Studies on Robust Social Influence Mechanisms

Incentives for Efficient Network Routing in Uncertain Settings

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Many of today's engineered systems are tightly interconnected with their users, and in many cases, system performance depends greatly on user behavior [1]. As a result, the traditional lines between engineering and the social sciences are becoming increasingly blurred, and analytical tools such as game theory are finding new applications in engineering [2], [3]. It is often insufficient to judge an engineered system on its technical merits alone, since strategic user behavior can lead to unpredictable and desirable results [4]. Of particular importance to this article are socially-integrated engineering problems in which users' strategic behavior has a significant impact on overall system performance. These types of systems appear in a variety of contexts in theory and practice: transportation networks [5], ridesharing applications [6], [7], supply-chain management [8], cloud computing [9], and electric power grids [10] are immediate examples. In all these settings, in addition to the merely-technical challenges they pose, an engineer may need to consider methods of influencing individual user behavior to effect positive change on aggregate system performance [11].

Fundamentally, any behavior-influencing mechanism requires information about the underlying system and the user population who are to be influenced, as depicted in Figure 1. For example, if a system planner desires to price a network resource to encourage efficient network usage, it may be desirable to characterize the sensitivity of the user population to pricing. If this information is difficult to gather or altogether unavailable, the planner may need to rely on crude estimates of user price-sensitivities, and the pricing design must take this uncertainty into account. At worst, a misunderstanding of informational dependencies can lead to “perverse incentives,” or incentives that exacerbate the very problems they were intended to solve.

Here, a theory of “robust social influence” is an attractive goal: how can behavior-influencing mechanisms be designed so that they are robust to a variety of mis-characterizations or variations in models of social behavior? Some natural questions in this context include

1) How robust are existing behavior-influencing methodologies to system mis-characterizations?
2) How “close” do behavior models need to approximate true behavior for an influencing mechanism to provide good performance?
3) How can perverse incentives be systematically avoided?

This article investigates the concept of robust social influence with regard to a well-studied model of network traffic routing in which drivers need to be routed across a congestion-sensitive network. It is known that if individual drivers make their own routing decisions to minimize their own experienced delays, overall network congestion can be considerably higher than if a central planner had the ability to explicitly direct traffic [12]. Accordingly, there has been a great deal of research on the application of road tolls for the purpose of influencing drivers to make routing choices that result in globally-optimal routing [13]–[19]. In much of this literature, a toll-designer is assumed to have a detailed characterization of the system: network topology, link congestion characteristics, and user demand structure are commonly assumed to be perfectly-known quantities.

If road tolls are designed to incentivize good performance for one instance of a routing problem, and then some detail of the routing problem changes (for example, a link is "removed" by a traffic accident or natural disaster), it would be desirable for the original tolls to incentivize good performance on the changed routing problem as well. That is, we would like to know if the performance guarantees provided by the original tolls are *robust* to changes or mischaracterizations in the underlying details of the system (e.g., network structure, traffic rate, user demands). Here, we distinguish between different degrees of robustness: a taxation mechanism is *strongly robust* to mischaracterizations of some system parameter if it incentivizes optimal behavior for variations of that parameter, whereas a mechanism is *weakly robust* if it merely incentivizes behavior that is no worse than the un-influenced behavior.

This article discusses some emerging results which evaluate the robustness of existing incentive methodologies to variations in network topology or mis-characterized user populations, and presents several studies which suggest new directions in this research area. The question of robustness in this context is non-trivial; some taxation methodologies prove to be highly sensitive to a wide variety of model variations, while some exhibit a high degree of robustness in some contexts and yet set up perverse incentives in others. We show that fixed tolls (i.e., tolls which are constant functions of traffic flow on each network link) cannot even be weakly robust unless they depend on some information about network structure. This is in stark contrast to the well-studied marginal-cost tolls, which for homogeneous populations are known to be strongly robust, despite requiring no information about network structure [20].

Traffic Routing Model and Toll Robustness

A classical example of an engineered systems whose performance depends heavily on the choices of its users is that of a transportation network; this is captured in the literature by a problem known as a "non-atomic routing game." The basic problem setup is this: there is a group of travelers who need to be routed through a congestion-sensitive network in a way that minimizes the users' average travel time. It is typically straightforward to compute an optimal routing profile (also called a *network flow*), but implementing a particular flow would require that a central planner had the ability to force every driver to take a specified route. Unfortunately, it is well-known that if each driver chooses his route in order to individually minimize his own travel time, the resulting aggregate behavior can be substantially less efficient than the centrally-computed optimal flow [21].

Since the system planner cannot direct traffic explicitly, he must use some indirect means of influencing users to behave optimally. Various methods which have been studied include financial incentives [22], providing drivers with specific information [23], access control [24], and socially-conscious network design [25]. All of these operate by indirectly modifying the drivers' preferences over their available routes. In this section we provide examples of the inefficiency resulting from self-interested behavior, and show that simple attempts to influence user behavior can have surprisingly negative results.

Pigou's Example: the Inefficiency of Self-Interested Behavior

Refer to Figure 2 for a simple example network as the routing model is introduced formally. There is a directed, acyclic network consisting of edge set E and vertex set V. A mass of r units of traffic needs to be routed from a source node to a destination node; in most settings, the traffic is considered to be inelastic – that is, all traffic must reach its destination (no driver can simply stay home). There are assumed to be infinitely-many drivers, each with an infinitesimally-small effect on traffic congestion. Each network edge (that is, link) $e \in E$ has associated with it a latency function $\ell_e(f_e)$, which describes the delay experienced by users of that edge when f_e units of traffic are using it. Throughout, latency functions are assumed to be nonnegative, differentiable, nondecreasing, and convex.

The symbol $\mathcal{P} \in 2^E$ denotes the set of *paths* available to the users, where a path $p \in \mathcal{P}$ is a collection of edges leading from the source to the destination. Finally, a *routing game* is specified by $G = \{V, E, \mathcal{P}, \{\ell_e\}_{e \in E}, r\}$.

For a given game G, a network flow may be expressed by a collection of *edge flows* $\{f_e\}_{e \in E}$ or alternatively as a collection of *path flows* $f = \{f_p\}_{p \in \mathcal{P}}$ where the flow on path p is expressed by $f_p = \sum_{e \in p} f_p$, and the path latency $\ell_p(f)$ is similarly the sum of all edge latencies on that path. The system-planner's hope is to minimize the *total latency*, which is a measure of aggregate network delay, expressed by the formula

$$L(f) = \sum_{e \in E} f_e \ell_e(f_e) = \sum_{p \in \mathcal{P}} f_p \ell_p(f). \tag{1}$$

An inherent challenge in this setting is that users' self-interested routing choices may lead to very different aggregate flows than those which minimize the total latency. One popular way to model the effects of users' self-interested choices is that of the Nash Flow (also called Wardrop Equilibrium). A Nash flow f^{nf} is a routing profile in which no individual user can change routes and decrease his or her latency; put differently, in a Nash flow every user's route choice is individually-optimal with respect to the choices of other users. In this setting, Nash flows always exist and are essentially unique: that is, for a given network, the total latencies of all Nash flows are equal [1]. One important property of Nash flows is that every path with positive traffic has equal latency:

$$\text{If } f^{\mathrm{nf}} \text{ is a Nash flow and } f_p^{\mathrm{nf}}, f_{p'}^{\mathrm{nf}} > 0 \text{ for some paths } p,\, p' \text{ then } \ell_p\left(f^{\mathrm{nf}}\right) = \ell_{p'}\left(f^{\mathrm{nf}}\right). \tag{2}$$

The first example, depicted in Figure 2, illustrates the basic problem that travelers' individual self-interested choices can lead to over-congested network flows. This illustration was first published in 1920 by the economist Arthur Pigou, and remains a centerpiece of work in this area [12]. The setting is as follows: there is a simple two-link network, in which 1 unit of travelers can choose between a linear-latency congestion-sensitive link and a constant-latency link. The first link offers a faster journey, provided that it is not chosen by too many users.

The optimal flow on this network as shown on the left in Figure 2 is to split the traffic evenly between the two links, so that a mass of $1/2$ experiences a latency of $1/2$ on the top link, and the remaining traffic experiences a latency of 1 on the lower link, giving a total latency of $L(f) = (1/2)^2 + 1/2 = 0.75$. However, this requires half the drivers to choose quite a long route; any individual driver has a compelling incentive to switch to the upper link and arrive at her destination in half the time. Unfortunately, if all drivers choose the route with the lowest latency, they will all crowd on to the upper link and establish a Nash flow as depicted on the right in Figure 2. The reader can verify that $(1, 0)$ is indeed a Nash flow, since in this configuration, both edges have an equal latency of 1, so no user can change routes and decrease their latency. In this Nash flow, the network's total congestion is 1, a factor of $4/3$ greater than the optimal total congestion.

Braess's Paradox: the Unintended Consequences of Naïve Influence

A second canonical example known as Braess's Paradox (first noted by Dietrich Braess [4]) illustrates that seemingly-innocuous attempts to influence user behavior can lead to unexpected and perverse consequences. Consider the network depicted in Figure 3(a); traffic can choose between two paths, each routing through its own intermediate node. As-is, the total congestion on the network is 1.5, since half the traffic uses the upper path and half uses the lower path.

Suppose now that the system planner adds a single zero-cost link to the network connecting the two intermediate nodes to one another, as depicted in Figure 3(b). Now, under the old flow in which users split evenly, any user at node (B) would prefer to take the new zero-cost link rather than continue on the upper path. This increases the lower path's congestion, causing more users at (A) to choose the upper path, but those users in turn will choose the new zero-cost link once they arrive at node (B). Ultimately, equilibrium is reached at the routing profile depicted in Figure 3(b), with a corresponding total congestion of 2. Here, this behavior-influencing mechanism (augmenting the network with a zero-cost link) backfired and caused a dramatic increase in total congestion.

Price of Anarchy: Worst-Case Equilibria

A common approach to quantifying the performance loss resulting from self-interested behavior is to divide the total latency of a Nash flow with that of an optimal flow. In each case above, this ratio was $4/3$ - the Nash flow in each instance was 33% worse than the corresponding optimal flow. This ratio between equilibrium and optimal costs has been extensively explored in the literature and is known as the *price of anarchy* [26]. It is typically evaluated in worst-case over classes of games; formally, for a game G in a class of games $\mathcal{G}$, writing $\mathcal{L}^{\mathrm{nf}}(G)$ to denote the Nash flow total latency for game G and $\mathcal{L}^*(G)$ to denote the respective optimal total latency, the price of anarchy of $\mathcal{G}$ is defined as

$$\mathrm{PoA}(\mathcal{G}) = \max_{G \in \mathcal{G}} \frac{\mathcal{L}^{\mathrm{nf}}(G)}{\mathcal{L}^*(G)}. \tag{3}$$

For example, if $\mathcal{G}$ is defined as the class of routing games where all edges have linear-affine latency functions, it is known that $\mathrm{PoA}(\mathcal{G}) = 4/3$ [27]. Thus, both Pigou's example and Braess's paradox are worst-case examples of linear-latency routing games. However, for networks with higher-degree polynomial latency functions, the price of anarchy can be arbitrarily high. This can be seen in Pigou's network by replacing the linear cost function on the first link with a polynomial cost function of $\ell_1(f_1) = (f_e)^d$; the price of anarchy is unbounded in d [28]. The price of anarchy has been evaluated for many different classes of games, with applications as diverse as network resource allocation [29], distributed control [30], and more. For more details, see "Computing Price of Anarchy Bounds."

Robust Social Coordination Using Tolls

It was shown above how self-interested behavior can lead to poor system performance; in both examples, the fundamental problem was that an optimal flow can only occur when some drivers experience a higher latency than others. If users were altruistic, willing to accept a personal degradation of service for the sake of the greater good, they might be expected to adopt this type of socially-optimal configuration. One way to influence users to choose this configuration is to charge tolls on over-congested links, hoping to increase their costs enough that a sufficient number of users will avoid them. The tolls are put in place essentially to induce an "artificial altruism" in the population; in fact, there are many parallels between the literature on altruism in congestion games and the literature on financial incentives in congestion games [31]. Of course, the toll-designer must take care in choosing the tolls; it has already been seen in Braess's Paradox that seemingly-innocuous approaches can have unexpected consequences, and the toll-designer must be certain not to fall into a similar trap. If tolls are too high on a particular edge, it may be that too many users will avoid that edge; if tolls are not properly balanced throughout the network, uneven and inefficient flow distributions could arise.

We write $\tau_e(f_e)$ to denote the (possibly flow-varying) toll on edge e; assigning tolls to network edges modifies the costs experienced by users on those edges, inducing a new game with new associated equilibria. The system-planner's goal is to levy tolls which induce the network's optimal flow as an equilibrium of the tolled game.

There has been a great deal of research on the topic of computing tolls to influence behavior in routing games, but much of this research has assumed that the toll-designer has a complete, detailed characterization of the routing system as a whole [32]–[34].

