Optimal Mechanisms for Robust Coordination in Congestion Games

Philip N. Brown and Jason R. Marden

***Abstract*—Uninfluenced social systems often exhibit suboptimal performance; specially-designed taxes can influence agents' choices and thereby bring aggregate social behavior closer to optimal. A perfect system characterization may enable a planner to apply simple taxes to incentivize desirable behavior, but system uncertainties may necessitate highly-sophisticated taxation methodologies. In this paper, we study the application of taxes to a network-routing game, assuming that the tax-designer knows neither the network topology nor the tax-sensitivities and demands of the agents. We show that it is possible to design taxes that guarantee that selfish network flows are arbitrarily close to optimal flows, despite the fact that agents' tax-sensitivities are unknown. In general, these taxes may be large; accordingly, for affine-cost parallel-network routing games, we explicitly derive the optimal bounded tolls and the best-possible performance guarantee as a function of a toll upper-bound. Finally, we restrict attention to simple fixed tolls and show that they fail to provide strong performance guarantees if the designer lacks accurate information about network topology or user sensitivities.**

I. Introduction

It is well-known that in systems that are driven by social behavior, agents' self-interested behavior can significantly degrade system performance. This poor performance is commonly referred to as the *price of anarchy*, defined as the ratio between the worst-case social welfare resulting from selfish behavior and the optimal social welfare [2]. This degradation of performance due to selfish behavior has been the subject of research in areas of network resource allocation [3], distributed control [4], traffic congestion [5]–[7], and others. As a result, there is a growing body of research geared at influencing social behavior to improve system performance [8]–[13].

To study the issues surrounding the problem of influencing selfish social behavior, we turn to a simple model of traffic routing: a unit mass of traffic needs to be routed across a network in such a way that minimizes the average network transit time. If a central planner has the ability to direct traffic explicitly, it is straightforward to compute the routing profile that minimizes total congestion. However, in real systems, it may not be possible to implement such direct centralized control: for example, if the network represents a city's road network, individual drivers make their own routing choices in response to their own personal objectives.

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P. N. Brown is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, pnbrown@ece.ucsb.edu. Corresponding author.

J. R. Marden is with the Department of Electrical and Computer Engineering, University of California, Santa Barbara, jrmarden@ece.ucsb.edu.

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Accordingly, we model this routing problem as a nonatomic congestion game, where the traffic can be viewed as a collection of infinitely-many users, each controlling an infinitesimally-small amount of traffic and seeking to minimize its own experienced transit time. We use the concept of a *Nash flow* (defined as a routing profile in which no user can switch to a different path and decrease her transit delay) to characterize the routing profile resulting from such self-interested behavior. It is widely known that Nash flows can exhibit considerably higher congestion than optimal flows. An important result in this setting states that a Nash flow on a network with linear-affine latency functions can be up to 33% worse than the optimal flow; that is, the price of anarchy in this setting is $4/3$. For networks with general latency functions, the price of anarchy can be unbounded [14].

A natural approach to mitigating this performance degradation is to charge monetary taxes for the use of network links, thereby modifying the users' individual costs and incentivizing a new, lower-cost Nash flow. Existing research has explored various methods of designing such optimal taxes given that the tax-designer has access to certain information regarding the system. In [15]–[17] it is shown that optimal "fixed" taxes (i.e., taxes are constant functions of traffic flow) can be computed for any routing game, but the computation requires precise characterizations of the network topology, user demands, and user tax-sensitivities. In contrast, [18], [19] derive optimal taxes known as "marginal-cost taxes" which require no knowledge of the network topology or user demands, but require that all users share a common tax-sensitivity. Furthermore, the taxation functions must be strictly flow-varying. In Section III, we survey these existing results in greater detail.

Thus, we see in the literature a hint of the relationship between a designer's capabilities (i.e., the tolling methodologies available to a designer), the information available at design-time, and the resulting performance guarantees. Table I compares several taxation mechanisms; note that optimal marginal-cost tolls require no information about the network topology or user demands, but must employ flow-varying tolling functions to compensate for this reduced informational dependency.

In this paper, we ask if it is possible to compute optimal taxes with no information about the system, and present several new results showcasing this relationship between sophistication, information, and achievable performance. Our main contribution is to derive a universal taxation mechanism that guarantees arbitrarily-good performance for any routing game while requiring no *a priori* knowledge of the specific network, user demand profile, or distribution of user sensitivities. This result holds for networks with general latency functions and

TABLE I

Toll Type	Information Available			Tolling Functions Required		
	Topology	Demands	Sensitivities	Flow-Varying	Unbounded	Performance Guarantee
Fixed [16], [17]	✓	✓	✓			100%
Marginal-Cost [18], [19]			✓	✓†		100%
Theorem 1: Universal				✓	✓‡	100%
Characterization Results for various tolling-function constraints						
Theorem 3: Bounded Affine				✓		Good, increasing in toll upper bound
Theorem 5: Sens-Agnostic Fixed	✓	✓				Poor; function of sensitivity knowledge

The relationship between allowable tolls, informational dependencies and performance guarantees for several taxation methodologies. Fixed tolls are simple constant functions of flow, but to guarantee optimality, they depend heavily on a precise system characterization. Marginal-cost tolls, though flow-varying, guarantee optimality while only requiring knowledge of the (homogeneous) user-sensitivities. In this paper, Theorem 1 defines tolls which require none of the above information, but are flow-varying and may be arbitrarily large. In Theorem 3, we disallow unbounded tolls and derive the optimal information-independent bounded tolls for a sub-class of networks, and guarantee performance that is increasing in the toll upper-bound. Finally, in Theorem 5, we disallow even flow-varying tolls and show that sensitivity-agnostic fixed tolls perform relatively poorly, while still relying on information about the network topology and user demands.

† We show the necessity of strictly flow-varying tolls in this setting in Theorem 4.

‡ The necessity of unbounded tolls in this setting is an immediate corollary of Theorem 3.

any topology. The third row of Table I summarizes this contribution in context with some existing results. Here, we see that if arbitrarily-high tolls are allowed, information about the users' toll-sensitivities is no longer required.

Since very high tolls may be impossible (or politically unpalatable) to implement, our second contribution is to explore the effect of an upper bound on the allowable tolling functions. To that end, for parallel networks with linear-affine latency functions, we derive the optimal tolling functions that minimize worst-case performance degradation for any unknown distribution of user sensitivities and toll upper bound, again requiring no *a priori* knowledge of the network topology. Surprisingly, these optimal tolls are simple affine functions of flow. We show that for parallel networks with linear-affine cost functions and simple user demands, the worst-case performance degradation strictly decreases with the toll upper bound. Our results suggest that we can compensate for a poor characterization of user sensitivities by charging higher tolls. Unfortunately, by imposing an upper bound on allowable taxation functions, we are no longer able to guarantee optimal behavior. Thus, this result additionally implies that unbounded tolls are *necessary* to enforce optimal flows if both the network topology and user sensitivities are unknown. The row of Table I labeled "Bounded Affine" pertains to this result.

In contrast to our first set of results, in Section VI we show that fixed tolls lack robustness to mischaracterizations of network topology and user sensitivity. Our first result here is that if the network topology is unknown, fixed tolls *cannot* enforce perfectly optimal routing, and we present a simple setting in which network performance can be arbitrarily bad if fixed tolls are not allowed to depend on the network structure. Finally, we show that even if fixed tolls are allowed to depend on the network topology and user demands, they provide relatively poor performance guarantees when the user sensitivities are unknown, as represented in the last row of Table I. Here, by reducing the designer's capability (by disallowing access to flow-varying taxation functions), we dramatically reduce the achievable performance guarantees. This is in sharp contrast to existing results which show that fixed tolls can induce optimal Nash flows when the user sensitivities are precisely known [15]–[17]. Our negative result here vividly demonstrates the need for a clear understanding of the robustness of incentive mechanisms to model imperfections.

II. Model and Summary of Contributions

A. Routing Game

Consider a network routing problem in which a unit mass of traffic needs to be routed across a network (V,E), which consists of a vertex set V and edge set $E \subseteq (V \times V)$. We call a source/destination vertex pair $(s^c,t^c) \in (V \times V)$ a *commodity*, and the set of all such commodities $\mathcal{C}$. For each $c \in \mathcal{C}$, there is a mass of traffic $r^c > 0$ that needs to be routed from s^c to t^c. We write $\mathcal{P}^c \subset 2^E$ to denote the set of *paths* available to traffic in commodity c, where each path $p \in \mathcal{P}^c$ consists of a set of edges connecting s^c to t^c. Let $\mathcal{P} = \cup\{\mathcal{P}^c\}$.

We write f_p^c to denote the mass of traffic from commodity c using path p, and $f_p \triangleq \sum_{c\in\mathcal{C}} f_p^c$. A *feasible flow* $f \in \mathbb{R}^{|\mathcal{P}|}$ is an assignment of traffic to various paths such that for each c, $\sum_{p\in\mathcal{P}^c} f_p^c = r^c$. Without loss of generality, we assume that $\sum_{c\in\mathcal{C}} r^c = 1$.

Given a flow f, the flow on edge e is given by $f_e = \sum_{p:e\in p} f_p$. To characterize transit delay as a function of traffic flow, each edge $e \in E$ is associated with a specific latency function $\ell_e : [0,1] \to [0,\infty)$. We adopt the standard assumptions that latency functions are nondecreasing, continuously differentiable, and convex. Note that latency functions are anonymous: all users affect network delay equally. The cost of a flow f is measured by the *total latency*, given by

$$\mathcal{L}(f) = \sum_{e\in E} f_e \cdot \ell_e(f_e) = \sum_{p\in\mathcal{P}} f_p \cdot \ell_p(f_p), \tag{1}$$

where $\ell_p(f) = \sum_{e\in p} \ell_e(f_e)$ denotes the latency on path p. We denote the flow that minimizes the total latency by

$$f^* \in \operatorname*{argmin}_{f \text{ is feasible}} \mathcal{L}(f). \tag{2}$$

Due to the convexity of ℓ_e, $\mathcal{L}(f^*)$ is unique.

A *routing problem* is given by the tuple $G = (V,E,\mathcal{C},\{\ell_e\})$. We write the set of all such routing problems as $\mathcal{G}$. We will often use shorthand notation such as $e \in \mathcal{G}$ to denote $(e \in G : G \in \mathcal{G})$.

