Homework 4

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Problem 1

Proof. Because $p = \frac{\lambda}{n}$

$$\begin{aligned}
&\lim_{n\to\infty} P(X=i) \\
&= \lim_{n\to\infty} \binom{n}{i} p^i (1-p)^{1-i} \\
&= \lim_{n\to\infty} \frac{n!}{i!(n-i)!} p^i (1-p)^{1-i} \\
&= \frac{\lambda^i}{i!} \lim_{n\to\infty} (1-\frac{\lambda}{n})^{n-i} \prod_{k=0}^{i} \frac{n-k}{n}
\end{aligned}$$

We have

$$\begin{aligned}
&\lim_{n\to\infty} (1-\frac{\lambda}{n})^{n-i} \\
&= \frac{\lim_{n\to\infty}(1-\frac{\lambda}{n})^n}{\lim_{n\to\infty}(1-\frac{\lambda}{n})^i} \\
&= \lim_{n\to\infty} (1-\frac{\lambda}{n})^n \\
&= e^{-\lambda}
\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{n-k}{n} = 1, k = 0, 1, ..., i < n$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} P(X=i) \\
&= \frac{\lambda^i}{i!} (\lim_{n\to\infty} (1-\frac{\lambda}{n})^{n-i}) \prod_{k=0}^{i} \lim_{n\to\infty} \frac{n-k}{n} \\
&= \frac{\lambda^i e^{-\lambda}}{i!}
\end{aligned}$$

$\square$

Problem 2

(a)
Denote X_i be the number of days between the $(i-1)$th and ith new toys. Because opening boxes are independent,

$$X_4 \sim Geom(\lambda = \frac{2}{5})$$

Thus,

$$E(X_4) = \frac{1}{\lambda} = \frac{5}{2}$$

(b)
Denote $X_s = \sum_{i=1}^{5} X_i$ be the total number of days used to collect all kinds of toys. Obviously,

$$\begin{aligned} E(X_s) &= E(\sum_{i=1}^{5} X_i) \\ &= \sum_{i=1}^{5} E(X_i) \\ &= \sum_{i=1}^{5} \frac{5}{6-i} \\ &= \frac{137}{12} \end{aligned}$$

Problem 3

Because

$$p_1 + p_2 + p_3 = 1$$
$$E(X) = p_1 + 2p_2 + 3p_3 = 2$$

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= p_1 + 4p_2 + 9p_3 - 4 \\ &= 2p_3 \end{aligned}$$

Considering $0 \leq p_1, p_2, p_3 \leq 1$

$$0 \leq p_3 \leq 1$$
$$0 \leq 1 - 2p_3 \leq 1$$

Thus

$$0 \leq p_3 \leq \frac{1}{2}$$

(a)
To maximize $Var(X)$, maximize p_3. Then,

$$(p_1, p_2, p_3) = (\frac{1}{2}, 0, \frac{1}{2})$$

(b)
To minimize $Var(X)$, minimize p_3. Then,

$$(p_1, p_2, p_3) = (0, 1, 0)$$

Problem 4

Based on the characteristic of P.D.F. and prerequisites,

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$\int_{-\infty}^{\infty} f_X(x)xdx = 0.5$$

Thus,

$$(\frac{b}{3}x^3 + \frac{a}{2}x^2)\Big|_0^1 = 1$$
$$(\frac{b}{4}x^4 + \frac{a}{3}x^3)\Big|_0^1 = \frac{1}{2}$$

We have $a = 6, b = -6$. Then,

$$\begin{aligned} Var(X) &= \int_{-\infty}^{\infty} f_X(x)(x-u)^2dx \\ &= \int_0^1 f_X(x)(x-0.5)^2dx \\ &= \frac{3}{2}\int_0^1 (4x^4 - 8x^3 + 5x^2 - x)dx \\ &= -\frac{1}{20}(24x^5 - 60x^4 + 50x^3 - 15x^2)\Big|_0^1 \\ &= \frac{1}{20} \end{aligned}$$

Problem 5

(a)

Proof.

$$\begin{aligned}
&E(X)\\
&= \sum_{x=1,2,...} xP(x)\\
&= P(1) + 2P(2) + 3P(3) + ... + iP(i) + ...\\
&= P(1) + P(2) + P(3) + ... + P(i) + ...\\
&\quad P(2) + P(3) + ... + P(i) + ...\\
&\quad P(3) + ... + P(i) + ...\\
&\quad ...\\
&= \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} P(j)\\
&= \sum_{i=1}^{\infty} P(X \geq i)
\end{aligned}$$

□

(b)
Let X be the number of games until one player wins two games in a row, we have

$$P(X \geq 1) = 1$$

$$P(X \geq 2) = 1$$

for $i = 3, 5, 7, ...$

$$\begin{aligned}
&P(X \geq i)\\
=&P(\text{no two wins in a row in i games})\\
=&P(ABA...A) + P(BAB...B)\\
=&2(p - p^2)^{\frac{n-1}{2}}
\end{aligned}$$

for $i = 4, 6, 8, ...$

$$\begin{aligned}
&P(X \geq i)\\
=&P(ABA...B) + P(BAB...A)\\
=&(p - p^2)^{\frac{n-2}{2}}
\end{aligned}$$

Thus,

$$
\begin{aligned}
&E(X)\\
=&\sum_{i=1}^{\infty} P(X \geq i)\\
=&1+1+3p(1-p)+3p^2(1-p)^2+...\\
=&2+\frac{3p(1-p)}{1-p(1-p)}\\
=&\frac{2+p(1-p)}{1-p(1-p)}
\end{aligned}
$$

Problem 6

Because the interval $[0, 1]$ is symmetric about 0.5, let's Suggest that $q \geq 0.5$. Denote $L \in [0, 1]$ be the length of the first segment, X be the length of the segment that contains Q. We have,

$$X = \begin{cases} {}^{1-L, L<q} \\ {}_{L, L>q} \end{cases}$$

Obviously,

$$f_L(l) = 1, l \in [0, 1]$$

for $x \in [0, 1-q]$, because $X > 1-q$, $f_X(x) = 0$
for $x \in (1-q, q]$, $X = 1-L$, $f_X(x) = f_L(1-x)/|-1| = 1$
for $x \in (q, 1]$, $X = 1-L$ or $X = L$, $f_X(x) = f_L(1-x)/|-1| + f_L(x)/|1| = 2$
Thus,

$$
\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} f_X(x)xdx\\
&= \int_{1-q}^{q} xdx + \int_{q}^{1} 2xdx\\
&= \frac{q^2-(1-q)^2}{2} + 1 - q^2\\
&= \frac{1-2q^2+2q}{2}
\end{aligned}
$$