It is the goal of robust social influence design to levy tolls that incentivize desirable behavior irrespective of changes or mischaracterizations of the underlying system. Figure 4, depicts several types of system changes which could potentially create problems for taxation methodologies. In these diagrams, the tolls were designed for the nominal system on the left, but after the respective change, these same tolls are effectively being applied to different networks than that for which they were designed. The hope is that the tolls designed for the original system provide comparable performance guarantees on the "new" systems; to this end, we will ask if each of several common taxation methodologies is robust to variation in parameters such as:

1) Network Changes: If tolls are designed to incentivize efficient flows for a particular network, and the network undergoes some change, do the original tolls still incentivize efficient flows for the new, changed network?
2) Traffic Rate: Do tolls designed for one traffic rate (i.e., one value of r) still incentivize efficient flows if the rate changes?
3) Demand Structure: If some users have access to different paths than others, how does this impact the design of the correct tolls?

To investigate the robustness of a particular tolling strategy to variations of a parameter, a system-planner can design tolls for a specific system realization, hold the tolls constant, and study the effect on total latency of varying the parameter in question. To formally define robustness, the tolls computed for game G by taxation mechanism τ are denoted by $\tau(G)$, and $\mathcal{L}^{\rm nf}(G, \tau(G^*))$ means the total latency for G induced by tolls that were designed for some (possibly different) game G^*. A taxation mechanism is said to be *strongly robust* to variations of a parameter if that taxation mechanism incentivizes optimal flows for all values of the parameter. That is, the strong robustness of τ implies that

$$\mathcal{L}^{\mathrm{nf}}(G, \tau(G^*)) = \mathcal{L}^*(G) \text{ for all } G, G^*. \tag{4}$$

This may be too strong a condition in some settings, so a taxation mechanism is said to be *weakly robust* to variations of a parameter if that mechanism never incentivizes Nash flows that are *worse* than the un-tolled flows given that parameter. That is, writing $\mathcal{L}^{\mathrm{nf}}(G, \emptyset)$ to denote the total latency of an un-tolled game, the weak robustness of τ implies that

$$\mathcal{L}^{\mathrm{nf}}(G, \tau(G^*)) \leq \mathcal{L}^{\mathrm{nf}}(G, \emptyset) \text{ for all } G, G^*. \tag{5}$$

Put differently, if tolls are weakly robust, they will not create perverse incentives. Thus, by the definitions presented in this section, assigning a toll of 0 to every link is always a weakly robust taxation mechanism, as it certainly cannot make Nash flows worse.

Homogeneous Toll-Response: Can Simplicity Confer Robustness?

Modeling the response of users to road tolls can seem a daunting challenge, so for simplicity, it is frequently assumed that all users respond to tolls in the same way; that is, all users trade off time and money equally when evaluating their possible routes through a network [35]. This is called the *homogeneous*-user model, with the cost experienced by any user on link e given by

$$J_e(f_e) = \ell_e(f_e) + \tau_e(f_e), \tag{6}$$

and the cost of path p given by $J_p(f) = \sum_{e \in p} J_e(f_e)$.

For the homogeneous model, given a set of edge tolls, a Nash flow f^{nf} is defined in the same way as before; i.e., f^{nf} is a flow in which no user can change routes and obtain a strictly lower latency. Again, paths with positive flow have equal costs [1]; i.e., for any Nash flow f^{nf},

$$\text{if } f_p^{\mathrm{nf}} > 0, f_{p'}^{\mathrm{nf}} > 0 \text{ for some paths } p,\, p' \text{ then } J_p\left(f^{\mathrm{nf}}\right) = J_{p'}\left(f^{\mathrm{nf}}\right). \tag{7}$$

Finally, the system-level costs are measured in the same way as before with the total latency:

$$L(f) = \sum_{e \in E} f_e \ell_e(f_e) = \sum_{p \in \mathcal{P}} f_p \ell_p(f).$$

Note that tolls are not included in the system cost – the system planner is still only attempting to minimize aggregate delay.

To see tolls in action, consider again Pigou's Example in Figure 2. Since the upper link is over-congested, it seems natural that charging the proper toll on that link would influence some users to deviate to the lower link. In the homogeneous model of user behavior, one set of edge tolls that enforces the optimal flow for Pigou's example is simply $\tau_1(f_1) = 0.5$, and $\tau_2(f_2) = 0$. Under these tolls, the optimal flow of $(1/2, 1/2)$ is a Nash flow, since in it all users experience a cost of 1.

Similarly, in Braess's Paradox (see Figure 3), one way to enforce optimal flows is to charge a toll of 1 on the center zero-cost link. This effectively removes the pathological link from the network, returning the network to its original optimal state as in Figure 3(a).

A Study on the Robustness of Fixed Tolls

For Pigou's Example and Braess's Paradox it is relatively straightforward to compute tolls that enforce optimal flows. These particular tolls are known as "fixed" tolls, since the tolling function on each edge is a constant function of edge flow. This fixed-tolling approach has been studied in general, and it is known that fixed tolls can be computed to enforce *any* feasible flow, provided that the system planner has a complete characterization of the system: network topology, user demand profile, and latency functions [33], [34]. In Figure 5(a), this fixed-toll approach is depicted as a block diagram; note in particular that tolls are computed by a single, centralized optimization problem that takes as inputs all system variables and outputs a list of edge tolls.

The following analysis is comprised of two examples which show that fixed tolls are strongly robust to neither network nor traffic-rate mischaracterizations. We then establish the following fact: fixed tolls cannot even be weakly robust to network changes unless they depend on some global information regarding network structure. That is, any fixed tolls that do not depend on network structure can be strictly harmful.

Fixed Tolls Cannot Be Strongly Robust

To begin, we ask if fixed tolls can ever be strongly robust to network changes, and show using a perturbed variant of Pigou's network that they cannot. In Pigou's network, a fixed toll of $1/2$ on the upper congestible link incentivized optimal routing. This fixed toll was computed for a network whose lower link latency function was given precisely by $\ell_2(f_2) = 1$; what if, instead of 1, the lower latency function was some unknown constant b? To be precise, let G_b represent Pigou's network with $\ell_2(f_2) = b$ so that G_1 represents the nominal Pigou network. To ascertain whether fixed tolls can be strongly robust on Pigou's network, consider the quantity $\mathcal{L}^{\rm nf}(G_b, \tau(G_1))$ from (4), which represents the total latency on perturbed network G_b resulting from tolls computed for nominal G_1. If fixed tolls $\tau(G_1)$ are strongly robust, then for each b, it will be true that $\mathcal{L}^{\mathrm{nf}}(G_b, \tau(G_1)) = \mathcal{L}^*(G_b)$. Thus, the robustness of $\tau(G_1)$ can be checked by varying b in the following price-of-anarchy-like expression:

$$\mathrm{PoA}(b) \triangleq \frac{\mathcal{L}^{\mathrm{nf}}\left(G_b, \tau(G_1)\right)}{\mathcal{L}^*(G_b)}. \tag{8}$$

In Figure 6, $\mathrm{PoA}(b)$ is plotted as b varies between 0 and 1.5 for two sets of tolls: the solid curve corresponds to tolls computed for $b = 1$, and the dashed curve corresponds to tolls equal to 0. Note that the fixed toll only incentivizes perfectly-efficient behavior exactly at $b = 1$ (that is, $\mathrm{PoA}(1) = 1$). For all other values, $\mathrm{PoA}(b) > 1$, which means that tolls lead to worse-than-optimal total latencies for $b \neq 1$, or $\mathcal{L}^{\mathrm{nf}}(G_b, \tau(G_1)) > \mathcal{L}^*(G_b)$, showing that these tolls are not strongly robust. Note that here we only checked the single fixed toll $\tau_1 = 1/2$, but it is easy to show that on this network, this toll is equivalent in all respects to any set of tolls for which $\tau_1 - \tau_2 = 1/2$, so assuming that $\tau_2 = 0$ is without loss of generality. Put differently, the optimal fixed tolls for G_1 are essentially unique. This implies that no fixed toll can be strongly robust to network variations, even on the simplified setting of Pigou's network.

Fixed Tolls and Rate-Dependence

We next investigate the robustness of fixed tolls along a different dimension: overall traffic rate, hoping to show that even if fixed tolls are not robust to network changes, perhaps they can be robust to variations of the traffic rate. To this end, we return to the Braess's Paradox network of Figure 3. Now, suppose that r, the total amount of traffic on the network, is not fixed at 1, but can take any value between 0 and 1. In the notation of (4), let G_r represent the Braess's Paradox network with r units of traffic, so the canonical version is simply given by

G_1. Let $\tau(G_1)$ simply be the toll design proposed earlier: a single fixed toll of 1 on the center zero-latency link. Note that unlike Pigou's example, this toll is far from unique – but since it incentivizes optimal flows on G_1, it is an adequate starting point.

As for Pigou, the robustness of $\tau(G_1)$ can be checked by varying r in the following price-of-anarchy-like expression:

$$\mathrm{PoA}(r) \triangleq \frac{\mathcal{L}^{\mathrm{nf}}\left(G_r, \tau(G_1)\right)}{\mathcal{L}^{*}(G_r)}. \tag{9}$$

I has already been shown that for $r = 1$, the optimal flow has no traffic on the center zero-cost link, so the proposed fixed toll achieves the goal of enforcing optimal flows. On the other hand, if $r \leq 0.5$, the optimal flow is to send *all* the traffic on the center zero-cost link, so that no traffic uses the constant-latency links. Unfortunately, the fixed toll on the center link is still boldly incentivizing all users to avoid the now-optimal center link. In Figure 7, the price of anarchy from (9) is plotted as a function of r with and without the fixed toll on the center link. Note that the tolled curve in Figure 7 increases rapidly as r approaches 0, quickly driving the price of anarchy above the un-tolled maximum of $4/3$; by setting r low enough, the price of anarchy in this instance can be made arbitrarily high.

Here, despite the unbounded price of anarchy, this does *not* immediately imply that fixed tolls are not weakly robust to rate changes, merely that *this particular toll* is not weakly robust. This example does demonstrate that great care must be taken with fixed tolls when the total traffic rate is varying or unknown, because fixed tolls designed for one demand profile can cause arbitrarily poor performance under a different demand profile.

Weakly Robust Fixed Tolls Must Depend on Network Structure

A fundamental problem with the fixed tolls applied to an uncertain Pigou network (as in Figure 6) was that the correct toll on the upper edge depended on the latency function of the lower edge; if the lower latency function was unknown, there was no way to compute an optimal toll on the upper edge, so fixed tolls could not be strongly robust. This prompts the question: could weakly robust fixed tolls be designed by letting the tolling function for edge e depend only on the local latency function ℓ_e? Such a taxation mechanism is called *network-agnostic*, prescribing tolling functions to edges without knowledge of how exactly the edges will be connected. Thus, a network-agnostic taxation mechanism τ^{na} is simply a mapping from latency functions to tolling functions so the toll on edge e is given by $\tau^{\mathrm{na}}(\ell_e)$. If a taxation mechanism is network-agnostic, then each edge toll depends only on local information, so any efficiency guarantees are automatically robust to changes in network structure.

Considering network-agnostic tolls, the robustness notion of "designing tolls for one network and applying them to another" must be slightly refined: as networks are varied, we assume that the tolling function τ_e "knows" the local latency function ℓ_e, but has no information about the location of e in the network or about the other latency functions in the network. Here, a toll-designer essentially pre-commits to a toll for each possible latency function without specific knowledge of which latency functions will appear in the final realization. Thus, the notation of (5) can be simplified by writing $\mathcal{L}^{\mathrm{nf}}(G,\tau)$ to mean the total latency of a Nash flow on network G resulting from the tolls generated by taxation mechanism τ, and $\mathcal{L}^{\mathrm{nf}}(G,\emptyset)$ to mean the total latency of a Nash flow on G with no tolls. This leads to the main result about the lack of robustness of fixed tolls:

Theorem 1: The only nonnegative network-agnostic fixed tolls that are weakly robust to network variations satisfy

$$\tau_e = 0 \tag{10}$$

for all possible network edges.

Theorem 1 shows that positive fixed tolls must in general require some information about network structure in order to ensure that they do not cause harm, even for the simple setting of homogeneous populations. Theorem 1 is proved with a series of simple example networks.

Proof: Let τ^{naft} be a network-agnostic taxation mechanism such that for any network G, $\mathcal{L}^{\text{nf}}(G, \tau^{\text{naft}}) \leq \mathcal{L}^{\text{nf}}(G, \emptyset)$. Write the toll assigned to an edge with latency function ℓ as $\tau^{\text{naft}}(\ell)$.

