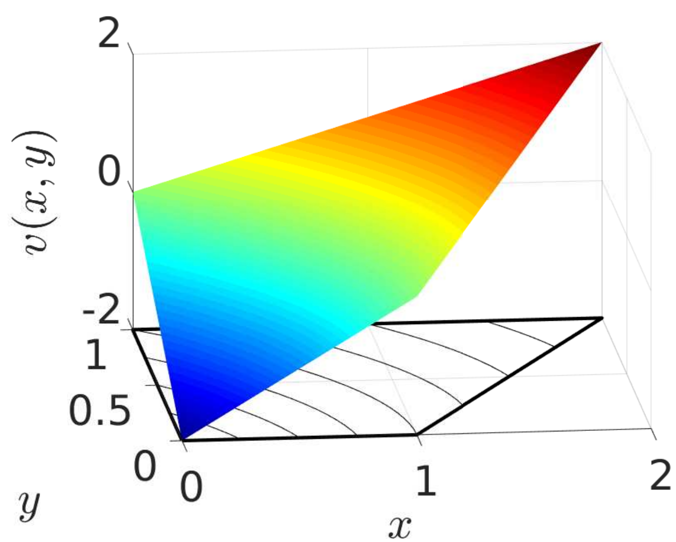




$$F\left(\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}\right) = \begin{bmatrix}\frac{\hat{x}+1}{2}\,\frac{\hat{y}+3}{2}\\ \frac{\hat{y}+1}{2}\end{bmatrix}$$

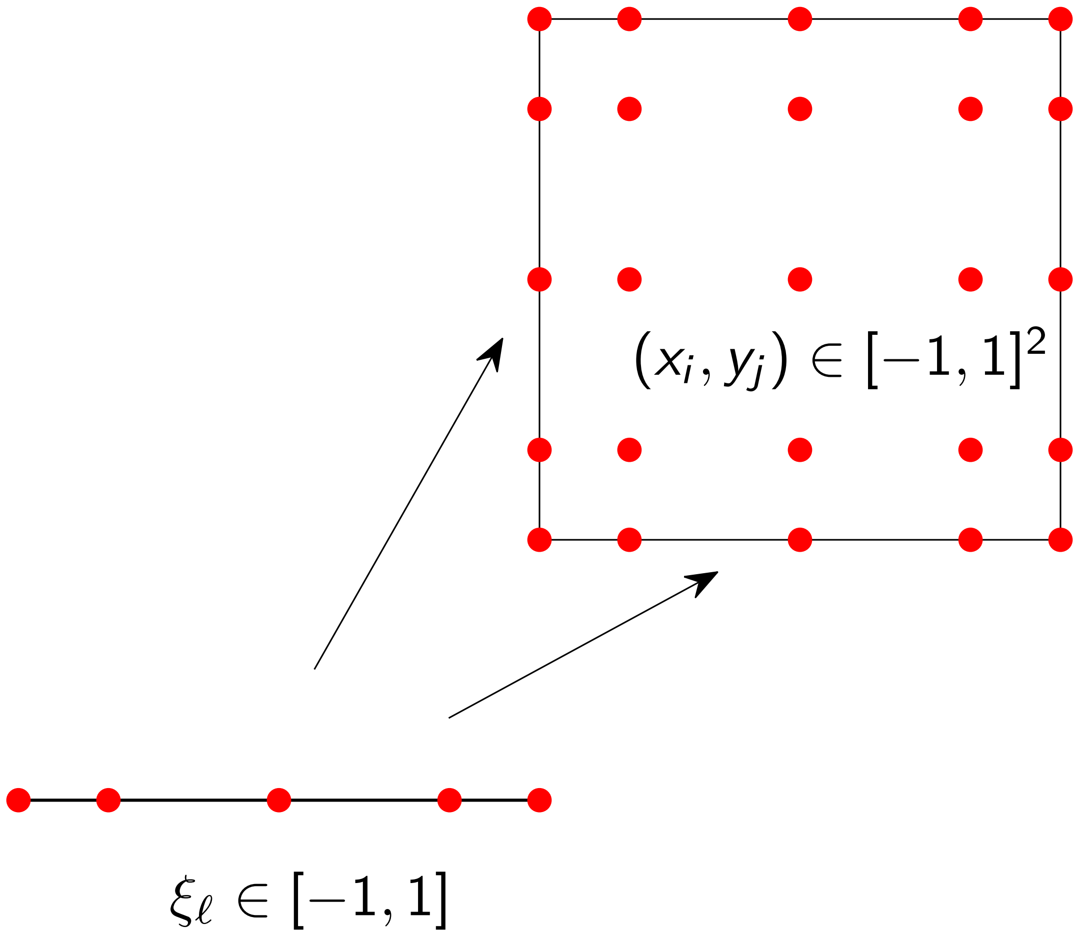
$$F^{-1}\left(\begin{bmatrix}x\\ y\end{bmatrix}\right) = \begin{bmatrix}\frac{2x-y-1}{y+1}\\ 2y-1\end{bmatrix}$$





$\hat{v}(\hat{x},\hat{y}) = \hat{x} + \hat{y} \in \mathbb{Q}_1(\hat{Q})$ $\qquad$ $v(x,y) = \frac{2x-y-1}{y+1} + 2y - 1 \notin \mathbb{Q}_1(Q)$

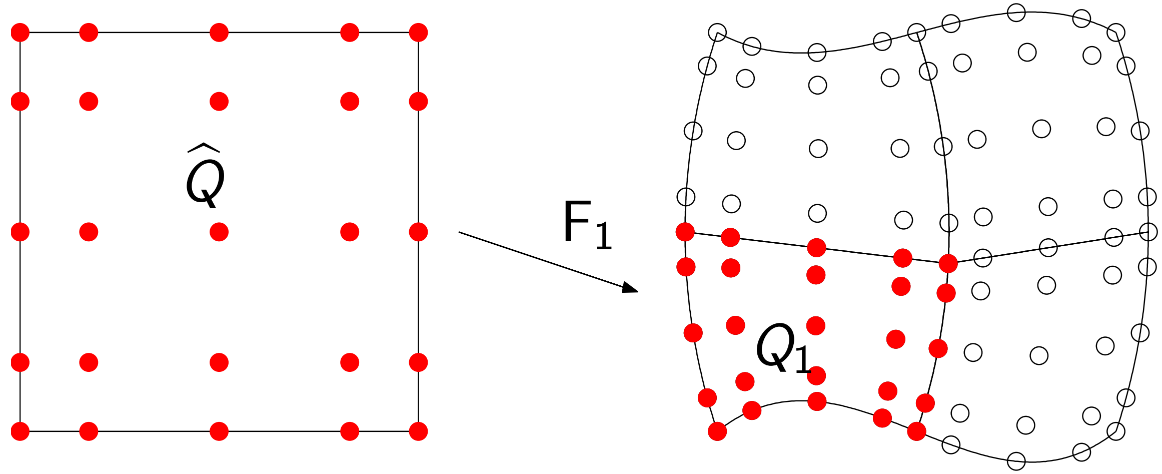
Tensorization of LGL nodes on $\hat{Q}$



Numerical integration ($\Omega \subset \mathbb{R}^2$)

$$F_k : \hat{x} \to x$$

$$J_k = \left[\frac{\partial x_i}{\partial \hat{x}_j}\right]_{i,j=1}^{d}$$



Local: LGL quadrature

$$\int_{Q_k} u(x)v(x)dx \simeq (u,v)_{\delta,Q_k} = \sum_{i,j=0}^{p} u(\xi_i,\xi_j)v(\xi_i,\xi_j)w_i w_j |\det(J_k(\xi_i,\xi_j))|$$

Global: composite LGL quadrature

$$\int_{\Omega} u(x)v(x)dx \simeq \sum_{k=1}^{Ne} (u,v)_{\delta,Q_k} = (u,v)_{\delta,\Omega}$$

Quadrature error: $\exists c = c(\Omega) > 0 : \ \forall f \in H^r(\Omega), \ r \geq 1, \ v_\delta \in X_\delta$

$$\left| \int_{\Omega} f v_\delta - (f, v_\delta)_{\delta,\Omega} \right| \leq c \ h^{\min(r,p)} \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)} \|v_\delta\|_{L^2(\Omega)}$$

