Profit Analysis of Queue with Advertising and Balking Based on Dynamic Evaluation by Clients

A DISSERTATION

Submitted in partial fulfillment of the
requirements for the award of the degree
of

INTEGRATED MASTER OF SCIENCE
in
APPLIED MATHEMATICS
by

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Under the Supervision
of
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ROORKEE- 247 667 (INDIA)
MAY 2018

Declaration

I hereby declare that the dissertation entitled "**Profit Analysis of Queue with Advertising and Balking Based on Dynamic Evaluation by Clients**", carried out in partial fulfillment of the requirement for the award of the degree of "**Integrated Master of Science in Applied Mathematics**", submitted in the Department of Mathematics, Indian Institute of Technology Roorkee, is an original and genuine work carried out during a period of January 2018 to May 2018 under the supervision of Dr. Madhu Jain, Department of Mathematics, Indian Institute of Technology Roorkee.

The matter presented in this dissertation has not been submitted by me for any other degree,

Date: May 7, 2018 Devendra Kumar

This is to certify that the statements made above by the student are correct to the best of my knowledge and belief.

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Acknowledgments

I would like to express my sincere gratitude towards my supervisor Dr. Madhu Jain who has motivated me to think about the topic of Queuing Theory. Her support and guidance have given me strength to complete my dissertation as I was very nascent to this area.

Besides my supervisor I would like to thank Dr. Kusum Deep who has helped me in designing the optimization paradigm for the problem related to the present work. Her knowledge and guidance played a significant role in completing this project.

I would like to thank my parents for all the moral support and encouragement they have given me over the years.

Devendra Kumar

Abstract

Queuing theory is the mathematical study of waiting lines or queues. A queuing model is constructed so that queue lengths and waiting time can be predicted. Queuing theory is generally considered a branch of Operations Research because the results are often used in making business decisions about the resources that are needed in order to a service.

Applications of queuing theory are increasing day by day, from computer networking problems to production management, various models with numerous modifications are in implementation. In this dissertation, attempts have been made to investigate M/M/1/K type of queuing models associated with the single server queue with advertising and balking with dynamic evaluation by clients.

The dissertation consists of four chapters. First chapter provides the introduction and familiarity with the queuing theory and various concepts that have been utilized in the dissertation. Second chapter will give the model derivation and the work that has been done earlier in the targeted problem. In the third chapter, we have presented an illustration and provided performance results. Chapter four will outline the conclusion and some potential application areas.

Contents

Chapter - 1

Introduction

1.1 Introduction to Queues

The study of queues, termed as queuing theory, is among the oldest and popularly used as quantitative analysis technique. Queues are frequent in occurrence, affecting people buying victuals in shops, making bank deposits, waiting for customer executive to pick up the phone, or waiting on the railway station for the ticket reservation of a particular train. Waiting lines, used term for the queues, can also take the form of accessories or machines waiting to be corrected, trains in line waiting to get go signal, or airplanes lined up on tracks waiting for permission to take off. The three important and basic components of a queuing process are arrival, service facilities, and queue (waiting line).

This chapter provides the basic information about the terms and some definition. We will also see the terms are closely defined with the practical sense. We will be looking at the characteristics of queue and will see how balking and advertisement affect the arrival and queue length and thus affecting the customer evaluation.

1.2 Waiting Cost

In general, queue problems are focused on the question of obtaining the better service that can be provided with relatively maximized profit. Large shops must decide how many cash counters positions they should open. Petrol pumps must decide how many pumps should be opened and how many caretakers should be there on duty. Our banks should decide on the number of service windows to keep open to serve customers during various hours of the day. Manufacturing industries must decide the correct number of mechanics to have on each shift to support, handle and repair machines that break down.

This level of service, in most cases, is a parameter and management has control on this parameter. An extra service window, for example, can be opened and an executive can be transferred or can be hired and trained easily if needed in situations. The same may not be the case always though. An industry may not be able to find or hire qualified and skilled mechanics to work with sophisticated machinery or industrial equipments.

The objective of any facility is to find a happy compromise between two points. On one side, a facility can keep a large staff (large number of servers) and provide parallel service facilities. This may lead to very good customer satisfaction, with few customers in a queue or waiting line. When customers are happy, the reputation of the facility automatically goes up. This, however, may get little expensive.

Having minimum possible number of servers and waiting lines can be another point. This keeps the total service cost low but can results in customer dissatisfaction and thus lowering the reputation of the facility. We ourselves don't like to go to the shop where only one counter is available and queue length get higher. As the length of the queue becomes larger and waiting time increases, customers and goodwill may get lost. Most servers (managers) identify the exchange that must be there in-between the price of providing good service and the price of waiting time of customer. They want waiting lines that are small in length so that customers can get served in a fast manner and stay happy. But they allow some waiting in the queue if it is evenhanded by the savings in the costs of service.

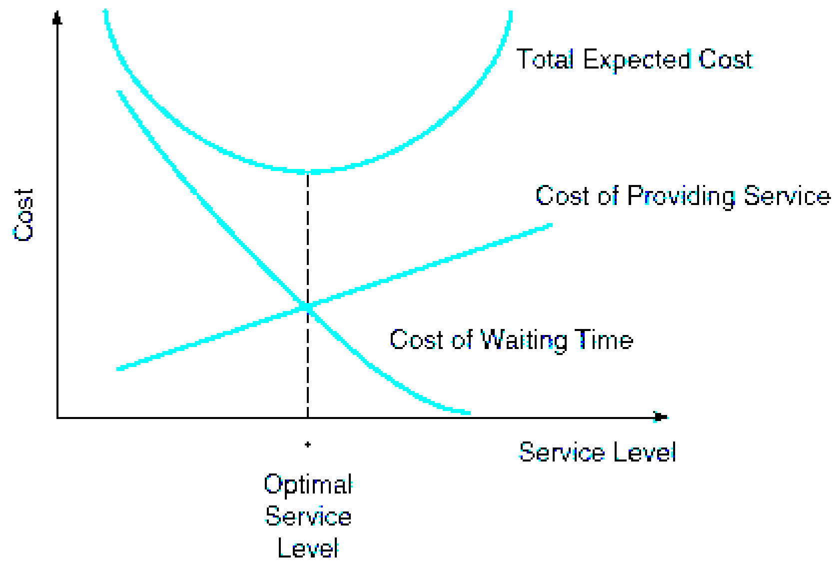

Figure 1.1 - Queuing Costs and Service Levels

One method of evaluating a service facility is to observe the total expected cost, a concept demonstrated as Figure 1.1. Total expected cost is the sum of expected service costs and expected waiting cost.

1.3 Characteristics of Queuing System

In this section, we will take some understanding of the three parts of a queuing system:

- The arrival or input to the system (also known as calling population)
- The queue or the waiting line
- The service facility

These components have certain properties that must be understood beforehand in mathematical modeling of any queuing system is developed.

Arrival Characteristics:

The source of input that generates customers for the system has three important characteristics or properties. These are as follows:

Size of population: Population size can be of two types. It can either be infinite (unlimited) or finite (limited or fixed). When the number of customers at any given time of instance is just a small portion of potential customers that can come, then the population is considered unlimited. Most models assume such an infinite population because it provides simplicity.
Models with finite population becomes complex. We have taken infinite population.

Pattern of coming customers at the system: Arrival of customers can be periodic in nature like getting a customer in every 20 min or it can be randomly. Arrival can be considered random when there is no way in which the rate can be predicted beforehand. Generally, in queuing problems, the number of customers arriving per unit of time can closely be approximated by a probability distribution called the Poisson distribution.
Behavior of arrivals: Nature of customers is really dynamic. In general, queuing models assume that customers are patient and they can wait till they get serviced. In reality, customers don’t want to wait. They value time and prefer to join another facility if they get a feel of waiting long.

Balking is the term used to refer the customer behavior of refusing a service or bouncing back after looking at the queue length, cost of service or poor service.
Reneging refers to the balking of customers after they join the queue. Initially the customer is patient but after some time he becomes impatient and leaves the queue.

Waiting Line Characteristics:
Waiting line is also an important component of queue. It can be finite or infinite. In case of infinite queue, the server can be assumed as to be connected with population source directly. Generally, models with fixed queue length are preferred because they simulate the real life conditions in a better way.
Queue is a FIFO structure. This means that whoever will enter the queue first will get the service first.

1.4 Customer Evaluation

Everything has some value. Evaluation of a service is the value that a customer gives to that service after observing the queue and service rate of the facility. We are taking this value in terms of monetary units.
In this dissertation, we have taken evaluation of service as some function of queue length. We will see how the profit of the facility changes as the nature of dependency of service evaluation to queue length changes.

Chapter - 2

Profit Modelling for Single Server Queue

This chapter mainly deals with the formulation of model and gives out the work that has been done by various mathematicians.

2.1 Background

The queuing model described by us assumes all the hypothesis of Van Ackere and Ninios's model [3] which can be thought as:

- A facility with one server.
- A cost of waiting is estimated for user per unit of time in the server.
- The customer comes, observe the length of the queue upon arrival and thinks whether he remain or not in the system (server).
- Customer who goes through the system pays the fixed rate.
- The manager makes efforts in attracting clients through advertisement or publicity.

Therefore, the arrival rate of users is variable and is dependent on the established publicity level and the pre-estimated price. There can be two different type of hypotheses based on publicity:

1. *Basic model*: The arrival rate depends on the publicity level directly.
2. *Word of mouth model*: The effect of expenses on advertisement is the function of the service quality which can be measured through the customers who join the service queue. If publicity is efficient, customers will be attracted and the system can get congested and which in turn may carry out a large refusal of customers to incorporate and a bad status of the service among potential customers.

2.2 Work and Literature

There is vast literature on the queuing theory and the applications are diversified from general inventory to computer networking. There are two significant papers that are extremely relevant to our subject matter. Let’s get some brief on them and see how our model got developed progressively from the contribution that these papers have given in this field.

2.2.1 Van Ackere and Ninios Work

Van Ackere and Ninios [3] model has been simulated and the price of maximum profit for the facility and the level of publicity has been obtained. They have used model with finite capacity whose arrival time is variable independent and distributed as per exponential distribution.

Profit model given by Van Ackere and Ninios [3] considered the fixed capacity and customer evaluation of service is assumed constant for all customers. The profit generated is per unit time.

The basic model was:

$$profit\,(facility) = \lambda_{ar}\,(1-P_x)\,F - C_{ads} \tag{2.1}$$

For basic model, they kept advertisement cost C_{ads} equals to the arrival rate λ_{ar} ,

$$profit\,(facility) = \lambda_{ar}\,(1-P_x)\,F - \lambda_{ar} \tag{2.2}$$

For word of mouth model, the renouncement becomes non-linear and

$$\lambda_{ar} = C_{ads}(1-P_x) \tag{2.3}$$

$$profit\,(facility) = \lambda_{ar}\,(1-P_x)\,F - \frac{\lambda_{ar}}{(1-P_x)} \tag{2.4}$$

Here F is the cost of service that is decided by the manager of the facility.

And P_x is the probability of the queue to have exactly x number of customers waiting in the queue including observer.

2.2.2 Atkinson Work

Atkinson [1] studied and analyzed the model given by Van Ackere and Ninios [3] again and produced the numerical results using search methods by taking queue length as integer variable and iterating it from 1 to the fixed value of customer evaluation. It is necessary to observe that he included customer evaluation of service as a parameter in profit function but kept it constant.

Atkinson's paper was the beginning of analysis and solution methods.

The same model was taken by Rommelfanger [2] also. He solved the same model by taking the mean time as a Triangular fuzzy number. He concluded that is much closer to practical life. Since the complexity of model and solution was high. The work of Rommelfanger [2] is not computationally efficient to be used in practical scenarios.

After Rommelfanger [2], Pardo and Fuento [4] established new method to optimize the fuzzy model of Rommelfanger [2].

