Fundamental Limits of Locally-Computed Incentives in Network Routing

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Abstract*— We ask if it is possible to influence social behavior with no risk of unintentionally incentivizing pathological behavior. In network routing problems, if network traffic is composed of many individual agents (such as drivers in a city's road network), it is known that strategic behavior among the agents can lead to suboptimal network congestion. To mitigate this, a system planner may charge monetary tolls for the use of network links in an effort to incentivize efficient routing choices by the users. We study situations in which these tolls are computed locally on each edge, as in the classical case of marginal-cost taxation, but that the users' sensitivity to tolls is not known. We seek locally-computed tolls that are guaranteed not to incentivize worse network routing than in the un-influenced case. Our results are twofold: first, we give a full characterization of all non-perverse locally-computed tolls for parallel networks with arbitrary convex delay functions, and show that they are all a generalized version of traditional marginal-cost tolls. Second, we exhibit a type of pathological network in which all locally-computed tolling functions can cause perverse incentives for heterogeneous price-sensitive user populations. That is, in general networks, the only locally-computed tolling functions that *do not* incentivize pathological behavior on some network are effectively zero tolls. Finally, we show that our results have interesting implications for the theory of altruistic behavior.

I. Introduction

Modern computational and infrastructure systems are becoming increasingly linked with the social systems that they serve. Accordingly, engineers must be aware of the ways in which social behavior affects the performance of engineered systems; this has spurred recent research on influencing social behavior to achieve engineering objectives [1]–[5]. Examples of interesting problems in this context include ridesharing systems [6], transportation networks [7], and power grids [8].

In this paper, we focus on a well-studied model of traffic congestion in networks known as a "non-atomic congestion game," in which traffic needs to be routed across a network from a source node to a destination node in a way that minimizes the average delay experienced by the traffic. If a central authority can control the traffic explicitly, it is typically straightforward to compute the optimal assignment of traffic; unfortunately, if the mass of traffic is composed of individual decision-makers, the aggregate network flows that emerge from individual localized decision-making may be far from optimal [9].

Accordingly, much research has focused on methods of influencing the routing choices made by individual users; one promising set of methodologies involves charging specially-designed tolls to network links in an effort to incentivize more-efficient network flows [10], [11]. In [12], [13] it is shown that if a special type of tolling function called a *marginal-cost* toll is levied on each network link, that this incentivizes optimal network routing – provided that all network users trade off time and money equally. An attractive feature of marginal-cost tolls is that they can be computed locally on each network link; that is, a link's toll depends *only* on that link's congestion characteristics and traffic flow. Thus, the optimality guaranteed by these tolls is intrinsically robust to variations of network structure. This local-computation property is known as *network agnosticity*; in essence, marginal-cost tolls only "know" their own edge – they are agnostic to global network specifications [1].

Weak robustness is defined in [14] as a guarantee that a given behavior-influencing mechanism never creates perverse incentives. Unfortunately, the authors of [14] also show that marginal-cost tolls are *not* weakly robust to variations of user toll-sensitivity. That is, if some users value their time more than others, networks exist on which the routing incentivized by marginal-cost tolls is worse than un-influenced routing.

Despite this negative result for traditional marginal-cost tolls, it has remained an open question whether some other network-agnostic taxation mechanism exists which can be weakly robust to variations of user toll-sensitivity. This question was partially addressed in [15] for the simplified case of parallel networks with linear-affine latency functions, where it was shown that any weakly-robust network agnostic taxation mechanism is essentially a generalization of traditional marginal-cost tolls. However, this restriction to parallel networks is not without loss of generality, as it is also proven that there is no weakly robust network-agnostic taxation mechanism for general asymmetric networks (that is, networks with more than one source and/or destination).

In this paper, we present an extension of the positive results of [15] to the case of all convex latency functions. First, in Theorem 4.1 we define the *generalized marginal-cost* taxation mechanism, and show that it is the only non-trivial network-agnostic taxation mechanism that is weakly robust on the class of parallel-path networks. Thus, a system planner can apply generalized marginal-cost tolls on any parallel network without fear of causing perverse incentives.

However, we also strengthen the negative results of [15], and exhibit a symmetric network (that is, having a single

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source/destination pair and all agents have access to the same set of paths) on which generalized marginal-cost taxes can never be weakly-robust. Thus, in Theorem 4.2 we rule out the possibility of a network-agnostic taxation mechanism being weakly-robust on general symmetric networks. Note that this does not mean that generalized marginal-cost tolls are not weakly-robust on *every* general network; merely that without *a priori* knowledge of network structure, the possibility of perverse incentives cannot be ruled out.

Finally, we show that our results imply corresponding facts about the behavior of altruistic network users. In particular, in the altruism model of [16], our characterization result in Theorem 4.1 implies that in every parallel network, users acting altruistically is socially beneficial. On the other hand, our negative result in Theorem 4.2 implies the somewhat paradoxical statement that there exist networks in which it can actually be socially harmful for some (but not all) users to act altruistically; we include these two results in Corollary 4.3.

II. Model and Summary of Contributions

A. Routing Game

Consider a network routing problem for a network (V, E) comprised of vertex set V and edge set E. A mass of r units of traffic needs to be routed from a common source $s \in V$ to a common destination $t \in V$. We write $\mathcal{P}$ to denote the set of *paths* available to the traffic, where each path $p \in \mathcal{P}$ consists of a set of edges connecting s to t. Note that this paper considers only the case of *symmetric* (or single-commodity) routing problems, in which all traffic can access the same set of paths. A network is called a *parallel-path* network if all paths are disjoint; i.e., for all paths $p, p' \in \mathcal{P}$, $p \cap p' = \emptyset$.

A *feasible flow* $f \in \mathbb{R}^{|\mathcal{P}|}$ is an assignment of traffic to various paths such that $\sum_{p\in\mathcal{P}} f_p = r$, where $f_p \geq 0$ denotes the mass of traffic on path p.

Given a flow f, the flow on edge e is given by $f_e = \sum_{p:e\in p} f_p$. To characterize transit delay as a function of traffic flow, each edge $e \in E$ is associated with a specific latency function $\ell_e : [0, 1] \to [0, \infty)$; $\ell_e(f_e)$ denotes the delay experienced by users of edge e when the edge flow is f_e. We adopt the standard assumptions that each latency function is nondecreasing, convex, continuous, and continuously differentiable. We measure the cost of a flow f by the *total latency*, given by

$$\mathcal{L}(f) = \sum_{e\in E} f_e \cdot \ell_e(f_e) = \sum_{p\in\mathcal{P}} f_p \cdot \ell_p(f_p), \tag{1}$$

where $\ell_p(f) = \sum_{e\in p} \ell_e(f_e)$ denotes the latency on path p. We denote the flow that minimizes the total latency by

$$f^* \in \operatorname*{argmin}_{f \text{ is feasible}} \mathcal{L}(f). \tag{2}$$

Due to the convexity of ℓ_e, $\mathcal{L}(f^*)$ is unique.