First, consider the network shown in Figure 8(a) in which the latency functions satisfy $\ell_1 + \ell_2 = \ell_3$. The flow $(1/2, 1/2)$ is both a Nash flow and an optimal flow; the only tolls which will always support this must satisfy $\tau^{\text{naft}}(\ell_1) + \tau^{\text{naft}}(\ell_2) = \tau^{\text{naft}}(\ell_3)$. This is the first condition on τ^{naft}:

$$\ell_1 + \ell_2 = \ell_3 \quad \Longrightarrow \quad \tau^{\text{naft}}(\ell_1) + \tau^{\text{naft}}(\ell_2) = \tau^{\text{naft}}(\ell_3). \tag{11}$$

Next, consider the network shown in Figure 8(b), a two-link parallel network with degree-d monomial cost functions $\ell_1(f_1) = \alpha(f_1)^d$ and $\ell_2(f_2) = \beta(f_2)^d$. It can be shown that the optimal flow on this network is equal to the untolled Nash flow for any $\alpha > 0, \beta > 0$, and $d \geq 1$. Thus, the tolls on each link must be equal; otherwise, the tolled flow will have a strictly higher total latency than the un-tolled flow. That is, all monomials of the same degree must be charged the same toll, regardless of the scale of the latency function: $\tau^{\text{naft}}\left(\alpha f^d\right) = \tau^{\text{naft}}\left(\beta f^d\right)$. In particular, letting $\beta = 2\alpha$ and appealing to (11), it holds that that $2\tau^{\text{naft}}\left(\alpha f^d\right) = \tau^{\text{naft}}\left(2\alpha f^d\right)$, which implies the second condition on τ^{naft}:

$$\text{for all } \alpha > 0 \text{ and } d \geq 1, \quad \tau^{\text{naft}}\left(\alpha f^d\right) = 0. \tag{12}$$

Using this fact regarding polynomials, positive fixed tolls on any other latency function can now be ruled out. Refer to Figure 8(c), another two-link parallel network. Given an arbitrary convex latency function ℓ (with derivative ℓ') on the upper link, it is possible to design a polynomial latency function for the lower link $\ell_2(f_2) = \alpha(f_2)^d$ to show that $\tau^{\text{naft}}(\ell) = 0$. Let the polynomial degree satisfy $d > \frac{\ell(1/2)}{2\ell'(1/2)}$, and coefficient satisfy $\alpha = 2^d\ell(1/2)$. Then by design, the un-tolled Nash flow on the network is $(1/2, 1/2)$, and at this flow, shifting any positive mass of traffic from the upper link to the lower link strictly increases the total latency on the network. To avoid this, the toll on the upper link must be zero: $\tau^{\text{naft}}(\ell) = 0$. Since ℓ is an arbitrary convex latency function, the theorem is proved. ■

Marginal-Cost Tolls: Strongly Robust Network-Agnostic Tolls

In view of the results of Theorem 1, one might ask if *any* network-agnostic taxation mechanism can be even weakly robust. Fortunately, if tolling functions are allowed to be flow-varying, such a mechanism exists in so-called *marginal-cost* tolls. To motivate the concept of marginal-cost tolls, note that in traffic routing, an agent's total cost can be viewed as being two-fold: the first component is the agent's own experienced delay, the second is the delay that the agent's presence imposes on others. A marginal-cost toll explicitly charges each agent for his imposition on other agents; in economic language, marginal-cost tolls internalize the agent's negative externalities [20], [35].

Marginal-cost tolls are computed by the formula

$$\tau_e^{\mathrm{mc}}(f_e) = f_e \cdot \frac{d}{df_e}\ell_e(f_e). \tag{13}$$

Note that each edge's toll depends only on that local edge's congestion properties and traffic flow; global information is not used. It is well-known that for homogeneous user populations, charging marginal-cost tolls enforces exactly-optimal network flows [20], [35]. That is, rephrased in the language of robustness, we have the following fact:

Theorem 2 (Beckmann, McGuire, Winsten, 1956 [20]): Marginal-cost tolls are strongly robust to variations of network structure, overall traffic rate, and driver demand structure.

This can be seen for Pigou's and Braess's networks in Figures 2 and 3, where marginal-cost tolls prescribe tolling functions of $\tau(f) = f$ for each of the linear-cost edges. For homogeneous users, the linear-cost edges have a resulting effective cost of $2f$, which enforces the desired optimal flows.

Furthermore, marginal-cost tolls incentivize optimal flows without requiring knowledge of the overall traffic rate. In particular, note that in Braess's Paradox, marginal-cost tolls do not prevent all drivers from using the center link when the overall rate is less than $1/2$. See Figure 5(b) for a depiction of the informational dependencies and localized optimization structure of marginal-cost tolls in comparison with that of the fixed-toll methods of [33], [34].

Open Questions for Marginal-Cost Tolls With Unknown Latency Functions

Marginal-cost tolls do not require knowledge of the other latency functions in the network, but note that the tolling function on edge e strongly depends on the specific latency function ℓ_e, as can be seen from (13). It remains an open question how mischaracterizations of local latency functions impact the congestion-minimizing guarantees of marginal-cost tolls. One possible approach for studying this could be to restrict attention to linear latency functions of the form $\ell_e(f_e) = a_e f_e + b_e$, and assume that the true coefficients a_e and b_e were only known to exist in some range $[\bar{a} - \delta, \bar{a} + \delta]$ and $[\bar{b} - \delta, \bar{b} + \delta]$ for some nominal $\bar{a}$ and $\bar{b}$. It seems reasonable to conjecture that if $\delta > 0$, marginal-cost tolls cease to be strongly robust, but it would be interesting to ask in what settings they remain weakly robust.

Paying for Optimality: Network-Agnostic Fixed Subsidies

Theorem 1 (regarding the lack of weak robustness of fixed tolls) carefully specified that the fixed tolls in question be nonnegative – and this prompts the question of negative tolls, i.e., subsidies. It turns out that for the special case of linear-latency networks, strongly robust network-agnostic fixed *subsidies* do exist. For a linear latency function of the form $\ell_e(f_e) = a_e f_e + b_e$, the corresponding network-agnostic fixed subsidy (represented as a negative toll) is given by

$$\tau_e^{\text{subsidy}} = -\frac{b_e}{2}. \tag{14}$$

By simply paying users half the constant-term cost on each link, optimal flows can be incentivized as Nash flows. As a side note, these subsidies are a special case of the "variable price schemes" of [35] with $\bar{\eta} = 1$.

To show the optimality of these subsidies, the cost functions resulting from these subsidies can be related to the cost functions resulting from standard marginal-cost tolls τ^{mc} (see (13)), given in this case by

$$\tau_e^{\text{mc}} = a_e f_e. \tag{15}$$

Recall that for homogeneous users, marginal-cost tolls are known to induce optimal Nash flows.

Under the homogeneous-user model, the cost functions resulting from the network agnostic fixed subsidies are

$$J_e^{\text{subsidy}}(f_e) = \underbrace{a_e f_e + b_e}_{\ell_e} - \underbrace{\frac{b_e}{2}}_{\tau_e} = a_e f_e + \frac{b_e}{2}, \tag{16}$$

while the cost functions resulting from marginal-cost tolls are

$$J_e^{\text{mc}}(f_e) = 2a_e f_e + b_e. \tag{17}$$

Since these two cost functions are related by a constant multiplicative factor for all agents (i.e., $J^{\text{mc}} = 2J^{\text{subsidy}}$), they induce the same optimal Nash flows, and these subsidies inherit the strong robustness of marginal-cost tolls. While this these strongly robust network-agnostic fixed subsidies are theoretically appealing, it is not clear that the concept generalizes beyond linear cost functions.

Model Variations and Alternative Mechanisms

Elastic Traffic

Up to this point, we have considered only the case of inelastic traffic; that is, all drivers have been required to arrive at their destination, regardless of cost. A more realistic model may consider the elastic demand case, recognizing that each user has a cost threshold; if the cost exceeds this threshold, he will simply decide to "stay home." It turns out that marginal-cost tolls preserve their robustness characteristics in the elastic model.

Under an elastic demand model, the system-planner's goal is to maximize a linear combination of aggregate travel benefit minus congestion costs; this objective is called *social welfare* maximization. For simplicity, consider the case of parallel networks. The elasticity of a user population is modeled with a continuous nonincreasing value-of-travel function $v(x)$; the utility of user x on edge e is given by

$$U_e(x) = v(x) - (\ell_e(f_e) + \tau_e(f_e)) \,. \tag{18}$$

If this quantity is negative for all edges in the network, user x will choose to stay home, as the total cost of commuting is higher than the user's value-of-travel. In economic language, $v(x)$ is known as an *inverse demand curve*. It is easily shown that in equilibrium, all users using the network have higher values than the users who stay home; write x^* to denote the lowest-value user who chooses to travel. The planner's objective, the social welfare W, is defined by the following:

$$W(x^*, f) = \int_0^{x^*} v(t)dt - L(f), \tag{19}$$

simply the difference between the total value obtained by users who choose to commute and the total latency of the resulting flow.

In [35], the author shows that in this elastic case, marginal-cost tolls maximize the social welfare for any valuation profile v without requiring that the system planner know anything about the specific valuation profile. In the language of robustness, a theorem from [35] can be rephrased as

Theorem 3 (Sandholm, 2002 [35]): Marginal-cost tolls are strongly robust to variations in user elasticity profiles v.

Thus, in addition to being robust to all the parameters previously discussed, marginal-cost toll are *also* robust to populations with unknown elasticity.

Budget-Balanced Tolls

Recent work has investigated the application of budget-balanced tolls to a network routing problem, tolls for which the planner has no net revenue but rather returns all tolls collected on some links as subsidies on the remaining links. This is an attractive approach when it is desirable to include the tolls paid as part of the system cost, since the net tolls paid will always be zero. Budget-balance is a condition which could lead to hopelessly complicated tolling functions for large networks if not properly computed: since everything paid on one edge must be returned to users on other edges, in principle the toll charged on one edge could actually be a function of the flows on *all* other edges. Fortunately, [36] provides an algorithm for computing budget-balanced tolls for any network which enforce optimal flows for homogeneous populations, and each edge's toll is a function of only that edge's flow.

For Pigou's network (refer to Figure 2), one set of tolls which enforces optimal flows is given by $\tau_1(f_1) = 1/4$ and $\tau_2(f_2) = (1 - 1/f_2)/4$. To see that these are in fact budget-balanced, it is necessary to distinguish between the *toll charged* on an edge and the *revenue collected* from that edge: the revenue collected is given by the toll multiplied by the flow. Combining this with the fact that $f_2 = 1 - f_1$, it is simple to verify that the edge revenues are equal and opposite, or $f_1 \cdot \tau_1(f_1) = -f_2 \cdot \tau_2(f_2)$, for all feasible flows. At the optimal flow of $(1/2, 1/2)$, all users on edge 1 pay a toll of $1/4$, and all users on edge 2 receive a subsidy of $1/4$, which the reader may verify constitutes a Nash flow.

The informational dependencies of these tolls are quite complex, requiring the designer to have quite as much information as in the design of optimal fixed tolls (i.e., network topology, demand profile, and latency functions). Characterizing the robustness of budget-balanced tolls is an open problem; later in this article we consider one simple approach to it.

Centralized Adaptive Tolls for Unknown Latency Functions

The foregoing discussion has assumed that the system-planner knows the edge latency functions perfectly, regardless of other uncertainties. What if the opposite were true, and a system-planner were given a network and demand profile, but did not have any information regarding the latency functions? This situation is considered in [37]. The authors assume that the population is homogeneous, with unit-sensitivity to tolls, that the demand profile is constant, and that the system-planner has a desired target flow that he wishes to enforce. They develop an iterative algorithm whereby a system planner levies tolls on the network, records the resulting Nash flow, updates the tolls accordingly, and so on – and it is proved that Nash flows resulting from this sequence of tolls converge to the planner's target flow. Interesting robustness questions in this context could be posed in a dynamic framework: does this algorithm converge quickly enough to ensure efficient behavior if the underlying system is changing rapidly?

Summary and Open Questions

Fixed Tolls

Using Pigou's example, it was shown that fixed tolls can never be strongly robust to network variations; using Braess's Paradox, it was suggested that fixed tolls are not strongly robust to rate variations either. Subsequently, Theorem 1 showed that fixed tolls cannot even be weakly robust if they do not depend on global network structure information. Given the fact that fixed tolls perform quite well when given perfect information, it would be interesting to investigate the relationship between information and robustness. One might ask "how much information is necessary to ensure the weak robustness of fixed tolls?" For example, are fixed tolls weakly robust if

1) Networks are known to be parallel?
2) All of the latency functions (but not their locations) in a network are known?
3) Network topology is known, but latency functions are not?

Marginal-Cost Tolls

Prior work was presented which proved that in the homogeneous-user model, marginal-cost tolls are strongly robust to variations of network structure, traffic rate, demand structure, and user elasticity profile (for elastic-demand models). A natural open question here is that of uncertainties in latency functions, which are assumed to be locally-known when computing marginal-cost tolls.

Heterogeneous Users

The foregoing analysis appears to crown marginal-cost tolls as the robustness winner for the homogeneous-user model. Can its robustness properties carry over to more complex models of user behavior? The second model of user behavior assumes that users are *heterogeneous* in their price-sensitivities, with different users valuing their time differently. To model this, the user population is represented by the interval $[0, r]$ and a *sensitivity* function $s : [0, r] \to \mathbb{R}_+$ that assigns a sensitivity value to each user. Given a flow f, cost experienced by user $x \in [0, r]$ using path $\tilde{p} \in \mathcal{P}$ as

$$J_x(f) = \sum_{e \in \tilde{p}} \left[\ell_e(f_e) + s_x \tau_e(f_e)\right]. \tag{20}$$

Intuitively, a user with a large s_x value is willing to use a high-latency route to avoid tolls, while a user with a low s_x value is willing to pay a high toll to use a low-latency route. The heterogeneous game specification now includes the population's sensitivity function as well as other parameters: $G = \{V, E, \mathcal{P}, \{\ell_e\}_{e \in E}, r, s\}$.