In our work, we have taken Atkinson model and made evaluation of service by customer as a function of queue length. The model is then solved by Particle Swarm Optimization.

The notations that we have used in the model and will be using in this whole document are given as follows:

λ_{ar} : Average rate of arrival to the system

t : Average time of service

ρ : Intensity of traffic

V : Customer valuation of the service provided by the system

F : Fee paid by user for using service (cost of service)

x : Capacity of the system

M_{basic} : Basic model

M_{wm} : Word of mouth model

P_{basic} : Level of expense in publicity in model basic

P_{wm} : Level of expense in publicity in word of mouth model

P_{ent} : Probability of entry into the system

S_{basic} : Function of standardized profit in model basic

S_{wm} : Function of standardized profit in word of mouth model.

μ_{serv} : Service rate of the server

In Atkinson [1] model, it is manager who decides the optimum use of a service by the customer. In this way, Atkinson considers that there is only one service to which clients apply as per Poisson distribution with arrival rate λ. The customer evaluation for the service is taken fixed for all the arrivals and is denoted by V monetary units. Atkinson considers that service time of the server is variable and uniformly distributed as per exponential distribution with mean service time t.

Atkinson [1] assumed that waiting cost of a customer for one service is one unit. So, if x customers are in the queue including the arriving customer, then the cost of waiting will be x. Therefore,

$$profit \ \ for\, customer = V - F - x \tag{2.5}$$

Customer will join the queue if he feels profitable. i.e.

$$V - F - x \geq 0$$

Hence the condition for a customer to join the queue will be

$$[V - F] \geq x$$

So Atkinson [1] model is very likely a finite capacity queuing model, *M/M/1/x* with capacity *k* equal to [*V* - *F*].

The capacity *x =[R-F]* is the capacity beyond which the customers should not be allowed to wait.

Atkinson [1] used the term standard profit which is the ratio of the profit of the facility to the valuation of the customer for the service in unit time.

That is,

$$S_{(.)} = \frac{profit(.)}{V\mu_{serv}} \tag{2.6}$$

In Atkinson's Basic model, publicity level affects the arrival rate in a direct way, so that the arrival rate is a linear function of the publicity expenditure,

P_{basic}, to be exact:

$$\lambda_{ar} \approx P_{basic}$$

The function of standardized profit to be maximized is:

$$S_{basic} = \rho P_{ent}\left(1-\frac{x}{V}\right)-\frac{\rho}{V} \tag{2.7}$$

Now, using equation (2.7) and placing

$$P_{ent} = \left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)$$

we have,

$$S_{basic} = \rho\left(1-\frac{x}{V}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V} \tag{2.8}$$

Word of mouth model is similar to that of Van Ackere and Ninios [3] model. It is given as:

$$S_{wm} = \rho\left(1-\frac{x}{V}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V}\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) \tag{2.9}$$

Here relation in publicity and arrival has been taken as

$$\lambda_{ar} \approx P_{wm} \times P_{ent}$$

2.3 Model with Dynamic Customer Evaluation

The value that a customer gives to a service is not constant in practical situation.

The customer evaluates the service of a facility by observing many things like service rate, queue length, congestion in the queue etc.

In the model we have presented here, we have taken the customer evaluation V as function of queue length x that the entering customer observes including him.

That is

$$V \approx V(x)$$

So, Atkinson model changes to the following form:

Basic Model:

$$S'_{basic} = \rho\left(1-\frac{x}{V(x)}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V(x)} \tag{2.10}$$

Word of mouth model:

$$S'_{wm} = \rho\left(1-\frac{x}{V(x)}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V(x)}\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) \tag{2.11}$$

$V(x)$ is decreasing function and since we have already said that everything has some value to a customer. Hence the value of $V(x)$ cannot be zero. Evaluation function $V(x)$ is decreasing because the evaluation that customer does is based on the expected satisfaction and if the waiting queue is long then it is obvious for the customer to feel low about the service. Shorter queue implies higher evaluation function value.

Next, we will use the same model and find out the optimum traffic intensity and queue length by taking some case study. We will compare our results obtained by changing the nature of $V(x)$.

Chapter - 3

Solution Methodology: Particle Swarm Optimization

3.1 Introduction

There have been attempts to simplify the complexity of the solution by various mathematicians in their produced literature. Van Ackere and Ninios [3] produced numerical results by using search methods, Rommelfanger [2] introduced method in which the model was exploiting the fuzzy optimization technique. Pardo and Fuento [4] developed method to optimize the fuzzy model of Rommelfanger [2]. Basically, all the methods were following traditional paradigm of optimization. They require some auxiliary information about the model.

We will be using particle swarm optimization for the same and we will observe that the same paradigm or program will work for the model with variations in evaluation function. There is no need to change the whole optimization steps to fit the new modified model.

3.2 Particle Swarm Optimization (PSO)

It is a computational method in which a population of solutions is taken as potential solution set. Every candidate from the population set is known as particle and the population collectively known as swarm. It is a nature inspired algorithm known for its fast convergence. In this optimization technique, initially particles are given values from feasible region randomly. The influence of particle's past experience and experience from its neighborhood is contributed in term of velocity of particle.

For implementing PSO, the following notations are used:

x : The particle number whose velocity is being calculated.

y : Indicates the component of the particle.

t : Iteration or generation number.

c_1, c_2 : Constants.

r_1^{xy}, r_2^{xy} : Random numbers scaled between 0 to 1.

$pbest$: Personal best of the corresponding particle.

$gbest$: Global best of the whole neighborhood.

V : Velocity of particle.

p^{xy} : Component of particle.

This velocity is given by:

$$V_{t+1}^{xy} = V_t^{xy} + c_1 r_1^{xy} (pbest_t^{xy} - p_t^{xy}) + c_2 r_2^{xy} (gbest_t^{y} - p_t^{xy}) \tag{3.1}$$

The equation (3.1) is also known as Velocity update equation. As the iteration goes on, the particles are updated by influence of the neighborhood and by themselves to exploit-explore the search space to find the best solution.

The particle update equation is given by:

$$p_{t+1}^{xy} = V_{t+1}^{xy} + p_t^{xy} \tag{3.2}$$

In our problem we have used Particle Swarm Optimization along with the Clerc constriction coefficient. Clerc constriction coefficient guarantees the convergence on global optima and avoids high exploration which sometimes results in divergence.

After using Clerc constriction coefficient, the velocity update equation (3.1) changes to

$$V_{t+1}^{xy} = \alpha_{clerk} [V_t^{xy} + c_1 r_1^{xy} (pbest_t^{xy} - p_t^{xy}) + c_2 r_2^{xy} (gbest_t^{y} - p_t^{xy})] \tag{3.3}$$

α is known as constriction coefficient given by:

$$\alpha_{clerk} = \frac{2.0 \times \kappa}{\left(|\,2.0 - \varphi - \sqrt{\varphi(\varphi - 4.0)}\,|\right)}$$

Here

$$\varphi = c_1 r_1^{xy} + c_1 r_2^{xy}$$

3.3 Computational Steps

- Initialize swarm by assigning a random position to each particle in feasible region.
- Evaluate fitness of each particle.
- For each individual particle compare particle's fitness value with its P_{best}.

  If current value is better than the P_{best} value, then set this value as the Pbest and the current particle's position, x_i, as p_i.

- Identify particle that has the best fitness value. The value of its fitness function is identified as P_{gbest} and its position as p_g.
- Update the velocities and positions of all particles.
- Repeat steps 2 to 5 until a stopping criterion is met (e.g., maximum number of iterations or a sufficiently good fitness value).

Chapter - 4

Numerical Results and Sensitivity Analysis

In this chapter, we will see the behavior of our model and we will compare the model based on constant customer evaluation with the model based on dynamic customer evaluation, the one proposed by us.

We will be taking the same illustration as taken by Atkinson [1] and Rommelfanger [2] and we will see how the method of particle swarm optimization produces the same results with much simplicity.

So, we have our models as described in equations (2.10) and (2.11)

$$S'_{basic} = \rho\left(1-\frac{x}{V(x)}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V(x)} \tag{4.1}$$

$$S'_{wm} = \rho\left(1-\frac{x}{V(x)}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{V(x)}\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) \tag{4.2}$$

The notations are usual as defined previously. Our objective is to maximize the standardized profit S'_{basic} and S'_{wm} and find out the traffic intensity and queue length that can be allowed in that scenario.

4.1 General Behavior of Standard Profit with Constant Customer Evaluation

In figure 4.1 (a) and figure 4.2 (a) we have the surface plot of our basic model and word of mouth model respectively with the customer evaluation $V(x)$ taken as constant with value equals to 15. We can see that the standard profit decrease as we take higher values of traffic intensity. It is also decreasing as we take higher values of queue length. There is very little portion of surface, that is depicted in figure 4.1 (b) and figure 4.2 (b) in red color, which is giving positive standard profit. The length of the queue, traffic intensity and customer evaluation all are very critical parameters to standard profit.