A *routing problem* is given by $G = (V, E, r, \{\ell_e\})$. The set of all routing problems is written $\mathcal{G}$, and we denote by $\mathcal{G}_p$ the set of all parallel-path routing problems.

To study the effect of taxes on self-interested behavior, we model the above routing problem as a non-atomic congestion game. We assign each edge $e \in E$ a flow-dependent taxation function $\tau_e : \mathbb{R}^+ \to \mathbb{R}^+$. To characterize users' taxation sensitivities, let each user $x \in [0, r]$ have a taxation sensitivity $s_x \in [S_{\rm L}, S_{\rm U}] \subseteq \mathbb{R}^+$, where $S_{\rm L} \geq 0$ and $S_{\rm U} \leq +\infty$ are lower and upper sensitivity bounds, respectively. Note that we allow $S_{\rm U}$ to take the value $+\infty$. If all users in s have the same sensitivity (i.e., $s_x = s_y$ for all $x, y \in [0, r]$), the population is said to be *homogeneous*; otherwise it is *heterogeneous*. Given a flow f, the cost that user x experiences for using path $\tilde{p} \in \mathcal{P}$ is of the form

$$J^x_{\tilde{p}}(f) = \sum_{e\in\tilde{p}} \left[\ell_e(f_e) + s_x \tau_e(f_e)\right], \tag{3}$$

and we assume that each user selects the lowest-cost path from the available source-destination paths. We call a flow f a *Nash flow* if all users are individually using minimum-cost paths given the choices of other users, or if for all users $x \in [0, r]$ we have

$$J^x(f) = \min_{p\in\mathcal{P}} \sum_{e\in p} \left[\ell_e(f_e) + s_x \tau_e(f_e)\right]. \tag{4}$$

It is well-known that a Nash flow exists for any non-atomic congestion game of the above form [17]. If the population is homogeneous or the taxes are constant functions, these flows are unique in cost (that is, all Nash flows have the same total latency) [18], [19].

We assume that the sensitivities of the agents are ordered increasing; that is, s_x is a monotone-increasing function of x. The set of possible sensitivity distributions is the set of monotone-increasing functions $\mathcal{S} = \{s : [0, r] \to [S_{\rm L}, S_{\rm U}]\}$.

B. Taxation Mechanisms and Robustness

To model locally-computed tolls, we consider so-called *network-agnostic* taxation mechanisms. Here, each edge's taxation function is computed using only locally-available information. That is, $\tau_e(f_e)$ depends only on ℓ_e, not on edge e's location in the network, the overall network topology, the overall traffic rate, or the congestion properties of any other edge. A network-agnostic taxation mechanism T is thus a mapping from latency functions to taxation functions, and any specific edge taxation function is given by

$$\tau_e(\cdot) = T(\ell_e). \tag{5}$$

To evaluate the performance of taxation mechanisms, we write $\mathcal{L}^{\rm nf}(G, s, T)$ to denote the total latency of a Nash flow for routing problem G and population s induced by taxation mechanism T. If more than one Nash flow exists, let $\mathcal{L}^{\rm nf}(G, s, T)$ denote the total latency of the worst Nash flow. We write $\mathcal{L}^{\rm nf}(G, \emptyset)$ to denote the total latency of an un-influenced Nash flow; note that when there are no tolls, the sensitivity distribution plays no role.

In the robustness framework of [14], taxation mechanism T is said to be *weakly robust* if for every network and sensitivity distribution, the total latency induced by T never exceeds the total latency of an un-influenced Nash flow; i.e., for all $G \in \mathcal{G}$,

$$\sup_{s\in\mathcal{S}} \mathcal{L}^{\rm nf}(G, s, T) \leq \mathcal{L}^{\rm nf}(G, \emptyset). \tag{6}$$

Loosely speaking, if a taxation mechanism is weakly robust, this means that it will never create perverse incentives on any routing problem. Note that at a minimum, the zero toll is weakly robust.

C. Summary of Our Contributions

This paper contains several contributions to the theory of robustly influencing social behavior. First, we provide a full characterization of the weakly-robust network agnostic taxation mechanisms for parallel-path congestion games with convex latency functions. Specifically, Theorem 4.1 defines the *generalized marginal-cost* taxation mechanism $T^{\rm gmc}$, which assigns edge tolls of

$$\tau_e^{\rm gmc}(f_e) = \kappa_1 \ell_e(f_e) + \kappa_2 f_e \ell_e'(f_e). \tag{7}$$

where $\kappa_1 > -1/S_{\rm U}$, and $0 \leq \kappa_2 \leq \kappa_1 + 1/S_{\rm U}$. Theorem 4.1 shows that $T^{\rm gmc}$ is the only weakly robust network-agnostic taxation mechanism for parallel-path congestion games. These tolling functions are a straightforward generalization of traditional marginal-cost tolls (given by $\tau_e^{\rm mc}(f_e) = f_e \ell_e'(f_e)$), and this means that we have proved that marginal-cost tolls are unique in the sense that any weakly-robust network-agnostic tolls always contains a marginal-cost component.

Subsequently, we strengthen the impossibility result of [15] regarding the non-existence of weakly-robust network agnostic taxes for general networks. Here, we exhibit symmetric networks on which $T^{\rm gmc}$ fails to be weakly robust, showing in Theorem 4.2 that even symmetry is not sufficient to guarantee the existence of weakly-robust tolls. Succinctly put, this means that the only network-agnostic tolls which never create perverse incentives are those which do nothing.

Finally, in Corollary 4.3 we show that our results can be interpreted in terms of the theory of altruistic behavior. Specifically, we show that altruism is always socially beneficial on parallel networks, but can actually be harmful on general networks.

III. Related Work and Examples

The following is a brief survey of relevant work on the robustness of taxation mechanisms in congestion games. Much of the tolling literature has focused on *network-dependent* taxation mechanisms, in which each edge toll is a function of the entire routing problem. Network-dependency allows taxation mechanisms to incentivize precisely targeted network flows, particularly in cases where the network is known to be immutable. One example of this is *fixed tolls*, which for any $e \in G$, $\tau_e(f_e) = q_e$ for some $q_e \geq 0$. If network, traffic-rate, and user sensitivity specifications are known precisely, it is possible to compute fixed tolls which induce optimal Nash flows on any network [10], [11]. However, if any of these pieces of information are unknown, fixed tolls fail to be strongly robust; if fixed tolls are further restricted to be network-agnostic, they fail even to be weakly-robust [14]. Other examples of network-dependent tolls include the restricted tolls of [4], [20], the piecewise-constant marginal-cost tolls of [21] and the dynamic fixed tolls of [22]. However, the robustness of these tolls to variations of user sensitivity has not been investigated explicitly.