How to represent $v_\delta \in X_\delta$

Np = total number of nodes in Ω

Nodal Lagrange basis functions $\{\varphi_i\}_{i=1}^{Np}$ w.r.t. the LGL nodes $\boldsymbol{\xi}_i$

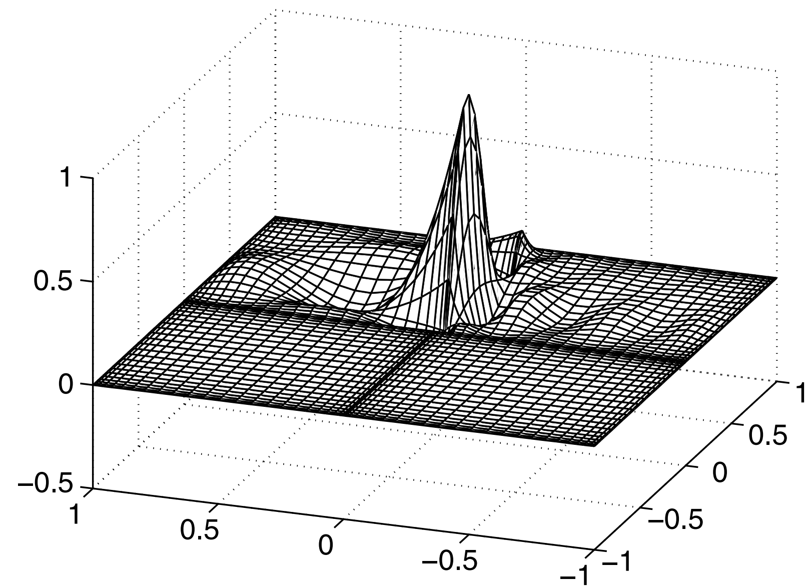
φ_i are globally continuous in $\overline{\Omega}$, and locally polynomials of degree p w.r.t. each variable $x_1, \ldots, x_d$.

$d = 1$,
$\varphi_i(\mathrm{x}) = \varphi_i^{(1)}(x)$

tensorial basis functions:

$d = 2$,
$\varphi_i(\mathrm{x}) = \varphi_{i1}^{(1)}(x_1)\varphi_{i2}^{(1)}(x_2)$,

$d = 3$,
$\varphi_i(\mathrm{x}) = \varphi_{i1}^{(1)}(x_1)\varphi_{i2}^{(1)}(x_2)\varphi_{i3}^{(1)}(x_3)$



$$v_\delta(\mathrm{x}) = \sum_{i=1}^{Np} v_\delta(\boldsymbol{\xi}_i)\varphi_i(\mathrm{x}) \quad \forall v_\delta \in X_\delta$$

SEM-GNI formulation

SEM-GNI $\quad ?u_\delta \in V_\delta : \quad a_\delta(u_\delta, v_\delta) = (f, v_\delta)_{\delta,\Omega} \quad \forall v_\delta \in V_\delta$

At the elements interfaces, u_δ is merely continuous
The continuity of the flux at interfaces is ensured only in the limit $p \to \infty$.
Expand u_δ w.r.t. the Lagrange basis:

$$u_\delta(\mathrm{x}) = \sum_{i=1}^{Np_0} u_\delta(\mathrm{x}_i)\varphi_i(\mathrm{x}) + \sum_{i=Np_0+1}^{Np} u_\delta(\mathrm{x}_i)\varphi_i(\mathrm{x})$$

($j = 1, \dots, Np_0$ functions corresponding to internal nodes, $j = Np_0 + 1, \dots, Np$ functions corresponding to boundary nodes)

and choose $v_\delta(\mathrm{x}) = \varphi_i(\mathrm{x})$ for any $i = 1, \dots, Np_0$.

SEM-GNI reads:

look for $\mathrm{u} = [u_\delta(\mathrm{x}_j)]_{j=1}^{Np}$, $u_\delta(\mathrm{x}_j) = 0$, for $j = Np_0 + 1, \dots, Np$, *and*

$$\sum_{j=1}^{Np} a_\delta(\varphi_j, \varphi_i)\mathrm{u}_j = (f, \varphi_i)_{\delta,\Omega} \qquad \text{for any } i = 1, \dots, Np_0$$

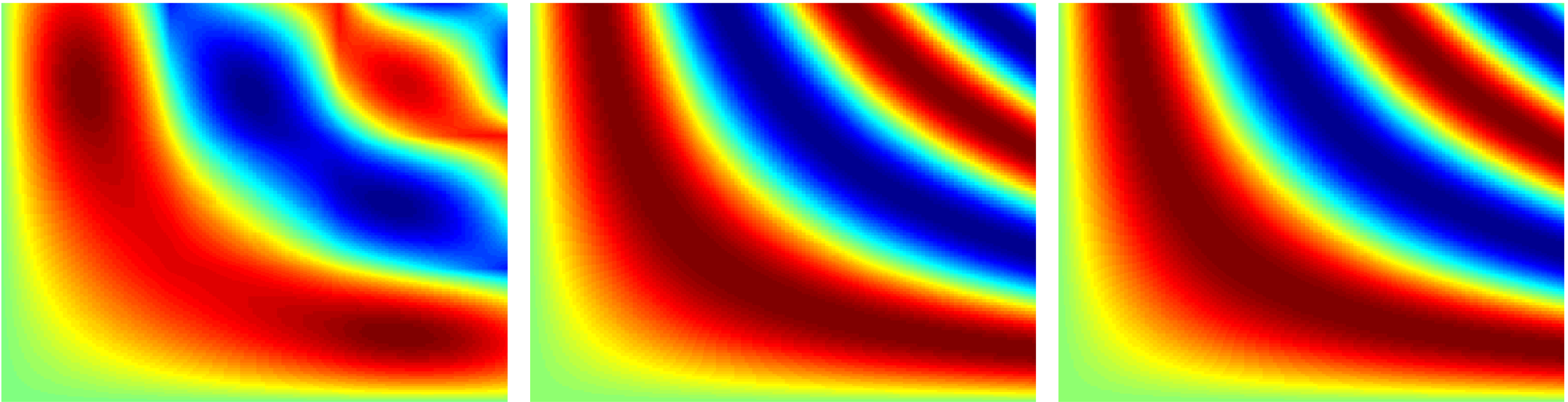
where $a_\delta(\varphi_j, \varphi_i) = (\mu\nabla\varphi_j, \nabla\varphi_i)_{\delta,\Omega} + (\sigma\varphi_j, \varphi_i)_{\delta,\Omega}$.

Convergence analysis for SEM-GNI

$$?u_\delta \in V_\delta : \quad a_\delta(u_\delta, v_\delta) = (f, v_\delta)_{\delta,\Omega} \qquad \forall v_\delta \in V_\delta$$

u_δ converges with spectral accuracy (with respect to p) to the exact solution when the latter and f are smooth:

$$\boxed{\begin{aligned}\|u - u_\delta\|_{H^1(\Omega)} \leq C(s)\Big(& h^{\min(p,s)} \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} \\ & + h^{\min(p,r)} \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)}\Big)\end{aligned}}$$



$h = 2/3$, $p = 2$ — $e_{H1} \simeq 3.77e\text{-}01$

$h = 2/3$, $p = 6$ — $e_{H1} \simeq 8.80e\text{-}04$

$h = 2/3$, $p = 16$ — $e_{H1} \simeq 3.64e\text{-}14$

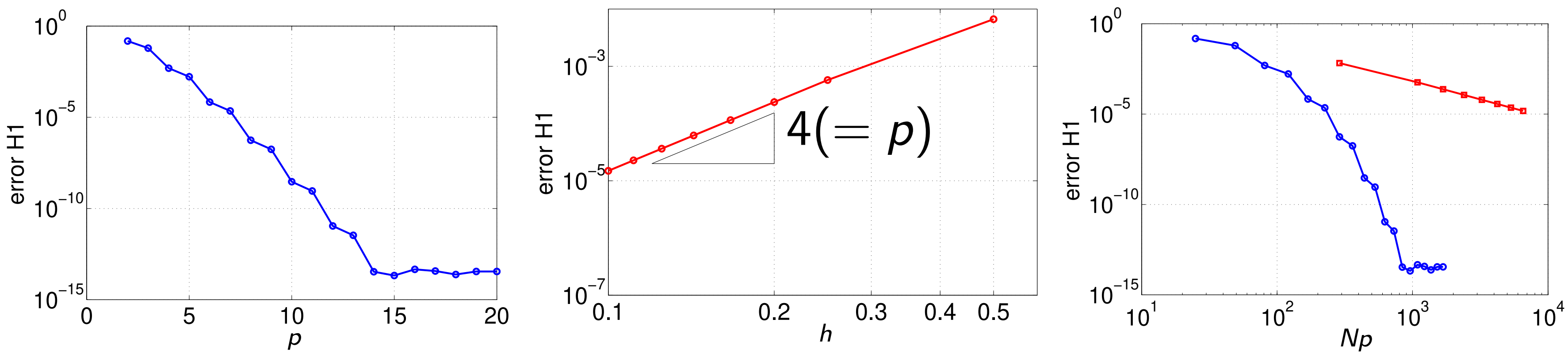
$\Omega = (0,2)^2$

Convergence rate

1. $s,\ r$ large $(s, r > p)$

$$\|u - u_\delta\|_{H^1(\Omega)} \le C\left(h^p \left(\frac{1}{p}\right)^s \|u\|_{H^{s+1}(\Omega)} + h^p \left(\frac{1}{p}\right)^r \|f\|_{H^r(\Omega)}\right)$$