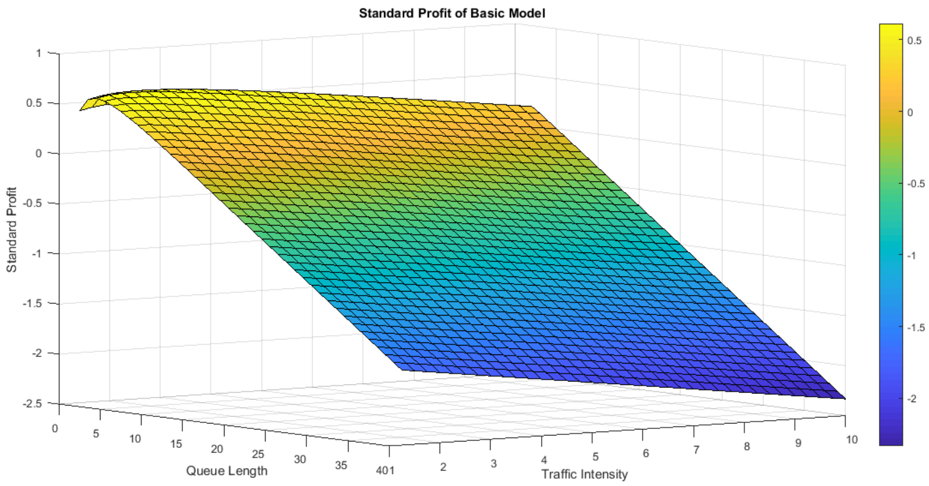

Figure 4.1 (a) - Basic Model

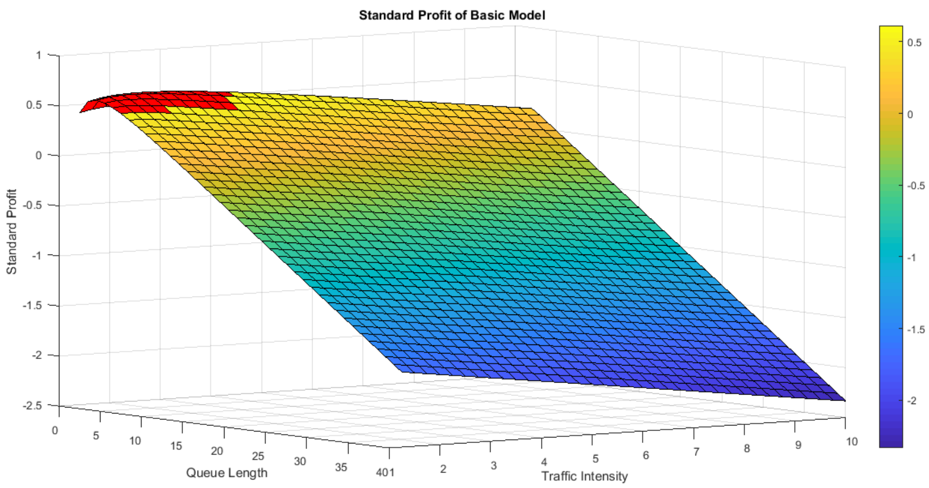

Figure 4.1 (b) - Basic Model

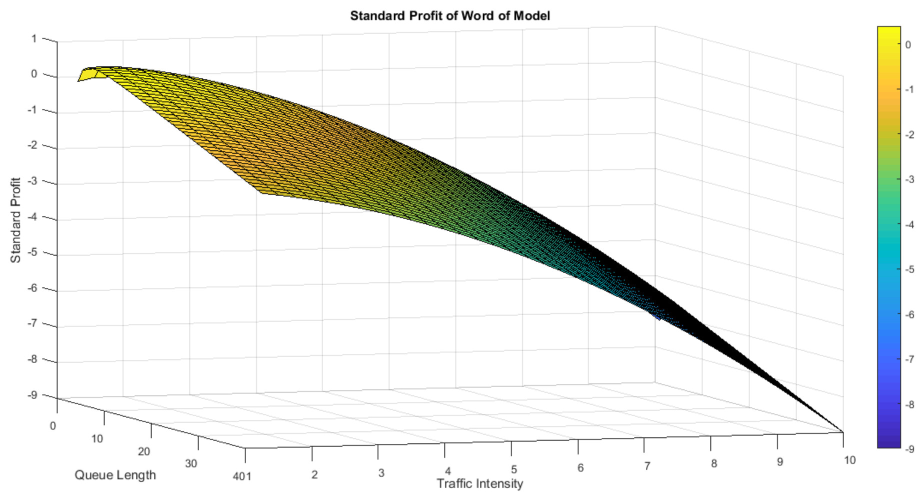

Figure 4.2 (a) - Word of Mouth Model

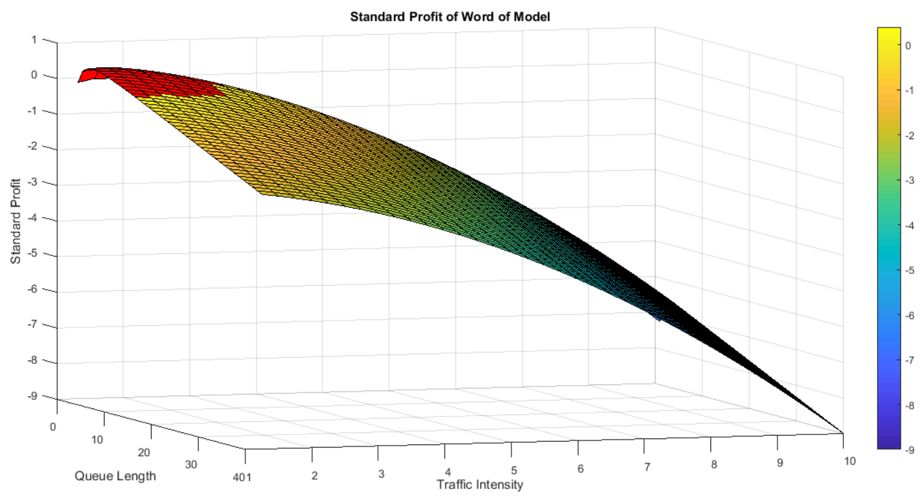

Figure 4.2 (b) - Word of Mouth Model

Figure 4.1(c) and figure 4.2 (c) show that the standard profit decreases as the queue length increases with constant flow of traffic in basic and word of mouth model respectively. Both the figures have various curves for different customer valuation. Decrease in the standard profit with increasing queue length is a result of increase in waiting cost. Figure 4.2 (c) shows faster decreasing nature because of the bad effect of the publicity in word of mouth model.

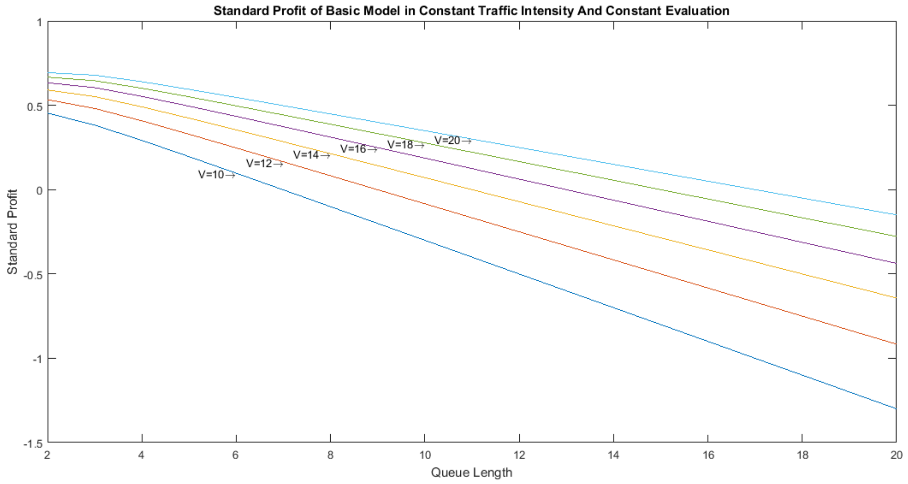

Figure 4.1 (c) - Basic Model with Constant Customer Valuation at $\rho=1.9$

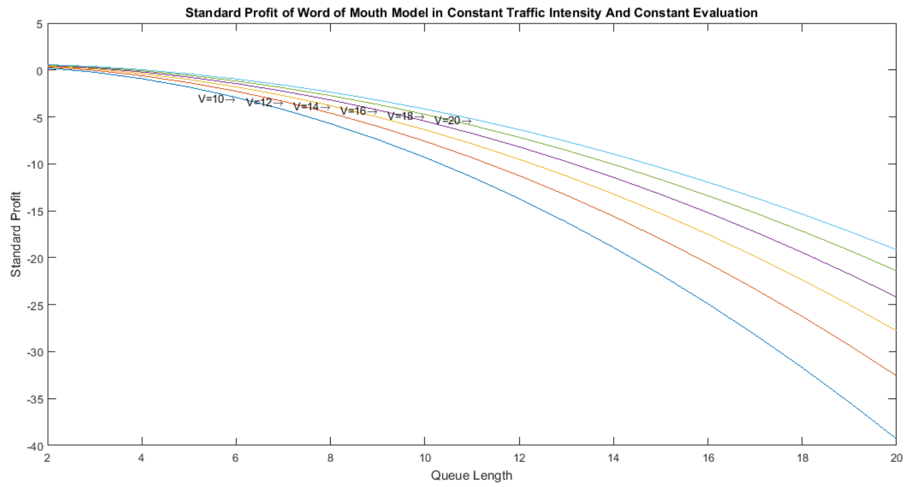

Figure 4.2 (c) - Word of Mouth Model with Constant Customer Valuation at $\rho=1.9$

Figure 4.1(d) and figure 4.2 (d), we see both the models, basic and word of mouth models respectively, show increase first then decrease in profit with increase in traffic intensity. This type of behavior is due to the congestion that arises because of the fixed queue length that is 3 here. For low ratio of arrival rate to service rate, queue accommodates easily but as the arrival rate increases faster the waiting cost and cost of publicity begin to contribute to loss.

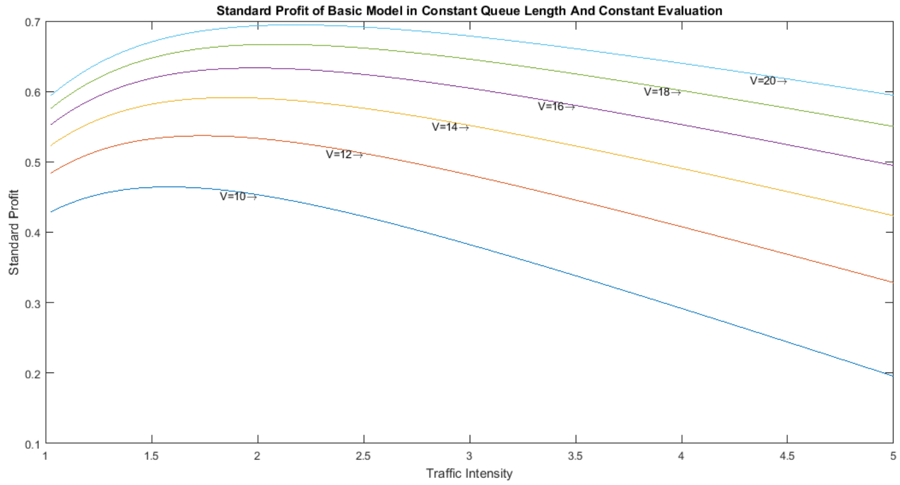

Figure 4.1 (d) - Basic Model with Constant Customer Valuation at $x = 3$

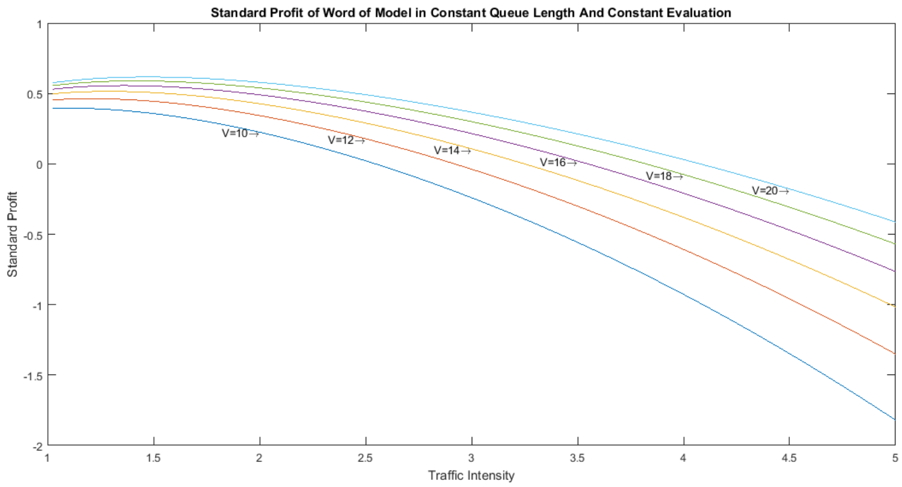

Figure 4.2 (d) - Word of Mouth Model with Constant Customer Valuation at $x = 3$

4.2 Illustration for Constant Customer Evaluation

Taking $V(x)$ equal to 15 (as taken by Atkinson [1] and Rommelfanger [2]):

From equation (4.1) and (4.2), we have,

$$S'_{basic} = \rho\left(1-\frac{x}{15}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{15} \tag{4.3}$$

$$S'_{wm} = \rho\left(1-\frac{x}{15}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right)-\frac{\rho}{15}\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) \tag{4.4}$$

Applying PSO to maximize the standard profit for basic model and get corresponding traffic intensity and queue length.

Table 1- Solution for Basic Model

Generation Number	ρ	x	S'_{basic}
1	1.941278	3.165769	6.135432e-01
8	1.930112	3.713150	6.135464e-01
15	**1.930344**	**3.806713**	**6.135465e-01**
23	**1.930343**	**3.806714**	**6.135465e-01**
30	**1.930344**	**3.806713**	**6.135465e-01**
37	**1.930344**	**3.806713**	**6.135465e-01**

Table 1 provide the global best, in our case its standard profit for basic model (expressed in equation 4.3), for different generation numbers. Looking at table and the swarm plots, we see that the global convergence is achieved at 0.61354. Figure 4.3 (a) to figure 4.3 (f) show the position of various particles in swarm in some chosen generations.

