Homework 2

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Problem 1

Proof. Let i represent the number of people selected from the first half of people. In order to select a total of n people, we need to select $n-i$ people from the second half.
Suggest for a given i, to select n people from 2n people with i people in the first half, the total combination number is A_i, Obviously,

$$\begin{aligned} A_i &= \binom{n}{i}\binom{n}{n-i} \\ &= \binom{n}{i}^2 \end{aligned}$$

Thus, to select n people from 2n people with i people in the first half, the total number

$$\begin{aligned} \binom{2n}{n} &= \sum_{i=1}^{n} A_i \\ &= \sum_{i=1}^{n} \binom{n}{i}^2 \end{aligned}$$

□

Problem 2

The number of total distributions is

$$5 \times 5 \times 4 = 100$$

A number $N = x_1 \times 100 + x_2 \times 10 + x_3$ is a multiple of 6, if and only if "$(x_1 + x_2 + x_3)$ is a multiple of 3" and "x_3 is a multiple of 2".
There are 20 numbers meeting the criteria: 102, 120, 210, 132, 312, 234, 324, 342, 432, 354, 534, 204, 240, 402, 420, 450, 504, 540, 150, 510. Thus,

$$P(\text{N is a multiple of 6}) = \frac{20}{100} = \frac{1}{5}$$

Problem 3

(a)
Consider putting every ball in a sequence. For each ball, there are r possible boxes, each will produce a different distribution. Thus, the total distribution number is

$$r^n$$

(b)
Because boxes are not empty, $n \geqslant r$
Arrange the n balls in a line, and separate them into r groups using boards. Put the balls in each group into each of the r boxes. Thus, an arrangement of boards will correspond to a distribution of balls.
Obviously, n balls will produce $n-1$ gaps, and in order to separate n balls into r groups, $r-1$ boards is needed. Thus, the total number of distributions is

$$\binom{n-1}{r-1}$$

(c)
Suggest $n, r > 0$. After distributing every ball, the set of all possible distributions is $A = \{(n_1, n_2, ..., n_r) : n_i = 0, 1, ..., n and \sum_{i=1}^{r} n_i = n\}$, where n_i is the number of balls in the ith box.

Let $B = \{(m_1, m_2, ..., m_r) : m_i = n_i + 1\}$, Obviously,

$$|B| = |A|$$

Because $m_i = n_i + 1$, we have $B = \{(m_1, m_2, ..., m_r) : m_i = 1, 2, ..., n + 1 and \sum_{i=1}^{r} m_i = n + r\}$ Thus, B is a representative of $n+r$ balls distributed to r boxes with no boxes empty. According to (b),

$$|A| = |B| = \binom{n+r-1}{r-1}$$

(d)
Consider putting all balls undistributed into a new box, then all balls will be distributed to $r+1$ boxes. Each of these distributions maps to one original distribution. Thus, the number of total outcomes with some balls undistributed is

$$\binom{n+r}{r}$$

Problem 4

Let A_i represent picking the ith box, B_b, B_w represent picking black and white balls. Accordingly, we have

$$P_(A_i) = \frac{1}{3}, i = 1, 2, 3$$

$$
\begin{aligned}
P(B_b|A_1) &= \frac{3}{5} \\
P(B_b|A_2) &= \frac{3}{4} \\
P(B_b|A_3) &= \frac{1}{2}
\end{aligned}
$$

(a)

$$
\begin{aligned}
P(B_b) &= \sum_{i=1,2,3} P(B_b|A_i)P(A_i) \\
&= \frac{3}{5} \times \frac{1}{3} + \frac{3}{4} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} \\
&= \frac{37}{60}
\end{aligned}
$$

(b)
According to Bayes' Law,

$$
\begin{aligned}
P(A_1|B_w) &= \frac{P(B_w|A_1)P(A_1)}{P(B_w)} \\
&= \frac{(1-\frac{3}{5})\frac{1}{3}}{1-\frac{37}{60}} \\
&= \frac{8}{23}
\end{aligned}
$$

Problem 5

(a)

$$
\begin{aligned}
P_1 &= 0 \\
P_2 &= \frac{1}{2}
\end{aligned}
$$

(b)

Proof. Let event B_n be that n people not having their own hats, and let $|B_n|$ be the number of equally likely possible outcomes. Thus,

$$
P_n = \frac{|B_n|}{n!}
$$

In all distributions, Suggest any person i takes the hat of person j. There are two mutually exclusive events:

$$
A : \text{j takes the hat of i}
$$

$$A^c: \text{j takes the hat of people other than i ad j}$$

In case of A, there are $n-1$ possible js, and each will result in $n-2$ undistributed hats. Thus,

$$|A| = (n-1)|B_{n-2}|$$

In case of A^c, There are also $n-1$ possible js. Because j will take the hat of another person other than i, and the hat of i will be taken by another person other than j, we can consider i and j together as a new node. Then, there will be $n-1$ hats to be distributed. Thus,

$$|A^c| = (n-1)|B_{n-1}|$$

Thus, we have

$$\begin{aligned}
P_n &= \frac{|B_n|}{n!} \\
&= \frac{(|A| + |A^c|)}{n!} \\
&= \frac{(n-1)|B_{n-2}| + (n-1)|B_{n-1}|}{n!} \\
&= \frac{1}{n}\frac{|B_{n-2}|}{(n-2)!} + \frac{n-1}{n}\frac{|B_{n-1}|}{(n-1)!} \\
&= \frac{(n-1)P_{n-1}}{n} + \frac{P_{n-2}}{n}
\end{aligned}$$

□

(c)

Proof. According to (b), we have

$$P_n - P_{n-1} = -\frac{1}{n}(P_{n-1} - P_{n-2})$$

Denote $C_n = P_n - P_{n-1}$. We have already know that $P_1 = 0, P_2 = \frac{1}{2}$. Thus,

$$C_2 = \frac{1}{2}$$

$$C_n = -\frac{C_{n-1}}{n}$$

Obviously,

$$\begin{aligned}
C_n &= -\frac{C_{n-1}}{n} \\
&= \frac{C_{n-2}}{n(n-1)} \\
&\dots \\
&= \frac{(-1)^n}{n!}, n \geqslant 2
\end{aligned}$$

Thus,

$$\begin{aligned}
P_n &= \frac{(-1)^n}{n!} + P_{n-1} \\
&= \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + P_{n-2} \\
... \\
&= \frac{1}{2!} - \frac{1}{3!} + ... + \frac{(-1)^n}{n!}, n \geqslant 2
\end{aligned}$$

$\square$