For the heterogeneous model, given a set of edge tolls, a Nash flow f^{nf} is defined in a similar way as before, with the exception that now the Nash condition must be checked for each user. That is, f^{nf} is a flow in which each user is individually choosing the lowest-cost route available, given the choices of other users. Formally, for each user $x \in [0, r]$,

$$J_x\left(f^{\mathrm{nf}}\right) = \min_{p \in \mathcal{P}} \left\{ \sum_{e \in p} \left[\ell_e\left(f_e^{\mathrm{nf}}\right) + s_x \tau_e\left(f_e^{\mathrm{nf}}\right)\right] \right\}. \tag{21}$$

Finally, the system-level costs are measured in the same way as before with the total latency:

$$L(f) = \sum_{e \in E} f_e \ell_e(f_e) = \sum_{p \in \mathcal{P}} f_p \ell_p(f).$$

Note that tolls are not included in the system cost – the system planner is still only attempting to minimize aggregate delay.

Returning to Pigou's example, it is clear that even if users have individual price-senitivities, an optimal fixed toll can still be found by charging exactly the price that would cause the most-sensitive half of the users to deviate to the lower link. Note also that since homogeneous populations are a subclass of heterogeneous populations, the negative results regarding fixed tolls immediately carry over to the heterogeneous case.

Marginal-Cost Taxes for Unknown-Sensitivity Homogeneous Populations

As a first step towards studying robust taxation mechanisms for heterogeneous users, the performance of "off-the-shelf" marginal-cost tolls is investigated on simple price-sensitive settings.

In Pigou's network, marginal-cost tolls assign a flow-varying toll of $\tau_1^{\mathrm{mc}}(f_1) = f_1$ to the upper link; for homogeneous users, this incentivized optimal routing. What if the users's sensitivities remained homogeneous, but took on some value other than 1? We can employ a parallel argument to that seen for fixed tolls in Figure 6: Write G_s to denote Pigou's network in which all users had sensitivity s. To ascertain whether marginal-cost tolls can be strongly robust to sensitivity variations, consider the quantity $\mathcal{L}^{\mathrm{nf}}(G_s, \tau^{\mathrm{mc}}(G_1))$ from (4), which represents the total latency on the perturbed population in G_s resulting from marginal-cost tolls computed for unit-sensitivity G_1. If marginal-cost tolls $\tau^{\mathrm{mc}}(G_1)$ are strongly robust, then for each s, it will be true that $\mathcal{L}^{\mathrm{nf}}(G_s, \tau^{\mathrm{mc}}(G_1)) = \mathcal{L}^*(G_s)$. Since the optimal flow on a network does not depend on the user sensitivities, this is simply equivalent to writing $\mathcal{L}^{\mathrm{nf}}(G_s, \tau^{\mathrm{mc}}(G_1)) = 0.75$.

In Figure 9, $\mathcal{L}^{\mathrm{nf}}(G_s, \tau^{\mathrm{mc}}(G_1))$ is plotted as s varies between 0 and 2. Note that the marginal-cost toll only incentivizes perfectly-efficient behavior exactly at $s = 1$. For all other values, the total latency is strictly greater than the optimal 0.75, showing that these tolls are not strongly robust. This implies that when considering user price-sensitivity, marginal-cost tolls are not strongly robust to network variations, even on the simplified setting of Pigou's network.

Perverse Marginal-Cost Tolls For Heterogeneous Users

Here, we ask a similar question of marginal-cost tolls in the heterogeneous model to that asked for fixed tolls in the homogeneous model: Despite lacking strong robustness to user sensitivities, is it at least possible to show that marginal-cost tolls are weakly robust? Unfortunately, in general settings, it is possible to show that even marginal-cost tolls can incentivize flows that are strictly worse than their un-tolled counterparts, thus lacking even weak robustness to heterogeneity. Consider the network and demand profile in Figure 10. This is essentially a three-link network; a population of mass 0.5 has access only to the upper two links, and a population of mass 1 has access only to the lower two links. The optimal flow has all users from the upper population using the center (congestible) link and all users from the lower population using the lower link. In the unique un-tolled Nash flow, half the traffic from the lower link has shifted to the center link so as to equalize the latencies of those two links.

Note that the un-tolled version bears a strong resemblance to Pigou's example if the upper link is ignored: self-interested users from the lower source have over-congested the center link and degraded its performance. To see how tolls can make things even worse, suppose that the upper population is toll-sensitive (with sensitivity factor 1), but the lower population is not (i.e., they have sensitivity factor close to 0). The insensitivity of the lower population effectively fixes the flow on the center link at 1, since this is the flow that equalizes the latencies of the lower two links. That is, regardless of what the upper population does, the center link flow will always be 1. Thus, marginal-cost tolls on this network serve only to force the upper population to choose the upper link, resulting in the flow depicted on the right in Figure 10. This pathological flow has a total latency of 1.75, which corresponds to a price of anarchy of 1.4. This is greater than the $4/3$ guaranteed as a worst-case on linear-latency networks, demonstrating that marginal-cost tolls lack even weak robustness for heterogeneous populations. In this case, instead of incentivizing altruism, marginal-cost tolls simply amplified the selfishness-induced inefficiency that was already present.

In this example, the most extreme pathology arises when the lower population has a sensitivity of 0, but perversities still arise for any low, positive sensitivity. That is, the poor performance in this example can still occur when every agent has a strictly positive price-sensitivity.

More-Sophisticated Taxation Mechanisms

The examples in the previous section demonstrate some of the diverse challenges in social coordination in uncertain environments, and they may leave the reader with the unsettling question "can we guarantee *anything* without knowing *everything?*" As the previous examples have shown, the answers to this question are complex and nuanced; ultimately, what is needed is a characterization of the fundamental relationship between the amount of information a taxation methodology requires, the sophistication of the mechanism, and the efficiency guarantees it can provide under uncertainty. Fixed tolls assign relatively "unsophisticated" tolling functions (being merely constant functions of flow), but appear to require a great deal of information to enforce optimal flows. Thus, they apparently sacrifice robustness for simplicity.

Marginal-cost tolls enforce optimal flows with a greater degree of robustness, doing so by allowing more-sophisticated flow-varying tolling functions. However, it is clear from the example of Figure 10 that they are not so sophisticated that they can prevent all pathologies from arising when agents are heterogeneous.

Fortunately, recent research has investigated methodologies which employ higher-complexity tolling functions to defeat some of the pathologies introduced by heterogeneity. Here, we survey three of these emerging methods: Large Universal Tolls, Discriminatory Pricing, and Budget-Balanced tolls. We discuss and provide simulations of each.

Large Universal Tolls

An issue in the pathological example in Figure 10 was that large sensitivity differences between agents led to large discrepancies in their reactions to tolls; a robust tolling methodology would have to find a way to reduce the effective inequity in users' sensitivities. It turns out that there is a natural way to do this, as shown in [5], where the authors prove the following theorem:

Theorem 4 (Brown and Marden, 2015): Consider any network G. If for some $S_{\mathrm{L}} > 0$ the toll-sensitivities of every agent x satisfy $s_x \geq S_{\mathrm{L}}$, and the tolling function on each edge is given by

$$\tau_e^{\mathrm{u}}(f_e) = \kappa \left(\ell_e(f_e) + f_e \cdot \frac{d}{df_e} \ell_e(f_e) \right), \tag{22}$$

then it will be true that

$$\lim_{\kappa \to \infty} \mathcal{L}^{\mathrm{nf}} \left(G, \tau^{\mathrm{u}} \right) = \mathcal{L}^*(G). \tag{23}$$

Note that the tolling functions in (22) are network-agnostic in that they do not depend on any instance-specific information other than latency functions; tolling functions can be designed with no knowledge of network topology, demand profile or distribution of user sensitivities, and computed locally at each edge using only local information. Thus, this shows that there is no network so pathological that nothing can be done about it with instance-agnostic tolls. In the language of robustness, (23) shows that these Universal Tolls are at least weakly robust to heterogeneous sensitivities, and in the large-κ limit, they get arbitrarily “close” to strong robustness.

Figure 11 shows these tolls applied to the pathological 3-link network surveyed earlier. The horizontal axis corresponds to the value of the κ factor from (22), and the vertical axis corresponds to the price of anarchy (in worst-case over sensitivity distributions) resulting from a particular value of κ. The three curves in this figure represent three different levels of precision with which the user population has been characterized. For the highest curve, all user sensitivities lie in somewhere in the interval $[0.01, 100]$; that is, the population is only characterized to within four orders of magnitude. Due to this uncertainty, it takes much larger tolls to guarantee efficient behavior for the poorly-characterized population than for a well-characterized one, e.g., a population with sensitivities in $[1, 2]$, represented by the lower curve in the figure.

Price-Discrimination and Robustness

As has been seen, problems can arise for marginal-cost tolls when users are highly differentiated in sensitivity, and the effect of the universal tolls of Theorem 4 is to obscure the differences between users. However, a conceptually-simple way of accomplishing the same thing would be for the tolls to depend on the identity and sensitivity of the payer. Suppose that each user x (with corresponding sensitivity s_x) is charged the following toll:

$$\tau_{e,x}(f_e) = \frac{1}{s_x} f_e \ell_e'(f_e). \tag{24}$$

Provided that no user has a sensitivity of exactly 0, substituting (24) into the general user cost function (20) yields an edge cost function for every user of

$$J_{e,x}(f_e) = \ell_e(f_e) + s_x \cdot \left(\frac{1}{s_x} f_e \ell_e'(f_e) \right) = \ell_e(f_e) + f_e \ell_e'(f_e), \tag{25}$$

which is precisely the cost function induced for a unit-sensitivity homogeneous population by marginal-cost tolls. Since marginal-cost tolls induce optimal Nash flows for homogeneous populations, these tolls would as well. This type of toll applies a principle known by economists as *perfect price-discrimination*, sometimes also called 1st-degree price discrimination. Perfect price-discrimination sidesteps the issue of individual price-sensitivities by charging each agent precisely the "correct" price for that agent. Of course, putting such a scheme into practice would be a formidable task, since it would require not only that the tax-designer *be able* to charge each agent an individualized price, but furthermore *know* the correct price for each agent. Nonetheless, it suggests an intriguing possibility: what if the tax-designer could price-discriminate coarsely? Rather than charge every user an individual price, what if the tax-designer could group users into several "bins" on the basis of their price-sensitivities, and charge a unique price to each bin?

The authors of [38] consider this possibility, and suggest that such a scheme could indeed yield significant efficiency gains. A procedure is proposed by which effective discriminatory pricing can be derived from simple pricing; this procedure works as follows: Suppose that it is possible to subdivide the the sensitivity interval $[S_{\rm L}, S_{\rm U}]$ into m individual bins $[S_{\rm L}, B_1]$, $[B_1, B_2]$, up through $[B_{m-1}, S_{\rm U}]$ and charge all users whose sensitivity lies in bin i a unique toll τ_i. How should the bin boundaries B_i be chosen, and what should the bin taxes be? In general, the answers to these questions depends on the specific form of taxation function, but in [38] is it proved that one way in particular works for *any* taxation functions and always weakly improves the worst-case routing efficiency. One simple robustness implication here is that if a weakly-robust simple pricing scheme is used with this procedure to choose discriminatory prices, those discriminatory prices will also be weakly robust.

Are Budget-Balanced Tolls Robust?

Budget-balanced tolls were introduced earlier in this article which collect no net revenue, instead returning all tolls collected on some network links as subsidies on other links. To date, we are not aware of any comprehensive study of the robustness of budget-balanced tolls in any setting, so here are presented a set of simulations of the budget-balanced tolls of [36] which suggest the possibility that a budget-balance constraint may yield significant robustness improvements.

The algorithm from [36] for optimal budget-balanced tolls prescribes a family of tolls for Pigou's Example (refer to Figure 2 for this example) parameterized by a number $q \geq 0$:

$$\tau_1(f_1) = q\,(2 - 1/f_1) + 1/4 \tag{26}$$

$$\tau_2(f_2) = q\,(2 - 1/f_2) + 1/4\,(1 - 1/f_2)\,. \tag{27}$$

The q parameter represents the degree to which the upper-link-toll is flow-varying. The simplest case has $q = 0$ (so that the upper link toll is simply $1/4$), but any nonnegative value induces optimal flows for homogeneous unit-sensitivity populations. As a preliminary investigation into the robustness of these tolls to mischaracterizations of user sensitivity, Nash flows are computed as a function of homogeneous sensitivity for various values of q. This plot is shown in Figure 12; naturally, when the sensitivity parameter is equal to 1, the price of anarchy is equal to 1, since this is the sensitivity for which the tolls were designed. Note that for $q = 1$, the price of anarchy is within 1% of optimal for a large range of sensitivities. Furthermore, larger values of q yield a lower price of anarchy for all sensitivity values plotted, suggesting that strongly flow-varying tolls may play an important role in sensitivity-robustness.