Swarm Movement for Basic Model

Population size taken =50

Number of generations taken= 50

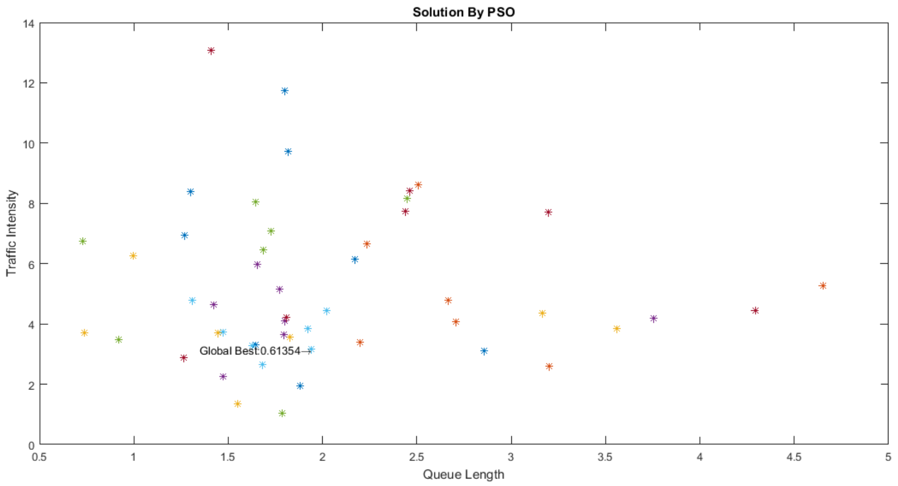

Figure 4.3 (a) - Standard Profit by PSO (Generation 1)

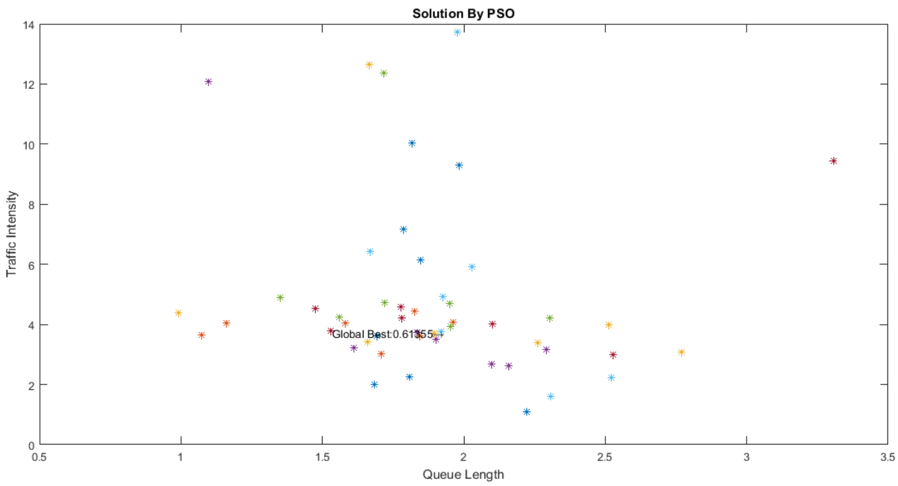

Figure 4.3 (b) - Standard Profit by PSO (Generation 8)

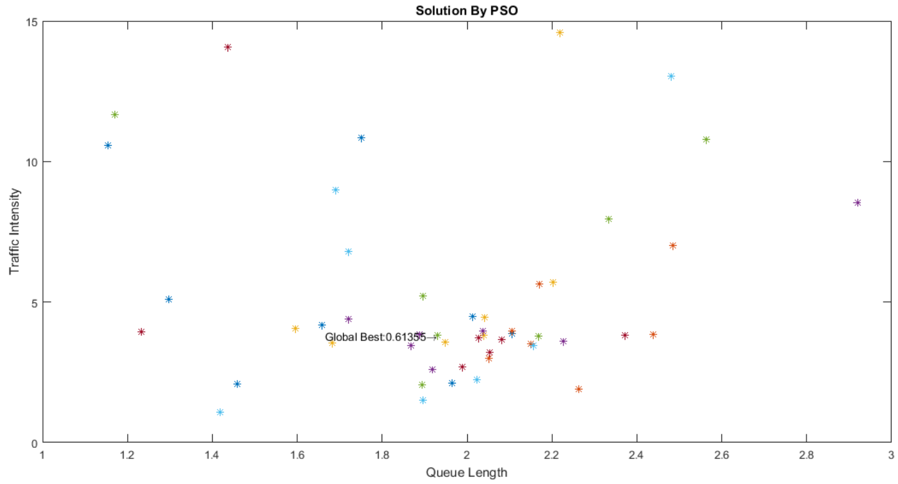

Figure 4.3 (c) - Standard Profit by PSO (Generation 15)

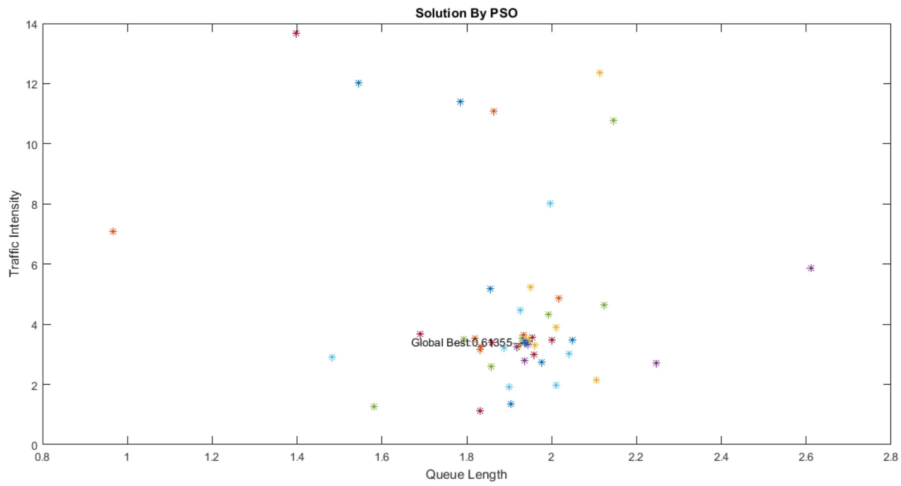

Figure 4.3 (d) - Standard Profit by PSO (Generation 23)

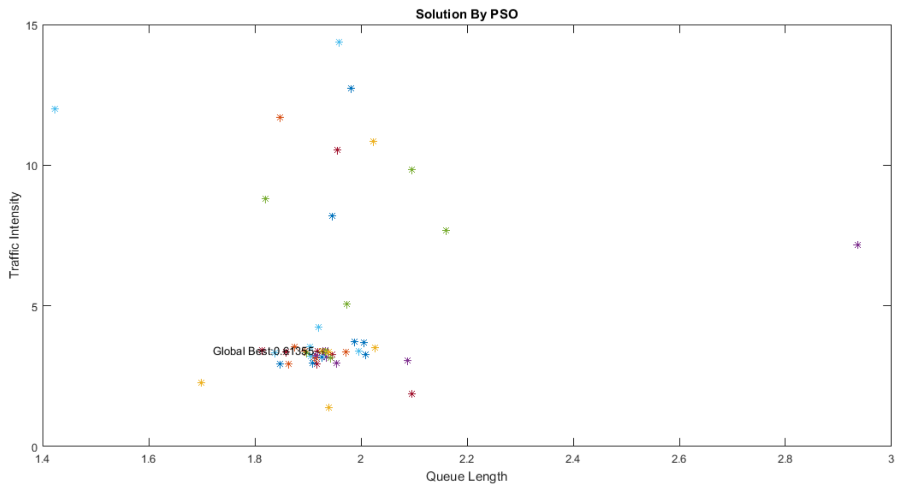

Figure 4.3 (e) - Standard Profit by PSO (Generation 30)

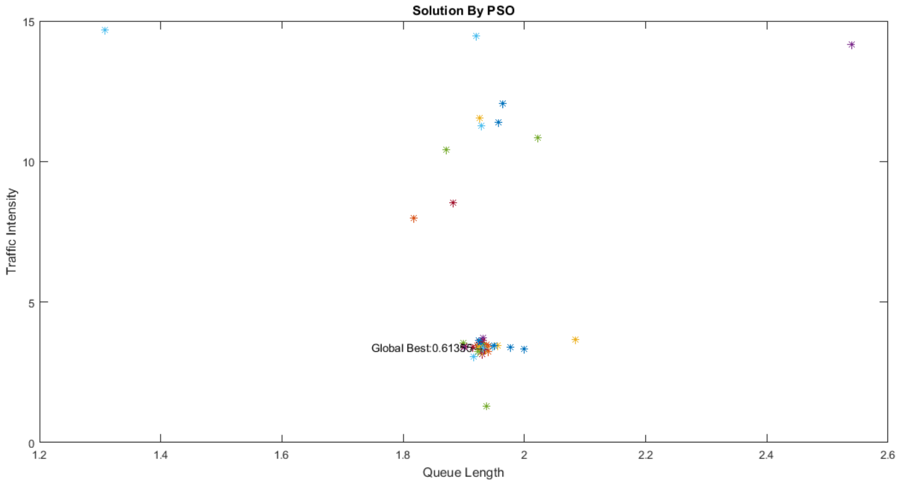

Figure 4.3 (f) Standard Profit by PSO (Generation 37)

Now let's apply PSO to maximize the standard profit for the word of mouth model and find out corresponding traffic intensity and queue length.

Table 2- Solution for Word of Mouth Model (Given by Equation- 4.4)

Generation Number	ρ	x	S'_{basic}
1	1.237792	3.801933	5.347855e-01
5	1.354691	3.372928	5.362668e-01
10	1.326520	3.591854	5.364443e-01
15	1.323518	3.471855	5.364423e-01
20	**1.325359**	**3.266006**	**5.364446e-01**
25	**1.325359**	**3.266006**	**5.364446e-01**
45	**1.325358**	**3.266025**	**5.364446e-01**

Similar to table 1, table 2 provides the global best, in our case its standard profit for word of mouth model (expressed in equation 4.4), for different generation numbers. Looking at table and the swarm plots, we see that the global convergence is achieved at 0.5364 in generation number 20. Figure 4.4 (a) to figure 4.4 (g) show the position of various particles in swarm in some chosen generations.

Swarm Movement for Word of Mouth Model

Population size taken =50

Number of generations taken= 50

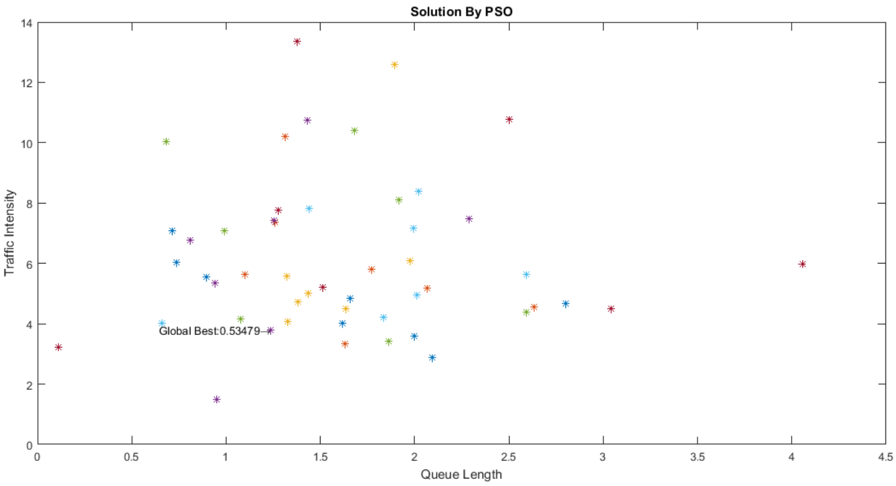

Figure 4.4 (a) - Standard Profit by PSO (Generation 1)