Moving past network-dependent tolls, the classical example of a network-agnostic taxation mechanism is that of the *marginal-cost* or *Pigovian* taxation mechanism $T^{\rm mc}$. For any edge e with latency function ℓ_e, the accompanying marginal-cost toll is

$$\tau_e^{\rm mc}(f_e) = f_e \cdot \ell_e'(f_e),\ \forall f_e \geq 0, \tag{8}$$

where ℓ' represents the flow derivative of ℓ. In [12] the authors show that for any $G \in \mathcal{G}$, it is true that $\mathcal{L}^*(G) = \mathcal{L}^{\rm nf}(G, s, T^{\rm mc})$, provided that all users have a sensitivity equal to 1.

Recent research has identified several new network-agnostic taxation mechanisms, which are all in some sense generalizations of $T^{\rm mc}$. For example, [23] exhibits a universal taxation mechanism which achieves weak robustness through large tolls. For the case of parallel networks, [1] studies scaled marginal-cost tolls for parallel networks under a utilization constraint. In [15], the authors show that for linear latency functions,

$$\tau_e(f_e) = \kappa_a a_e f_e + \kappa_b b_e, \tag{9}$$

can be weakly-robust for parallel-path routing games if constants $\kappa_a \geq 0$ and $\kappa_b \geq 0$ are chosen carefully. However, [15] contains an impossibility result showing that *no* non-trivial network-agnostic taxation mechanism can be weakly robust on general asymmetric networks, but posit that perhaps weak robustness may still be possible on some class of symmetric networks.

Here, we exhibit a new type of pathology for marginal-cost tolls. In contrast to the pathology reported in [15] for asymmetric networks, our example here occurs for symmetric networks as well.

Example 3.1: Consider the network depicted in Figure 1, consisting of the well-known Braess's Paradox network [24] in parallel with a single constant-latency edge. Consider the case that marginal-cost tolls are charged on the network according to (8); that is, edges e_1 and e_4 are each charged a flow-varying toll of $\tau_e(f_e) = f_e$. If the user population has 2 units of traffic and a homogeneous toll sensitivity of $s \in [0, 1]$, it is easy to verify that that unique Nash flow on this network is the one labeled "Efficient Nash Flow" in Figure 1. In this flow, all agents are experiencing a cost of $2 + s$; deviating to the zig-zag path would yield a larger cost of $2 + 2s$, and deviating to the constant-latency link would yield a cost of 3. Since there are 2 units of traffic experiencing a delay of 2 each, the total latency is $2 \cdot 2 = 4$.

Now consider a heterogeneous population in which 1 unit of traffic has a sensitivity of $s_1 = 0$ (the orange traffic in Figure 1), and 1 unit of traffic has a sensitivity of $s_2 = 1$. In this case, a new Nash flow emerges: one in which all the insensitive traffic uses the zig-zag path, and all the sensitive traffic uses the constant-latency link; this is labeled "Inefficient Nash Flow" in Figure 1. In this flow, any agent on the zig-zag path has a delay of 2, but any agent on the constant-latency path has a delay of 3, for a total latency of $2 + 3 = 5$, which is significantly greater than the un-tolled total latency of 4.

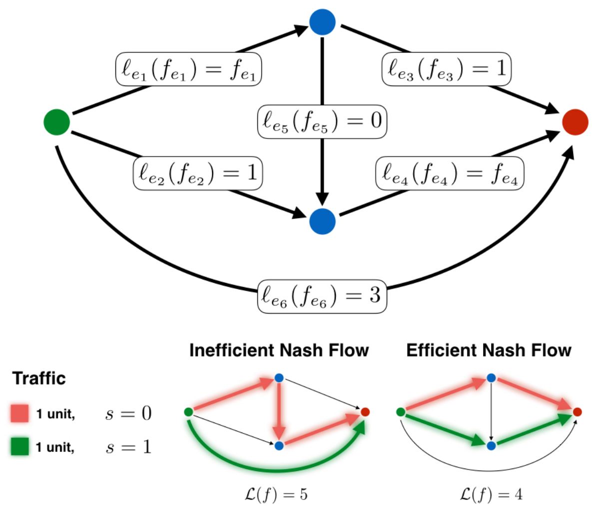

Fig. 1. Example 3.1: A network demonstrating that marginal-cost tolls are not weakly robust to user heterogeneity. This network consists of the well-known Braess network in parallel with a single constant-latency edge. The user population has total mass 2, half of which is insensitive to tolls (i.e., their sensitivity value $s = 0$), half of which trades off time and money equally (i.e., their sensitivity value $s = 1$). On this network, with this population, marginal-cost tolls induce more than one Nash flow; two of these Nash flows are exhibited in the figure. On the left, all of the insensitive traffic (orange) is using the zig-zag path, experiencing a latency of 2; all the sensitive traffic (green) is using the constant-latency edge, experiencing a latency of 3 – for a total latency of 5. On the right flow, all traffic is experiencing a latency of 2, for a total latency of 4. Here, *any* homogeneous population using this network has the right-hand flow as a unique Nash flow. Note that this example exhibits perverse incentives even in the case that user sensitivities do not actually take the value 0; all that is required is a sufficient level of heterogeneity in the user population.

IV. Our Contributions

A. Weakly Robust Network Agnostic Taxation Mechanisms for Parallel Networks

Theorem 4.1 characterizes the space of weakly-robust network-agnostic taxation mechanisms for parallel-path networks and arbitrary admissible latency functions. Specifically, we show that all weakly-robust network-agnostic taxation mechanisms can be expressed as a simple generalization of classical marginal-cost tolls. Thus, perverse incentives can be systematically avoided on parallel-path networks by applying $T^{\rm gmc}$.

Theorem 4.1: A network agnostic taxation mechanism is weakly robust on $\mathcal{G}_p$ if and only if it assigns *generalized marginal-cost* tolls

$$\tau_e^{\rm gmc}(f_e) = \kappa_1 \ell_e(f_e) + \kappa_2 f_e \ell_e'(f_e), \tag{10}$$

where $\kappa_1 > -1/S_{\rm U}$, $\kappa_2 \geq 0$, and $\kappa_2 \leq \kappa_1 + 1/S_{\rm U}$.

The proof of Theorem 4.1 appears in the appendix.

B. Impossibility for General Symmetric Networks

Given the characterization of weakly-robust tolls for parallel networks given in Theorem 4.1, it is an attractive goal to extend the analysis beyond parallel networks. Unfortunately, one need not go far before even $T^{\rm gmc}$ fails to be weakly robust. In particular, we show in Theorem 4.2 that even the relatively restrictive condition of symmetry (all agents have access to the same set of paths) is not sufficient to guarantee the existence of nontrivial weakly-robust taxation mechanisms. This means that if network structure is unknown, the *only* way to avoid perverse incentives is effectively to do nothing.