In this paper we study taxation mechanisms for influencing the emergent collective behavior resulting from self-interested price-sensitive users. To that end, we model the above routing problem as a non-atomic game. We assign each edge $e \in E$ a flow-dependent, nondecreasing taxation function $\tau_e : [0,1] \to \mathbb{R}^+$. We characterize the taxation sensitivities of the users in commodity c with a monotone, nondecreasing function $s^c : [0, r^c] \to [S_\mathrm{L}, S_\mathrm{U}]$, where each user $x \in [0, r^c]$ has a taxation sensitivity $s^c_x \in [S_\mathrm{L}, S_\mathrm{U}] \subseteq \mathbb{R}^+$ and $S_\mathrm{U} \geq S_\mathrm{L} \geq 0$ denote upper and lower sensitivity bounds, respectively. Given a flow f, the cost that user $x \in [0, r^c]$ experiences for using path $\tilde{p} \in \mathcal{P}^c$ is of the form

$$J^c_x(f) = \sum_{e \in \tilde{p}} \left[\ell_e(f_e) + s^c_x \tau_e(f_e)\right], \tag{3}$$

and we assume that each user prefers the lowest-cost path from the available source-destination paths. We call a flow f a *Nash flow* if for all commodities $c \in \mathcal{C}$ and all users $x \in [0, r^c]$ we have

$$J^c_x(f) = \min_{p \in \mathcal{P}^c} \left\{ \sum_{e \in p} \left[\ell_e(f_e) + s^c_x \tau_e(f_e)\right] \right\}. \tag{4}$$

It is well-known that a Nash flow exists for any non-atomic game of the above form [20]; further, such Nash flows are essentially unique.

In our analysis, we assume that each sensitivity distribution function s^c is unknown; for a given routing problem G and $S_\mathrm{U} \geq S_\mathrm{L} \geq 0$ we define the set of possible sensitivity distributions as the set of monotone, nondecreasing functions $\mathcal{S}_G = \{s^c : [0, r^c] \to [S_\mathrm{L}, S_\mathrm{U}]\}_{c \in \mathcal{C}}$. We write $s \in \mathcal{S}_G$ to denote such a specific collection of sensitivity distributions, which we term a *population*.

For a given routing problem $G \in \mathcal{G}$, we gauge the efficacy of a collection of taxation functions $\tau = \{\tau_e\}_{e \in E}$ by comparing the total latency of the resulting Nash flow and the total latency associated with the optimal flow, and then performing a worst-case analysis over all possible user populations. Let $\mathcal{L}^*(G)$ denote the total latency associated with the optimal flow, and $\mathcal{L}^\mathrm{nf}(G, s, \tau)$ denote the total latency of the Nash flow resulting from taxation functions τ and population s. The worst-case system cost associated with this specific instance is captured by the *price of anarchy* which is of the form

$$\mathrm{PoA}(G, \tau) = \sup_{s \in \mathcal{S}_G} \left\{ \frac{\mathcal{L}^\mathrm{nf}(G, s, \tau)}{\mathcal{L}^*(G)} \right\} \geq 1.$$

B. Summary of Our Contributions

Our results can be broadly grouped into two categories: in Theorems 1, 2, and 3, we present a series of positive results regarding the effectiveness of large tolls for situations in which the toll-designer has little knowledge of specific system information. Second, Theorems 4 and 5 are a pair of negative results regarding fixed tolls, showing that fixed tolls generally perform poorly unless the toll-designer has accurate characterizations of the system.

Specifically, in Theorem 1 we prove that if each edge's taxation function is given by the universal expression

$$\tau^\mathrm{u}_e(f_e) = \kappa \left(\ell_e(f_e) + f_e \frac{d}{df_e} \ell_e(f_e) \right), \tag{5}$$

for $\kappa \in \mathbb{R}^+$, the price of anarchy converges to 1 as κ approaches infinity, for any user population and network topology. Thus, the toll designer can enforce arbitrarily-good performance simply by charging these tolls with sufficiently high κ. Note that these tolls are universal in the sense that they have no dependence on the specific network or sensitivity distribution.

In Theorem 2, we provide some insight into how high κ must be to guarantee a particular price of anarchy. Here, we prove that if users are homogeneous in price-sensitivity, the price of anarchy of Theorem 1's universal taxation mechanism is given by

$$\sup_{G \in \mathcal{G}} \mathrm{PoA}\left(G, \tau^\mathrm{u}(\kappa)\right) \leq \frac{1 + \kappa S_\mathrm{L}}{\kappa S_\mathrm{L}}. \tag{6}$$

Naturally, in some situations it may be impractical to charge very high tolls; for example, it may be politically unpalatable, or there may be a degree of elasticity in network demand. Accordingly, in Theorem 3, we investigate the effect of an upper bound T on allowable tolling functions for single-commodity parallel networks in which each ℓ_e is linear-affine. Though one might expect that the optimal tolling functions in this situation would be equal to the toll presented in Theorem 1 for some value of κ, this is not generally the case.

Theorem 3 derives functions $\kappa_1(\mathcal{G}, S_\mathrm{L}, S_\mathrm{U}, T)$ and $\kappa_2(\mathcal{G}, S_\mathrm{L}, S_\mathrm{U}, T)$ such that if an edge's latency function is $\ell_e(f_e) = a_e f_e + b_e$, the optimal tolling function is given by

$$\tau_e(f_e) = \kappa_1(\mathcal{G}, S_\mathrm{L}, S_\mathrm{U}, T) a_e f_e + \kappa_2(\mathcal{G}, S_\mathrm{L}, S_\mathrm{U}, T) b_e, \tag{7}$$

and we derive expressions for the price of anarchy when using this tolling methodology. Since $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ do not depend on instance-specific parameters, these tolls can be applied without *a priori* knowledge of the specific routing instance. Thus, these performance guarantees are robust to a wide variety of mischaracterizations of the routing scenario.

We compare the efficacy of the Theorem 1 and 3 taxation mechanisms in Figure 1. Though both mechanisms guarantee optimal flows in the large-toll limit, the tolls from Theorem 3 are far more effective when the toll bound is low. This shows that the universal guarantees made by Theorem 1 come at a price: if we have additional information about the specific class of networks, we may be able to guarantee significantly lower system cost for a given toll upper bound.

We now turn to our negative results regarding fixed tolls. If the network topology is unknown, (strictly) flow-varying tolls are sufficient to guarantee a price of anarchy of 1, either in the case of homogeneous sensitivities [18] or large tolls (Theorem 1). In Theorem 4, we ask if there exist fixed tolls which guarantee a price of anarchy of 1 for unknown network topologies, and prove that there do not. Specifically, we prove that if the network topology is unknown, strictly flow-varying tolls are indeed *necessary* to enforce optimal routing, even for homogeneous user populations.

Though Theorem 4 shows that fixed tolls are ineffective when the network topology is unknown, it gives no insight into the performance of fixed tolls when user sensitivities are unknown. Accordingly, Theorem 5 investigates the performance of fixed tolls when the network topology is known, but user

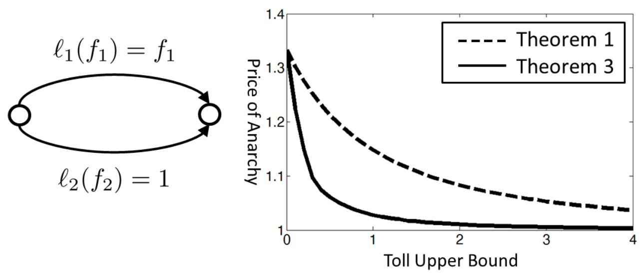

Fig. 1. Price of Anarchy plot contrasting the Universal toll result from Theorem 1 (dashed line) with the optimal toll result from Theorem 3 (solid line). For both price of anarchy curves, the user sensitivities satisfy $S_{\rm L} = 1$ and $S_{\rm U} = 10$. Note that the price of anarchy of either taxation mechanism converges to 1 as the toll upper bound increases, but the solid line converges much more quickly. This is because Theorem 3 gives the optimal tolls for a specific class of networks (parallel networks), but the universal tolls from Theorem 1 are designed to work on *all* classes of networks.

sensitivities are not. We show that the routing performance resulting from fixed tolls can be quite poor in general, and significantly worse than that guaranteed by the flow-varying result in Theorem 3. This quantifies the intuitive principle that performance guarantees are heavily dependent on the capabilities of the tax designer: flow-varying taxation functions can help compensate for poor system characterizations. Figure 4 compares the effectiveness of fixed tolls with that of optimal flow-varying tolls.

III. Related Work

The following is a brief overview of the existing literature on taxation mechanisms in this context. A taxation mechanism simply computes edge tolls as a function of some set of information about the system; here, we focus in particular on the informational dependencies of several well-studied taxation approaches.

– *Omniscient taxation mechanisms:* These taxation mechanisms are assumed to have access to complete information regarding the routing game. For edge $e \in G$, with sensitivity distribution $s \in \mathcal{S}$, the edge tolling function takes the following form: $\tau_e\left(f_e; G, s\right)$. That is, each edge's taxation function can depend on the entire routing problem G and the population sensitivities s. Recent results have identified taxation mechanisms of this form that assign fixed tolls (i.e., for any $e \in G$, $\tau_e(f_e) = q_e$ for some $q_e \geq 0$) that can enforce any feasible flow [16], [17], thus guaranteeing a price of anarchy of 1. However, the robustness of these mechanisms to variations or mischaracterizations of network topology and user sensitivities is heretofore unknown [21].

– *Network-agnostic taxation mechanisms:* This type of taxation mechanism is agnostic to network specifications: each taxation function is derived from locally-available information only. Here, a system designer essentially commits to a taxation function for each potential edge $e \in \mathcal{G}$, and any network realization $G \in \mathcal{G}$ merely employs a subset of these predefined taxation functions. An edge's toll cannot depend on any *other* edge's cost or location in the network.

A commonly-studied network-agnostic taxation mechanisms is the marginal-cost (or Pigovian) taxation mechanism $\tau^{\rm mc}$, which is of the following form: for any $e \in \mathcal{G}$ with latency function ℓ_e, the accompanying taxation function is

$$\tau^{\rm mc}\left(f_e\right) = f_e \cdot \frac{d}{df_e}\ell_e(f_e),\ \forall f_e \geq 0. \tag{8}$$

In [18] it is shown that for any $G \in \mathcal{G}$ we have $\mathcal{L}^*(G) = \mathcal{L}^{\rm nf}\left(G, s, \tau^{\rm mc}\right)$ provided that all users have a sensitivity exactly equal to 1. Hence, irrespective of the underlying network structure, a marginal-cost taxation mechanism always ensures the optimality of the resulting Nash flow, provided that all users share a common known sensitivity.