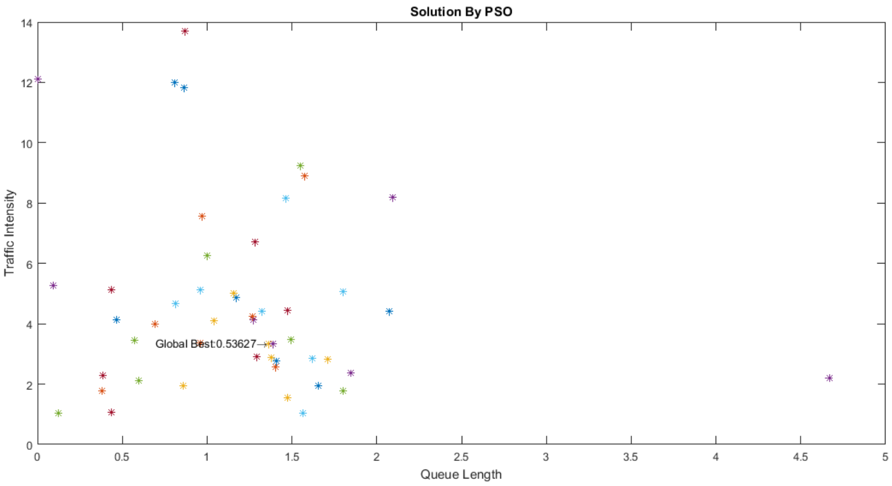

Figure 4.4 (b) - Standard Profit by PSO (Generation 5)

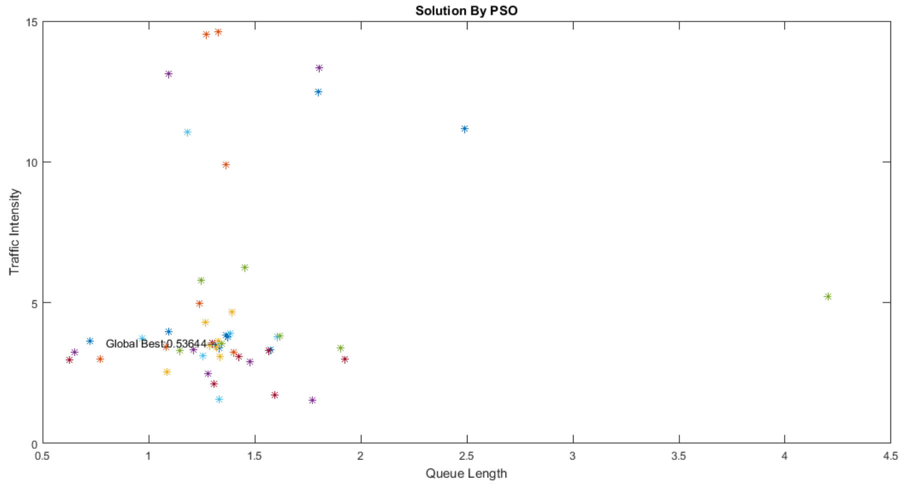

Figure 4.4 (c) - Standard Profit by PSO (Generation 10)

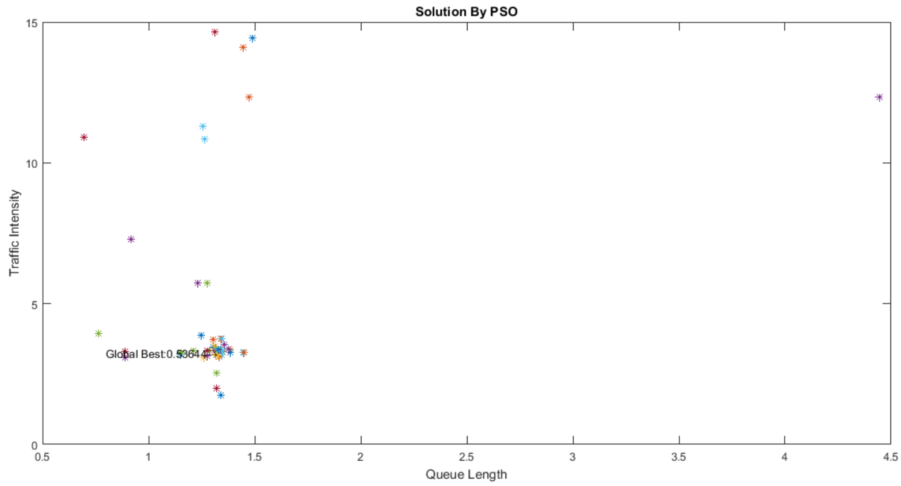

Figure 4.4 (d) - Standard Profit by PSO (Generation 15)

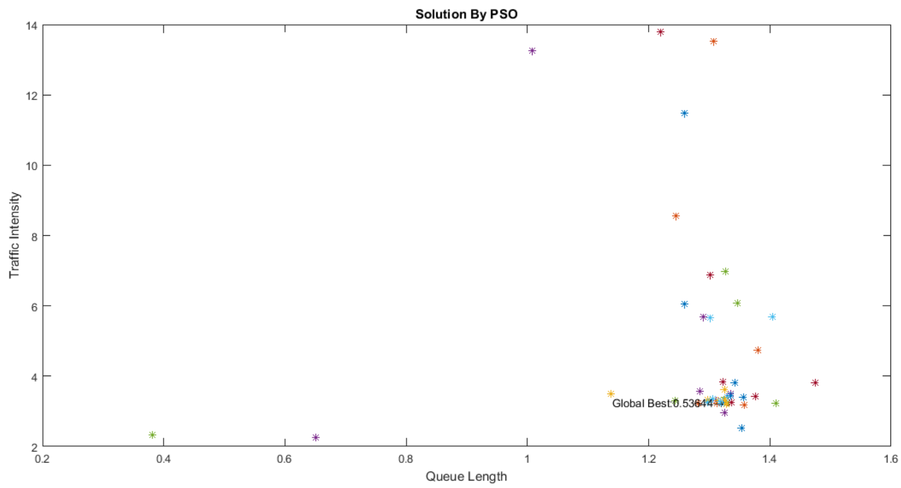

Figure 4.4 (e) - Standard Profit by PSO (Generation 20)

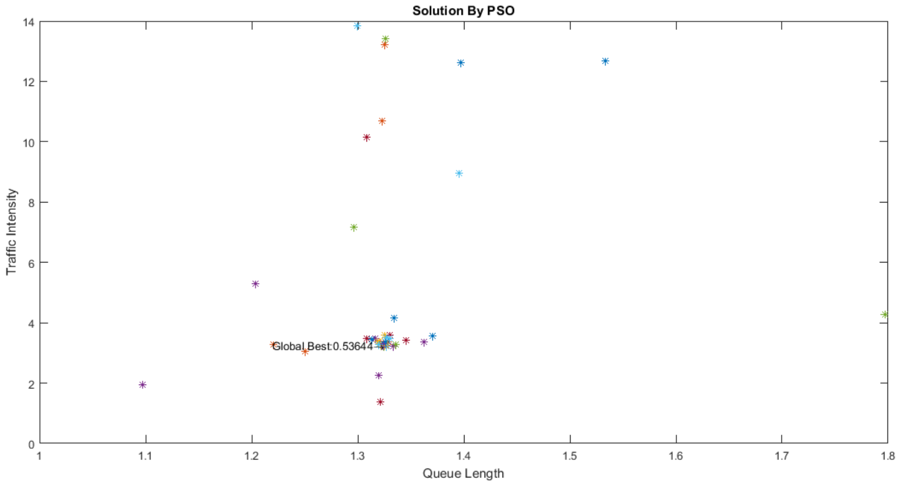

Figure 4.4 (f) - Standard Profit by PSO (Generation 25)

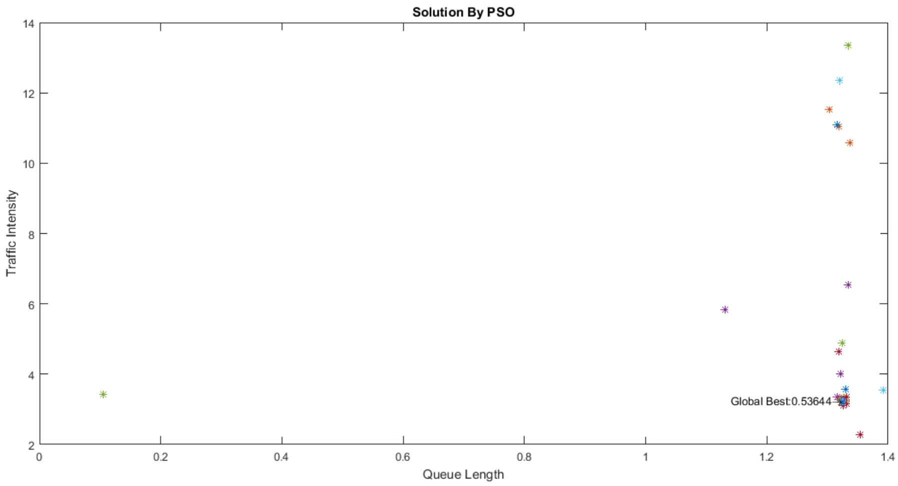

Figure 4.4 (g) - Standard Profit by PSO (Generation 45)

Results:

For *basic model* with fixed evaluation of 15 monetary units:

Traffic intensity for maximum profit, $\rho_{bs}* = 1.93034$

Allowed waiting queue length for maximum profit, $x_{bs}* = 3.8067 \approx 3$ customers

Maximum Standard Profit, $S'_{basic} = 6.135465\text{e-}01$

For *word of mouth model* with fixed evaluation of 15 monetary units:

Traffic intensity for maximum profit, $\rho_{wm}*$ = 1.32535

Allowed waiting queue length for maximum profit, $x_{wm}*$ = 3.2660 $\approx$ 3 customers

Maximum Standard Profit, S'_{wm} = 5.364446e-01

4.2.1 Result Comparison

Table 3 - Comparison: Basic Model

S.N.	Methodology by	$\rho_{bs}*$	$x_{bs}*$	S'_{basic}	Method Used
1	Atkinson	1.933	3	0.61354	Direct Search
2	Rommelfanger	1.9416	3	0.6446	Fuzzy Optimization
3	This work	1.93034	3	0.61355	Particle Swarm Optimization

Table 4 - Comparison: Word of Mouth Model

S.N.	Methodology by	$\rho_{bs}*$	$x_{bs}*$	S'_{basic}	Method Used
1	Atkinson	1.3251	3	0.5361	Direct Search
2	Rommelfanger	0.9329	3	0.5522	Fuzzy Optimization
3	This work	1.3254	3	0.5364	Particle Swarm Optimization

From table 3 and table 4, we can say that although PSO approach is easy to apply and much more simple than traditional search and fuzzy optimization method, it is powerful in getting optimum results. When the problem becomes large, say for example for a customer evaluation of 50 units, the time consumed by the search method grows larger and the complexity of fuzzy method also make them undesirable. PSO is independent of the values parameters, hence can be applied for any large value of customer evaluation.

4.3 General Behavior of Model with Dynamic Customer Evaluation

Function $V(x)$ can be taken as any decreasing function and is left for the manager's choice. Valuation will depend on the customers in the queue and will decrease as the queue length increases.

We are taking $V(x)$ as decay function with decay rate Γ. The customer evaluation $V(x)$ is dependent on queue length x as:

$$V(x) = V \times e^{-\Gamma x} \tag{4.5}$$

Here, V is the evaluated cost by the customer when, there is no one waiting in the queue.

Using equation (4.1) and (4.2) and substituting $V(x)$ from equation (4.5), we get the results as follows:

For basic model,

$$S'_{basic} = \rho\left(1 - \frac{x}{V \times e^{-\Gamma x}}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) - \frac{\rho}{V \times e^{-\Gamma x}} \tag{4.6}$$

For word of mouth model,

$$S'_{wm} = \rho\left(1 - \frac{x}{V \times e^{-\Gamma x}}\right)\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) - \frac{\rho}{V \times e^{-\Gamma x}}\left(\frac{1-\rho^{x}}{1-\rho^{x+1}}\right) \tag{4.7}$$

V is as defined for equation (4.5) and Γ is the valuation decay rate and can be decided by the server.