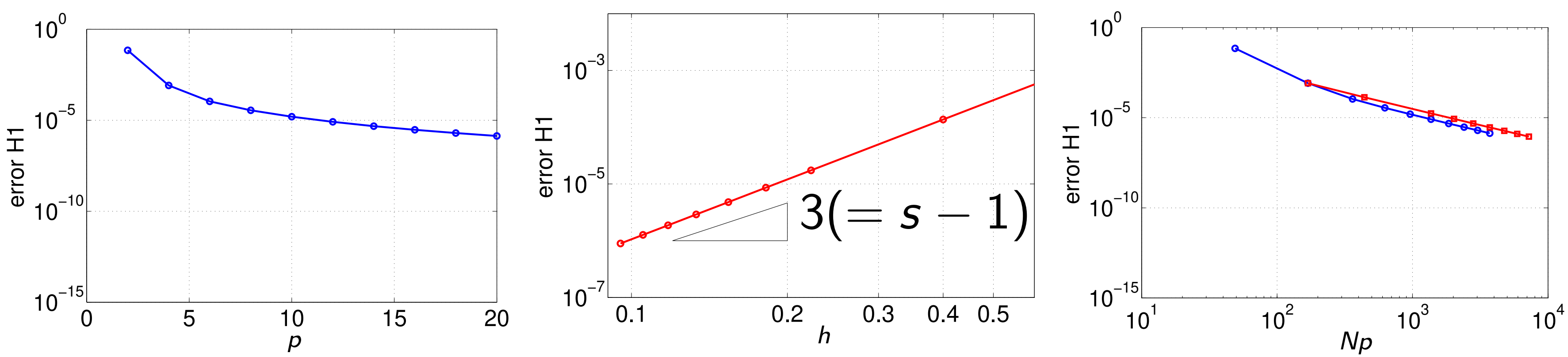
$s, r = \infty$



2. s small $(s \le p)$

$$\|u - u_\delta\|_{H^1(\Omega)} \le C\left(\frac{h}{p}\right)^s \|u\|_{H^{s+1}(\Omega)}$$

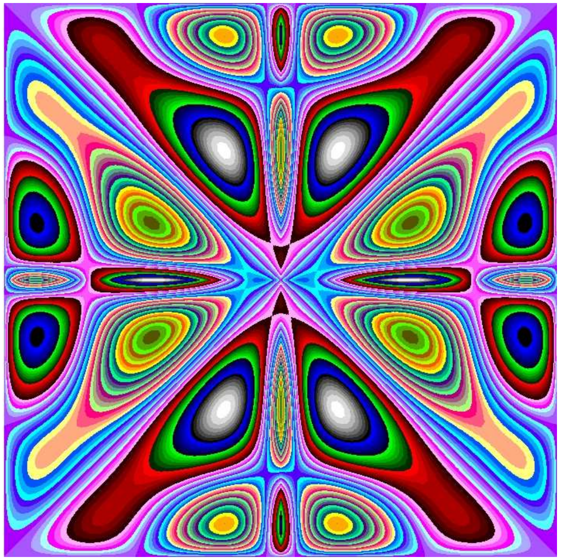
$s = 4$, $r = 2$, f composite $\mathbb{Q}_2$, null quadrature error on f, when $p > 2$

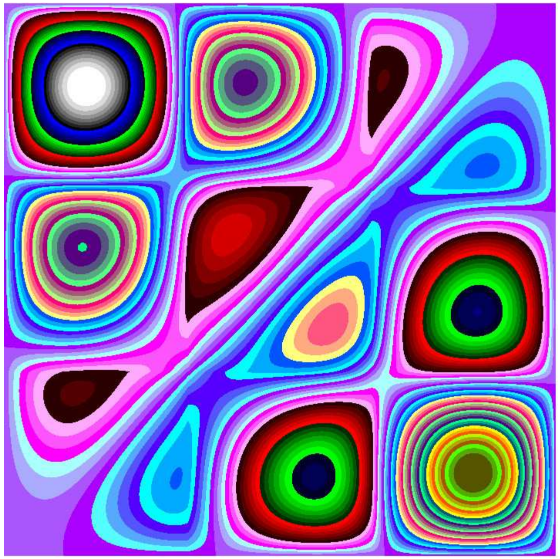


SEM-GNI eigenfunctions for the Laplace operator

A = SEM-GNI stiffness matrix on $\Omega = (-1,1)^2$, $u = 0$ on $\partial\Omega$. M = SEM-GNI mass matrix. v_i = ith eigenfunction of A, w_i = ith eigenfunction of $M^{-1}A$.

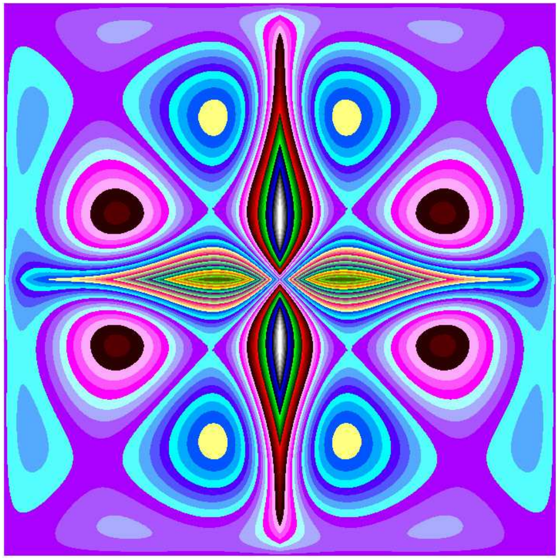
v_{45},
$p = 4$
2×2 elem.

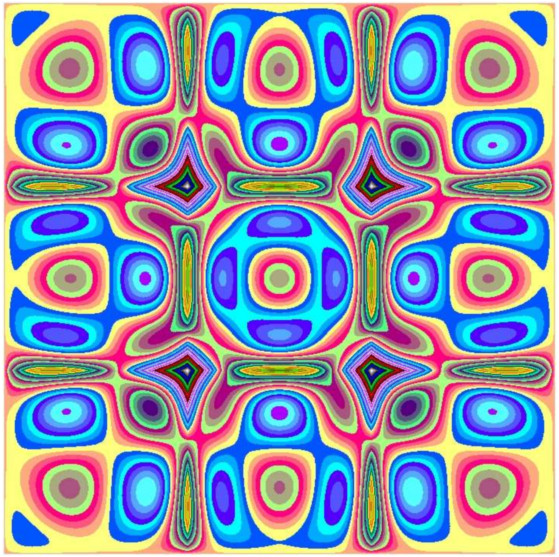




v_{19},
$p = 4$
3×3 elem.

w_{37},
$p = 4$
2×2 elem.





w_{101},
$p = 4$
3×3 elem.