There are many other results in this area; for example, in [22] the authors investigate the price of anarchy of various types of tolling functions with built-in upper bounds. In [23], it is shown that if taxes can be computed in a centralized fashion, any feasible flow can be enforced even if the central planner does not know the network's latency functions. For affine-cost parallel networks, [24] derives omniscient, flow-varying taxation mechanisms for applications where the total traffic rate is unknown. Finally, in [8], the authors show that marginal-cost taxes scaled by $\sqrt{S_{\rm L}S_{\rm U}}$ do possess a degree of robustness to mischaracterizations of user sensitivities for affine-cost parallel networks.

IV. A Universal Taxation Mechanism

In this paper, we prove that network- and sensitivity-agnostic tolls exist which can drive the price of anarchy to 1 for general networks and latency functions. We term these "universal" because they take the same form and provide the same performance guarantee regardless of which particular routing scenario they are applied to. Using this taxation mechanism, we show in Theorem 1 that for any network, regardless of network topology, user demands, or price-sensitivity functions, the price of anarchy can be made arbitrarily close to 1 if we allow edge tolls to be sufficiently high.

Theorem 1. *For any network edge $e \in \mathcal{G}$ with convex, nondecreasing, continuously differentiable latency function ℓ_e, define the universal taxation function on edge e as*

$$\tau^{\rm u}(\kappa) = \kappa\left(\ell_e(f_e) + f_e \cdot \frac{d}{df_e}\ell_e(f_e)\right). \tag{9}$$

Then for any routing problem $G \in \mathcal{G}$ and any $S_{\rm U} \geq S_{\rm L} > 0$,

$$\lim_{\kappa\to\infty} {\rm PoA}\left(G, \tau^{\rm u}(\kappa)\right) = 1. \tag{10}$$

That is, on *any* network being used by *any* population of users, the total latency can be made arbitrarily close to the optimal latency, and each individual link toll is a simple continuous function of that link's flow. The reason for this is that as κ increases, the original latency function has a smaller and smaller relative effect on the users' cost functions; in the large-toll limit, the only cost experienced by the users is the tolling function itself which is specifically designed to induce optimal Nash flows.

Proof. Using a sequence of tolls, we construct a sequence of Nash flows that converges to an optimal flow. Let κ_n be an unbounded, increasing sequence of tolling coefficients.

For any routing problem $G \in \mathcal{G}$ and price-sensitivities $s \in \mathcal{S}_G$, let $f^n = \left(f_p^n\right)_{p\in\mathcal{P}}$ denote the Nash flow resulting from the tolling coefficient κ_n. For each commodity c, let $\mathcal{P}_n^c \subseteq \mathcal{P}^c$ denote the set of paths that have positive flow in f^n. For any $p \in \mathcal{P}_n^c$, there must be some user $x \in [0, r^c]$ using p with sensitivity s_x^c; the cost experienced by this user is given by

$$J_x^c(f^n) = \sum_{e\in p}\left[\ell_e(f_e) + \kappa_n s_x^c\left(\ell_e(f_e) + f_e \cdot \frac{d}{df_e}\ell_e(f_e)\right)\right].$$

Define $\gamma_{n,x} \triangleq \frac{\kappa_n s_x^c}{1+\kappa_n s_x^c}$. Let $\ell_e^*(f_e) = f_e \cdot \frac{d}{df_e}\ell_e(f_e)$; then for any other path $p' \in \mathcal{P}^c \setminus p$, user x must experience a lower cost on p than on p', or

$$\sum_{e\in p}\ell_e(f_e) - \sum_{e\in p'}\ell_e(f_e) \leq \gamma_{n,x}\left[\sum_{e\in p'}\ell_e^*(f_e) - \sum_{e\in p}\ell_e^*(f_e)\right]. \quad (11)$$

Therefore, for any $n \geq 1$, f^n must satisfy some set of inequalities defined by (11). Note that for all $c \in \mathcal{C}$ and any $x \in [0, r^c]$, $\lim_{\kappa_n\to\infty}\gamma_{n,x} = 1$, so because all the functions in (11) are continuous, f^n converges to a set F^* of feasible flows that satisfy

$$\sum_{e\in p}\ell_e(f_e) - \sum_{e\in p'}\ell_e(f_e) \leq \left[\sum_{e\in p'}\ell_e^*(f_e) - \sum_{e\in p}\ell_e^*(f_e)\right] \quad (12)$$

for all c, all $p \in \mathcal{P}^{c*}$, and $p' \in \mathcal{P}^c$, where $\mathcal{P}^{c*} \subseteq \mathcal{P}^c$ is some subset of paths. But inequalities (12) (combined with the feasibility constraints on f) also specify a Nash flow for G for a unit-sensitivity population with marginal-cost taxes as specified by (8). Any such Nash flow must be optimal [18]; that is, any $f \in F^*$ is a minimum-latency flow for G. Thus, since $\mathcal{L}(f)$ is a continuous function of f,

$$\lim_{n\to\infty}\mathcal{L}\left(f^n\right) = \mathcal{L}^*\left(G\right), \quad (13)$$

obtaining the proof of the theorem. □

Example 1 [An Application of Theorem 1] Consider again the simple two-link "Pigou's Example" network depicted on the left in Figure 1. An un-tolled Nash flow on this network has all traffic using the upper congestion-sensitive link (with a total latency of 1), while the optimal flow has the traffic split evenly between link 1 and link 2 (with a total latency of 0.75), for a price of anarchy of 4/3.

Suppose $S_\mathrm{L} = 1$ and the tax-sensitivities of the user population are only known to within 10% (i.e., $S_\mathrm{U} = 10$), and we wish to design tolls that reduce the price of anarchy as close to 1 as possible. On this network, Theorem 1 assigns tolling functions $\tau_1(f_1) = 2\kappa f_1$ and $\tau_2(f_2) = \kappa$; we simply need to set κ high enough to achieve our desired performance.

Figure 2 shows plots of the Nash flows and price of anarchy as a function of κ. For a two-link network, a network flow is uniquely determined by the edge-1 flow f_1. For any $\kappa \geq 0$, the gray-shaded area highlights all Nash flows that could result from some sensitivity distribution in $[1, 10]$. Note that if $\kappa = 0$, the figure shows a Nash flow with $f_1 = 1$, but that as $\kappa \to \infty$, all Nash flows in the shaded area converge to $f_1 = 1/2$ (i.e., the optimal flow). The dashed horizontal lines show the price of anarchy associated with a flow at that level. Note that the price of anarchy decreases rapidly with κ, and by the time κ is greater than 10, the price of anarchy is already below 1.01.

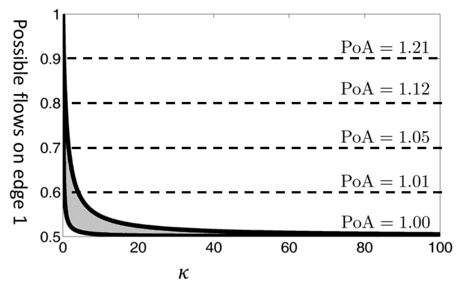

Fig. 2. Figure for Example 1. The plot shows the Nash flows and price of anarchy as a function of κ on Pigou's network (depicted in Figure 1). The bold curves represent the possible edge-1 Nash flows as a function of κ for extreme-sensitivity homogeneous populations S_L and S_U. The gray-shaded area highlights all Nash flows that could result from some heterogeneous population $s \in [1, 10]$. Note that if $\kappa = 0$, the figure shows a Nash flow with $f_1 = 1$, but that the Nash flows in the shaded area converge to $f_1 = 1/2$ as $\kappa \to \infty$. The dashed horizontal lines show the price of anarchy that results from a flow at that level. Note that the price of anarchy decreases rapidly with κ, and by the time κ is greater than 10, the price of anarchy for this network is already well below 1.01.

A. Price of Anarchy Bounds for Homogeneous Populations

The result in Theorem 1 is encouraging since it ensures that no routing game or user population is so pathological that we cannot enforce optimal routing with sufficiently-high tolls, but it gives no indication of *how high* these tolls must be. In our next result in Theorem 2, we show that for homogeneous price-sensitive populations (i.e., all users have the same non-zero price sensitivity), the performance degradation is uniformly bounded in all games by a simple expression.

Theorem 2. *If all users have (unknown) homogeneous price-sensitivity $s \geq S_\mathrm{L} > 0$, the price of anarchy induced by $\tau^\mathrm{u}(\kappa)$ is given by*

$$\sup_{G\in\mathcal{G}}\mathrm{PoA}\left(G, \tau^\mathrm{u}(\kappa)\right) \leq \frac{1+\kappa S_\mathrm{L}}{\kappa S_\mathrm{L}}. \quad (14)$$

Thus, by introducing the additional assumption of homogeneity, we can considerably strengthen the result of Theorem 1 by proving a sense of uniformity of the price of anarchy over all games.

Proof. We employ an argument similar to the so-called "(λ, μ)-smoothness" approach introduced in [25] and extended in [22]. For a population with price-sensitivity $s \geq S_\mathrm{L}$, the cost function seen on edge e by any agent is given by $(1 + s\kappa)\ell_e(f_e) + s\kappa f_e\ell_e'(f_e)$. We can scale this cost function uniformly by $(1 + s\kappa)$ without changing the underlying Nash flows, so defining $\gamma \triangleq \frac{s\kappa}{1+s\kappa} < 1$, the effective cost seen by any user is given by

$$c_e^\gamma(f_e) \triangleq \ell_e(f_e) + \gamma f_e\ell_e'(f_e). \quad (15)$$

Note that $\gamma = 0$ corresponds to un-tolled cost functions (either $\kappa = 0$ or $s = 0$) and $\gamma = 1$ corresponds to exact marginal-cost functions (the limiting case as either κ or s approach infinity).