General Behavior of Standard profit functions (4.6 and 4.7) with respect to queue length x *and traffic intensity* ρ *:*

Taking Γ *= 0.05 and* V *=15,*

We have the surface plot from equation (4.6) and equation (4.7) as

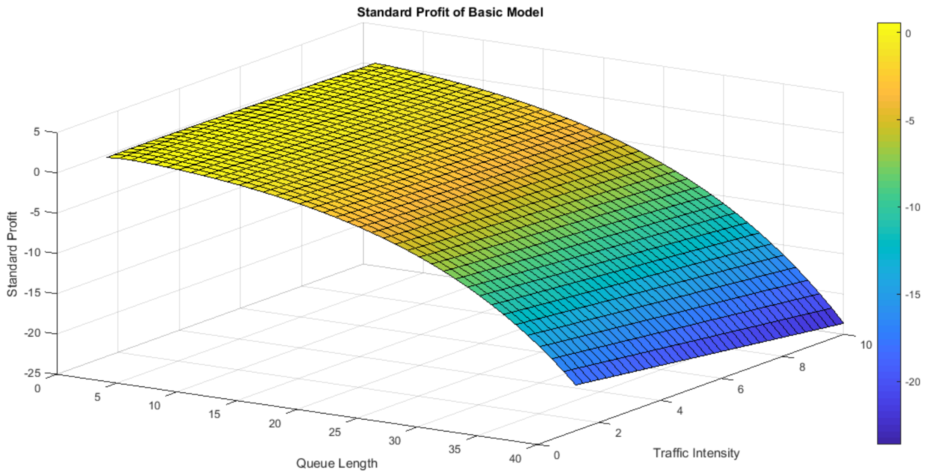

Figure 4.5 (a) - Basic Model

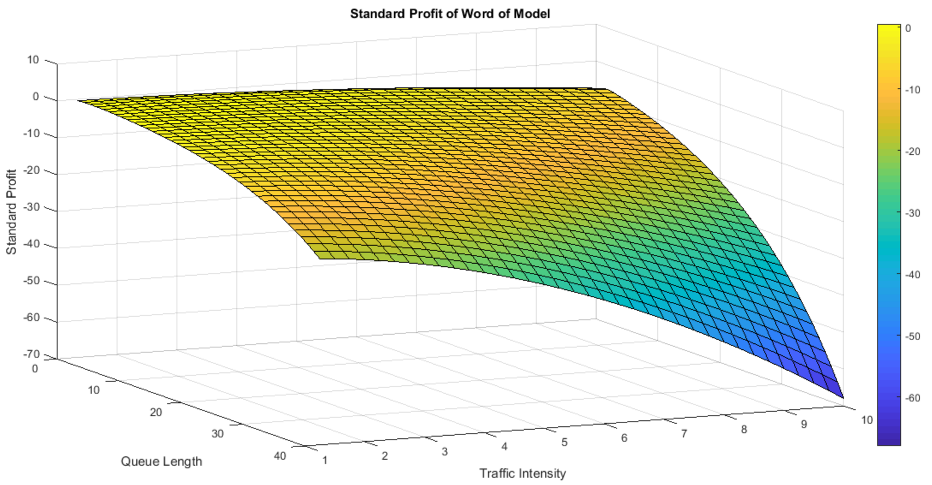

Figure 4.6 (a) - Word of Mouth Model

Figure 4.5 (a) and figure 4.6 (a) both are similar to figure 4.1 (a) and figure 4.2 (a). The only difference is that the profit is slightly less in the present case. This less profit is due to the decrease in value that customer gives to the service.

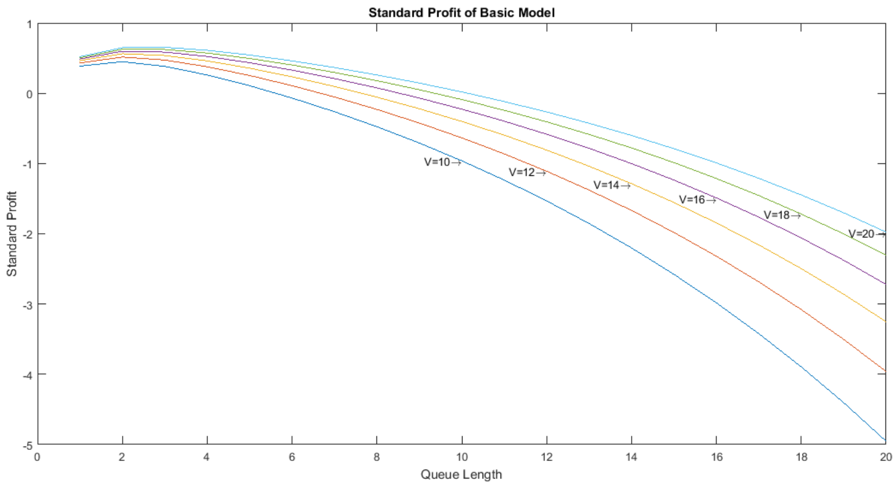

Figure 4.5 (b) – Basic Model with Dynamic Customer Valuation at $\rho=1.9$

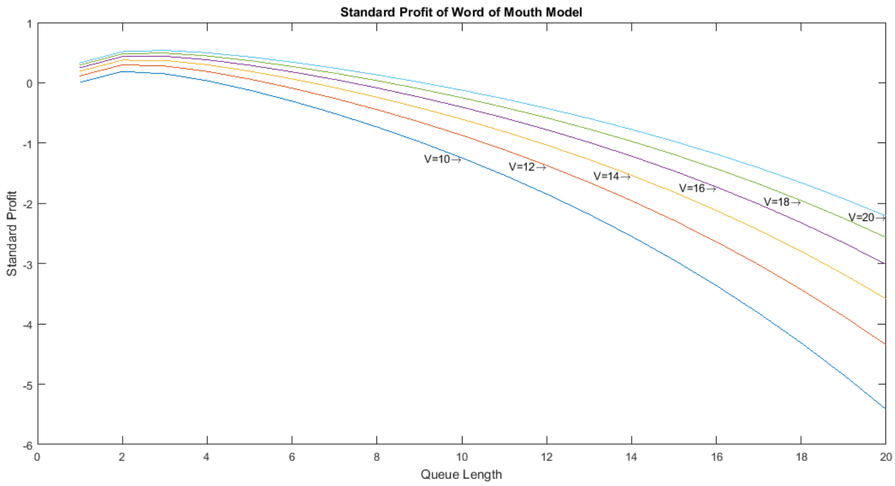

Figure 4.6 (b) - Word of Mouth Model with Dynamic Customer Valuation at $\rho=1.9$

Figure 4.5 (b) and figure 4.6 (b), like figure 4.1 (c) and figure 4.2 (c), show that the standard profit decreases as the queue length increases with constant flow of traffic in basic model as well as in word of mouth model respectively. Both the figures have various curves for different initial customer valuation and the rate of decay in evaluation is fixed as 0.05. Decrease in the standard profit with increasing queue length is a result of increase in waiting cost. Figure 4.2 (c) shows faster decreasing nature because of the bad effect of the publicity in word of mouth model.

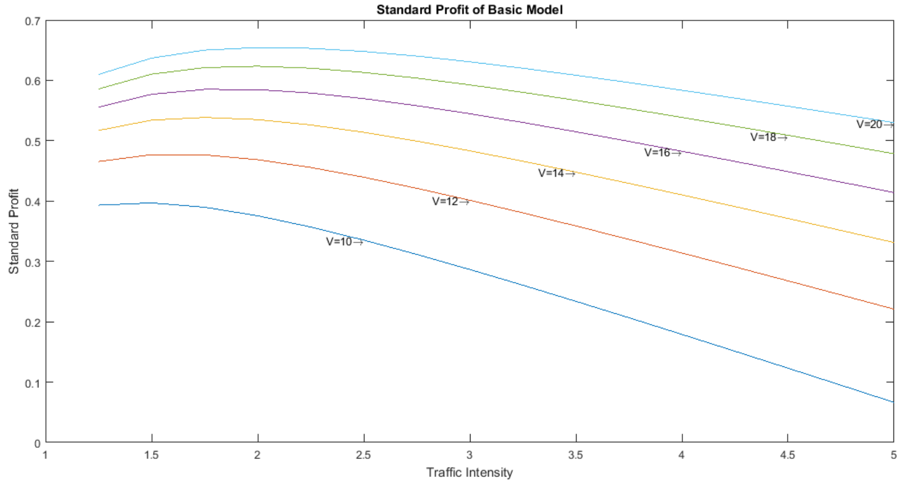

Figure 4.5 (c) - Basic Model with Dynamic Customer Valuation at $x = 3$

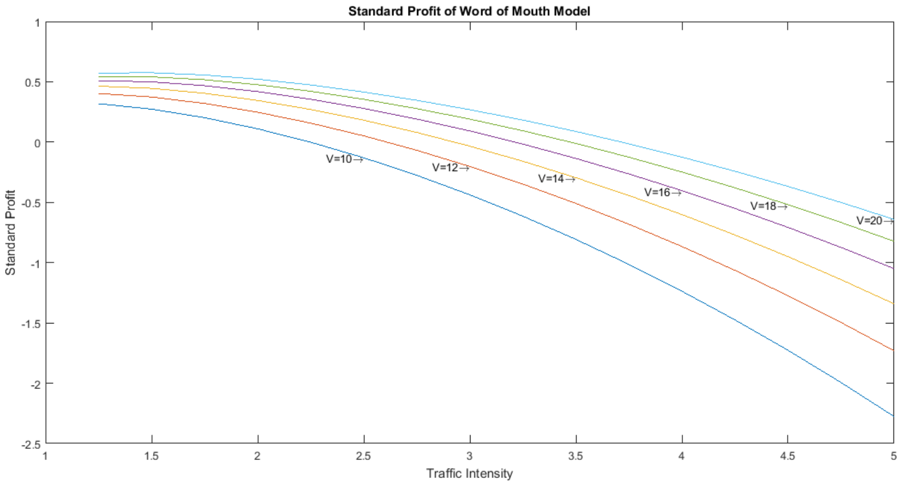

Figure 4.6 (c) - Word of Mouth Model with Dynamic Customer Valuation at $x = 3$

Figure 4.5(c) and figure 4.6 (c), we again see both the model, basic and word of mouth model respectively, showing very slight increase first then decrease in profit with increase in traffic intensity. This type of behavior is again due to the congestion that arises because of the fixed queue length that is 3 here. For low ratio of arrival rate to service rate, queue accommodates easily but as the arrival rate increases faster the waiting cost and cost of publicity begin to contribute in hampering of profit.

4.4 Illustration for Dynamic Customer Evaluation

Let us take the same case as taken in section 4.2 with $V = 15$ as we took in solving the model for initial constant evaluation, but this time we will take dynamic nature of customer evaluation with evaluation decay rate $\Gamma = 0.05$.

Let's apply Particle Swarm Optimization to maximize the standard profit for basic model:

Table 5- Solution for Basic Model with dynamic $V(x)$

Generation Number	ρ	x	S'_{basic}
1	1.569109	3.429087	5.526845e-01
5	2.251538	2.053935	5.834875e-01
10	2.119964	2.007739	5.843585e-01
20	2.167670	2.002477	5.845300e-01
40	**2.166664**	**2.000060**	**5.845437e-01**
80	**2.166706**	**2.000010**	**5.845437e-01**

Table 5 provide the global best, in our case its standard profit for basic model (expressed in equation 4.6), for different generation number. Looking at table and the swarm plots, we see that the global convergence is achieved at 40 with value 0.58454. Figure 4.7 (a) to figure 4.7 (c) show the position of various particles in swarm in some chosen generations.