Derivatives computation (let us work on $\widehat{Q}$)

$$(\mu\nabla\varphi_j, \nabla\varphi_i)_{\delta,\widehat{Q}} = \sum_{m,n=0}^{p} \mu(\hat{\xi}_m, \hat{\xi}_n)\nabla\varphi_j(\hat{\xi}_m, \hat{\xi}_n) \cdot \nabla\varphi_i(\hat{\xi}_m, \hat{\xi}_n)\hat{w}_m\hat{w}_n$$

We need to know derivatives at quadrature nodes (=interpolation nodes), then (recalling that $\varphi_j(\mathrm{x}) = \varphi_{j1}^{(1)}(x_1)\varphi_{j2}^{(1)}(x_2)$)

$$\boxed{\begin{aligned} \frac{\partial\varphi_j}{\partial\hat{x}_1}(\hat{\xi}_m, \hat{\xi}_n) &= \frac{\partial\varphi_{j1}^{(1)}}{\partial\hat{x}_1}(\hat{\xi}_m)\varphi_{j2}^{(1)}(\hat{\xi}_n) = D_{m,j1}\delta_{n,j2} \\ \frac{\partial\varphi_j}{\partial\hat{x}_2}(\hat{\xi}_m, \hat{\xi}_n) &= \varphi_{j1}^{(1)}(\hat{\xi}_m)\frac{\partial\varphi_{j2}^{(1)}}{\partial\hat{x}_2}(\hat{\xi}_n) = \delta_{m,j1}D_{n,j2} \end{aligned}}$$

spectral derivative matrix

$$D_{ij} = \left[\begin{array}{c|c|c} \ldots & \varphi_j'(\hat{\xi}_i) & \ldots \end{array} \right]$$

$$\varphi_j(\xi) = -\frac{1}{p(p+1)}\frac{(1-\xi^2)}{\xi-\xi_j}\frac{L_p'(\xi)}{L_p(\xi_j)}$$

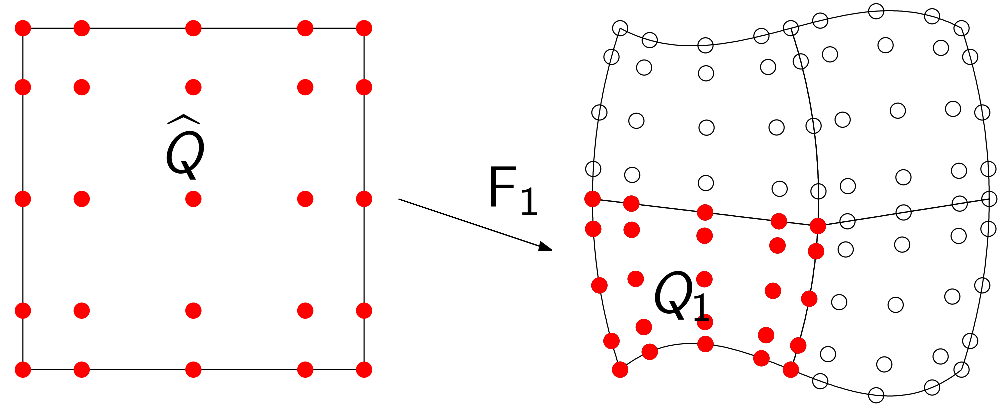
$$D_{ij} = \begin{cases} \dfrac{L_p(\hat{\xi}_j)}{L_p(\hat{\xi}_i)}\dfrac{1}{\hat{\xi}_j - \hat{\xi}_i} & j \neq i \\ -\dfrac{p(p+1)}{4} & j = i = 0 \\ \dfrac{p(p+1)}{4} & j = i = p \\ 0 & \text{otherwise} \end{cases}$$

Derivatives on $Q_k = \mathrm{F}_k(\widehat{Q})$

Standard arguments:

$$\mathrm{F}_k : \hat{\mathrm{x}} \to \mathrm{x}$$

$$J_k(\hat{\mathrm{x}}) = \left[\frac{\partial x_i}{\partial \hat{x}_j}(\hat{\mathrm{x}})\right]_{i,j=1}^{d}$$



$$(\varphi_j(\mathrm{x}) = \hat{\varphi}_j(\hat{\mathrm{x}}))$$

$$\left[\begin{array}{c} \dfrac{\partial \varphi_j}{\partial x_1}(\boldsymbol{\xi}_i) \\ \dfrac{\partial \varphi_j}{\partial x_2}(\boldsymbol{\xi}_i) \end{array}\right] = \frac{1}{\det J_k(\hat{\boldsymbol{\xi}}_i)} \left[\begin{array}{cc} \dfrac{\partial x_2}{\partial \hat{x}_2}(\hat{\boldsymbol{\xi}}_i) & -\dfrac{\partial x_2}{\partial \hat{x}_1}(\hat{\boldsymbol{\xi}}_i) \\ -\dfrac{\partial x_1}{\partial \hat{x}_2}(\hat{\boldsymbol{\xi}}_i) & \dfrac{\partial x_1}{\partial \hat{x}_1}(\hat{\boldsymbol{\xi}}_i) \end{array}\right] \left[\begin{array}{c} \dfrac{\partial \hat{\varphi}_j}{\partial \hat{x}_1}(\hat{\boldsymbol{\xi}}_i) \\ \dfrac{\partial \hat{\varphi}_j}{\partial \hat{x}_2}(\hat{\boldsymbol{\xi}}_i) \end{array}\right]$$

$$\nabla \varphi_j(\boldsymbol{\xi}_i) = J_k^{-T}(\hat{\boldsymbol{\xi}}_i) \hat{\nabla} \hat{\varphi}_j(\hat{\boldsymbol{\xi}}_i)$$

Algebraic aspects of SEM-GNI

Recall that $a_\delta(\varphi_j, \varphi_i) = (\mu \nabla \varphi_j, \nabla \varphi_i)_{\delta,\Omega} + (\sigma \varphi_j, \varphi_i)_{\delta,\Omega}$

$$
\begin{array}{lll}
A \in \mathbb{R}^{Np \times Np}: & a_\delta(\varphi_j, \varphi_i) & \\
M \in \mathbb{R}^{Np \times Np}: & M_{ij} = (\varphi_j, \varphi_i)_{\delta,\Omega} & \text{mass matrix} \\
K \in \mathbb{R}^{Np \times Np}: & K_{ij} = (\nabla \varphi_j, \nabla \varphi_i)_{\delta,\Omega} & \text{stiffness matrix}
\end{array}
$$

Let us define the Spectral condition number

$$
cond(A) := \frac{\max_j |\lambda_j(A)|}{\min_j |\lambda_j(A)|} \qquad \forall A \in \mathbb{R}^{n \times n} \text{ non-singular}
$$

It is responsible for Conjugate-Gradient (in general Krylov) iterations:

$$
\#it \simeq \sqrt{cond(A)}
$$

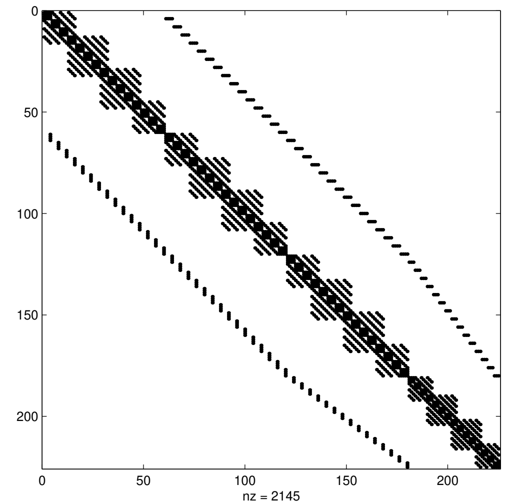
Mass matrix M [not for exam]

- diagonal for any $d \geq 1$
- $M_{ii} > 0$
- $\lambda_{min}(M) = \mathcal{O}(p^{-2d} h^d)$, $\lambda_{max}(M) = \mathcal{O}(p^{-d} h^d)$
- $cond(M) = \mathcal{O}(p^d)$ (Bernardi, Maday '92)
- $\widetilde{M} = [(\sigma\varphi_j, \varphi_i)_{\delta,\Omega}]_{i,j=1}^{Np}$ is diagonal even when $\sigma = \sigma(\mathrm{x})$.