For a sense of why a budget-balance constraint in this particular case may confer sensitivity-robustness, consider carefully the tolling functions in (26) and (27). Note that near the optimal flow of $(1/2, 1/2)$, both tolling functions are rather gently flow-varying, but that near either extreme point of $(0, 1)$ or $(1, 0)$, users on the low-flow link are being paid extremely large subsidies. This is a simple consequence of a budget-balance constraint on a two-link network: a large number of people paying a small toll on one road implies a small number of people receiving a large subsidy on the other road. From a robustness standpoint, this effect serves to buffer the damage that extremely-high or extremely-low sensitivity agents can cause. Of course, this is far from a comprehensive study on the robustness of budget-balanced tolls, and the apparent robustness seen here could simply be a artifact of the simplified setting of Pigou's network.

Optimal Weakly Robust Tolls for Designers With More Information

The Large Universal and Discriminatory approaches both assumed that the toll designer knew very little about the design environment; what if more information were available? For example, if the designer knows that the networks under consideration are parallel networks with affine latency functions, this information can be exploited to improve the robustness of taxation methodologies [39]. Considering this simplified setting, it turns out that a toll designer can guarantee considerably improved performance as compared with some of the foregoing complex-toll results.

Theorem 4 proves that near-optimality can be achieved with large tolls, but it gives no indication as to how high tolls must be. Indeed, when the universal tolls were applied to the pathological network in Figure 11, the worst of the three curves does not drop below $4/3$ until the tolls are above 6; this means that the most-sensitive users (those with a sensitivity value of 100) may be experiencing costs roughly 600 times what they would be without tolls. Aside from such "fairness" issues, there may be settings where this high-toll approach would not be possible for technical or political reasons, either of which could impose an upper limit on allowable tolling functions.

This raises the following question: if a toll-designer cannot charge more on any link than some upper bound T, what are the optimal tolling functions, and what can be said about their robustness? For parallel networks (that is, networks consisting of only two nodes) with linear-affine latency functions in which all users have access to all links, the answer is found in [5]. The basic approach is this: in affine-latency networks, every edge has a latency function of the form

$$\ell_e(f_e) = a_e f_e + b_e, \tag{28}$$

where a_e and b_e are nonnegative edge-specific constants. In [5], the authors prove that for any population and upper bound T, the optimal instance-agnostic tolls are always of the form

$$\tau_e(f_e) = \kappa_1 a_e f_e + \kappa_2 b_e, \tag{29}$$

where κ_1 and κ_2 are related by the formula

$$\kappa_2 = \max\left\{0, \frac{\kappa_1^2 S_{\mathrm{L}} S_{\mathrm{U}} - 1}{S_{\mathrm{L}} + S_{\mathrm{U}} + 2\kappa_1 S_{\mathrm{L}} S_{\mathrm{U}}}\right\}. \tag{30}$$

For bounded tolls, the approach is to set κ_1 and κ_2 as high as possible with respect to the upper bound.

Theorem 5 (Brown and Marden, 2015 [5]): For parallel-network, linear-latency routing games in which all users have access to all network edges, tolls as defined in (29) and (30) are weakly robust to variations of heterogeneous user sensitivities.

This shows that the universal tolls of Theorem 4 are in fact *not* the best a toll-designer can do. To compare, Figure 13 contains plots of the price of anarchy on Pigou's network resulting from Theorem 4's universal tolls and Theorem 5's optimal affine tolls, and shows that the latter approach yields considerably better performance than the universal approach. This is a vivid illustration of the value of additional information; if a designer knows the network to be parallel with affine cost functions, he can use the tolls of Theorem 5 and achieve considerable performance gains compared with the instance-agnostic universal tolls.

Furthermore, the Universal Tolls of Theorem 4 are only weakly robust for large values of κ, but in the optimal affine approach of Theorem 5, the additional information regarding the class of networks confers weak robustness to *all* affine tolls computed according to (29) and (30).

Elasticity and Robustness

This final study investigates the robustness of marginal-cost tolls to variations of a user population's elasticity characteristics. For the majority of this article, it has been assumed that traffic is inelastic; that is, drivers have no option to simply "stay home" if travel costs are too high. In general, this may be an unrealistic expectation, particularly in the results of Theorems 4 and 5 which prescribe large tolls. In this section, we initiate a study on the robustness of tolls for price-sensitive elastic populations.

Recall that in an elastic demand model, the social planner desires to maximize the social welfare as defined in (19); that is, maximize the total value obtained by commuting less the total cost incurred by congestion. To investigate the robustness of marginal-cost tolls to variations in elasticity for heterogeneous populations, we adopt the simple setting of Pigou's network (see Figure 2) and study the performance of a variety of scaled marginal-cost tolls for a specific family of inverse demand curves. The inverse demand curves in question satisfy

$$v(x) = \bar{v} - x. \tag{31}$$

Tolls of κf_1 are applied to the congestible link for various values of $\kappa \in [0, 1]$. For each $\bar{v} \in [1, 2]$ and $\kappa \in [0, 1]$, the worst-case social welfare $W^{\max}(\bar{v}, \kappa)$ is found by searching over the space of (possibly heterogeneous) populations with price sensitivities between $S_{\mathrm{L}} = 0.1$ and $S_{\mathrm{U}} = 10$. This worst-case social welfare is then normalized by the optimal social welfare for the corresponding $\bar{v}$, and plotted in Figure 14.

In Figure 14, the blue dots correspond to the optimal choice of κ for any $\bar{v}$. Note that the optimal κ depends on $\bar{v}$, and that furthermore, no combination of κ and $\bar{v}$ achieves the optimal social welfare (that is, a normalized social welfare of 1); this implies that scaled marginal-cost tolls are not strongly robust to mischaracterizations of elasticity in a heterogeneous-user model. However, the question of weak robustness is still open; this study was for a very simple class of networks and elasticity profiles and it may be that increasingly complex settings will further demonstrate that these tolls are not even weakly robust.

Summary and Open Questions for the Heterogeneous Model

With Pigou's example, the strong robustness of marginal-cost tolls for price-sensitive users was disproved; on the new example depicted in Figure 10, it was shown that marginal-cost tolls cannot even be weakly robust to heterogeneous price sensitivities in general. Following this, several emerging taxation methodologies were surveyed and their robustness properties discussed. Large Universal Tolls can be made weakly robust for any game simply by setting a parameter κ large enough, and simulations were performed for budget-balanced tolls suggesting possible weak robustness to price-sensitivity uncertainty.

It was shown that if a toll-designer has access to additional information about the system to be influenced, additional robustness may be attainable. In particular, parallel networks allow for robustness that has not been proved for general networks. Finally, simulations were performed for price-sensitive elastic-population models which showed that marginal-cost tolls lose their strong robustness to demand elasticity when user populations are price-sensitive.

In the heterogeneous setting, there are many open questions; the most fundamental of these seek to understand the fundamental connections between the robustness of a taxation mechanism and the information that mechanism has access to. In some settings, simply giving the taxation mechanism information regarding the *class* of considered networks (see Figure 11) conferred enormous robustness benefits; further research should endeavor to find the "best" information to give the mechanism. Furthermore, this article has barely scratched the surface in studying the robustness of budget-balanced tolls; the apparent robustness showcased in this setting in Figure 12 may warrant further research.

Conclusion

This article has presented a variety of recent and new results showcasing the emerging research area of robust social influence applied to the routing game setting. We have endeavored to demonstrate that robustness depends greatly on the particular style of taxation mechanism under consideration: for example, the examples show that perverse incentives can be substantially easier to avoid if a system planner employs flow-varying tolls rather than the simpler, less-robust fixed-tolls. We demonstrated the importance of properly characterizing the informational dependencies of taxation mechanism through examples showing that misunderstanding these dependencies can have severe consequences.

Beyond routing games, this style of analysis has applications in many areas of engineering design; anywhere that social systems mesh with engineering systems seems fertile ground for further study on robust social influence.

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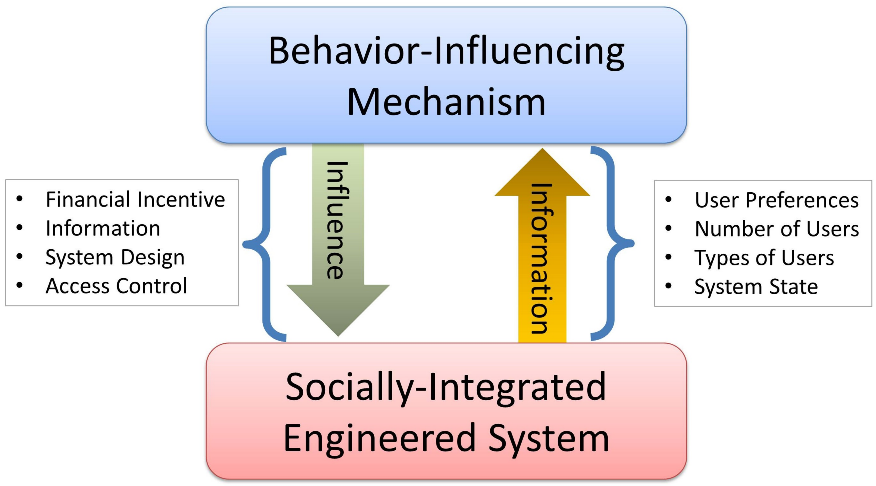


Figure 1. Relationship between social system and behavior-influencing mechanism. In order to exert influence (whether via financial incentives, access control, system design, or providing specific information to users), a behavior-influencing mechanism typically requires some information about the social system to be influenced. Recent research has investigated the relationship between the quality of this information and the resulting effectiveness of the influence mechanism. This article highlights recent efforts to characterize this informational dependence.

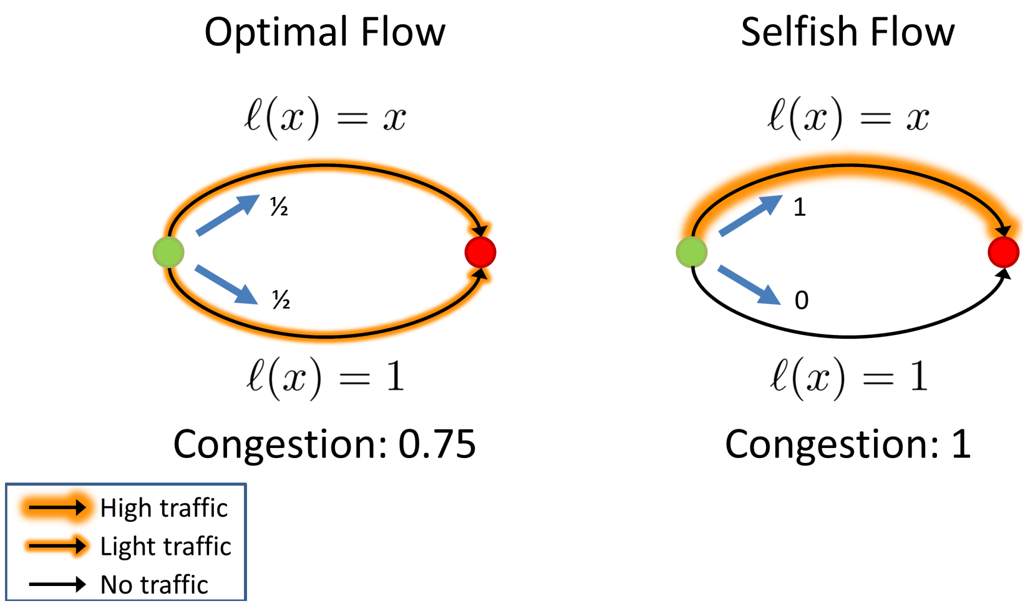


Figure 2. Pigou's Network, illustrating the negative effects of selfish behavior. In this network, drivers can choose between the upper, congestion-sensitive link and the lower, constant-latency link. The image on the left depicts a congestion-minimizing routing profile in which the traffic is split evenly between the two links. However, in this optimal flow, agents on the lower link experience a latency of 1, and (individually) could decrease their travel time by switching to the upper link. Unfortunately, this self-interested behavior can have negative consequences for system performance. The image on the right depicts a routing profile arising when every driver chooses the path with lowest delay; here, drivers have crowded on to the upper link, degrading its performance. A central problem is that no driver has an incentive to choose the lower, more-efficient path.