Swarm Movement for Basic Model

Population size taken =50

Number of generations taken= 150

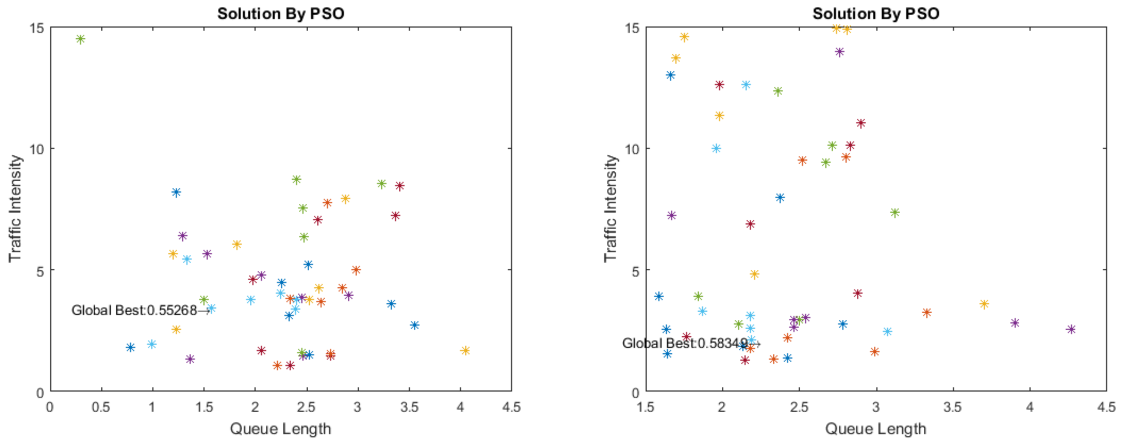

Figure 4.7 (a) - Standard Profit by PSO at Generation 1 (Left) and Generation 5 (Right)

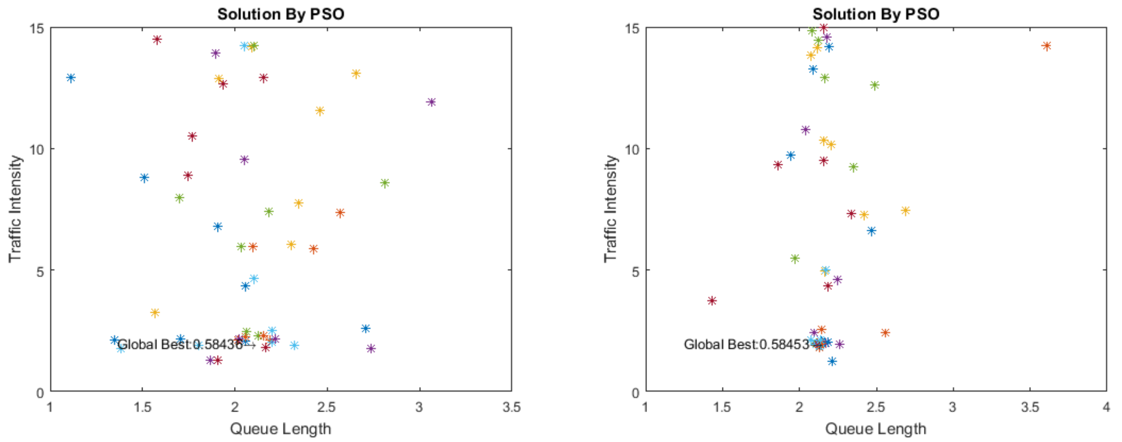

Figure 4.7 (b) - Standard Profit by PSO at Generation 10 (Left) and Generation 20 (Right)

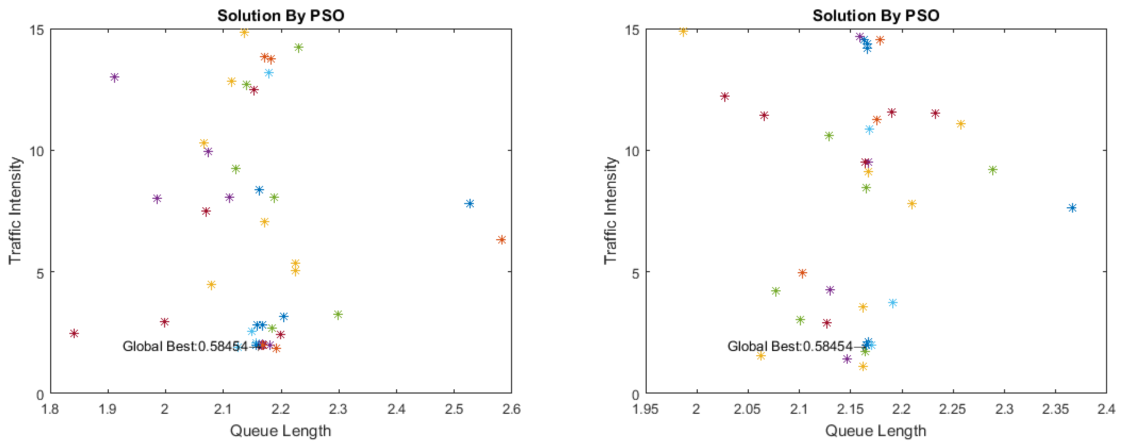

Figure 4.7 (c) - Standard Profit by PSO at Generation 40 (Left) and Generation 80 (Right)

Similarly, applying Particle Swarm Optimization to maximize the standard profit for word of mouth model (as expressed in equation 4.7) gives the result as table 6:

Table 6 - Solution for Word of Mouth Model with dynamic $V(x)$

Generation Number	ρ	x	S'_{basic}
1	1.447128	3.172051	4.760084e-01
5	1.302127	3.064245	4.862418e-01
10	1.231038	3.031832	4.874879e-01
20	1.262612	3.011657	4.878180e-01
40	1.248065	3.000048	4.880752e-01
80	**1.246359**	**3.000000**	**4.880765e-01**
100	**1.246369**	**3.000000**	**4.880765e-01**

Figure 4.8 (a) to figure 4.8 (f) show the position of various particles in swarm in some generations.

Swarm Movement for Word of Mouth Model

Population size taken =50

Number of generations taken= 150

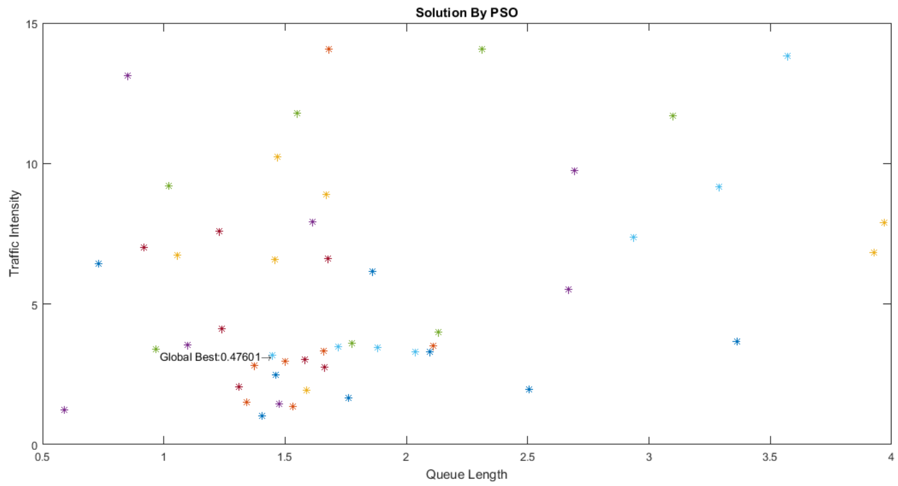

Figure 4.8 (a): Standard Profit by PSO (Generation 1)

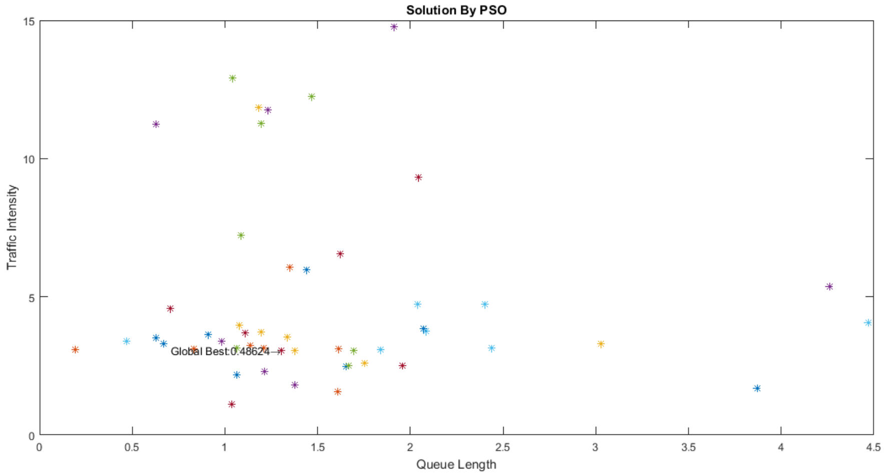

Figure 4.8 (b): Standard Profit by PSO (Generation 5)

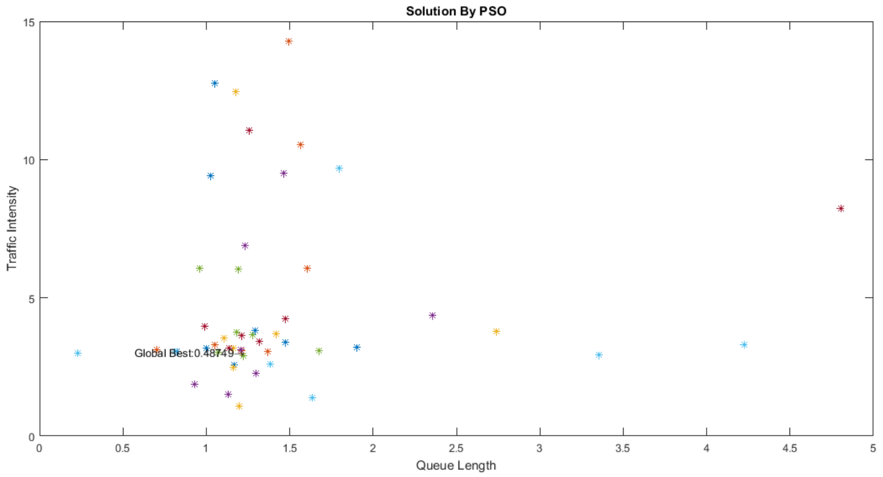

Figure 4.8 (c): Standard Profit by PSO (Generation 10)

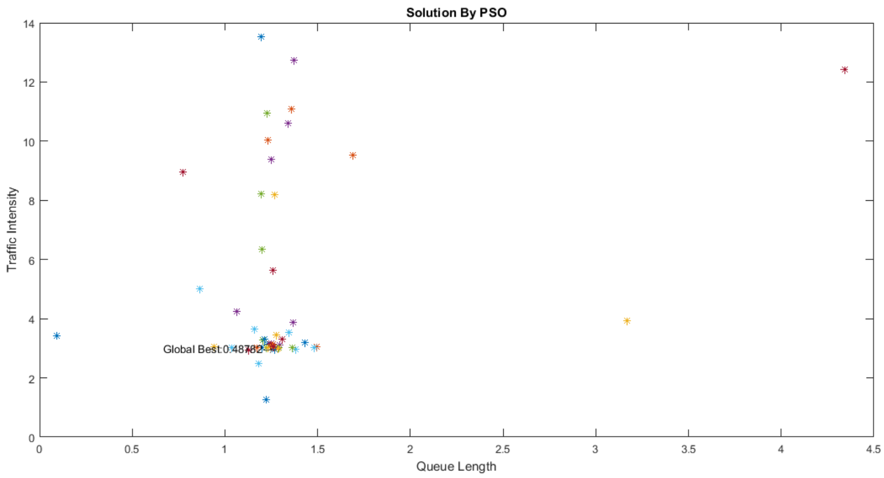

Figure 4.8 (d): Standard Profit by PSO (Generation 25)