For any routing problem G, given a Nash flow f for G, following the argument in [26], it is true that for any feasible flow $\bar{f}$ we have

$$\sum_{e\in G} f_e c_e^\gamma(f_e) \leq \sum_{e\in G} \bar{f}_e c_e^\gamma(f_e). \tag{16}$$

Thus, for any feasible flow $\bar{f}$,

$$\begin{aligned}\mathcal{L}(f) &= \sum_{e\in G} f_e \left[\ell_e(f_e) + \gamma f_e \ell_e'(f_e) - \gamma f_e \ell_e'(f_e)\right] \\ &\leq \sum_{e\in G} \left[\bar{f}_e \ell_e(f_e) + \gamma f_e \ell_e'(f_e)\left(\bar{f}_e - f_e\right)\right]. \end{aligned}\tag{17}$$

The convexity of each ℓ_e implies that $\ell_e'(f_e)\left(\bar{f}_e - f_e\right) \leq \ell\left(\bar{f}_e\right) - \ell_e(f_e)$, so we can bound the expression (17) by

$$\mathcal{L}(f) \leq \sum_{e\in G} \left[\bar{f}_e \ell_e(\bar{f}_e) + \left(\bar{f}_e - \gamma f_e\right)\left(\ell_e(f_e) - \ell_e(\bar{f}_e)\right)\right].$$

Here, note that if $\bar{f}_e > f_e$, $\left(\bar{f}_e - \gamma f_e\right)\left(\ell_e(f_e) - \ell_e(\bar{f}_e)\right) < 0$, which is subsumed by the case in which $f_e \geq \bar{f}_e$, when it can be shown that

$$\mathcal{L}(f) \leq \sum_{e\in G} \left[\bar{f}_e \ell_e\left(\bar{f}_e\right) + (1-\gamma) f_e \ell_e(f_e)\right]. \tag{18}$$

That is, for any Nash flow f and any feasible flow $\bar{f}$, $\mathcal{L}(f) \leq \mathcal{L}\left(\bar{f}\right) + (1-\gamma)\mathcal{L}(f)$, or equivalently that $\mathcal{L}(f)/\mathcal{L}\left(\bar{f}\right) \leq 1/\gamma$, which immediately implies the statement of the theorem. □

V. Theorem 3: Optimal Bounded Tolls

Of course, it may be impractical or politically infeasible to charge extremely high tolls. For example, if network demand is elastic, very large tolls could induce some users to avoid travel altogether. Therefore, in Theorem 3, we analyze the effect of placing an upper bound on the allowable tolling functions. For parallel networks with affine cost functions in which every edge has positive flow in an un-tolled Nash flow, we explicitly derive the optimal bounded taxation mechanism, and then provide an expression for the price of anarchy. These optimal tolls are simple affine functions of flow, and the price of anarchy is strictly decreasing in the toll upper bound. Formally, we say a taxation mechanism is *bounded* if it never assigns taxation functions that exceed some upper bound:

Definition 1. *Taxation mechanism τ is* bounded *by T on a class of routing problems $\bar{\mathcal{G}}$ if for every edge $e \in \bar{\mathcal{G}}$, τ assigns a (possibly flow-varying) tolling function that satisfies*

$$\tau_e : [0,1] \to [0,T]. \tag{19}$$

We write the set of taxation mechanisms bounded by T on $\bar{\mathcal{G}}$ as $\mathfrak{T}\left(T, \bar{\mathcal{G}}\right)$.

For the following results, let $\mathcal{G}^p \subseteq \mathcal{G}$ represent the class of all single-commodity, parallel-link routing problems with affine latency functions. That is, for all $e \in \mathcal{G}^p$, the latency function satisfies

$$\ell_e(f_e) = a_e f_e + b_e \tag{20}$$

where $a_e \geq 0$ and $b_e \geq 0$ are edge-specific constants. "Single-commodity" implies that all traffic has access to all network edges. Furthermore, we assume that every edge has positive flow in an un-tolled Nash flow.[1] In order to meaningfully discuss uniform toll bounds on a broad class of networks, it is necessary to describe classes of networks with bounded latency functions. To this end, we define $\mathcal{G}\left(\bar{a}, \bar{b}\right) \subset \mathcal{G}^p$ as the set of parallel, affine-cost networks such that for every $e \in \mathcal{G}\left(\bar{a}, \bar{b}\right)$, the latency function coefficients satisfy $a_e \leq \bar{a}$ and $b_e \leq \bar{b}$.

Definition 2. *For every edge $e \in \mathcal{G}$ with latency function ℓ_e a* network-agnostic *taxation mechanism is a mapping $\tau^{\mathrm{na}} : [0,1] \times \{\ell_e\}_{e\in\mathcal{G}} \to \{\tau_e\}$ that assigns the following flow-dependent taxation function to edge e:*

$$\tau_e(f_e) = \tau^{\mathrm{na}}\left(f_e; \ell_e\right) \tag{21}$$

where $\tau^{\mathrm{na}}\left(f, \ell\right)$ satisfies the following additivity condition:[2] *for all $e, e' \in \mathcal{G}$ and $f \in [0,1]$,*

$$\tau^{\mathrm{na}}\left(f; \ell_e + \ell_{e'}\right) = \tau^{\mathrm{na}}\left(f; \ell_e\right) + \tau^{\mathrm{na}}\left(f; \ell_{e'}\right). \tag{22}$$

Thus, both marginal-cost tolls (8) and universal tolls (9) are network-agnostic according to Definition 2. Network-agnostic taxation mechanisms are attractive because any performance guarantees associated with them are robust to network changes by construction.

Our goal is to derive the bounded network-agnostic taxation mechanism that minimizes the worst-case selfish routing on $\mathcal{G}^p$. We define the price of anarchy with respect to class of routing problems $\mathcal{G}$ and bound T as the best price of anarchy we can achieve on $\mathcal{G}$ with a taxation mechanism bounded by T:

$$\mathrm{PoA}_T\left(\mathcal{G}\right) \triangleq \inf_{\tau\in\mathfrak{T}(T,\mathcal{G})} \left\{\sup_{G\in\mathcal{G}} \mathrm{PoA}\left(G, \tau\right)\right\}. \tag{23}$$

Theorem 3. *Let $\mathcal{G}(\bar{a}, \bar{b}) \subset \mathcal{G}^p$ be some subset of parallel, affine-cost networks with finite $\bar{a}$ and $\bar{b}$. For any toll bound T and $S_{\mathrm{U}} \geq S_{\mathrm{L}} > 0$, define the set of universal parameters by the tuple $U_T = \left(S_{\mathrm{L}}, S_{\mathrm{U}}, \bar{a}, \bar{b}\right)$. Then there exist functions $\kappa_1\left(U_T\right)$ and $\kappa_2\left(U_T\right)$ such that the optimal network-agnostic taxation mechanism bounded by T on $\mathcal{G}(\bar{a}, \bar{b})$ assigns tolling functions*

$$\tau_e(f_e) = \kappa_1(U_T) a_e f_e + \kappa_2(U_T) b_e. \tag{24}$$

Furthermore, the price of anarchy $\mathrm{PoA}_T\left(\mathcal{G}\left(\bar{a}, \bar{b}\right)\right)$ is given by the following:

$$\begin{cases} \frac{4}{3}\left(1 - \frac{\kappa_1(U_T)S_{\mathrm{L}}}{(1+\kappa_1(U_T)S_{\mathrm{L}})^2}\right) & \text{if } \kappa_1(U_T) < \frac{1}{\sqrt{S_{\mathrm{L}}S_{\mathrm{U}}}} \\ \frac{4}{3}\left(1 - \frac{(1+\kappa_1(U_T)S_{\mathrm{L}})\left(\frac{S_{\mathrm{L}}}{S_{\mathrm{U}}} + \kappa_1(U_T)S_{\mathrm{L}}\right)}{\left(1+2\kappa_1(U_T)S_{\mathrm{L}} + \frac{S_{\mathrm{L}}}{S_{\mathrm{U}}}\right)^2}\right) & \text{if } \kappa_1(U_T) \geq \frac{1}{\sqrt{S_{\mathrm{L}}S_{\mathrm{U}}}}. \end{cases} \tag{25}$$

[1] This is essentially a regularity condition that prevents the creation of unrealistic, highly-pathological networks. For example, if a network contains an edge with a very high constant latency function, tolling functions could cause highly-sensitive users to divert to this edge, causing gross network "inefficiencies." Note that we can always assign infinite tolls to such unused edges to ensure that the regularity condition is met.

[2] The additivity condition in Definition 2 is a natural assumption which simply ensures that two edges connected in series will be assigned the same taxation function as if they were replaced by a single edge whose latency function is the sum of the underlying latency functions.

For the reader's convenience, we include a closed-form expression for $\kappa_1(\cdot)$ in the appendix as (49), and for $\kappa_2(\cdot)$ in the proof of Theorem 3 as (33). It is evident from these expressions that $\kappa_1(\cdot)$ and $\kappa_2(\cdot)$ are both nondecreasing and unbounded in T; among other things, this implies that $\lim_{T\to\infty} \mathrm{PoA}_T\left(\mathcal{G}\left(\bar{a},\bar{b}\right)\right)=1$. Qualitatively, it is important to note that they depend only on parameters that are common to all network edges. Thus, the above price of anarchy expression is universal in the sense that it applies to all networks in the class $\mathcal{G}\left(\bar{a},\bar{b}\right)$.

We now proceed with the proof of Theorem 3, which relies on two supporting lemmas. For our first milestone, we restrict attention to simple affine taxation functions:

Lemma 2.1. *Let $\tau^A(\kappa_1,\kappa_2)$ denote an affine taxation mechanism that assigns tolling functions $\tau_e(f_e)=\kappa_1 a_e f_e+\kappa_2 b_e$. For any $\kappa_{\max}\geq 0$, the optimal coefficients κ_1^* and κ_2^* satisfying*

$$(\kappa_1^*,\kappa_2^*)\in\underset{\kappa_1,\kappa_2\leq\kappa_{\max}}{\arg\min}\left\{\sup_{G\in\mathcal{G}^p}\mathrm{PoA}\left(G,\tau^A(\kappa_1,\kappa_2)\right)\right\} \tag{26}$$

are given by

$$\kappa_1^*=\kappa_{\max}, \tag{27}$$

$$\kappa_2^*=\max\left\{0,\ \frac{\kappa_{\max}^2 S_{\mathrm{L}}S_{\mathrm{U}}-1}{S_{\mathrm{L}}+S_{\mathrm{U}}+2\kappa_{\max}S_{\mathrm{L}}S_{\mathrm{U}}}\right\}. \tag{28}$$

Furthermore, for any $G\in\mathcal{G}^p$, $\mathrm{PoA}\left(G,\tau^A(\kappa_1^*,\kappa_2^*)\right)$ is upper-bounded by the following expression:

$$\begin{cases}\frac{4}{3}\left(1-\frac{\kappa_{\max}S_{\mathrm{L}}}{(1+\kappa_{\max}S_{\mathrm{L}})^2}\right) & \text{if}\quad \kappa_{\max}<\frac{1}{\sqrt{S_{\mathrm{L}}S_{\mathrm{U}}}}\\ \frac{4}{3}\left(1-\frac{(1+\kappa_{\max}S_{\mathrm{L}})\left(\frac{S_{\mathrm{L}}}{S_{\mathrm{U}}}+\kappa_{\max}S_{\mathrm{L}}\right)}{\left(1+2\kappa_{\max}S_{\mathrm{L}}+\frac{S_{\mathrm{L}}}{S_{\mathrm{U}}}\right)^2}\right) & \text{if}\quad \kappa_{\max}\geq\frac{1}{\sqrt{S_{\mathrm{L}}S_{\mathrm{U}}}}.\end{cases} \tag{29}$$

See the Appendix for the proof of Lemma 2.1.