Theorem 4.2: Let $\mathcal{G}$ denote the class of all symmetric networks. If $S_{\rm L} = 0$ and $S_{\rm U} > 0$, a network-agnostic taxation mechanism T is weakly robust on $\mathcal{G}$ if and only if it is trivial; that is, for every network $G \in \mathcal{G}$ and every population s it satisfies

$$\mathcal{L}^{\rm nf}(G, s, T) = \mathcal{L}^{\rm nf}(G, \emptyset). \tag{11}$$

Note that the "trivial tolls" of Theorem 4.2 are any tolls satisfying $\kappa \ell_e(f_e)$ for $\kappa > 0$; these tolls have no effect on any Nash flow, and are thus strategically equivalent to tolls of $\tau_e(f_e) = 0$.

Proof: Lemma 5.1 rules out all taxation mechanisms other than those satisfying the conditions of Theorem 4.1, so suppose we are given a taxation mechanism assigning taxes of $\tau_e(f_e) = \kappa_1 \ell_e(f_e) + \kappa_2 f_e \ell_e'(f_e)$, where $\kappa_1 > -1/S_{\rm U}$, and $\kappa_2 \leq \kappa_1 + 1/S_{\rm U}$. If $\kappa_2 = 0$, this taxation mechanism satisfies (11) trivially, so let $\kappa_2 > 0$. Our task is to create a user population s (that is, a distribution of tax-sensitivities) and a network G such that $\mathcal{L}^{\rm nf}(G, s, T) > \mathcal{L}^{\rm nf}(G, \emptyset)$. We will do this with a population having two sensitivity values $s_1 < s_2$ and a network resembling that in Figure 1. Let s_2 satisfy $0 < s_2 \leq S_{\rm U}$, and let $s_1 = 0$ to model the extreme case.[1] Construct the population as follows: let a unit mass of users have sensitivity s_1 and a unit mass have s_2, for a total of 2 units of traffic. Define $\gamma_2 \triangleq \frac{s_2 \kappa_2}{1 + s_2 \kappa_1} \in (0, 1]$, so any agent with sensitivity s_2 sees an effective cost function[2] on edge e of

$$J_e(f_e) = \ell_e(f_e) + \gamma_2 f_e \ell_e'(f_e). \tag{12}$$

Now, let G be the network depicted in Figure 1, but let the latency function on edge e_6 be $\ell_{e_6}(f_{e_6}) = 2 + \gamma_2$. Enumerate the paths as follows: denote the "zig-zag" path $p_1 = \{e_1, e_5, e_4\}$, the remaining two paths in the upper subnetwork $p_2 = \{e_1, e_3\}$ and $p_3 = \{e_2, e_4\}$, and the isolated constant-latency path $p_4 = \{e_6\}$; and denote the path flow of p_i by f_i. We will refer to paths p_1, p_2, and p_3 in the upper subnetwork as the "Braess subnetwork."

On this network, this population has at least two distinct Nash flows, corresponding to the two Nash flows depicted in Figure 1. We will write the inefficient flow (in which only half the agents use the Braess subnetwork) as $f^{\rm perverse} \triangleq (1, 0, 0, 1)$ and the efficient flow (in which all agents use the Braess subnetwork) as $f^{\rm efficient} \triangleq (0, 1, 1, 0)$. Note that in either flow, the delay experienced by agents on the Braess subnetwork is 2. However, in $f^{\rm perverse}$, half of the agents (those on p_4) are experiencing a delay of $2 + \gamma_2 > 2$. Thus, $\mathcal{L}\left(f^{\rm perverse}\right) = 4 + \gamma_2$, while $\mathcal{L}\left(f^{\rm efficient}\right) = 4$.

It can easily be verified that if we remove tolls, only $f^{\rm efficient}$ remains as a Nash flow, which means that

[1] Here, it is also possible to show perversities by letting s_1 be some small positive number; we let $s_1 = 0$ for the sake of parsimony.

[2] See the argument in the proof of Lemma 5.2 in the Appendix.

$\mathcal{L}\left(f^{\text{efficient}}\right) = \mathcal{L}^{\text{nf}}(G,\emptyset)$, or

$$\mathcal{L}^{\text{nf}}(G,s,T) > \mathcal{L}^{\text{nf}}\left(G,\emptyset\right) \quad (13)$$

and the considered tolls are not weakly-robust. ■

C. Implications for Altruistic Behavior

All of the foregoing has assumed that users are selfish and act with the sole objective of minimizing personal cost. However, real users may act altruistically, with the public good in mind. Recent research has investigated this in the α-altruism model, which assigns each user x an *altruism level* $\alpha_x \in [0,1]$; a user with $\alpha = 0$ is totally selfish, whereas a user with $\alpha = 1$ is totally altruistic [16]. This is modeled by assuming that user x (with corresponding altruism level α_x) on edge e experiences a cost of

$$\begin{aligned} J_e^x(f_e) &= (1-\alpha_x)\ell_e(f_e) + \alpha_x \frac{d}{df_e}\left(f_e\ell_e(f_e)\right) \\ &= \ell_e(f_e) + \alpha_x f_e \ell_e'(f_e). \end{aligned} \quad (14)$$

In other words, a totally-altruistic user fully accounts for the marginal effects that his actions have on those around him.

By comparing the cost functions induced by marginal-cost tolls (8) with the cost functions experienced by altruistic players (14), it is clear that there is a deep connection between this model of α-altrusim and the theory of marginal-cost taxation. In essence, marginal-cost taxes are designed to induce artificial altruism in the user population.

The authors of [16] exhibit two contexts in non-atomic congestion games in which worst-case performance improves with increasing levels of altruism: the first is in general networks with homogeneous altruism, and the second is in parallel networks with heterogeneous altruism. In both cases, if the average level of altruism in the population increases, worst-case performance improves.

Given the equivalence of marginal-cost taxation and altruism, our Corollary 4.3 strengthens the parallel-network result of [16], showing that *on any network*, the worst-cast flows are realized by a low-altruism homogeneous population. On the other hand, given our impossibility result in Theorem 4.2, Corollary 4.3 shows that increased altruism does not, in general, improve performance. That is, on the network in Figure 1, a totally-selfish population is associated with the efficient Nash flow, but a partially-altruistic population is associated with the inefficient Nash flow.