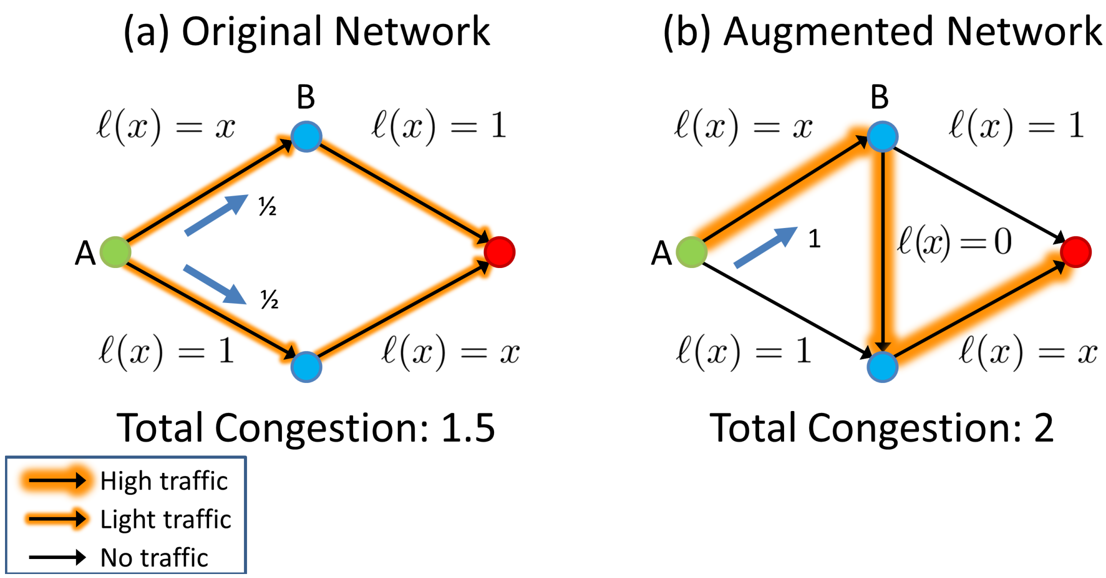


Figure 3. Braess's network, depicting an unintended consequence of attempting to influence social behavior. The image on the left in (a) depicts a transportation network and its associated Nash flow. A well-meaning traffic engineer, hoping to improve network congestion, adds a new link to the network connecting the two intermediate nodes. Despite the fact that this link's cost is zero, its addition to the network leads to the setting on the right in (b). Any user on the upper path can take the new link without increasing his cost, but in doing so, he increases the cost of the lower path, which in turn leads to more users taking the upper path. The ultimate effect of augmenting the network with a zero-cost link is that every driver's travel time increases by 33%.

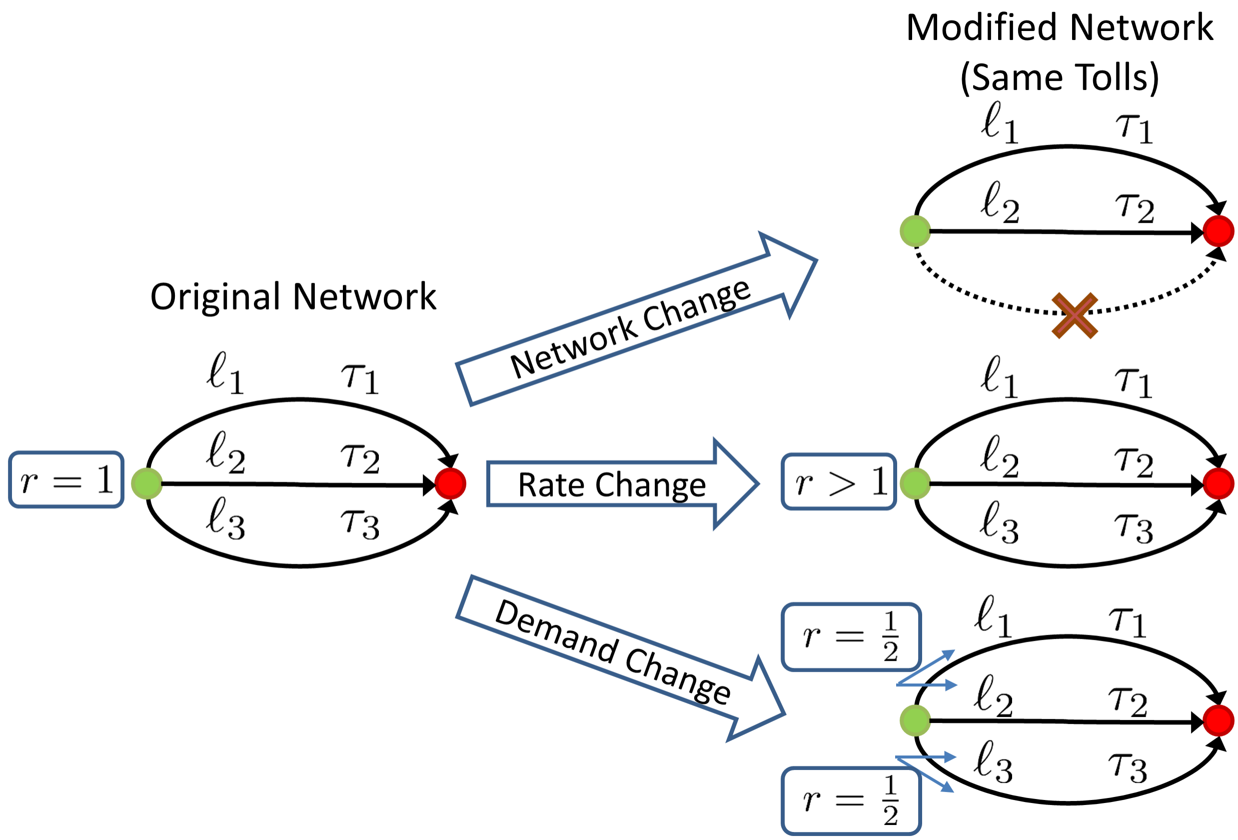


Figure 4. Diagram depicting possible network changes a designer may consider in robustness analysis. Suppose tolls τ_1, τ_2, andτ_3 are designed for the network on the left. To analyze the robustness of this toll design, the designer may subject the routing problem to various hypothetical changes: topology (deleting link 3), traffic rate (increasing r above its original value), and demand structure (restricting half the traffic to the upper two links, the other half of the traffic to the lower two links), while keeping the tolls designed for the original network. If the tolls are able to incentivize efficient behavior on the "new" networks despite having been designed for the original network, they are called robust.

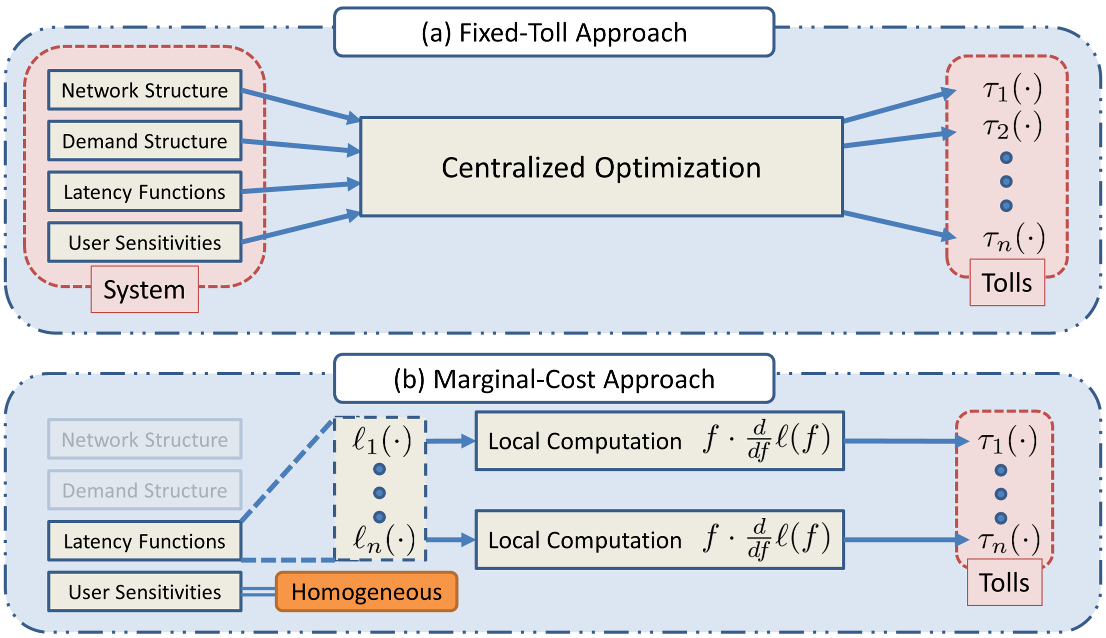


Figure 5. Depiction of informational dependencies of various taxation mechanisms. Subfigure (a) depicts the optimization strategy and informational dependencies of the fixed-toll mechanisms of [33], [34], in which a network's optimal fixed tolls are calculated as the output of a single centralized optimization problem that takes a complete system specification as input. In contrast, (b) is a corresponding diagram for the marginal-cost toll approach of [20], [35]. First, note that network structure and user demands are not required for the computation of optimal marginal-cost tolls. Second, note that the tolling function for any given edge e is computed locally - $\tau_e(\cdot)$ depends only on edge e's latency function $\ell_e(\cdot)$ and traffic flow f_e. This implies that the efficiency guarantees provided by marginal-cost tolls are robust to variations in network and demand structure simply by merit of their functional form.

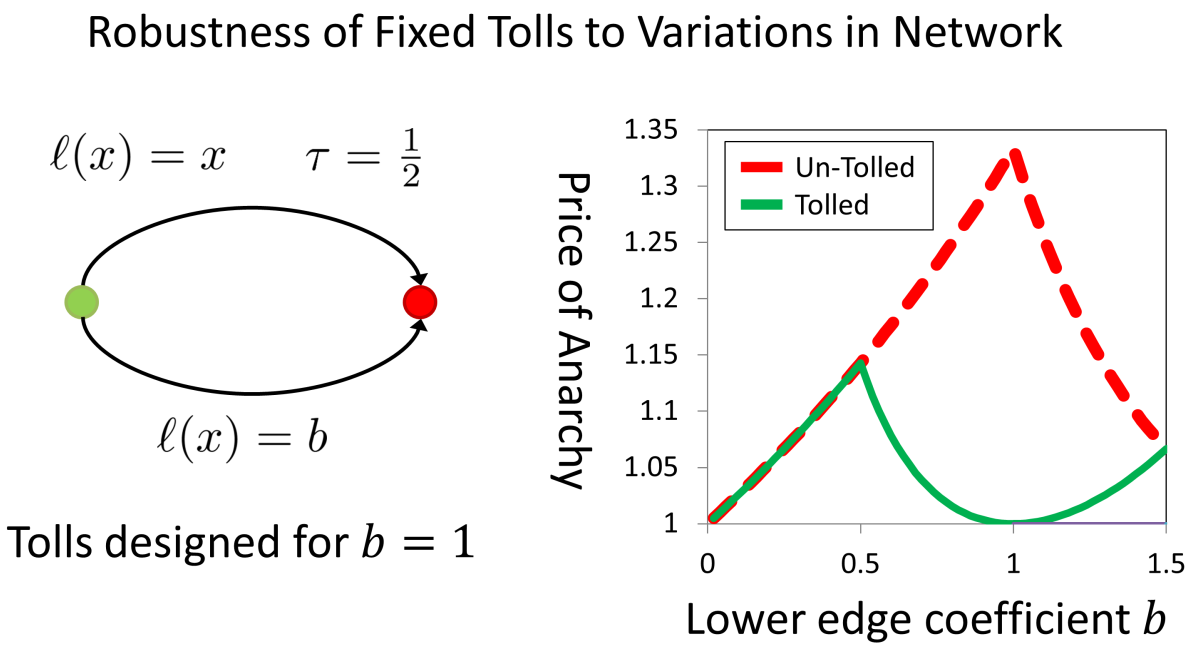


Figure 6. Pigou's Network: fixed tolls applied to the "wrong" network. Depicted here is an analysis of the robustness of fixed tolls to network variations. On the left, a Pigou-style network has a fixed toll of $1/2$ charged to the upper link; this toll incentivizes optimal behavior when the lower latency function satisfies $\ell(x) = 1$. To investigate the robustness of fixed tolls to network variations, the toll is held constant while the lower-link latency function b is allowed to vary between 0 and 1.5. For each value of b, the total latency of tolled and un-tolled Nash flows as well as the optimal total latency for that particular value of b are recorded. Finally, the price of anarchy curves shown are generated by dividing the Nash latencies by the respective optimal latency for each b. Note that the tolled price of anarchy is only 1 when $b = 1$; that is, this fixed toll only incentivizes optimal behavior on the specific network for which it was designed. The fact that the toll does not incentivize optimal behavior for all networks shows that it is not strongly robust to network variations.