Next, in Lemma 2.2, we investigate the possibility that some other taxation mechanism could perform better than the affine $\tau^A(\kappa_1^*,\kappa_2^*)$ while still respecting the bound T. To that end, we assume that some arbitrary taxation mechanism outperforms affine tolls, and deduce various properties of these hypothetical tolls. We show that this hypothetical "better" taxation mechanism must universally charge higher tolls than our optimal affine tolls.

Lemma 2.2. *Let τ^* be any network-agnostic taxation mechanism such that for $\kappa_{\max}\geq 0$*

$$\sup_{G\in\mathcal{G}^p}\mathrm{PoA}\left(\mathcal{G}^p,\tau^*\right)<\sup_{G\in\mathcal{G}^p}\mathrm{PoA}\left(\mathcal{G}^p,\tau^A(\kappa_1^*,\kappa_2^*)\right). \tag{30}$$

Then τ^* must charge strictly higher tolls than $\tau^A(\kappa_1^*,\kappa_2^*)$ on every edge in every network:

$$\forall\ e\in\mathcal{G}^p,\quad \forall\ f_e\in(0,1],\quad \tau_e^*(f_e)>\tau_e^A(f_e). \tag{31}$$

The proof of Lemma 2.2 appears in the Appendix.

Proof of Theorem 3. For any non-negative κ_1 and κ_2, $\tau^A(\kappa_1,\kappa_2)$ is tightly bounded by $\left(\kappa_1\bar{a}+\kappa_2\bar{b}\right)$ on $\mathcal{G}\left(\bar{a},\bar{b}\right)$. Note that for κ_1^* and κ_2^* as defined in Lemma 2.1, $\left(\kappa_1^*\bar{a}+\kappa_2^*\bar{b}\right)$ is a strictly increasing, continuous function of $\kappa_{\max}$. Thus, for any $T\geq 0$, there is a unique $\kappa_{\max}^*\geq 0$ for which $\tau^A(\kappa_1^*,\kappa_2^*)$ is tightly bounded by T on $\mathcal{G}\left(\bar{a},\bar{b}\right)$. We define the function $\kappa_1(U_T)$ as the maximal $\kappa_{\max}^*$ for any $T\geq 0$, given $S_{\mathrm{L}}, S_{\mathrm{U}}, \bar{a}$, and $\bar{b}$. That is, $\kappa_1(U_T)$ is defined implicitly as the unique function satisfying

$$\kappa_1(U_T)\bar{a}+\max\left\{0,\ \frac{\left(\kappa_1^2(U_T)S_{\mathrm{L}}S_{\mathrm{U}}-1\right)\bar{b}}{S_{\mathrm{L}}+S_{\mathrm{U}}+2\kappa_1(U_T)S_{\mathrm{L}}S_{\mathrm{U}}}\right\}=T. \tag{32}$$

For completeness, in the appendix we include a closed-form expression for $\kappa_1(U_T)$ as (49). We define $\kappa_2(U_T)$ as

$$\kappa_2(U_T)=\max\left\{0,\ \frac{\kappa_1^2(U_T)S_{\mathrm{L}}S_{\mathrm{U}}-1}{S_{\mathrm{L}}+S_{\mathrm{U}}+2\kappa_1(U_T)S_{\mathrm{L}}S_{\mathrm{U}}}\right\}. \tag{33}$$

Let $e'\in\bar{\mathcal{G}}$ be an edge with latency function $\ell_{e'}(f_{e'})=\bar{a}f_{e'}+\bar{b}$. By construction, the tolling function assigned by $\tau^A(\kappa_1(U_T),\kappa_2(U_T))$ to e' satisfies bound T with equality: $\tau_{e'}^A(1)=T$.

Now let τ^* be any taxation mechanism with a strictly lower price of anarchy than $\tau^A(\kappa_1(U_T),\kappa_2(U_T))$. By Lemma 2.2, τ^* assigns higher tolling functions than $\tau^A(\kappa_1(U_T),\kappa_2(U_T))$ on *every* edge for every flow rate. In particular, on edge e', $\tau_{e'}^*(1)>\tau_{e'}^A(1)=T$, violating bound T and proving the optimality of $\tau^A\left(\kappa_1(U_T),\kappa_2(U_T)\right)$ over the space of all network-agnostic taxation mechanisms bounded by T. By substituting $\kappa_1(U_T)$ for $\kappa_{\max}$ in expression (29), we obtain the complete price of anarchy expression (25). □

VI. Negative Results for Fixed Tolls

Theorem 3 showed that simple affine tolling functions are sufficient to achieve the best-possible price of anarchy for network-agnostic bounded taxation mechanisms. It is natural to ask what guarantees are possible for an even simpler class of taxation functions, the constant functions. There are practical benefits to such fixed tolls, foremost among which is the simplicity and predictability they offer to network users.

It has long been known that flow-varying tolls are sufficient to optimize network routing in cases when the network topology is unknown [18]. We ask here if fixed tolls can provide the same guarantee; i.e., we ask if (strictly) flow-varying tolls are also *necessary* to optimize routing in these settings. In Theorem 4, we prove this necessity, which immediately implies that the network-agnostic price of anarchy of fixed tolls is bounded away from 1.

Theorem 4. *If for every $G\in\mathcal{G}$ and unit-sensitivity homogeneous population s, network-agnostic taxation mechanism τ satisfies*

$$\mathcal{L}^{\mathrm{nf}}(G,s,\tau)=\mathcal{L}^*(G), \tag{34}$$

then it must be the case that τ assigns strictly flow-varying taxation functions to some network edges.

Proof. We prove Theorem 4 by contradiction. Let τ^{na} be a network-agnostic fixed tolling mechanism for which $\mathcal{L}^{\mathrm{nf}}(G,s,\tau^{\mathrm{na}})=\mathcal{L}^*(G)$; that is, it is a mapping from latency functions to non-negative constant taxation functions that enforces optimal routing on *every* network. Consider the two-path network shown in Figure 3(a); we denote this network G_n. The upper path is composed of n copies of the same link in series; network-agnosticity requires that τ^{na} charges the

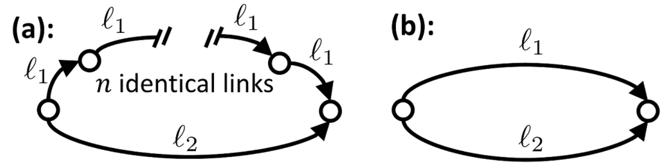

Fig. 3. Networks for Theorem 4 Proof. In both networks, $\ell_1(f_1) = f_1$ and $\ell_2(f_2) = b$, with $b > 0$. (a): Network with n copies of the same link in series on the upper path. Since every upper edge has the same latency function, network-agnostic tolls must charge the same amount to each edge. (b): The same network with $n = 1$; if network-agnostic tolls τ_1 and τ_2 were designed for network (a), they can cause highly inefficient performance on network (b).

same toll to every copy of that link. For a total traffic mass of r, the optimal routing profile for this network is $f_1^* = b/2$ and $f_2^* = r - b/2$. For a unit-sensitivity homogeneous population, optimal fixed tolls τ_1 and τ_2 must satisfy the following expression:

$$\tau_2 = n\tau_1 - b/2. \tag{35}$$

Since these tolls are network-agnostic, τ_1 cannot be a function of b, so there exists some universal constant $\beta > 0$ for which $\tau_1 = \beta$ and $\tau_2 = n\beta - b/2$. It is straightforward to show that for any n and any choice of β, these tolls induce optimal routing on the network for a unit-sensitivity homogeneous user population. That is, $\mathcal{L}^{\rm nf}(G_n, s, \tau^{\rm na}) = \mathcal{L}^*(G_n)$

Our hope is that these tolling functions would optimize routing when applied to *any* network; i.e., that we could apply $\tau_1 = \beta$ to any edge with latency function $\ell_e(f_e) = f_e$, and τ_2 to any edge with latency function b and still get optimal performance. To test this, we apply the same tolls to the network in Figure 3(b), which we denote G_1. Here, we find that $\tau_2 = n\beta - b/2$ is now much too high; if the total traffic rate is high enough, these tolls induce a flow with $f_1 = \beta(n-1) + b/2$ and $f_2 = 0$, even though the optimal flow has $f_1 = b/2$. This allows us to compute a lower bound on the price of anarchy for these tolling functions:

$$\frac{\mathcal{L}^{\rm nf}(G_1, s, \tau^{\rm na})}{\mathcal{L}^*(G_1)} \geq \frac{\left(\beta(n-1) + \frac{b}{2}\right)^2}{b\left(\beta(n-1) + \frac{b}{4}\right)}, \tag{36}$$

which is unbounded in both n and β, generating a contraction to our hypothesis that for all G, $\mathcal{L}^{\rm nf}(G, s, \tau^{\rm na}) = \mathcal{L}^*(G)$. □

In light of this negative result, in Theorem 5, we ask what guarantees are possible with fixed tolls if we know the network structure but do not know the user sensitivities; refer to the last row of Table I for a quick summary of the setting we investigate here. Since we are allowing these fixed tolls to depend on network structure (e.g., the number of edges in the network), we denote such taxation functions by $\tau^{\rm ft}(G) = \{\tau_e^{\rm ft}(G)\}_{e \in G}$. The following theorem demonstrates that any network-dependent fixed-toll taxation mechanism generally provides poor performance guarantees when compared with the optimal bounded taxation mechanism from Theorem 3.