In the following, $\mathcal{L}^{\text{nf}}_{\text{alt}}(G,\alpha)$ denotes the worst-case Nash flow total latency on G for a given altruism distribution α, where users in α take altruism levels in the interval $[A_{\text{L}}, A_{\text{U}}] \subseteq [0,1]$. A homogeneous altruism distribution in which all users have value A_{L} is denoted α^{L}.

Corollary 4.3: For any $G \in \mathcal{G}_p$,

$$\mathcal{L}^{\text{nf}}_{\text{alt}}(G,\alpha) \leq \mathcal{L}^{\text{nf}}_{\text{alt}}\left(G,\alpha^{\text{L}}\right). \quad (15)$$

However, if $G \notin \mathcal{G}_p$, there may exist an altruism distribution α satisfying

$$\mathcal{L}^{\text{nf}}_{\text{alt}}(G,\alpha) > \mathcal{L}^{\text{nf}}_{\text{alt}}\left(G,\alpha^{\text{L}}\right). \quad (16)$$

Proof: Any Nash flow induced by T^{gmc} is a Nash flow for some altruism distribution (see, e.g., the argument in the proof of Lemma 5.2 in the Appendix). Thus, Corollary 4.3 is implied by Theorems 4.1 and 4.2. ■

V. Conclusion

This paper has fully-characterized the weakly-robust network-agnostic taxation mechanisms for parallel networks, and ruled them out entirely for general networks. We have shown that except in very limited settings (e.g., parallel networks), local computation of incentives carries a risk of causing harm. Among other things, this seems to indicate that information about the structure of the network is crucial for avoiding perverse incentives; characterizing these types of informational dependencies is the subject of ongoing work.

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APPENDIX: PROOF THEOREM 4.1

As a first step toward proving Theorem 4.1, we prove in Lemma 5.1 that all weakly-robust network-agnostic taxation mechanisms are essentially a generalized form of marginal-cost tolls. This lemma shows that tolls of the $T^{\rm gmc}$ form are necessary for weak robustness.

Lemma 5.1: If network-agnostic taxation mechanism T is weakly robust on $\mathcal{G}_p$, then for every edge e, it assigns taxation functions satisfying the conditions of Theorem 4.1.

Proof: We shall exhibit example networks on which various tolls are perverse, thus eliminating all but tolls of the form in (10). First, consider the network in Figure 2(a).

This network has two paths in parallel; the first path is a pair of edges in series with arbitrary latency functions ℓ_1 and ℓ_2, the second path consists of a single edge with latency function ℓ_3 satisfying $\ell_1 + \ell_2 = \ell_3$. For any such network, any nominal Nash flow $f^{\rm nf}$ is optimal; thus, a weakly-robust taxation mechanism T would need to incentivize a flow f^T that satisfies $f^T = f^{\rm nf} = f^{\rm opt} = (r/2, r/2)$ by charging tolls satisfying $\tau_1(r/2) + \tau_2(r/2) = \tau_3(r/2)$. That is, T is additive: if $\ell_1 + \ell_2 = \ell_3$, it is true that $T(\ell_1) + T(\ell_2) = T(\ell_1 + \ell_2)$. Note that this also implies that $T(0) = 0$, since any latency function ℓ_1 can be written as $\ell_1 + 0$.

Next we show that $T(\ell)$ is constant when ℓ is constant. Consider again the network in Figure 2(a) when $\ell_1(f_1) = 0$, $\ell_2(f_2) = b_2$, and $\ell_3(f_3) = b_3$. It is clear that if $b_2 < b_3$, the unique Nash flow routes all traffic on the upper path, and this flow is also optimal. Writing $T(b_2)(\cdot)$ as the tolling function assigned to $\ell_2(f_2) = b_2$ by T, it follows that for all f, $T(b_2)(f) < T(b_3)(0)$. If this were not the case, there would exist perverse Nash flows for large r which route a positive mass of traffic on the lower path. Since this must hold for all b_2 and b_3, it implies that for all b, $T(b)(\cdot)$ is nonincreasing function of flow. By an opposite argument, it must be that for all f, $T(b_2)(0) < T(b_3)(f)$, implying that $T(b)(\cdot)$ must be a nondecreasing (and thus constant) function of flow. Because T is simply a mapping from $\mathbb{R}$ to $\mathbb{R}$ for constant functions, its additivity implies linearity:[3] $T(b) = \kappa_1 b$. Finally, for $b_2 < b_3$, it must always be true for any possible agent sensitivities $s \in [S_{\rm L}, S_{\rm U}]$ that $(1 + \kappa_1 s)b_2 < (1 + \kappa_1 s)b_3$, or that $\kappa_1 > -1/S_{\rm U}$.

Next, we show that degree-d monomial latency functions must be assigned degree-d tolling functions. The network in

[3] Of course, this is provided that we require T to be Lebesgue-measurable.

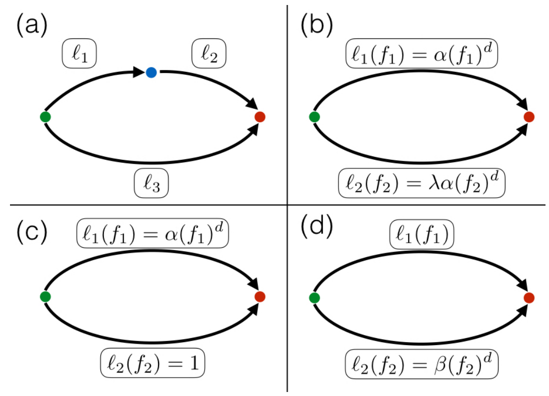

Fig. 2. Example networks used to prove Lemma 5.2.

Figure 2(b) has two edges in parallel with latency functions $\ell_1(f_1) = \alpha(f_1)^d$ and $\ell_2(f_2) = \lambda\alpha(f_2)^d$, where $\alpha > 0$, $\lambda > 0$, and $d \geq 1$. For any such network, the unique un-influenced Nash flow is optimal; thus, a weakly-robust T would need to induce a flow f^T that satisfies $f^T = f^{\rm nf} = f^{\rm opt}$. It can be shown that for any $r > 0$, this flow is

$$f_1^T = \frac{(\lambda\alpha)^{1/d} r}{(\alpha)^{1/d} + (\lambda\alpha)^{1/d}}, \quad f_2^T = \frac{(\alpha)^{1/d} r}{(\alpha)^{1/d} + (\lambda\alpha)^{1/d}}.$$

Since f^T is a nominal Nash flow, $\ell_1(f_1^T) = \ell_2(f_2^T)$; and $f^T = f^T$ implies that $\tau_1(f_1^T) = \tau_2(f_2^T)$. In the following, let $r = (\alpha)^{1/d} + (\lambda\alpha)^{1/d}$, so for all α, λ, $\tau_1\left((\lambda\alpha)^{1/d}\right) = \tau_2\left((\alpha)^{1/d}\right)$.