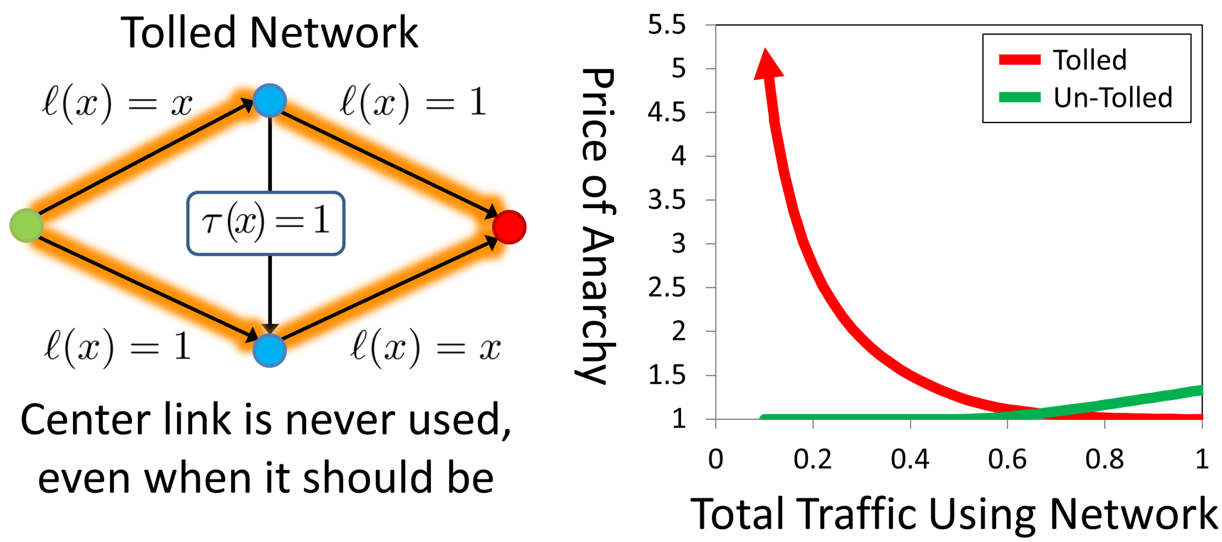


Figure 7. Braess' network: perverse incentives piled on top of unintended consequences. Following on the classical Braess's paradox (see Figure 3), this figure depicts an attempt to redeem the pathological network augmentation with a simple fixed toll. If a toll of 1 is levied on the link connecting the two intermediate nodes, it is simple to show that no traffic will ever use that link. Unfortunately, for low traffic rates (particularly when the total mass of traffic is less than 0.5), it is actually *optimal* for traffic to use the center link, but the rigid fixed toll prevents this. On the right is plotted the price of anarchy as a function of the total traffic rate; note that under the influence of tolls, the price of anarchy can become arbitrarily large as traffic approaches 0 - despite the fact that with no tolls, the price of anarchy can never exceed $4/3$.

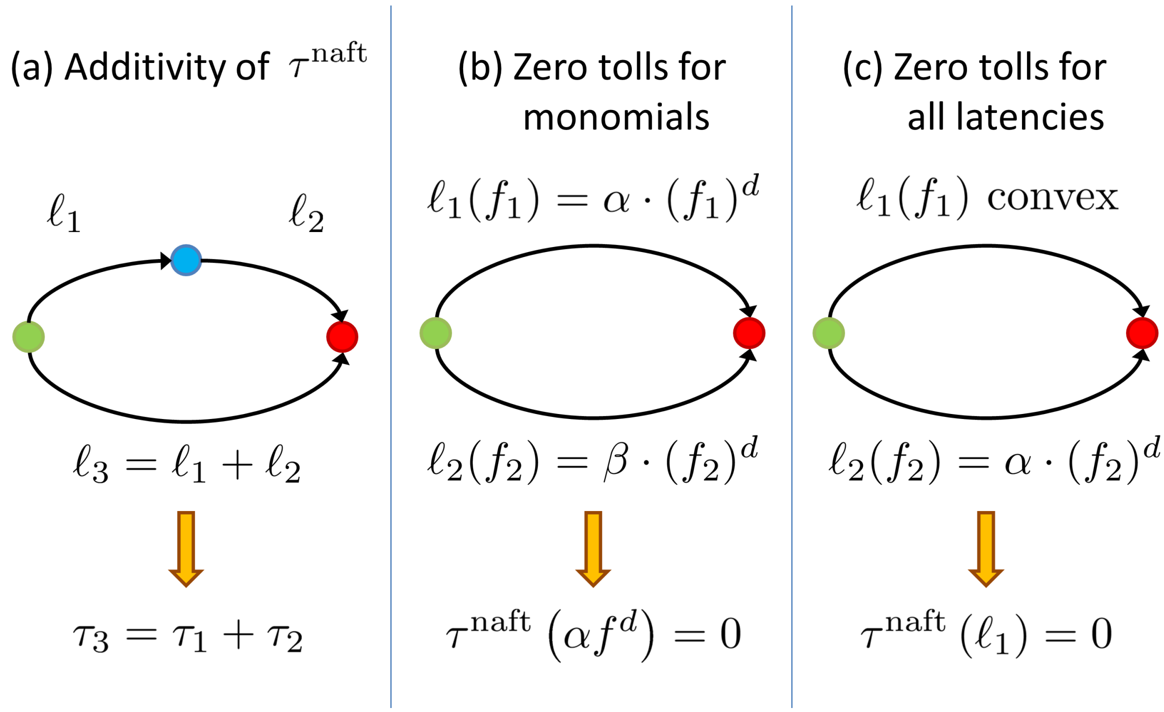


Figure 8. Networks used to prove Theorem 1, showing that any network-agnostic fixed-toll taxation mechanism must charge taxes of 0 on every link. The network in (a) is used to show that network-agnostic fixed-toll taxation mechanism τ^{naft} is additive in latency functions. Next, (b) is used to show in conjunction with (a) that monomial cost functions of any degree must be assigned a zero toll. Finally, in (c), for any general latency function ℓ_1, it is shown how to create a polynomial latency function ℓ_2 which is more congestible at equilibrium than ℓ_1, so that a positive toll charged on ℓ_1 would increase the total latency of the network. Thus it is shown that network-agnostic fixed tolls much charge 0 on every edge or they risk increasing the total latency compared to un-tolled networks.

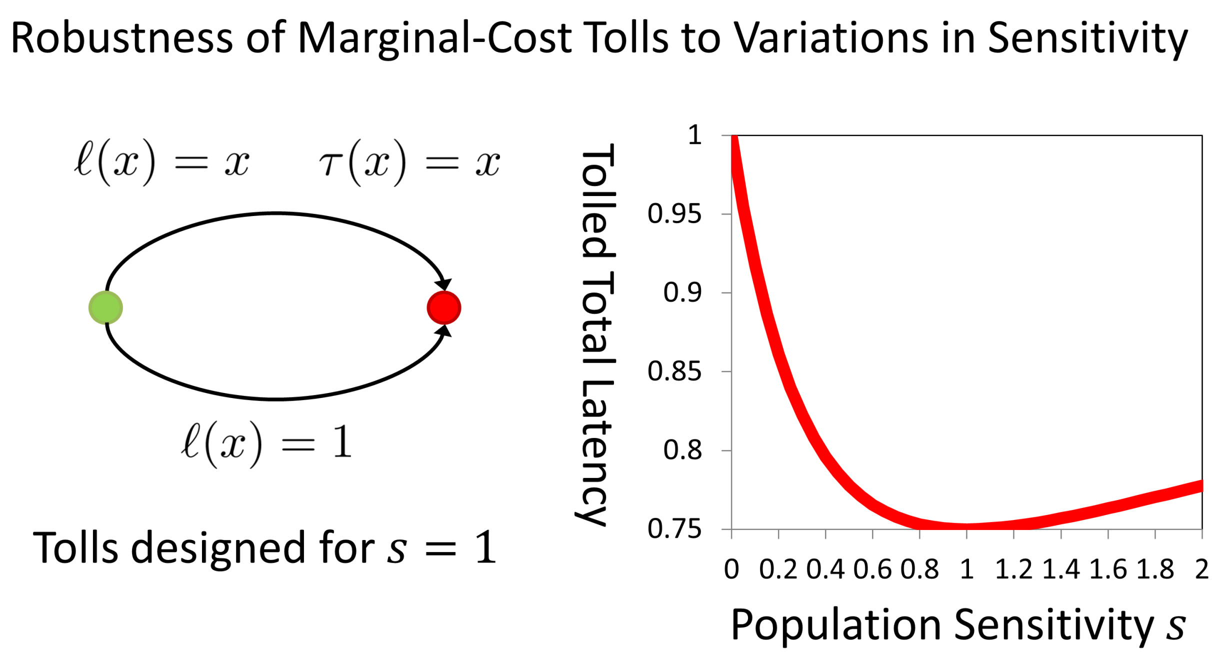


Figure 9. A setting in which marginal-cost tolls are shown not to be strongly robust to variations of user sensitivity. Depicted is the canonical Pigou Network with marginal-cost toll $\tau^{\mathrm{mc}}(x) = x$ assigned to the top link. If marginal-cost tolls were strongly robust to variations of user sensitivities, they will incentivize optimal flows for every user sensitivity profile. To check this, tolls are chosen without *a priori* knowledge of the user sensitivities, and then the population's homogeneous sensitivity is varied from 0 to 2 and the resulting total latency is plotted. It is evident from the plot that the optimal total latency is only obtained when the user sensitivities are exactly 1; all other sensitivity values incentivize some suboptimal total latency $L(f) > 0.75$. Thus, marginal-cost tolls are not strongly robust to variations of user sensitivity.

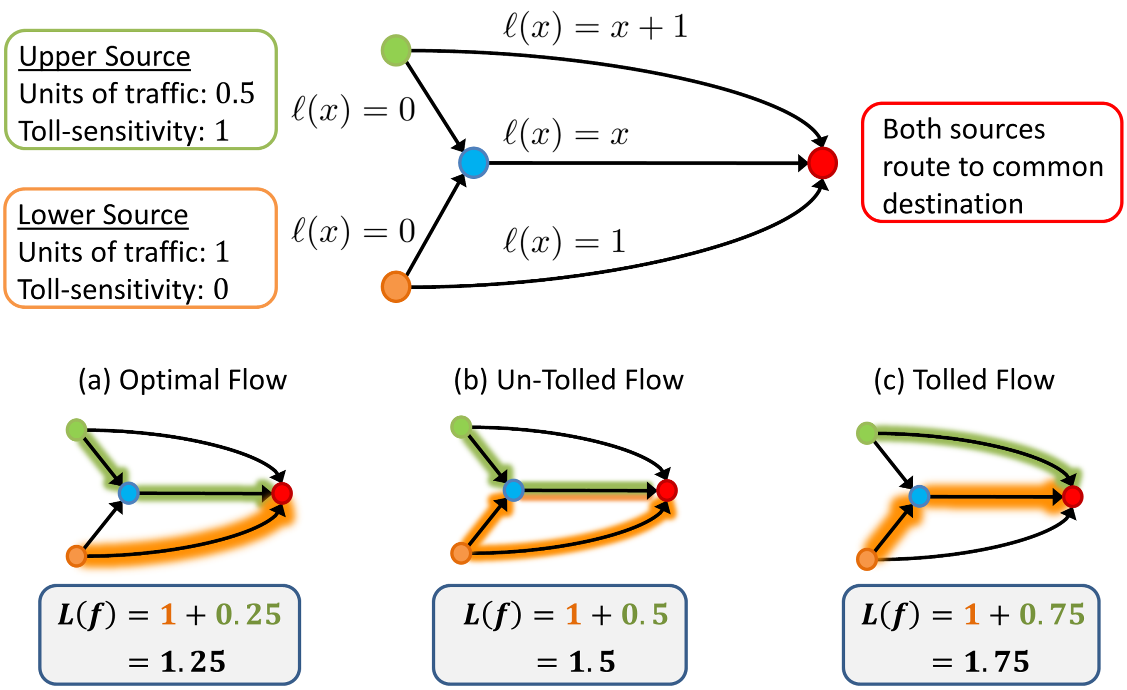


Figure 10. A network demonstrating the marginal-cost tolls are not weakly robust to user heterogeneity. This figure depicts a simple two-source network in which 0.5 units of traffic route from the upper (green) source, and 1 unit of traffic routes from the lower (orange) source. If traffic from the upper source trades off time and money equally (i.e., $s \equiv 1$), but traffic from the lower source cares only about time (i.e., $s \equiv 0$), marginal-cost tolls result in a price of anarchy of 1.4 on this network. The optimal flow here requires all of the traffic from the lower source to use the lower, constant-latency link. However, only the traffic from the upper source responds to tolls; when marginal-cost tolls are levied, all of the upper-source (green traffic) moves to the inefficient upper path, and the lower-source (orange) traffic moves to replace it on the middle path, as depicted on the right.