Theorem 5. *Consider any network-dependent fixed-toll taxation mechanism $\tau^{\rm ft}$. For any network $G \in \mathcal{G}^p$,*

$$\sup_{s \in \mathcal{S}} \mathcal{L}^{\rm nf}\left(G, s, \tau^{\rm ft}(G)\right) \geq \sup_{s \in \mathcal{S}} \mathcal{L}^{\rm nf}\left(G, s, \tau^A(1/S_{\rm U}, 0)\right), \tag{37}$$

with affine tolls $\tau^A(\cdot)$ as defined in Lemma 2.1. Thus,

$$\begin{aligned}\sup_{G \in \mathcal{G}} \mathrm{PoA}\left(G, \tau^{\rm ft}\right) &\geq \sup_{G \in \mathcal{G}^p} \mathrm{PoA}\left(G, \tau^A(1/S_{\rm U}, 0)\right) \\ &= \frac{4}{3}\left(1 - \frac{S_{\rm L}/S_{\rm U}}{\left(1 + S_{\rm L}/S_{\rm U}\right)^2}\right). \end{aligned} \tag{38}$$

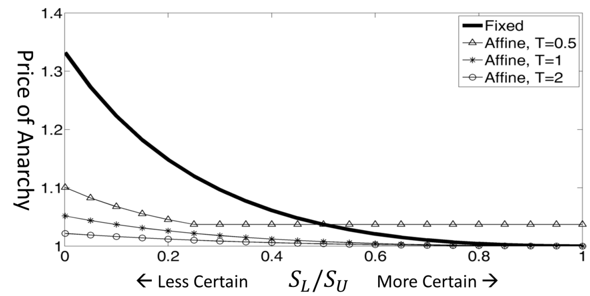

Fig. 4. Comparison of Price of Anarchy guaranteed by Theorems 3 and 5. All plots are for $S_{\rm L} = 1$ and $\bar{a} = \bar{b} = 1$. The horizontal axis represents the level of certainty in price-sensitivity; note that most taxation mechanisms guarantee a price of anarchy of 1 for complete certainty unless they are restricted by a very low upper-bound. The solid line represents the price of anarchy resulting from fixed tolls (according to (38)), and the marked lines represent the price of anarchy resulting from optimal flow-varying affine tolls for a given toll bound (according to (25)). Note that for a very low toll bound, fixed tolls slightly outperform affine tolls for well-characterized popuations; this is due to the fact that the fixed tolls are allowed to depend on network structure, but the affine tolls are not.

We point out that the right-hand side of (38) represents the price of anarchy due to network-agnostic affine tolls for a *very low toll upper bound.* For example, in the canonical Pigou network depicted in Figure 1, if $S_{\rm U} = 10$, affine tolls prescribed by $\tau^A(1/S_{\rm U}, 0)$ imply a toll upper-bound of just 0.1. As shown in Figure 1, the price of anarchy for optimal affine tolls is steeply decreasing in the toll upper-bound, so a designer wishing to exploit the simplicity of fixed tolls may need to accept dramatically lower performance guarantees as a result.

Furthermore, it is important to note that Theorem 5 shows that τ^A, a *network-agnostic* tolling mechanism, provides better performance guarantees (even for moderately low tolls) than $\tau^{\rm ft}$, a *network-dependent* tolling mechanism. This starkly shows the power of Theorem 3's taxation mechanism: given *less* information, it performs *better* than any fixed-toll taxation mechanism.

See Figure 4 for a comparison of the price of anarchy afforded by Theorems 3 and 5, and note that fixed tolls only outperform flow-varying affine tolls when both uncertainty and the toll upper bound are low. In all other situations, uncertainty-optimal affine tolls provide better performance guarantees.

In the proof of Theorem 5, we begin by considering homogeneous sensitivity distributions and then extend to heterogeneous. We use the notation $f^{\rm ft}(G, s, \tau)$ to denote a Nash flow induced by fixed tolls $\tau \in \mathbb{R}^n$ on network G, with homogeneous sensitivity $s \in [S_{\rm L}, S_{\rm U}]$. We write the total latency of this flow as $\mathcal{L}^{\rm nf}(G, s, \tau)$. Similarly, we write the total latency of a Nash flow resulting from affine tolls $\tau^A(\kappa_1, \kappa_2)$ as $\mathcal{L}^{\rm nf}\left(G, s, \tau^A(\kappa_1, \kappa_2)\right)$.

We define the optimal fixed tolls τ^* as those satisfying the following expression:

$$\tau^* \in \arg\min_{\tau \in \mathbb{R}^n} \max_{s \in [S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}(G, s, \tau). \tag{39}$$

That is, τ^* is in the set of edge tolls that minimize the total latency for the worst possible user sensitivity.

In Lemma 5.1, we see that there is a curious relationship between the total latencies of Nash flows resulting from fixed tolls and those resulting from affine tolls $\tau^A(1/S_{\rm U}, 0)$. That is, the optimal fixed tolls guarantee the same worst-case performance as affine tolls with extremely low coefficients.

Lemma 5.1. *For any $G \in \mathcal{G}^p$, for a homogeneous population, the worst-case total latency resulting from the optimal fixed tolls τ^* is equal to the worst-case total latency resulting from $\tau^A(1/S_{\rm L}, 0)$:*

$$\max_{s \in [S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}\left(G, s, \tau^*\right) = \max_{s \in [S_{\rm L}, S_{\rm U}]} \mathcal{L}^{\rm nf}\left(G, s, \tau^A(1/S_{\rm U}, 0)\right). \tag{40}$$

The proof of Lemma 5.1 appears in the appendix.

Proof of Theorem 5. Since the set of homogeneous populations is a strict subset of the set of heterogeneous ones, we can only make things worse by extending from homogeneous to heterogeneous populations, so the bound in (38) must hold. The expression in (38) is obtained by substituting $1/S_{\rm U}$ in for $\kappa_{\max}$ in the first part of expression (29). □

VII. Conclusion

In this paper we have explored several avenues for influencing social behavior when aspects of the underlying system are uncertain. We showed in Theorem 1 that in principle, it is possible to charge tolls that induce arbitrarily-efficient Nash flows without requiring knowledge of the network topology, user demands, or user sensitivities, but that the required tolls may be very high. To make this more realistic, in Theorem 3 we investigated the effect of an upper bound on the allowable tolling functions for affine-cost parallel networks. We showed that affine tolls are sufficient to achieve the lowest price of anarchy over the space of all possible network-agnostic tolling functions for this class of networks, and derived the price of anarchy as an explicit function of the upper bound on tolling coefficients. This neatly demonstrates the principle that the more we can charge, the better performance we can guarantee.

Finally, in Theorems 4 and 5, we further restrict our space of allowable taxation mechanisms to the class of fixed tolls, and show that when network topologies or user sensitivities are unknown, fixed tolls offer markedly worse performance guarantees than those offered by the optimal flow-varying tolls of Theorem 3.

Avenues for future work include extending the conclusions of Theorem 3 to less restrictive classes of networks and cost functions. Furthermore, in this paper we have assumed that user demands were inelastic; an interesting extension would be to model a degree of elasticity, allowing users to simply "stay home" if the total network travel cost is too high.

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Appendix: Proofs of Supporting Lemmas

To prove Lemma 2.1, we analytically relate the Nash flows induced by affine tolls with coefficients κ_1 and κ_2 to the Nash flows induced by marginal-cost tolls scaled by κ_1 for some other sensitivity distribution s'. We can then use known analytical techniques for scaled marginal-cost tolls to derive the optimal κ_1 and κ_2. We make use of the following theorem:

Theorem 6 (Brown and Marden, [8])**.** *For any routing problem $G \in \mathcal{G}^p$ satisfying the assumptions of Theorem 3, the scaled marginal-cost taxation mechanism $\tau^{\rm smc}(\kappa)$ assigns the following tolls to any edge $e \in \mathcal{G}^p$ for $\kappa \geq 0$:*

$$\tau_e^{\rm smc}(f_e) = \kappa a_e f_e. \tag{41}$$

The unique cost-minimizing marginal-cost toll scalar is

$$\kappa^* = \frac{1}{\sqrt{S_{\rm L} S_{\rm U}}} = \underset{\kappa \geq 0}{\arg\min} \ \{{\rm PoA}(G, \tau^{\rm smc}(\kappa))\}. \tag{42}$$

Finally, for any $G \in \mathcal{G}^p$, for $q = S_L/S_U$, the price of anarchy resulting from the optimal scaled marginal-cost taxation mechanism is

$${\rm PoA}\left(G, \tau^{\rm smc}(\kappa^*)\right) \leq \frac{4}{3}\left(1 - \frac{\sqrt{q}}{\left(1+\sqrt{q}\right)^2}\right). \tag{43}$$

Proof of Lemma 2.1

Let $G \in \mathcal{G}^p$ and $\kappa_1 \geq \kappa_2 \geq 0$.[3] For user $x \in [0,1]$ with sensitivity $s_x \in [S_{\rm L}, S_{\rm U}]$, the cost of edge $e \in G$ given flow f under affine tolls is given by

$$J_x^e(f) = (1+\kappa_1 s_x) a_e f_e + (1+\kappa_2 s_x) b_e.$$

Note that we may scale $J_x^e(f)$ by any factor without changing the underlying preferences of agent x, provided that the scale factor is the same for all edges. Thus, without loss of generality, we may write

$$J_x^e(f) = \frac{1+\kappa_1 s_x}{1+\kappa_2 s_x} a_e f_e + b_e. \tag{44}$$

Now, define sensitivity distribution s' by the following: for any $x \in [0,1]$, s'_x satisfies

$$s'_x = \frac{s_x(\kappa_1 - \kappa_2)}{\kappa_1(1+\kappa_2 s_x)}. \tag{45}$$

By a series of algebraic manipulations, we may combine (44) and (45) to obtain

$$J_x^e(f) = \left(1+\kappa_1 s'_x\right) a_e f_e + b_e, \tag{46}$$

which is simply the cost resulting from scaled marginal-cost tolls (41). Thus, for any sensitivity distribution s, we may model a Nash flow resulting from affine tolls with coefficients κ_1 and κ_2 as a Nash flow for sensitivity distribution s' resulting from scaled marginal-cost tolls with $\kappa = \kappa_1$.

Thus, by Theorem 6, assuming first that $\kappa_{\max}$ is sufficiently high, our optimal choice of κ_1 is that which satisfies

$$\kappa_1 = \frac{1}{\sqrt{S'_{\rm L} S'_{\rm U}}}, \tag{47}$$

where $S'_{\rm L}$ and $S'_{\rm U}$ are computed according to (45).

We may combine (45) and (47) to obtain the following characterization of the optimal κ_2 with respect to κ_1, for $\kappa_{\max} \geq \left(S_{\rm L} S_{\rm U}\right)^{-1/2}$:

[3] Here, the requirement that $\kappa_1 \geq \kappa_2$ is without loss of generality; later analysis shows that $\kappa_2 > \kappa_1$ would always result in a Nash flow with higher congestion than the un-tolled case.

$$\kappa_2 = \frac{\kappa_1^2 S_{\rm L} S_{\rm U} - 1}{S_{\rm L} + S_{\rm U} + 2\kappa_1 S_{\rm L} S_{\rm U}}. \tag{48}$$

We compute the price of anarchy resulting from optimal affine tolls by evaluating (43) at $q = S'_{\rm L}/S'_{\rm U}$ to for this high-$\kappa_{\max}$ case, verifying the second part of (29) as the correct expression for ${\rm PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right)$.