First let $\lambda = 2$, so that $\ell_2(f_2) = 2\ell_1(f_2)$. Then additivity ensures that $\tau_2(f_2) = 2\tau_1(f_2)$. That is, $\tau_1\left((2\alpha)^{1/d}\right) = 2\tau_1\left((\alpha)^{1/d}\right)$. Since this must hold for any α, it implies either that $\tau_1(f) \equiv 0$, or that $\tau_1(f) = \eta_1 f^d$ for some $\eta_1 > 0$.

To find η, we need only substitute f^T into $\tau_1(f_1^T) = \tau_2(f_2^T)$ and solve, yielding $\eta_1\lambda = \eta_2$. Due to the fact that η_1 cannot be a function of λ, the above is only satisfied when all tolling functions are given by $K\alpha_e(f_e)^d$ for some $K \geq 0$. K is a constant that does not depend on e (but may depend on d).

To find K, consider Figure 2(c). This network has $\ell_1(f_1) = \alpha(f_1)^d$ in parallel with a constant latency function $\ell_2(f_2) = 1$. Here, if $r \leq (1/(\alpha(d+1)))^{1/d}$, the uninfluenced Nash and optimal flow on this network is $(r, 0)$. Thus, $\tau_1(f)$ must be small enough that it does not incentivize *any* user to use edge 2 when r is low. Precisely, keeping in mind that $\tau_1(f_1) = K\alpha(f_1)^d$, we require that for all sensitivities $s \in [S_{\rm L}, S_{\rm U}]$, $\alpha(f_1)^d + sK\alpha(f_1)^d \leq 1 + s\kappa_1$, or, substituting the appropriate f_1, that

$$\alpha(1 + sK)\left(\left(\frac{1}{\alpha(d+1)}\right)^{1/d}\right)^d \leq 1 + s\kappa_1. \tag{17}$$

This implies that $sK \leq s\kappa_1 d + s\kappa_1 + d$. This simplifies nicely if we write $K = \kappa_1 + \kappa_2 d$ (where $\kappa_2 \in \mathbb{R}$), in which case it follows that $\kappa_2 \leq \kappa_1 + 1/s$ for all d and s, or that $\kappa_2 \leq \kappa_1 + 1/S_{\rm U}$. Writing $\tau_1(f_1)$ in terms of κ_1 and κ_2 gives the nice decomposition in terms of latency function ℓ_1 and marginal-cost $f_1 \cdot \ell_1'$:

$$\tau_1(f_1) = \kappa_1 \ell_1(f_1) + \kappa_2 f_1 \cdot \ell_1'(f_1).$$

Finally, consider the network in Figure 2(d). This network has some arbitrary admissible latency function ℓ_1 on edge 1 and a monomial latency function $\ell_2(f_2) = \beta(f_2)^d$ on edge 2. We will choose ℓ_2 such that the optimal and Nash flows coincide on this network for some $r > 1$ when $f_2 = 1$. Due to the additivity of T, we can assume without loss of generality that $\ell_1(0) = 0$ because any nonzero intercept can be "canceled" by adding an equal constant term to ℓ_2.

Let $\beta = \ell_1(f_1)$ and $d = f_1\ell_1'(f_1)/\ell_1(f_1) \geq 1$. Then $\ell_1(f_1) = \ell_2(1)$ and $f_1\ell_1'(f_1) = \ell_2'(1)$; i.e., both the latencies and the marginal costs of the edges are equal, which means that $(f_1, 1)$ is both a Nash and an optimal flow. Since ℓ_2 is a monomial, we can write its tolling function as $\tau_2(f_2) = \kappa_1\beta(f_2)^d + \kappa_2 d\beta(f_2)^d$, where $\kappa_2 \leq \kappa_1 + 1/S_{\mathrm{U}}$. Using this, we can simply derive the first-link tolling function $T(\ell_1)(f_1)$ using the following:

$$\ell_1(f_1) + T(\ell_1)(f_1) = (\beta + \kappa_1\beta + \kappa_2 d\beta)\,(f_2)^d.$$

Substituting the definitions of β and d and canceling similar terms, we obtain that $T(\ell_1)$ satisfies (10) as desired. ■

Next, Lemma 5.2 shows that Nash flows on parallel-path networks behave very nicely under the influence of T^{gmc}. Specifically, Lemma 5.2 proves that the worst-case total latency on a parallel network with T^{gmc} is realized by a low-sensitivity homogeneous population.

Lemma 5.2: Let s^{L} denote a homogeneous population in which every user has sensitivity $S_{\mathrm{L}} \geq 0$, and denote by T^{gmc} a taxation mechanism satisfying the conditions of Lemma 5.1. For any $G \in \mathcal{G}_p$ and any population s in which every user has a sensitivity no less than S_{L},

$$\mathcal{L}^{\mathrm{nf}}\left(G, s^{\mathrm{L}}, T^{\mathrm{gmc}}\right) \geq \mathcal{L}^{\mathrm{nf}}\left(G, s, T^{\mathrm{gmc}}\right). \tag{18}$$

Proof: For every user x, T^{gmc} induces cost functions of the form

$$J_e^x(f_e) = (1 + s_x\kappa_1)\ell_e(f_e) + s_x\kappa_2 f_e\ell_e'(f_e). \tag{19}$$

Since we can scale these costs functions by any user-specific positive scalar without changing the underlying Nash flows, these cost functions are equivalent to the following:

$$J_e^x(f_e) = \ell_e(f_e) + \frac{s_x\kappa_2}{1 + s_x\kappa_1} f_e\ell_e'(f_e). \tag{20}$$

Given the conditions $\kappa_2 \geq 0$ and $\kappa_1 \geq \kappa_2 - 1/S_{\mathrm{U}}$, the expression $\frac{s_x\kappa_2}{1+s_x\kappa_1} \in [0,1]$ and is monotone increasing in s_x. Thus, analysis can be simplified by assuming that $\kappa_1 = 0$, $\kappa_2 = 1$ and cost functions are simply given by

$$J_e^x(f_e) = \ell_e(f_e) + s_x f_e\ell_e'(f_e), \tag{21}$$

where $s_x \in [0, 1]$ for all x.

For convenience, we write $\ell_e^*(f_e) \triangleq f_e\ell_e'(f_e)$. When describing the cost experienced by a particular agent whose sensitivity is $s \in \mathbb{R}_+$, we write

$$\ell_e^s(f_e) \triangleq \ell_e(f_e) + s\ell_e^*(f_e), \tag{22}$$

and we write $\ell_e^{\mathrm{mc}}(f_e) \triangleq \ell_e^1(f_e)$ to denote the marginal-cost function associated with edge e.

The following proposition gives important information about the structure of Nash flows induced by T^{gmc}.