Stiffness matrix K

[not for exam]

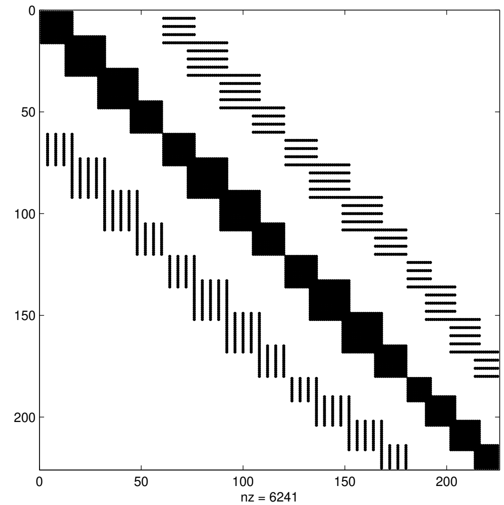
$\Omega = (-1,1)^2$



$p = 4$

4×4 elem

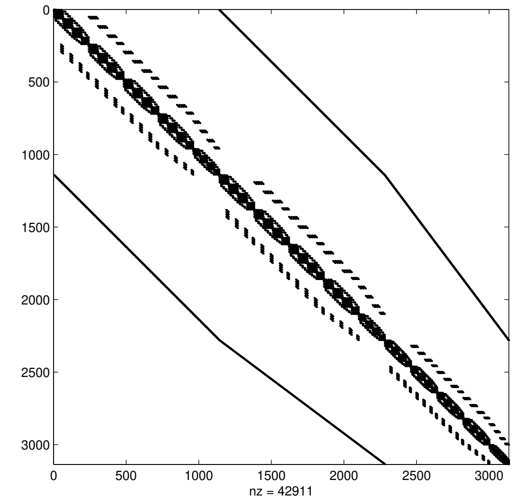
Ω skew quadrilateral



$p = 4$

4×4 elem

$\Omega = (-1,1)^3$



$p = 4$

$3 \times 4 \times 5$ elem

Sparsity

$\Omega = (-1,1)^2$	Ω skew quadrilateral	$\Omega = (-1,1)^3$
4%	12%	0.4%

- s.p.d.
- $\lambda_{min}(K) = \mathcal{O}(p^{-d} h^d)$, $\lambda_{max}(K) = \mathcal{O}(p^{3-d} h^{d-2})$
- $cond(K) = \mathcal{O}(p^3 h^{-2})$ (Bernardi, Maday '92 for Dir. b.c., Melenk 2002 for Neu. b.c. Bank, Scott '89 procedure for regular meshes)
- pattern of A is like that of K, even when $\mu = \mu(x)$

SEM-GNI Algebraic System

[not for exam]

Let us consider the differential problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d, \ (d = 2, 3) \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

SEM-GNI:

$$?\mathrm{u} = [u_\delta(\mathrm{x}_j)]_{j=1}^{Np} : \quad \sum_{j=1}^{Np} a_\delta(\varphi_j, \varphi_i)\mathrm{u}_j = (f, \varphi_i)_{\delta,\Omega} \quad i = 1, \ldots, Np$$

Since $A = K$, by setting $\mathrm{f} = [f(\mathrm{x}_j)]_{j=1}^{Np}$,
the algebraic system reads

$$K\mathrm{u} = M\mathrm{f} \qquad \text{weak form}$$

or equivalently

$$M^{-1}K\mathrm{u} = \mathrm{f} \qquad \text{strong (or collocation) form}$$

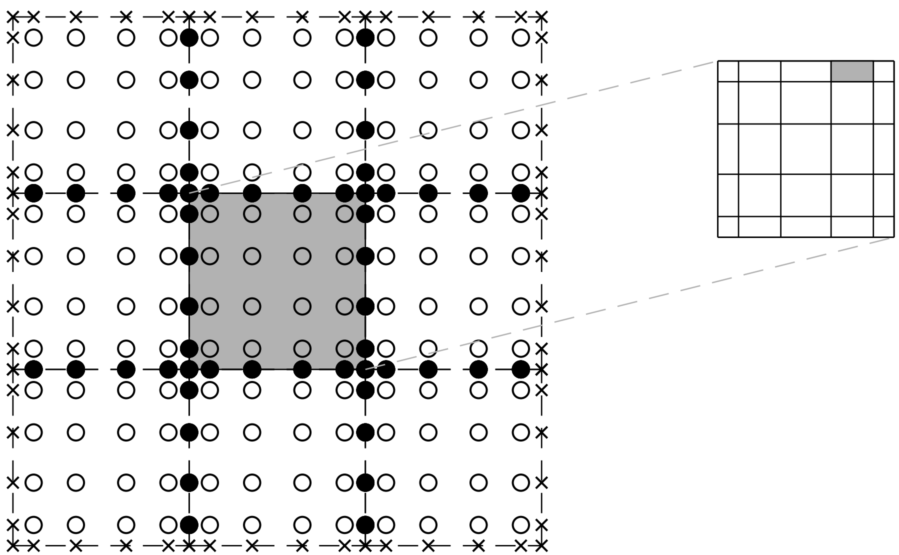
$$\begin{aligned} cond(K) &\simeq c_1 p^3 h^{-2} \\ cond(M^{-1}K) &\simeq c_2 p^4 h^{-2} \end{aligned}$$

Preconditioning on quads by low-order FEM

[not for exam]

Preconditioning by low-order Finite Element Matrices

The LGL nodes in each spectral element Q_k induce a mesh of bilinear quadrilateral elements $\mathbb{Q}_1$.



Build the $\mathbb{Q}_{1,NI}$ local stiffness matrix on each spectral element Q_k and then assemble the local matrices on the whole domain.

Laplace operator preconditioning

$$\begin{cases} Lu = -\Delta u & \text{in } \Omega = (-1,1)^d, \ d = 2,3 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

- Primal discretization: A = SM-GNI stiffness matrix (one spectral element).

- Preconditioner: $P = K_{\mathbb{Q}_{1,NI}}$, i.e. the global $\mathbb{Q}_{1,NI}$ stiffness matrix on the LGL local grid.[3]

Theorem (For a single spectral element)

$$cond(P^{-1}A) \leq 2.5 \qquad \forall p, \text{ for } d = 1,2,3$$

[3] $\mathbb{Q}_{1,NI}$ stands for $\mathbb{Q}_1$ with numerical integration (instead of exact integration). Numerical integration = trapezoidal rule.

$cond(P^{-1}A)$ for the Laplace operator

Numerical results for SEM

$A =$ SEM-GNI stiffness matrix. $P = K_{\mathbb{Q}_{1,NI}}$

Total number of spectral elements $= Ne^d$, $\#$dof $= (p \cdot Ne - 1)^d$.

d=2

p	$Ne = 1$	$Ne = 2$	$Ne = 4$	$Ne = 8$
4	1.55	2.69	2.69	2.70
8	1.95	3.07	3.07	3.07
12	2.10	3.26	3.26	3.26

d=3

p	$Ne = 1$	$Ne = 2$	$Ne = 4$
4	1.46	1.59	1.59
6	1.41	1.47	1.47
8	1.35	1.38	1.38

The condition number is bounded independently of both p and Ne.

Essential bibliography (books)

BM C. Bernardi, Y. Maday. *Approximations Spectrales de Problèmes aux Limites Elliptiques.* Springer Verlag (1992)

KS G.E. Karniadakis, S.J. Sherwin. *Spectral/hp Element Methods for Computational Fluid Dynamics*, 2nd ed. Oxford University Press (2005)

CHQZ2 C. Canuto, M.Y. Hussaini, A. Quarteroni, T. Zang. *Spectral Methods. Fundamentals in Single Domains*. Springer (2006)

CHQZ3 C. Canuto, M.Y. Hussaini, A. Quarteroni, T. Zang. *Spectral Methods. Evolution to Complex Geometries and Applications to Fluid Dynamics*. Springer (2007)

Simplices (Triangles in 2D, Tetrahedra in 3D)

[not for exam]

What happens on triangles

We recall the strong points of SEM-GNI on quads:

1. nodal (Lagrange) basis
2. the interpolation nodes are the Legendre Gauss Lobatto (LGL) nodes
3. the **quadrature nodes** are exactly the **interpolation nodes** and Lagrange basis is orthogonal w.r.t. the discrete L^2 inner product (induced by quadrature) $\Longrightarrow$ diagonal mass matrices (in $\mathbb{R}^d$, $d \geq 1$)
4. tensorial structure of the basis functions in $\mathbb{R}^d$ (with $d \geq 2$) $\Longrightarrow$ high computational efficiency

> Unfortunately, it is not possible to preserve simultaneously all these upsides on simplices

Nodal basis and tensorial structure are incompatible in T, then two alternatives are possible:

1. preserve tensorization and use the modal basis

or

2. preserve nodal basis and lose tensorization

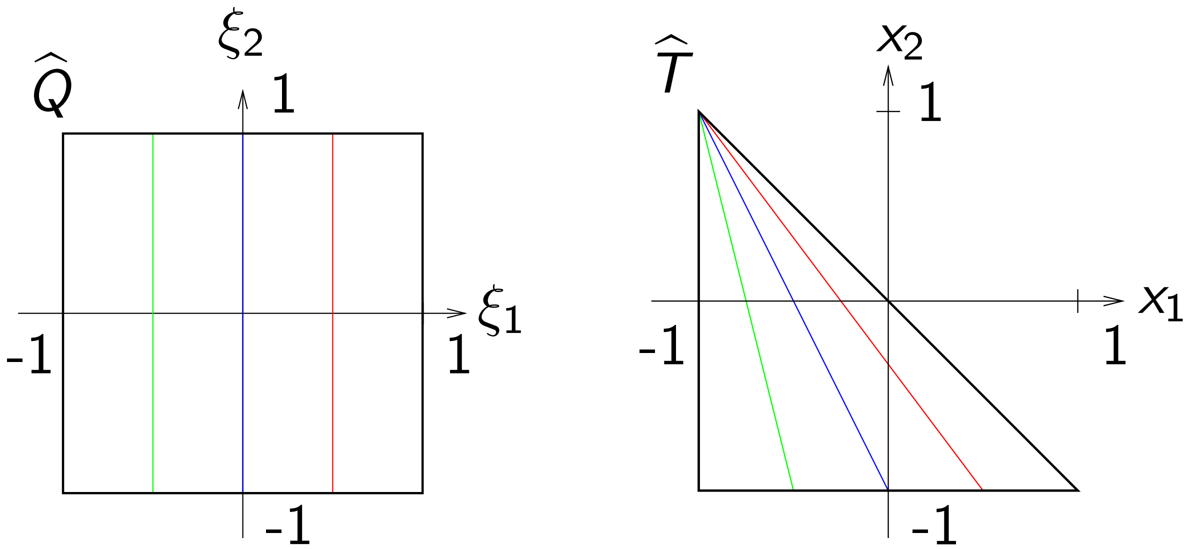
SEM on triangles with modal basis

How to set up an adapted modal basis functions on triangles in order to exploit tensorization?

1. Collapsed Cartesian coordinates
2. Warped tensorial basis functions

Collapsed Cartesian coordinates

First,
collapse the reference square into the reference triangle by the map $\widehat{\mathrm{F}}$:



$\widehat{Q} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \ : \ -1 < \xi_1, \xi_2 < 1\}$ is the reference square
$\widehat{T} = \{(x_1, x_2) \in \mathbb{R}^2 \ : \ -1 < x_1, x_2 \ ; \ x_1 + x_2 < 0\}$ is the reference triangle

$$\widehat{\mathrm{F}}\left(\left[\begin{array}{c} \xi_1 \\ \xi_2 \end{array}\right]\right) = \left[\begin{array}{c} \frac{1}{2}(1+\xi_1)(1-\xi_2) - 1 \\ \xi_2 \end{array}\right] = \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

is a bijective map, singular at the upper vertex of the triangle.
Nevertheless it stays bounded as one approaches the vertex.

Warped tensorial basis

1. Given the polynomial degree p, let $\mathbb{P}_p(\widehat{T})$ be the space of polynomials of global degree p, $\quad \dim(\mathbb{P}_p(\widehat{T})) = \dfrac{(p+1)(p+2)}{2} = nb.$

2. $\varphi^{(1)}_{k1}(\xi_1)$, for $-1 \leq \xi_1 \leq 1$ and $k1 = 0, \ldots, p$, are the boundary adapted 1D basis functions along the 1st coordinate

3. $\varphi^{(2)}_{k1,k2}(\xi_2)$, for $-1 \leq \xi_2 \leq 1$ are the boundary adapted 1D basis functions along the 2nd coordinate. Each polynomial depends on the index $k2$, but also on $k1$.

$$
\varphi^{(2)}_{k1,k2}(\xi_2) = \begin{cases}
\varphi^{(1)}_{k2}(\xi_2) & k1 = 0,\ 0 \leq k2 \leq p \\
\left(\dfrac{1-\xi_2}{2}\right)^{k1+1} & 1 \leq k1 \leq p-1,\ k2 = 0 \\
\left(\dfrac{1-\xi_2}{2}\right)^{k1+1} \left(\dfrac{1+\xi_2}{2}\right) P^{(2k1+1,1)}_{k2-1}(\xi_2) & 1 \leq k1 \leq p-1, \text{ and} \\
 & \quad 1 \leq k2 \leq p-k1-1 \\
\varphi^{(1)}_{k2}(\xi_2) & k1 = p,\ 0 \leq k2 \leq p-1
\end{cases}
$$

where $P^{(\alpha,\beta)}_k$ is the Jacoby polynomial of degree k.

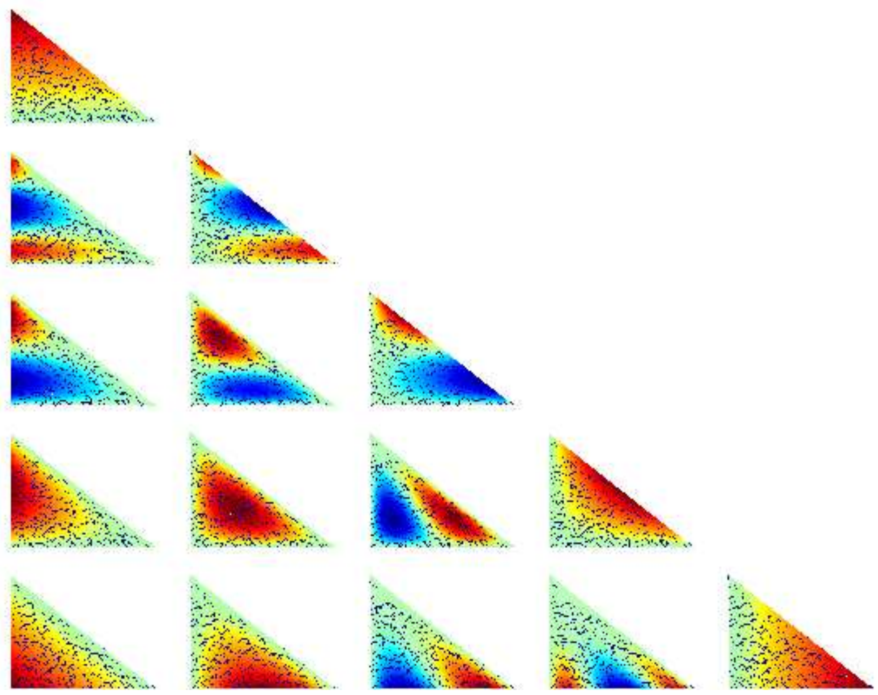
Warped tensorial basis

Let $k = (k1, k2)$ a bijection to use 1-index ordering. The **boundary adapted modal basis in 2d on** $\widehat{T}$ (also named **modified** C^0 **modal expansion**) reads

$$\phi_k(x_1, x_2) = \varphi_k(\xi_1, \xi_2) = \varphi^{(1)}_{k1}(\xi_1)\varphi^{(2)}_{k1,k2}(\xi_2)$$

where $(x_1, x_2) = \widehat{\mathrm{F}}(\xi_1, \xi_2), \; -1 \leq \xi_1, \xi_2 \leq 1.$

but at the corner point $V_3(-1, 1): \; \phi_3(x_1, x_2) = \varphi_3(\xi_1, \xi_2) = \frac{1+\xi_2}{2}$