Finally, we must consider the case when $\kappa_{\max} < \left(S_{\rm L} S_{\rm U}\right)^{-1/2}$. Now, (48) would prescribe a negative value for κ_2, so the optimal choice is to let κ_2 saturate at 0. Now, we are precisely applying scaled marginal-cost tolls with $\kappa = \kappa_1$, so we apply the fact shown in Lemma 1.2 of [8] that on this class of networks, if $\kappa \leq \left(S_{\rm L} S_{\rm U}\right)^{-1/2}$, the worst-case total latency of a Nash flow always occurs for the extreme low-sensitivity homogeneous sensitivity distribution given by $s_x \equiv S_{\rm L}$ for all $x \in [0,1]$.

Equation (35) in [8] gives the total latency of a Nash flow for a homogeneous population with sensitivity $S_{\rm L}$ as

$$\mathcal{L}^{\rm nf}(G, S_{\rm L}, \kappa) = L_R - \frac{\kappa S_{\rm L}}{\left(1+\kappa S_{\rm L}\right)^2}\Theta, \tag{50}$$

where L_R and Θ are positive constants depending only on G, satisfying $\Theta \leq L_R$. It is easy to verify that the above expression is minimized on a subset of $\left[0, \left(S_{\rm L} S_{\rm U}\right)^{-1/2}\right]$ by maximizing κ, and using the fact that $\Theta \leq L_R$, we may verify that the price of anarchy for $\kappa_{\max} < \left(S_{\rm L} S_{\rm U}\right)^{-1/2}$ is given by the first part of (29), completing the proof of Lemma 2.1. $\square$

Proof of Lemma 2.2

Here, we derive properties of any taxation mechanism that outperforms $\tau^A(\kappa_1^*, \kappa_2^*)$. We define the set of routing problems $\mathcal{G}^0$ as follows: $G \in \mathcal{G}^0$ is a parallel network consisting of two edges, with $\ell_1(f_1) = cf_1$ and $\ell_2(f_2) = c$.

Let $G \in \mathcal{G}^0$. For any c, the optimal flow on G is $(f_1^*, f_2^*) = (1/2, 1/2)$ and the optimal total latency is $\mathcal{L}^*(G) = 3c/4$, but the un-tolled Nash flow has a total latency of $\mathcal{L}^{\rm nf}(G, s, \emptyset) = c$, so the un-tolled price of anarchy is $4/3$. It is straightforward to show furthermore that if $S_{\rm U} > S_{\rm L} \geq 0$, for any $\kappa_{\max} > 0$, this network constitutes a worst-case example and the price of anarchy bound of this particular network is tight; i.e., it equals the expression given in (29): ${\rm PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right) = \sup_{G \in \mathcal{G}^p} {\rm PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right)$. Thus, if our hypothetical τ^* outperforms τ^A in general, it must specifically outperform τ^A on any network $G \in \mathcal{G}^0$, or

$${\rm PoA}\left(G, \tau^*\right) < {\rm PoA}\left(G, \tau^A(\kappa_1^*, \kappa_2^*)\right). \tag{51}$$

Now, we investigate the performance of the hypothetical tolling mechanism τ^* on networks in $\mathcal{G}^0$. Given a network $G \in \mathcal{G}^0$, τ^* assigns edge tolling functions $\tau_1^*(f_1)$ and $\tau_2^*(f_2)$. Recall that since τ^* is network-agnostic, there is some function $\tau^*(f; a, b)$ such that an edge $e \in E$ with latency function $\ell_e(f_e) = a_e f_e + b_e$ is assigned tolling function $\tau^*(f_e; a_e, b_e)$. By analyzing networks in $\mathcal{G}^0$, we can deduce properties of the function with the 2nd and 3rd arguments set to 0, since $\tau_1^*(f_1) = \tau^*(f_1; c, 0)$ and $\tau_2^*(f_2) = \tau^*(f_2; 0, c)$.

Now we show that τ^* must assign higher tolls than $\tau^A(\kappa_1^*, \kappa_2^*)$. Let $S_{\rm U} > S_{\rm L}$. By design, the worst-case Nash

$$\kappa_1(U) = \min\left\{\frac{T}{\bar{a}},\ \frac{2TS_{\rm L}S_{\rm U} - (S_{\rm L}+S_{\rm U})\bar{a} + \sqrt{\left((S_{\rm L}+S_{\rm U})\,\bar{a} + 2TS_{\rm L}S_{\rm U}\right)^2 + 4\bar{b}S_{\rm L}S_{\rm U}\left(2\bar{a}+\bar{b}+T\left(S_{\rm L}+S_{\rm U}\right)\right)}}{2S_{\rm L}S_{\rm U}\left(2\bar{a}+\bar{b}\right)}\right\} \tag{49}$$

Fig. 5. Closed-form expression for $\kappa_1(U)$ used in Theorem 3. Note that it is a continuous, unbounded, strictly increasing function of T.

flows resulting from $\tau^A(\kappa_1^*, \kappa_2^*)$ occur for homogeneous populations with $s = S_{\rm L}$ and $s = S_{\rm U}$. Since any network $G \in \mathcal{G}^0$ has only 2 links, we can characterize a Nash flow simply by the flow on edge 1; accordingly, let $f_{\rm L}(c)$ denote the flow as a function of c on edge 1 in the Nash flow resulting from sensitivity distribution $s = S_{\rm L}$, and $f_{\rm H}(c)$ the corresponding edge 1 flow for $s = S_{\rm U}$. These flows are the solutions to the following equations:

$$cf_{\rm L}(c)\left(1+\kappa_1^* S_{\rm L}\right) = c\left(1+\kappa_2^* S_{\rm L}\right), \tag{52}$$
$$cf_{\rm H}(c)\left(1+\kappa_1^* S_{\rm U}\right) = c\left(1+\kappa_2^* S_{\rm U}\right). \tag{53}$$

Summing (52) and (53) yields

$$\kappa_1^*\left(f_{\rm L}(c) - f_{\rm H}(c)\right) = \frac{f_{\rm H}(c)}{S_{\rm U}} - \frac{f_{\rm L}(c)}{S_{\rm L}} + \frac{1}{S_{\rm L}} - \frac{1}{S_{\rm U}}. \tag{54}$$

It is always true that $f_{\rm H}(c) < f_{\rm L}(c)$. By design, $\mathcal{L}(f_{\rm L}(c)) = \mathcal{L}(f_{\rm H}(c))$. Note that $\mathcal{L}$ is simply a concave-up parabola in the flow on edge 1.

Now, let $f_{\rm L}^*(c)$ and $f_{\rm H}^*(c)$ be defined as the Nash flows resulting from τ^* for a given value of c; i.e., the solutions to

$$cf_{\rm L}^*(c) + \tau_1^*(f_{\rm L}^*(c))S_{\rm L} = c + \tau_2^*(1-f_{\rm L}^*(c))S_{\rm L}, \tag{55}$$
$$cf_{\rm H}^*(c) + \tau_1^*(f_{\rm H}^*(c))S_{\rm U} = c + \tau_2^*(1-f_{\rm H}^*(c))S_{\rm U}. \tag{56}$$

Since τ^* guarantees better performance than $\tau^A(\kappa_1^*, \kappa_2^*)$, it must do so in particular for these homogeneous sensitivity distributions $s = S_{\rm L}$ and $s = S_{\rm U}$. Since $\mathcal{L}$ is a parabola, this means that for any c, $f_{\rm H}(c) < f_{\rm H}^*(c) < f_{\rm L}^*(c) < f_{\rm L}(c)$.

Define the nondecreasing function $\Delta^*(f) = \tau_2^*(f) - \tau_1^*(1-f)$ (which is implicitly also a function of c), so equations (55) and (56) can be combined and rearranged to show

$$\begin{aligned}\Delta^*(f_{\rm L}^*(c)) - \Delta^*(f_{\rm H}^*(c)) &> c\left[\frac{f_{\rm H}(c)}{S_{\rm U}} - \frac{f_{\rm L}(c)}{S_{\rm L}} + \frac{1}{S_{\rm L}} - \frac{1}{S_{\rm U}}\right] \\ &= \kappa_1^* c\left(f_{\rm L}(c) - f_{\rm H}(c)\right)\end{aligned} \tag{57}$$

The above inequality can be further loosened by replacing $f_{\rm L}^*(c)$ with $f_{\rm L}(c)$ and $f_{\rm H}^*(c)$ with $f_{\rm H}(c)$, and substituting from (54) and rearranging, we finally obtain

$$\frac{\Delta^*(f_{\rm L}(c)) - \Delta^*(f_{\rm H}(c))}{f_{\rm L}(c) - f_{\rm H}(c)} > \kappa_1^* c. \tag{58}$$

Since this must be true for any $c > 0$, the average slope of $\Delta^*(f)$ must be greater than $\kappa_1^* c$ for all $f > 0$. Since $\tau_2^*(f) \geq 0$ this implies that $\tau_1^*(f) > \kappa_1^* cf$ for all $f > 0$, or that

$$\tau^*(f; a, 0) > \tau^A(f; a, 0) \tag{59}$$

for all positive f and a.
Now consider the following rearrangement of (56):

$$\begin{aligned}\tau_2^*(1-f_{\rm H}^*(c)) &= \left[cf_{\rm H}^*(c) + \tau_1^*(f_{\rm H}^*(c)) - cS_{\rm U}\right]\cdot\frac{1}{S_{\rm U}} \\ &> c\left[\left(1+\kappa_1^* S_{\rm U}\right)f_{\rm H}(c) - 1\right]\cdot\frac{1}{S_{\rm U}} \\ &= \kappa_2^* c S_{\rm U} = \tau_2^A(f).\end{aligned} \tag{60}$$

This implies that $\tau_2^*(f) > \kappa_2^* c$ for all $f > 0$, or that

$$\tau^*(f; 0, b) > \tau^A(f; 0, b) \tag{61}$$

for all positive f and b.