Proposition 5.3: If f^{nf} is a Nash flow on $G \in \mathcal{G}_p$ for population s under the influence of T^{gmc}, the following facts hold for any two paths satisfying $\ell_i(0) \leq \ell_j(0)$, $f_j^{\mathrm{nf}} > 0$, and where a user x is on p_i and user y on p_j:

1) $\ell_i^{\mathrm{mc}}(f_i^{\mathrm{nf}}) \geq \ell_j^{\mathrm{mc}}(f_j^{\mathrm{nf}})$
2) $s_x \leq s_y$.

When $\ell_i(0) < \ell_j(0)$, inequality (1) is strict.

Proof: Order the paths so that $\ell_i(0) \leq \ell_{i+1}(0)$ for all $i < n$, and take two adjacent paths p_i and p_{i+1} such that $f_i^{\mathrm{nf}} > 0$ and $f_{i+1}^{\mathrm{nf}} > 0$. Because this is a Nash flow, any agent y using path p_{i+1} experiences a (weakly) lower cost than he would on path p_i, or

$$\ell_i(f_i) + s_y\ell_i^*(f_i) \geq \ell_{i+1}(f_{i+1}) + s_y\ell_{i+1}^*(f_{i+1}). \tag{23}$$

Any latency function can be uniquely decomposed into its 0-flow latency and its flow-varying part in the following way:

$$\ell(f) = \tilde{\ell}(f) + \ell(0). \tag{24}$$

It is always true that $f\tilde{\ell}_i'(f) = f\ell_i'(f)$, so (23) and $\ell_{i+1}(0) - \ell_i(0) \geq 0$ imply that

$$s_y\left(\ell_i^*(f_i) - \ell_{i+1}^*(f_{i+1})\right) \geq \tilde{\ell}_{i+1}(f_{i+1}) - \tilde{\ell}_i(f_i). \tag{25}$$

In the same Nash flow, consider some user x using path p_i. For this user, a similar argument shows that

$$s_x\left(\ell_i^*(f_i) - \ell_{i+1}^*(f_{i+1})\right) \leq \tilde{\ell}_{i+1}(f_{i+1}) - \tilde{\ell}_i(f_i). \tag{26}$$

Combining (25) and (26) yields

$$0 \leq (s_y - s_x)\left(\ell_i^*(f_i) - \ell_{i+1}^*(f_{i+1})\right), \tag{27}$$

meaning that $s_y \geq s_x$ implies that $\ell_i^*(f_i) \geq \ell_{i+1}^*(f_{i+1})$. That is, higher-sensitivity agents use higher-index paths (paths with higher zero-flow latencies), proving item (2).

This means that for each pair of paths, if we define s_i as the number satisfying

$$\ell_i(f_i) + s_i\ell_i^*(f_i) = \ell_{i+1}(f_{i+1}) + s_i\ell_{i+1}^*(f_{i+1}), \tag{28}$$

it will be the case that each $s_i \leq s_{i+1}$ and that $s_i \leq 1$. Finally, it follows from $\ell_i^*(f_i) \geq \ell_{i+1}^*(f_{i+1})$ and (28) that for any $s_i < 1$, we have $\ell_i(f_i) + \ell_i^*(f_i) > \ell_{i+1}(f_{i+1}) + \ell_{i+1}^*(f_{i+1})$, proving item (1). ■

The basic proof approach is to exploit this ordering of marginal costs, and show that reducing agents' sensitivities (thereby making the population "more homogeneous") shifts agents from low marginal-cost paths to high marginal-cost paths, increasing the total latency. Formally, we define a mapping $\Sigma : [0,1] \times \mathcal{S} \to \mathcal{S}$. For any starting population s^0 and any α, we will define $\Sigma\left(\alpha; s^0\right)$ as a right-shift of s^0 by α units. The sensitivity of user x in population $\Sigma(\alpha, s^0)$ is given by

$$\Sigma(\alpha, s^0)_x = \begin{cases} s_0(0) & \text{if } x \leq \alpha \\ s_0(x - \alpha) & \text{if } x > \alpha. \end{cases} \tag{29}$$

Because s is defined to be an increasing function, this is equivalent to converting a mass of α of the most-sensitive users to a mass α of the least-sensitive users.

Proposition 5.3 allows us to assume without loss of generality that any user population s has a finite number of sensitivity types; to see this, simply note that if users with distinct sensitivities are using the same path in a Nash flow, one sensitivity may be exchanged for the other without perturbing either agent's preferences. To be precise, given a Nash flow $f^{\rm nf}$, we will assume for each path $p_i \in \mathcal{P} \setminus p_1$, each user has the *minimally-indifferent* sensitivity; that is, the sensitivity satisfying (28).

For notational brevity, we will typically write $f^{\rm nf}(\alpha)$ to represent $f^{\rm nf}(\Sigma(\alpha; s_0))$. Our central goal will be to characterize the effect of marginal increases in α on the Nash flow. We express this marginal effect as $\frac{\partial}{\partial\alpha} f^{\rm nf}(\alpha)$.

The following definition will be helpful in the proof:

Definition 1: In a Nash flow $f^{\rm nf}$, paths p_i and p_j with $i < j$ are said to be *strategically coupled* if s_i satisfies $\ell_i^{s_i}(f_i^{\rm nf}) = \ell_j^{s_i}(f_j^{\rm nf})$. That is, agents on the lower-order path are indifferent between the two paths. We write $\mathcal{P}_i(f^{\rm nf})$ to denote the set of paths that are strategically coupled to path p_i in $f^{\rm nf}$.[4]

First, we show that the primary effect of an increase in α is to shift traffic from $\mathcal{P}_n$ to $\mathcal{P}_1$.

Proposition 5.4: For every path $p_i \in \mathcal{P}_1$, $\frac{\partial}{\partial\alpha} f_i^{\rm nf}(\alpha) \geq 0$. For every path $p_j \in \mathcal{P}_n$, $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) \leq 0$.

Proof: Let s_1 denote the sensitivity of agents using p_1 in $f^{\rm nf}$. Increasing α changes the sensitivity of a small fraction of high-sensitivity users to s_1. By Definition 1 and Proposition 5.3, these users strictly prefer the paths in $\mathcal{P}_1$ to any other paths, so a marginal increase in α induces a marginal increase in flow on $\mathcal{P}_1$. That is, at least one path in $p_i \in \mathcal{P}_1$ has $\frac{\partial}{\partial\alpha} f_i^{\rm nf}(\alpha) > 0$. An implication of Proposition 5.3 is that all paths in $\mathcal{P}_1$ have strictly flow-varying cost functions, so an increase on flow on p_i induces an increase in flow on all paths in $\mathcal{P}_1$, proving the first statement.