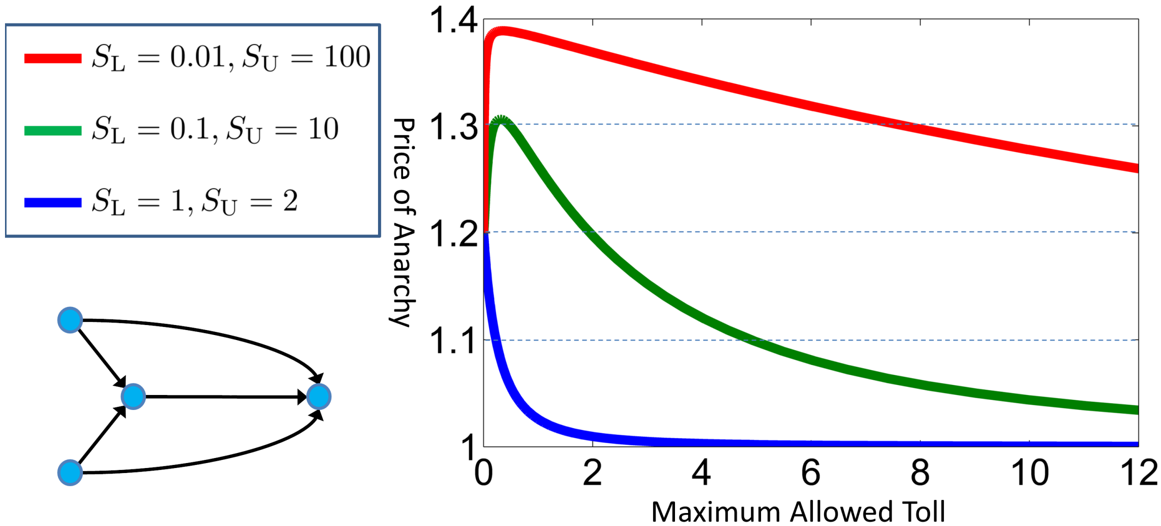


Figure 11. Applying large universal tolls to a pathological network. The network depicted is the same as that shown in more detail in Figure 10, in which asymmetry and heterogeneity lead to highly inefficient tolled behavior. Here, Universal Tolls of (22) are applied to the network, and the resulting worst-case congestion for various populations is plotted. In general, the better a toll-designer's characterization of the population, the better performance the designer will be able to guarantee; when $S_{\mathrm{L}} = 0.01$ and $S_{\mathrm{U}} = 100$ (see the red curve above), much higher tolls are required to enforce efficient flows than when $S_{\mathrm{L}} = 1$ and $S_{\mathrm{U}} = 2$ (as in the blue curve above).

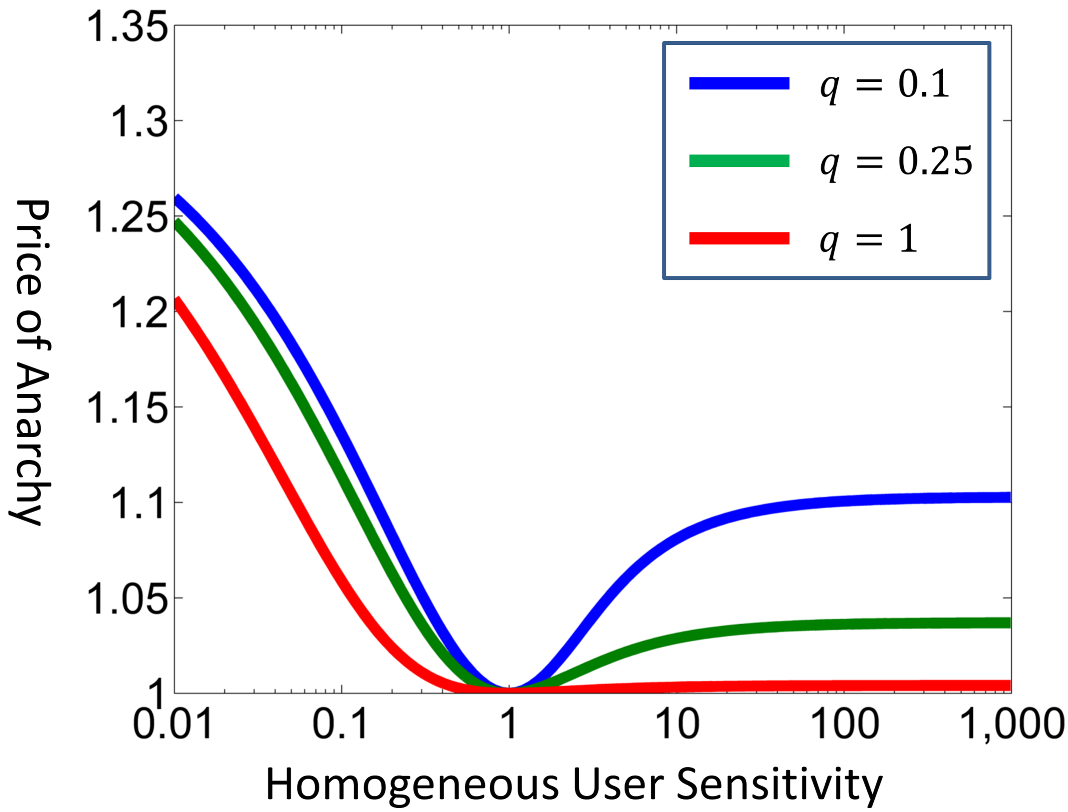
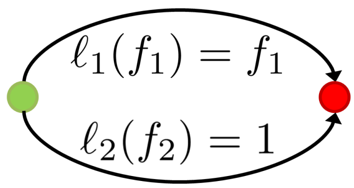


Figure 12. Empirical study on the sensitivity-robustness of budget-balanced tolls. Budget-balanced tolls were computed for a unit-sensitivity homogeneous population on Pigou's network (refer to Figure 2) by the procedure prescribed in [36]; the q parameter describes to what extent the tolling function on edge 1 is flow-varying. Once computed, the population's price sensitivity was varied from 0.01 to 1000 and the resulting price of anarchy was plotted. For $q = 1$, the best case, the price of anarchy is within 1% of optimal for all sensitivities plotted greater than about 0.33, indicating a high level of robustness to mischaracterizations of user sensitivity. Note that for all sensitivities plotted, larger values of q correspond to lower values for the price of anarchy; indicating that strongly flow-varying tolls may play an important role in robustness.

Latency Function	Universal Toll	Optimal Toll
f_1	$2\kappa_u f_1$	$\kappa_1 f_1$
1	κ_u	κ_2



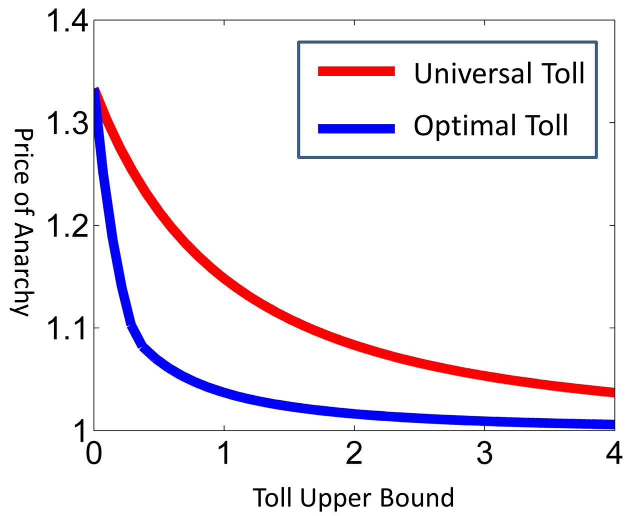


Figure 13. Comparison of universal tolls with optimal affine tolls on Pigou's network. Universal tolls are so named because for any network they induce optimal routing if parameter κ_u is high enough; however, if more information is known, they can be fine-tuned and offer considerably enhanced performance. This figure depicts the enhanced performance possible if the considered network is known to be parallel with linear-affine latency functions, as is the classical Pigou's network reproduced on the left (see also Figure 2). The table on the upper-left show the functional form of the tolls prescribed by the two different mechanisms; for universal tolls, κ_u is set as high as a toll upper bound permits. For the optimal affine tolls, κ_2 is chosen according to (30), and then κ_1 is set as high as the upper bound permits. The graph on the right shows how much more effective the optimal affine tolls are in mitigating network inefficiencies than the universal tolls, despite the fact that for both approaches, the price of anarchy converges to one in the large-toll limit.

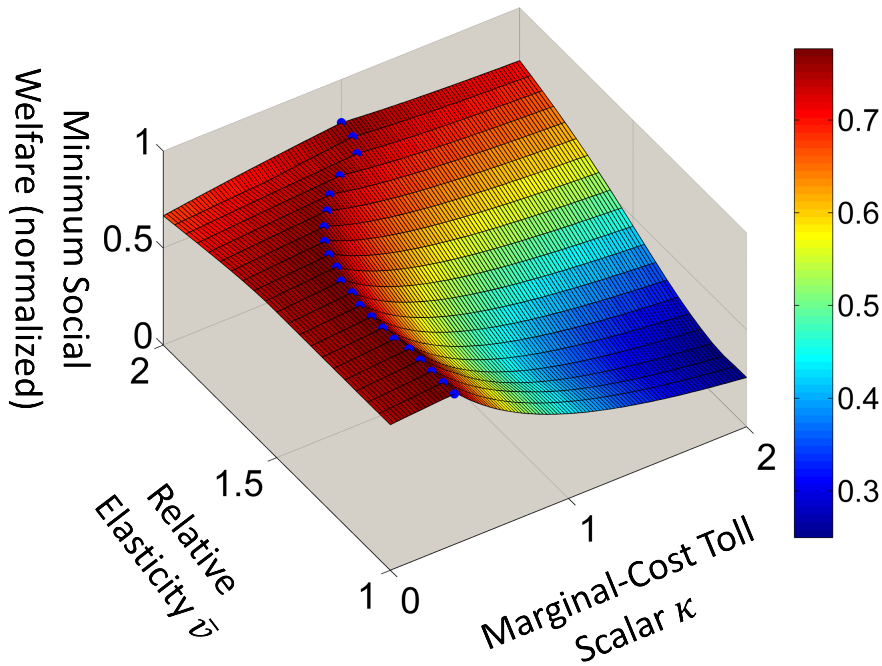


Figure 14. Empirical results for finding the optimal robust tolls for elastic traffic. Depicted are the results of simulating scaled marginal-cost tolls applied to Pigou's Network (see Figure 2) when traffic is both elastic *and* price-sensitive. The elasticity is modeled by inverse demand curve $v(x) = \bar{v} - x$; if the minimum cost of travel on the network exceeds $v(x)$, then user x will stay home. The right horizontal axis represents a scalar κ on marginal-cost tolls and the left horizontal axis represents the relative elasticity of the population $\bar{v}$ (high values of $\bar{v}$ correspond to highly-inelastic traffic). The surface of the plot is the minimum social welfare (divided by the optimal social welfare) on the network resulting from a given toll scalar and elasticity parameter. The key thing to note on this plot is that the optimal choice of toll scalar κ (denoted on the plot by small blue circles) depends on the elasticity parameter $\bar{v}$ (that is, tolls depend on the inverse demand curve); this is not true in the homogeneous-user model of [35] in which marginal-cost tolls optimize social welfare for all elasticity profiles. This indicates that if users are heterogeneous, marginal-cost tolls are no longer robust to mischaracterizations of a population's elasticity.

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Sidebar 1: Computing Price of Anarchy Bounds

The concept known as "price of anarchy," a measure of the inefficiency of selfish behavior, was first studied in the area of routing games in [27], and has found far-reaching applications in fields as diverse as supply-chain management [40], auction theory [41], telecommunication systems [42], and others. Price of anarchy is defined generally as follows: In a class of games $\mathcal{G}$, suppose a k-player game $G \in \mathcal{G}$ has outcomes denoted by x and some global cost function to be minimized $C(x) = \sum_{i=1}^{k} C_i(x)$. The price of anarchy is defined as

$$\mathrm{PoA}(\mathcal{G}) \triangleq \max_{G \in \mathcal{G}} \frac{C(x^{\mathrm{ne}})}{\min_{x} C(x)}, \tag{S1}$$

where x^{ne} is a Nash equilibrium. That is, price of anarchy is a worst-case measure over all games in $\mathcal{G}$ of the performance degradation due to selfish behavior.

There is no universal technique for computing exact worst-case equilibria for an arbitrary game, but there do exist methods which can simplify the process of upper-bounding the worst-case performance of equilibria. One such method is known as (λ, μ)-*smoothness*. The approach is as follows: Suppose it can be shown for games in $\mathcal{G}$ that the following is true for some $\lambda > 0$ and $\mu < 1$ and all outcomes x and x^*:

$$\sum_{i=1}^{k} C_i\left(x_i^*, x_{-i}\right) \leq \lambda C(x^*) + \mu C(x), \tag{S2}$$

where the notation (x_i^*, x_{-i}) indicates that player i is playing according to action profile x^*, and all players different from i are playing according to action profile x. In this case, the following bound holds on the price of anarchy of $\mathcal{G}$:

$$\mathrm{PoA}(\mathcal{G}) \leq \frac{\lambda}{1-\mu}. \tag{S3}$$

Thus, computing an upper bound on the price of anarchy reduces to finding the (λ, μ) parameters which make the inequality in (S3) as tight as possible. While this may seem a daunting task, it has proved useful in computing price of anarchy bounds in many settings, including mechanism design [43], machine scheduling [44], and a variety of network routing problems [45]. In the case of affine-cost routing games, it can be shown that $\lambda = 1$ and $\mu = 1/4$, implying a price of anarchy of $4/3$.