Finally, note that the additivity assumption of Definition 2 implies that $\tau^*(f; a, b)$ is additive in its second and third arguments. That is, we may add inequalities (59) and (61) to conclude that for all nonnegative f, a, and b, it is true that

$$\tau^*(f; a, b) > \kappa_1^* af + \kappa_2^* b, \tag{62}$$

or that a necessary condition for $\sup_{G\in\mathcal{G}^p} \mathrm{PoA}(G, \tau^*) < \sup_{G\in\mathcal{G}^p} \mathrm{PoA}(G, \tau^A)$ is that τ^* must charge higher tolls on *every* edge in *every* network. $\square$

Proof of Lemma 5.1

We first derive a simple expression for a Nash flow for a homogeneous population as a linear function of the tolls τ.

Claim 4.1.1. *A Nash flow on $G \in \mathcal{G}$ for sensitivity $s \in \mathcal{S}_1$ and fixed tolls $\tau \in \mathbb{R}^n$ that has positive traffic on all links can be described by the following linear function:*

$$f^{\rm ft}(G, s, \tau) = R + H(b + s\tau), \tag{63}$$

where $R \in \mathbb{R}^n$ and $H \in \mathbb{R}^{n\times n}$ are constant matrices depending only on G. The total latency of this flow is given by the following convex quadratic in τ:

$$\mathcal{L}^{\rm ft}(G, s, \tau) = L_R + s\tau^T H^T(2AH + I)b + s^2\tau^T H^T AH\tau. \tag{64}$$

Proof. Since all users share the same sensitivity, all links have equal cost to all agents in a Nash flow, so when all network edges have positive flow, for any $e_i, e_j \in E$,

$$a_i f_i + b_i + s\tau_i = a_j f_j + b_j + s\tau_j.$$

Similar to the approach in the proof of Lemma 1.2 in [8], a Nash flow $f^{\rm ft}(G, s, \tau)$ is a solution f to the linear system

$$\underbrace{\begin{bmatrix} a_1 & -a_2 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}}_{P} f = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{r} + \underbrace{\begin{bmatrix} -1 & 1 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{X} (b + s\tau). \tag{65}$$

P is invertible, so letting $H = P^{-1}X$ and $R = P^{-1}r$, a Nash flow is given by the linear equation (63).

The following observations will be helpful to our proof:

Observation 4.1. *The matrices H and R possess the following properties for any $G \in \mathcal{G}$:*

$$\mathbf{1}^{\mathrm{T}} H b = \mathbf{0}^{\mathrm{T}}, \tag{66}$$

$$\mathbf{1}^{\mathrm{T}} R = 1, \tag{67}$$

$$AR \in \mathrm{sp}\{\mathbf{1}\}, \tag{68}$$

$$b^T H^T A H b = -M^T b. \tag{69}$$

Finally, every column of $(AH + I)$ is in $\mathrm{sp}\{\mathbf{1}\}$.

These facts follow algebraically from the fact that by definition, $f^{\mathrm{ft}}(G, s, \tau)$ satisfies (65). Subsuting (63) into (1) and simplifying using the facts in Observation 4.1 yields (64). □

Next, we establish a necessary condition for a set of fixed tolls to be optimal in the sense of (39).

Claim 4.1.2. *Fixed tolls τ^* satisfying* (39) *must also satisfy*

$$H\left(\tau^* + \frac{b}{S_{\mathrm{L}} + S_{\mathrm{U}}}\right) = \mathbf{0}. \tag{70}$$

Proof. By (64) the total latency due to fixed tolls is a concave parabola in s, so for any τ, the maximum total latency on $[S_{\mathrm{L}}, S_{\mathrm{U}}]$ occurs at either S_{L} or S_{U}. Since $\mathcal{L}^{\mathrm{ft}}(G, s, \tau)$ is continuous and convex in τ, this means that τ^* must satisfy

$$\mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = \mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*). \tag{71}$$

Thus, for any optimal fixed tolls τ^*, $\mathcal{L}^{\mathrm{ft}}(G, s, \tau^*)$ is a parabola centered at $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$:

$$\underset{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]}{\operatorname{argmin}} \ \mathcal{L}^{\mathrm{ft}}(G, s, \tau^*) = (S_{\mathrm{L}} + S_{\mathrm{U}})/2. \tag{72}$$

Our goal is to find the parabola with minimum as in (72) which minimizes the values in (71).

Equation (64) implies that for all $\tau, \tau' \in \mathbb{R}^n$, $\mathcal{L}^{\mathrm{ft}}(G, 0, \tau) = \mathcal{L}^{\mathrm{ft}}(G, 0, \tau')$; that is, the $s = 0$ endpoint of the parabola has the same value for all tolls. Thus, for τ satisfying (72), $\mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) < \mathcal{L}^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau)$ if and only if $\mathcal{L}^{\mathrm{ft}}\left(G, \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}, \tau^*\right) < \mathcal{L}^{\mathrm{ft}}\left(G, \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}, \tau\right)$.

Thus, by concavity, any tolls which result in globally optimal routing for $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$ will also be optimal in the sense of (39). It is easily verified that for a known homogeneous sensitivity s, any tolls τ which satisfy

$$H\left(\tau + b/(2s)\right) = \mathbf{0} \tag{73}$$

result in globally optimal routing. The proof of this is obtained by substituting (73) into the gradient (with respect to τ) of $\mathcal{L}^{\mathrm{ft}}(G, s, \tau)$ and applying the facts from Observation 4.1.

Therefore, any τ which satisfies (73) with $s = \frac{S_{\mathrm{L}} + S_{\mathrm{U}}}{2}$ will be uncertainty-optimal. That is, τ^* satisfies (70). □

Evaluating (63) with tolls satisfying (70) yields an expression for a Nash flow induced by τ^* as a function of s:

$$f^{\mathrm{ft}}(G, s, \tau^*) = R + Hb\left(S_{\mathrm{L}} + S_{\mathrm{U}} - s\right) / \left(S_{\mathrm{L}} + S_{\mathrm{U}}\right). \tag{74}$$

The above implies that $(R + Hb)$ specifies an un-tolled Nash flow. For parallel networks, it is easy to show that every element of R is non-negative; thus, since $\left(\frac{S_{\mathrm{L}} + S_{\mathrm{U}} - s}{S_{\mathrm{L}} + S_{\mathrm{U}}}\right) \in [0, 1]$, it must be that $(R + Hb\alpha)$ represents a feasible flow.

There are two possible worst-case flows using fixed toll τ^*; one when the sensitivity is S_{U}, the other when the sensitivity is S_{L}. In terms of (74), we write these flows as:

$$f_-^{\mathrm{ft}} = f^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = R + Hb\left(S_{\mathrm{U}} / (S_{\mathrm{L}} + S_{\mathrm{U}})\right). \tag{75}$$

$$f_+^{\mathrm{ft}} = f^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*) = R + Hb\left(S_{\mathrm{L}} / (S_{\mathrm{L}} + S_{\mathrm{U}})\right). \tag{76}$$

Next we show that f_-^{ft} and f_+^{ft}, the worst-case flows for optimal fixed tolls, are actually exactly equal to worst-case flows achievable with *scaled marginal-cost* tolls (41) with a particular scalar. The machinery of Claim 4.1.1 describes the Nash flows $f^{\mathrm{smc}}(G, s, \kappa)$ resulting from homogeneous sensitivity s and marginal-cost tolls scaled by $\kappa > 0$:

$$f^{\mathrm{smc}}(G, s, \kappa) = R + Hb / \left(1 + s\kappa\right). \tag{77}$$

The derivation of this is straightforward; it is detailed in [8].

The *worst* worst-case flows occur when the sensitivity of the population has been grossly over- or under-estimated; for example, if a population with sensitivity S_{U} is using a network with $\kappa = 1/S_{\mathrm{L}}$ (and vice-versa). There are two such flows:

$$f_-^{\mathrm{smc}} = R + \frac{Hb}{1 + S_{\mathrm{L}}/S_{\mathrm{U}}} \quad \text{and} \quad f_+^{\mathrm{smc}} = R + \frac{Hb}{1 + S_{\mathrm{U}}/S_{\mathrm{L}}}.$$

Comparing the above to (75) and (76), we see that $f_-^{\mathrm{smc}} = f_-^{\mathrm{ft}}$ and $f_+^{\mathrm{smc}} = f_+^{\mathrm{ft}}$. Thus, since

$$f^{\mathrm{ft}}(G, S_{\mathrm{L}}, \tau^*) = f^{\mathrm{smc}}(G, S_{\mathrm{L}}, 1/S_{\mathrm{U}}),$$
$$f^{\mathrm{ft}}(G, S_{\mathrm{U}}, \tau^*) = f^{\mathrm{smc}}(G, S_{\mathrm{U}}, 1/S_{\mathrm{L}}),$$

it must be true that (re-writing now in terms of affine tolls)

$$\mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{L}}, \tau^*) = \mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{L}}, \tau^A(1/S_{\mathrm{U}}, 0)), \tag{78}$$

$$\mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{U}}, \tau^*) = \mathcal{L}^{\mathrm{nf}}(G, S_{\mathrm{U}}, \tau^A(1/S_{\mathrm{L}}, 0)). \tag{79}$$

By design, (78) equals (79), so we have that

$$\max_{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]} \mathcal{L}^{\mathrm{nf}}\left(G, s, \tau^A(1/S_{\mathrm{U}}, 0)\right) = \max_{s \in [S_{\mathrm{L}}, S_{\mathrm{U}}]} \mathcal{L}^{\mathrm{nf}}\left(G, s, \tau^*\right).$$

□

Philip N. Brown is a PhD student in the Department of Electrical and Computer Engineering at the University of California, Santa Barbara. Philip received the Bachelor of Science in Electrical Engineering in 2007 from Georgia Tech and the Master of Science in Electrical Engineering in 2015 from the University of Colorado at Boulder under the supervision of Jason R. Marden, where he received the University of Colorado Chancellor's Fellowship.

Jason R. Marden is an Assistant Professor in the Department of Electrical and Computer Engineering at the University of California, Santa Barbara. Jason received the Bachelor of Science in Mechanical Engineering in 2001 from UCLA, and the PhD in Mechanical Engineering in 2007, also from UCLA, under the supervision of Jeff S. Shamma, where he was awarded the Outstanding Graduating PhD Student in Mechanical Engineering. After graduating from UCLA, he served as a junior fellow in the Social and Information Sciences Laboratory at the California Institute of Technology until 2010, and then as an Assistant Professor at the University of Colorado until 2015. Jason is a recipient of an ONR Young Investigator Award (2015), NSF Career Award (2014), AFOSR Young Investigator Award (2012), SIAM CST Best Sicon Paper Award (2015), and the American Automatic Control Council Donald P. Eckman Award (2012). Jason's research interests focus on game theoretic methods for the control of distributed multiagent systems.