Next, let s_n denote the sensitivity of agents using p_n in $f^{\rm nf}$; Definition 1 and Proposition 5.3 shows that these agents weakly prefer $\mathcal{P}_n$. Increasing α shifts some of these users to $\mathcal{P}_1$, so at least one path in $p_i \in \mathcal{P}_n$ has $\frac{\partial}{\partial\alpha} f_i^{\rm nf}(\alpha) < 0$. If $\mathcal{P}_n$ contains a path with a constant latency function, then this is the path which the flow leaves; otherwise, the flow would deviate to a non-Nash flow. On the other hand, if all paths in $\mathcal{P}_n$ are strictly flow-varying, then every path flow in $\mathcal{P}_n$ must decrease, proving the second statement. ■

Proposition 5.5: For any α, if $p_j \notin \mathcal{P}_1(\alpha)$ and $p_j \notin \mathcal{P}_n(\alpha)$, it holds that $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) = 0$.

Proof: First, let p_i be the lowest-index path such that $p_i \notin \mathcal{P}_1$ (that is, $p_{i-1} \in \mathcal{P}_1$). Definition 1 means that for any $p_j \in \mathcal{P}_1$, $\ell_j^{s_j}(f_j) < \ell_i^{s_j}(f_i)$. Since the inequality is strict, the fact from Proposition 5.4 that $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) \geq 0$ means that marginally no agent on $\mathcal{P}_1$ will switch to p_i.

However, since $f^{\rm nf}(\alpha)$ is a Nash flow, it is true that $\ell_j^{s_i}(f_j) \geq \ell_i^{s_i}(f_i)$. Here, $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) \geq 0$ implies that $\ell_j^{s_i}(f_j)$ can only increase, so no agent on p_i will be incentivized to switch to any path in $\mathcal{P}_1$. Thus, the flow on p_i is not influenced by the changes in flow on any lower-index path, so if its flow changes, the influence must come from some higher-index path.

Now, let p_i be the highest-index path such that $p_i \notin \mathcal{P}_n$ (that is, $p_{i+1} \in \mathcal{P}_n$). Definition 1 means that for any $p_j \in \mathcal{P}_n$, $\ell_i^{s_i}(f_i) < \ell_j^{s_i}(f_j)$. Since the inequality is strict, the fact that $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) \leq 0$ means that (marginally) no agent on p_i will be incentivized to switch to any path in $\mathcal{P}_n$. However, since $f^{\rm nf}(\alpha)$ is a Nash flow, it is true that $\ell_j^{s_j}(f_j) \leq \ell_i^{s_j}(f_i)$. Here, $\frac{\partial}{\partial\alpha} f_j^{\rm nf}(\alpha) \leq 0$ implies that $\ell_j^{s_j}(f_j)$ can only decrease, so no agent on any path in $\mathcal{P}_n$ will be incentivized to switch to p_i. Thus, the flow on p_i is not influenced by the changes in flow on any higher-index path.

This argument may then be repeated with all remaining paths that are not in $\mathcal{P}_1$ or $\mathcal{P}_n$ to show that the only path flows that may change in response to α are those in $\mathcal{P}_1$ and $\mathcal{P}_n$, obtaining the proof of the proposition. ■

Proof of Lemma 5.2: We can now quantify the effect of an increase in α on total latency. In the following, $\nabla_f L(f)$ represents the gradient vector of L with respect to flow f given by $\{\ell_p^{\rm mc}\}_{p\in\mathcal{P}}$, which by Proposition 5.3 is ordered descending. Let p_j be the highest-index path in $\mathcal{P}_1$, and p_k be the lowest-index path in $\mathcal{P}_n$:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha} L\left(f^{\rm nf}(\alpha)\right) &= \nabla_f L\left(f^{\rm nf}(\alpha)\right) \cdot \frac{\partial}{\partial\alpha} f^{\rm nf}(\alpha) \\
&= \sum_{i\in\mathcal{P}_i\cup\mathcal{P}_n} \ell_i^{\rm mc}\left(f_i^{\rm nf}(\alpha)\right) \frac{\partial}{\partial\alpha} f_i^{\rm nf}(\alpha) \\
&\geq \left[\ell_j^{\rm mc}\left(f_j^{\rm nf}(\alpha)\right) - \ell_k^{\rm mc}\left(f_k^{\rm nf}(\alpha)\right)\right] \geq 0.
\end{aligned}
$$

Since at every Nash flow $f^{\rm nf}(\alpha)$ it is true that $\frac{\partial}{\partial\alpha} L\left(f^{\rm nf}(\alpha)\right) \geq 0$, the definition of $\Sigma(\alpha, s_0)$ implies that for any initial sensitivity distribution s_0,

$$
\mathcal{L}\left(f^{\rm nf}\left(\Sigma\left(1, s_0\right)\right)\right) \geq \mathcal{L}\left(f^{\rm nf}\left(\Sigma\left(0, s_0\right)\right)\right), \tag{30}
$$

or that $\mathcal{L}^{\rm nf}\left(G, s^{\rm L}, T^{\rm gmc}\right) \geq \mathcal{L}^{\rm nf}\left(G, s, T^{\rm gmc}\right)$. ■

Proof of Theorem 4.1: Let $G \in \mathcal{G}_p$ be a parallel-path network, s be any arbitrary sensitivity distribution, $s^{\rm L}$ be a homogeneous population in which all users have sensitivity $S_{\rm L}$, and let taxation mechanism $T^{\rm gmc}$ satisfy (10). Lemma 5.2 ensures that

$$
\mathcal{L}^{\rm nf}\left(G, s^{\rm L}, T^{\rm gmc}\right) \geq \mathcal{L}^{\rm nf}\left(G, s, T^{\rm gmc}\right). \tag{31}
$$

Let s^0 denote a totally-insensitive homogeneous population; that is, all agents have sensitivity 0. Note that s^0 is itself a low-sensitivity homogeneous population and that s is a population in which all users have sensitivity no less than 0; thus, we may simply apply Lemma 5.2 a second time to obtain

$$
\mathcal{L}^{\rm nf}\left(G, s^0, T^{\rm gmc}\right) \geq \mathcal{L}^{\rm nf}\left(G, s^{\rm L}, T^{\rm gmc}\right). \tag{32}
$$

The left-hand side of (32) is simply the un-tolled total latency on G, so combining inequalitites (31) and (32), we obtain

$$
\mathcal{L}^{\rm nf}(G, \emptyset) \geq \mathcal{L}^{\rm nf}\left(G, s, T^{\rm gmc}\right). \tag{33}
$$

Since G and s were arbitrary, this implies that $T^{\rm gmc}$ is weakly-robust on $\mathcal{G}_p$. ■

[4]When clear from context, we write $\mathcal{P}_i(f^{\rm nf})$ simply as $\mathcal{P}_i$.