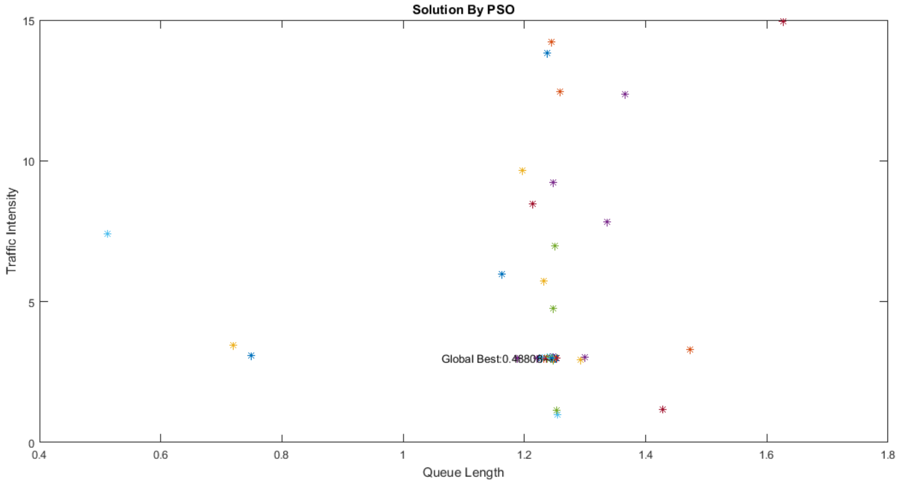

Figure 4.8 (e): Standard Profit by PSO (Generation 40)

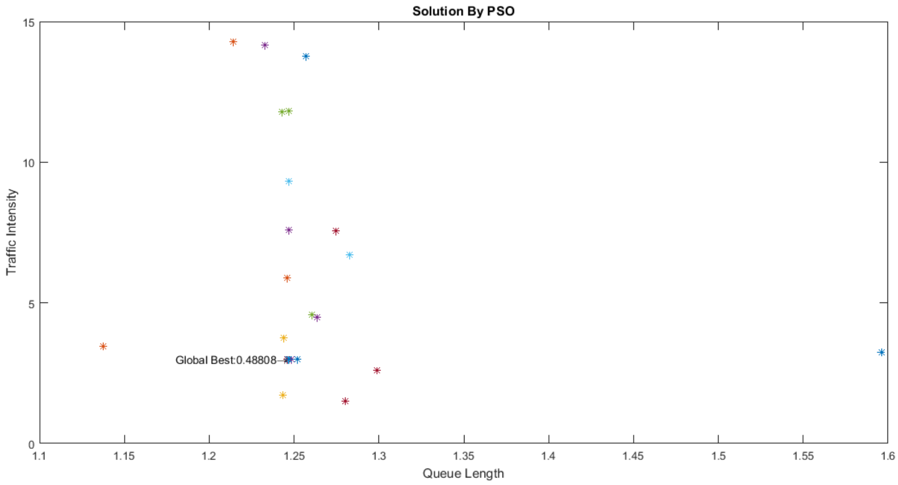

Figure 4.8 (f): Standard Profit by PSO (Generation 80)

Results:

For *basic model* with initial evaluation of 15 monetary units and evaluation decay rate of 0.05 per customer:

Traffic intensity for maximum profit, $\rho_{bs}* = 2.167$

Allowed waiting queue length for maximum profit, $x_{bs}* = 2.000 \approx 2$ customers

Maximum Standard Profit, $S'_{basic} = 0.5845$

For *word of mouth model* with initial evaluation of 15 monetary units and evaluation decay rate of 0.05 per customer:

Traffic intensity for maximum profit, $\rho_{wm}* = 1.246$

Allowed waiting queue length for maximum profit, $x_{wm}* = 3.000 \approx 3$ customers

Maximum Standard Profit, $S'_{wm} = 0.4881$

4.5 Results: Constant Evaluation Vs Dynamic Evaluation

Now, on comparing various results of models with constant and dynamic customer evaluation, we can see that the queue length recommended for the facility following the basic model is less in case of dynamic customer evaluation. The profit is also reduced because of the reduced customer valuation. Although the recommended queue length in case of word of mouth model did not change, but the profit is reduced as we see in table 8.

Table 7 - Comparison: Basic Model

S.N.	Model	$\rho_{bs}*$	$x_{bs}*$	S'_{basic}
1	With Constant Customer Evaluation	1.933	3	0.61354
2	With Dynamic Customer Evaluation	2.167	2	0.5845

Table 8 - Comparison: Word of Mouth Model

S.N.	Model	$\rho_{wm}*$	$x_{wm}*$	S'_{wm}
1	With Constant Customer Evaluation	1.325	3	0.5364
2	With Dynamic Customer Evaluation	1.246	3	0.4881

Since, customer evaluation in a market is very dynamic and largely depend on the crowd and congestion of the shop. Some incentives or reward system should be developed to compensate the deceasing valuation. This will help in binding customers and increasing profit.

4.6 Conclusion

Firstly, we saw that the traditional search methods of optimization, that are mostly dependent on the auxiliary information of objective function, are lengthy and inefficient as compared to Particle Swarm Optimization method.

Rommelfager [3] gave the solution using fuzzy optimization which is further simplified by Pardo and Fuente [4]. The complexity that arises in fuzzy model can also be avoided by using

Particle Swarm Optimization. The results have been shown for both publicity and renouncement model provide insights for more realistic queueing scenario.

Secondly, we see, in real world, there is very competitive market. You can easily find multiple places to purchase a single item. Customer value to the service facility plays a key role in deciding whether he will go for the particular facility or shop or he will bounce to another nearby shop.

After closely looking at the results and graphs, we see that even a little factor of dissatisfaction or bad feeling about a particular facility can lead the facility to face loss as compared to others in the market.

We have come to conclusion that the feeling of less value for any service, provided by any facility, based on the queue length only can hamper the profit to significant level. To compensate the customer value that is being lost, the facility should provide incentives and rewards. This can make the customers feel compensated. These inceptives should be queue length dependent and should not exceed the actual service price for facility to avoid lost from facility side itself. A restaurant can provide some discount on waiting for service. Railways can provide free meals for waiting for a late train.

For any system where the demand of customers is high so that there may be long queue, our results for the model may be helpful to prevent the customers from renouncing and going back.

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- https://en.wikipedia.org/wiki/Queueing_theory

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[2] H. Rommelfanger, R. Hanuscheck, J. Wolf, Linear programming with fuzzy objectives, Fuzzy Sets and Systems 29 (1989) 31-48.

[3] Van Ackere, P. Ninios, Simulation and queuing theory applied to a single-server queue with advertising and balking, Journal of the Operational Research Society 44 (1993) 407-414.

[4] M.J. Pardo, D. de la Fuente, A new technique to optimize the functions of fuzzy profit of queuing models application to a queuing model with publicity and renouncement, Computers and Mathematics with Applications 57 (2009) 850-864.

[5] M. Clerc, J. Kennedy, The particle swarm: explosion, stability, and convergence in a multi-dimensional complex space, IEEE Transactions on Evolutionary Computation, 6 (2002) 58-73.

Appendix
Source Code for Particle Swarm Optimization

```
%============================================================
%The program is created by Mr. Devendra kumar (13312005)
%For the purpose of this dissertation work
%No content has been copied from any sources
%============================================================


%variables declaration which are crucial to control algorithm
%+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%n= number of candidates on swarm or population size
n=15;
%i= number of iterations
i=15;
%varn = the dimension of the objective function
varn=2;
%c1 and c2= constants used in velocity update equation
c1=2.05;
c2=2.05;
%fitness array for particles in the swarm
fitness= zeros(n,1);
%x= particles locations nxvarn dimension array
x= zeros(n,varn);
%v= velocities nxvarn dimension array
v= zeros(n,varn);
%feasibleL and feasibleU array will have
%the feasible region for the variable used in the
%problem. L denotes lower bound and U denotes the upper bound
feasible= zeros(varn,2);
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

%Initialization
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

feasible(1,1)=0;
feasible(1,2)=7;
feasible(2,1)=1;
feasible(2,2)=15;

%random initialization of swarm
%+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
for swarmid= 1:n
    for varindex= 1:varn
        x(swarmid,varindex)= feasible(varindex,1)+rand()*(feasible(varindex,2)-
feasible(varindex,1));
        v(swarmid,varindex)= 0.0001*rand();

    end
end
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
```

```
%calculation of objective function
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
for swarmid= 1:n

    %In maximization, fitness is same as objective function
    %In minimization, reciprocal of (1+ objective function) can be taken

    fprintf('\nx values:  %d  %d ', x(swarmid,1),x(swarmid,2));
    fitness(swarmid,1)= (objecfun(x(swarmid,:)));
    fprintf('\n\tfitness : %d', fitness(swarmid,1));
end




[gbest,gbestx]= max(fitness);
gbestx= x(gbestx,:);
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

%Calculation of pbest for the particles
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%For first iteration it is same as the fitness of the function at particle location
pbest=fitness;
pbestx=x;
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

%initial steps are completed now, lets move to iteration
%velocity and position will be calculated and the c=2 has been taken

%Generation loop starts here
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%gen= the current generation number, i is the last generation number

for gen= 2:i
        fprintf('\n#######################################################' );

        %Particle update loop starts here
        %++++++++++++++++++++++++++++++++++++++++++++++++++++++
        for swarmid= 1:n
                fprintf('\nswarmid: %d',swarmid);
                fprintf('\n++++++++++++++++++++++++++++++++++++++++');
                %++++++++++++++++++++++++++++++++++++++++++++++

                %Particles are getting updated vairiable by variable. Loop starts
                %++++++++++++++++++++++++++++++++++++++++++++++
                for varindex= 1:varn
                        fprintf('\nx%d:',varindex);
                        phai= c1+c2;
                        %Chai calculation from Clerc and Kennedy [5]
                        chai= 2/(abs(2-phai-sqrt(phai*phai-4*phai)));
                        %Velocity update equation

                        v(swarmid,varindex)=chai*(v(swarmid,varindex)
                                +(c1*rand()*(pbestx(swarmid,varindex)
                                        -x(swarmid,varindex))
                                        + c2*rand()*(gbestx(varindex)
                                        -x(swarmid,varindex))));

                        %Position update equation
```

```
                        x(swarmid,varindex)= x(swarmid,varindex)
                                                 + v(swarmid,varindex);

                        %velocity clamping introduced
                        %combined with particle relocation

                        if(x(swarmid,varindex)>feasible(varindex,2))
                                x(swarmid,varindex)=feasible(varindex,1)
                                +rand()*(feasible(varindex,2)
                                -feasible(varindex,1));
                        end
                        if(x(swarmid,varindex)<feasible(varindex,1))
                                x(swarmid,varindex)=feasible(varindex,1)
                                +rand()*(feasible(varindex,2)
                                -feasible(varindex,1));
                        end
                        %Display of some information
                        fprintf('\t pbest = %d', pbestx(swarmid,varindex));
                        fprintf('\tnew x%d = %d',varindex, x(swarmid,varindex));
                        fprintf('\t gbest = %d', gbestx(varindex));
                        fprintf('\t Velocity = %d', v(swarmid,varindex));
                end
                %+++++++++++++++++++++++++++++++++++++++++++++++++
                %particles update loop ends here

                %Generating a particle plot for population
                plot(x(swarmid,1),x(swarmid,2),'*');
                hold on;
                fitness(swarmid,1)= objecfun(x(swarmid,:));
                fprintf('\n\tfitness : %d', fitness(swarmid,1));
                %Updating Pbest
                if pbest(swarmid,1) <= fitness(swarmid,1)
                        pbest(swarmid,1)= fitness(swarmid,1);
                        pbestx(swarmid,:) = x(swarmid,:);
                end
        end
        %Particle update loop ends here
    hold off;
    %updating Gbest
    if gbest< max(fitness)
        [gbest,gbestx]= max(fitness);
        gbestx= x(gbestx,:);
    end
    %Displaying best result of the geneartion
    fprintf('\nx values:  %d  %d  %d', gbestx(1),(gbestx(2)));
    fprintf('\ngbest : %d', gbest );
    %Waiting for key press to move to next generation
    w = waitforbuttonpress;
end
%++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
%Program is ended here.
```