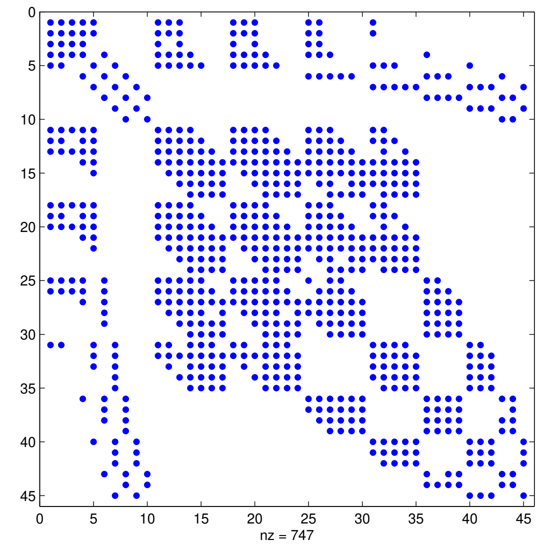


Mass matrix on one triangle

You can exploit the tensorial structure of the basis functions:

$$M_{ij} = \int_{\hat{T}} \phi_j \phi_i d\hat{T} = \int_{-1}^{1} \varphi_{j1}^{(1)}(\xi_1)\varphi_{i1}^{(1)}(\xi_1)d\xi_1 \cdot \int_{-1}^{1} \varphi_{j1,j2}^{(2)}(\xi_2)\varphi_{i1,i2}^{(2)}(\xi_2)\frac{1-\xi_2}{2}d\xi_2$$

(recall that $\varphi_3(\xi_1,\xi_2) = \dfrac{1+\xi_2}{2}$)



Quadrature formulas with $(p+1)$ nodes in $[-1,1]$:

	degree of exactness	at the end-points
Legendre-Gauss	$2p+1$	open – open
Legendre-Gauss-Radau	$2p$	closed – open
Legendre-Gauss-Lobatto	$2p-1$	closed – closed

Set the nodes and weights in $\widehat{Q}$ and collapse them on $\widehat{T}$ by F.

Stiffness matrix on one triangle

You can exploit the tensorial structure of the basis functions to compute derivatives, but not to compute integrals for a generic triangle.

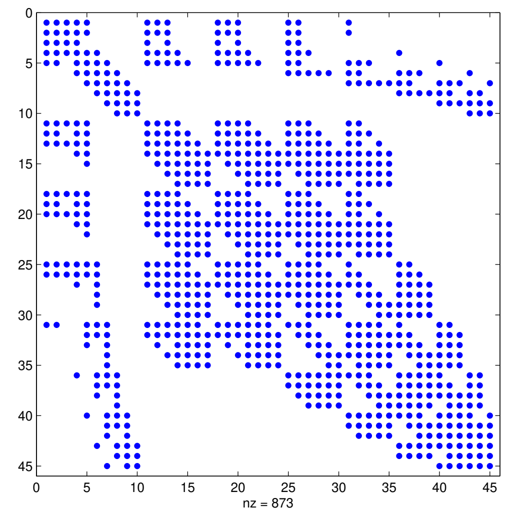
$$\frac{\partial \varphi_j}{\partial x_1}(x_1, x_2) = \frac{\partial \varphi_{j1}^{(1)}}{\partial x_1}(x_1)\varphi_{j1,j2}^{(2)}(x_2) \qquad \frac{\partial \varphi_j}{\partial x_2}(x_1, x_2) = \varphi_{j1}^{(1)}(x_1)\frac{\partial \varphi_{j1,j2}^{(2)}}{\partial x_2}(x_2)$$

and, as usual,

$$K_{ij} = \int_{\hat{T}} \nabla\phi_j \cdot \nabla\phi_i d\hat{T} = \int_{\hat{Q}} \left(\frac{J^{cof}}{detJ_F}\nabla\varphi_j\right) \cdot \left(\frac{J^{cof}}{detJ_F}\nabla\varphi_i\right) detJ_F d\widehat{Q}$$

$detJ_F = 0$ at $V_3(-1,1)$, thus you can use:

Legendre-Gauss-Lobatto along $x-$ direction
(quadrature nodes are in $[-1,1]$)
Legendre-Gauss-Radau along $y-$ direction
(quadrature nodes are in $[-1,1[$)



Dirichlet boundary conditions or global C^0

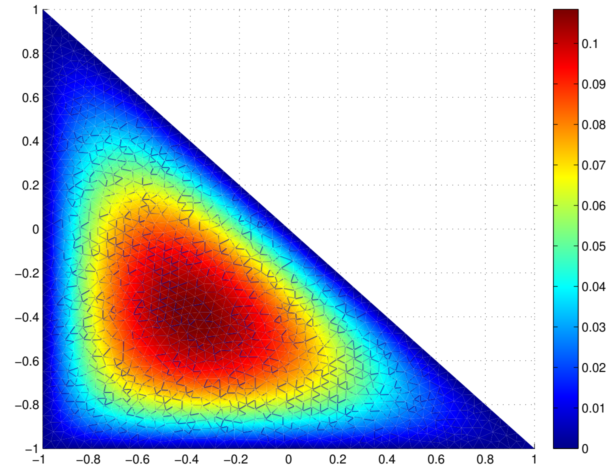
To impose Dirichlet b.c. or the continuity across adjacent elements, replace the $3p$ equations associated to the boundary modes with

$$\mathbb{P}_p \ni u_p(\mathrm{x}_\ell) = \sum_{k=1}^{nb} \tilde{u}_k \phi_k(\mathrm{x}_\ell) = g(\mathrm{x}_\ell) \qquad \ell = 1, \ldots, 3p$$

where g is a known function and x_ℓ on each edge are the image, through $\widehat{\mathrm{F}}$, of the $p+1$ Legendre-Gauss-Lobatto nodes.

$$\begin{cases} -\Delta u + u = 1 & \text{in } \Omega = \widehat{T} \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$A = K + M$, $\mathrm{f} = [(1, \phi_i)_{L^2(\Omega)}]_{i=1}^{nb}$
Modify $3p$ equations to impose Dir b.c. and solve $A\widetilde{\mathrm{u}} = \mathrm{f}$.



The numerical solution is $u_p(\mathrm{x}) = \sum_{k=1}^{nb} \tilde{u}_k \phi_k(\mathrm{x})$

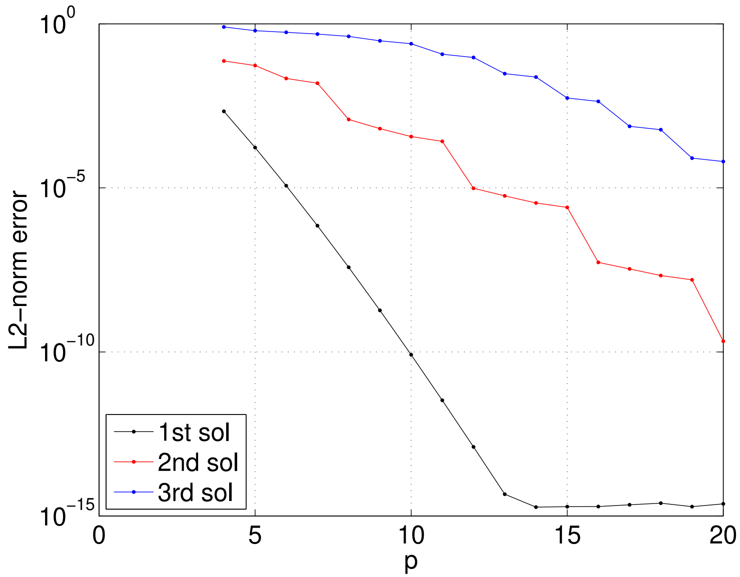
For a general partition, standard arguments for assembling matrices and ordering the modes/nodes can be applied.

Triangular SEM with boundary adapted modal basis

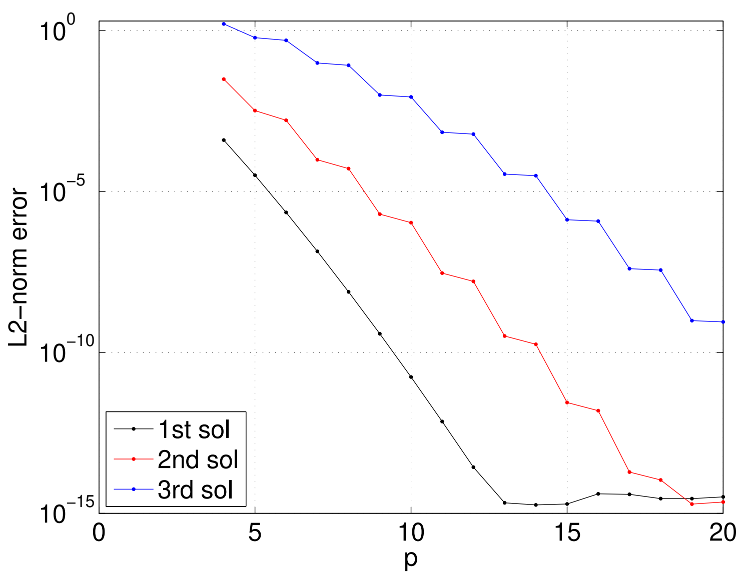
Condition number of the stiffness matrix (Laplace operator) is $cond(A) = \mathcal{O}(p^3 h^{-2})$

Preconditioners designed by Babuska et al. (1991) for $\mathbb{P}_p$ can be used: coarse $\mathbb{P}_1$, and coarse average.
For both: $cond(P^{-1}A) = \mathcal{O}(\log^2 p)$, independently of h, but it does depend on the interior angles and aspect ratios of the elements.

Convergence analysis: spectral accuracy in both L^2- and H^1-norm versus p:
$u(x,y) = e^{-x-y}$, $u(x,y) = sin(\pi xy) + 1$, $u(x,y) = sin(2\pi x)\cos(2\pi y)$



Triangular SEM on $\Omega = \widehat{T}$



Quad SEM on $\Omega = \widehat{Q}$

Triangular SEM with nodal basis

Given p, $\mathbb{P}_p(\widehat{T})$, $\dim(\mathbb{P}_p(\widehat{T})) = \dfrac{(p+1)(p+2)}{2} = nb$.

- Nodal Lagrange basis $\{\varphi_i(\mathrm{x})\}$ (on $\hat{T}$) associated with a set of interpolation nodes $\{\mathrm{x}_i\}$ (on $\hat{T}$): $\varphi_j(\mathrm{x}_i) = \delta_{ij}$
- choose the set of interpolation nodes $\{\mathrm{x}_i\}$ so that:
  - it includes LGL nodes on the edges (to use triangles in conjuction with quads)
  - the interpolation is stable (small Lebesgue constant)

  $\Longrightarrow$ electrostatic points (Hesthaven (1998)),
  Fekete points (Bos (1983), Chen & Babuska (1995), Taylor et al. (2000))

  Both sets are not known explicitely, but computable by suitable algorithms. They provide very poor quadrature formulas.
- choose a set of quadrature nodes and weights.
  A good choice: Gaussian quadrature formulas on $\widehat{Q}$ and collapse the nodes on $\widehat{T}$ by $\widehat{\mathrm{F}}$

Lagrange basis on $\widehat{T}$

While the Lagrange polynomials have an explicit form in $[-1,1]$

$$\varphi_j(\xi) = -\frac{1}{p(p+1)} \frac{(1-\xi^2)}{\xi-\xi_j} \frac{L'_p(\xi)}{L_p(\xi_j)}$$

and this is the keypoint to compute efficiently derivatives,

there is not a closed-form expression for the Lagrange polynomials associated with an arbitrary set of points in T.

$\Longrightarrow$ express the Lagrange polynomials in terms of another polynomial basis, e.g. the orthogonal modal basis (Dubiner) polynomials $\{\psi_k(\boldsymbol{\xi})\}$ in $\widehat{Q}$ and, if $\boldsymbol{\xi}_i = \widehat{\mathrm{F}}^{-1}(\mathrm{x}_i)$ (nb interpolation nodes),

$$\psi_k(\boldsymbol{\xi}) = \sum_{j=1}^{nb} \underbrace{\psi_k(\boldsymbol{\xi}_j)}_{\mathrm{V}_{jk}} \varphi_j(\boldsymbol{\xi}), \qquad \varphi_j(\boldsymbol{\xi}) = \sum_{k=1}^{nb} (\mathrm{V}^{-1})_{kj} \psi_k(\boldsymbol{\xi}) \qquad j,k = 1,\dots,nb$$

V is the matrix of basis change, also known as **generalized Vandermonde matrix**.

Triangular SEM with nodal basis

Derivatives: V and V^{-1} are used to compute derivatives of basis functions.

Analogous matrices are used for Quadrature:

$$\widetilde{\mathrm{V}}_{\ell,k} = \psi_k(\boldsymbol{\eta}_\ell), \qquad k = 1, \dots, nb, \quad \ell = 1, \dots, nq$$

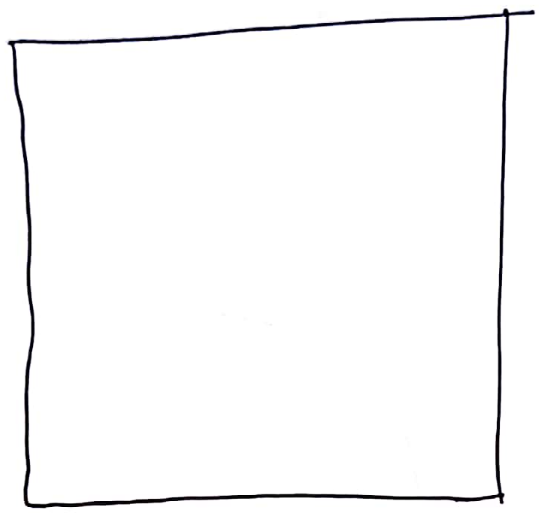
In general $nq \geq nb$ and $\widetilde{\mathrm{V}}$ is rectangular

Condition number. V and $\widetilde{\mathrm{V}}$ affect the condition number of both mass and stiffnesss matrices.
Numerical results show that $cond(A) = C(h)(p^4)$ when A is the stiffness matrix of the Laplace operator. (Pasquetti & Rapetti (2004)).

Preconditiong: $\mathbb{P}_1$ FEM stiffness matrix (induced by either Fekete and electrostatic meshes) is not an optimal preconditioner, contrary to what happens for quads. $cond(P^{-1}A) = \mathcal{O}(p)$ (independent of h) (Warburton, Pavarino, & Hesthaven (2000))

Convergence rate: numerical results show spectral accuracy, only if quadrature formulas are adeguate. (Warburton & Pavarino & Hesthaven (2000), Pasquetti & Rapetti (2004), (2006), (2010))

Still about
Spectral method 1 domain → FE preconditioners: why?



$\Omega = (-1,1)^2$

global polynomial

$$u_N(x_1,x_2) = \sum_{k,m=0}^{N} a_{km}\, L_k(x_1)\, L_m(x_2) \in \mathbb{Q}_N^2$$

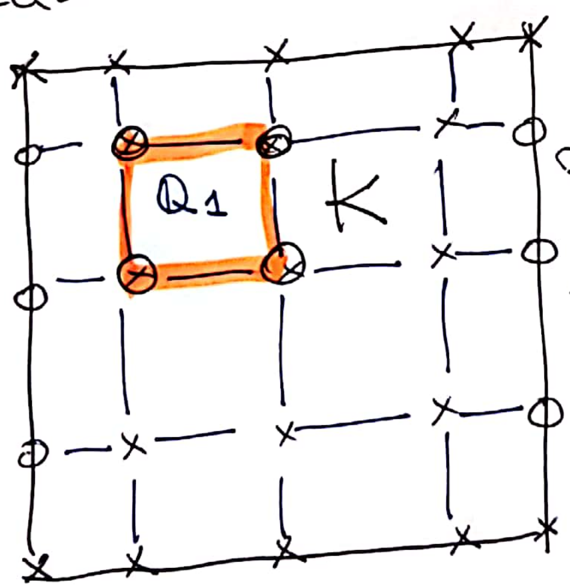
$$\psi_k(x_1)\, \psi_m(x_2)$$

Stiffness: $(A_{sp})_{ij} = a(\underbrace{\psi_j, \psi_i}_{\text{global bases}})$ full matrix

$\neq$

sparse

LGL nodes



$\mathcal{T}_{h,sp}$ $\quad (A_{fe})_{\ell m} = a(\chi_\ell, \chi_m)$

h $\quad \{\chi_\ell\}$ basis of the FE space

$$X_h^1 = \{v_h \in C^0(\bar{\Omega}),\ v_h|_K \in \mathbb{Q}^1(K)\ \forall K \in \mathcal{T}_{h,sp}\}$$

$$\Delta_{sp}\,\vec{u}_{sp} = \vec{f}_{sp}$$

sparse

$$A_{fe}\left(\vec{u}_{sp}^{\,k+1} - u_{sp}^{\,k}\right) = \omega\cdot\Big(\underbrace{\vec{f}_{sp} - A_{sp}\vec{u}_{sp}^{\,(k)}}_{\text{spectral residuals}}\Big)$$

1 it of Richardson

u_N^{sp}, N Large

Ω