Homework 3

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Problem 1

Let A denote a person to be a liar, and B denote the test identifies the person to be a liar. Obviously, we have

$$P(A) = 0.05$$

$$P(B|A) = 0.94$$

$$P(B|A^c) = 0.08$$

According to Bayes' Law,

$$\begin{aligned}
P(A|B) &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c))} \\
&= \frac{0.94 \times 0.05}{0.94 \times 0.05 + 0.08 \times 0.95} \\
&= 0.382
\end{aligned}$$

Problem 2

Proof. Because A and B are conditionally independent given any C_i,

$$\begin{aligned}
P(AB|C_i) &= \frac{P(AB)}{P(C_i)} \\
&= P(A|C_i)P(B|C_i), i = 1, 2, ..., M
\end{aligned} \tag{1}$$

Because B and C_i are independent,

$$P(B|C_i) = P(B) \tag{2}$$

Combining (1) and (2), we have

$$
\begin{aligned}
P(AB) &= \sum_{i=1}^{M} P(AB|C_i) \\
&= \sum_{i=1}^{M} P(A|C_i)P(B|C_i) \\
&= P(B)\sum_{i=1}^{M} P(A|C_i) \\
&= P(A)P(B)
\end{aligned}
$$

$\square$

Problem 3

Let A, B, C denote a day to be rainy, heavy traffic and being late respectively. Thus,

$$
\begin{aligned}
P(A) &= \frac{1}{3} \\
P(B|A) &= \frac{1}{2} \\
P(B|A^c) &= \frac{1}{4} \\
P(C|AB) &= \frac{1}{2} \\
P(C|A^cB^c) &= \frac{1}{8} \\
P(C|AB^c) = P(C|A^cB) &= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
&P(C) \\
=&P(C|AB)P(AB) + P(C|A^cB^c)P(A^cB^c) + P(C|AB^c)P(AB^c) + P(C|A^cB)P(A^cB) \\
=&P(C|AB)P(B|A)P(A) + P(C|A^cB^c)(1-P(A))(1-P(B|A^c))+ \\
&P(C|AB^c)P(A)(1-P(B|A)) + P(C|A^cB)(1-P(A))P(B|A^c) \\
=&\frac{1}{2}\times\frac{1}{2}\times\frac{1}{3}+\frac{1}{8}\times\frac{2}{3}\times\frac{3}{4}+\frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}+\frac{1}{4}\times\frac{2}{3}\times\frac{1}{4} \\
=&\frac{11}{48}
\end{aligned}
$$

Problem 4

a.

Proof. Denote C_n be the event that $A \subseteq B$ and D_i represent the event that the ith number $\{i \in A, i \in B \quad \text{or} \quad i \notin A, i \notin B \quad \text{or} \quad i \notin A, i \in B\}$
Obviously,

$$C_n = D_1 \cap D_2 \cap ... \cap D_n$$

D_i is independent of D_j, $i \neq j, i, j = 1, 2, ..., n$, and

$$P(D_i) = \frac{3}{4}$$

Thus,

$$\begin{aligned} P(C_n) &= P(D_1 \cap D_2 \cap ... \cap D_n) \\ &= P(D_1)P(D_2)...P(D_n) \\ &= (\frac{3}{4})^n \end{aligned}$$

□

b.

Proof. Similarly, denote C_n be the event that $A \cap B = \emptyset$ and D_i represent the event that the ith number $\{i \in A, i \notin B \quad \text{or} \quad i \notin A, i \notin B \quad \text{or} \quad i \notin A, i \in B\}$
Accordingly, We have

$$C_n = D_1 \cap D_2 \cap ... \cap D_n$$

D_i is independent of D_j, $i \neq j, i, j = 1, 2, ..., n$, and

$$P(D_i) = \frac{3}{4}$$

Thus,

$$\begin{aligned} P(C_n) &= P(D_1 \cap D_2 \cap ... \cap D_n) \\ &= P(D_1)P(D_2)...P(D_n) \\ &= (\frac{3}{4})^n \end{aligned}$$

□

Problem 5

Proof.

$$
\begin{aligned}
P(A|A \cup B) &= \frac{P(A \cap (A \cup B))}{P(A \cup B)} \\
&= \frac{P(A)}{P(A \cup B)} \\
&= \frac{P(A \cap B) + P(A \cap B^c)}{P(A \cap (A \cup B)) + P(A^c \cap (A \cup B))} \\
&= \frac{P(A \cap B) + P(A \cap B^c)}{P(A \cap B) + P(A \cap B^c) + P(A^c \cap B)}
\end{aligned}
$$

Lemma 0.1. *For any* $a, b, c \geqslant 0$

$$
\begin{aligned}
&\frac{a+b}{a+b+c} \geqslant \frac{b}{b+c} \\
\Leftrightarrow &(a+b)(b+c) \geqslant b(a+b+c) \\
\Leftrightarrow &ab + bc + ac + b^2 \geqslant ab + bc + b^2 \\
\Leftrightarrow &ac \geqslant 0
\end{aligned}
$$

Let $a = P(A \cap B), b = P(A \cap B^c), c = P(A^c \cap B)$. We have

$$
\begin{aligned}
&P(A|A \cup B) \\
=&\frac{P(A \cap B) + P(A \cap B^c)}{P(A \cap B) + P(A \cap B^c) + P(A^c \cap B)} \\
\geqslant&\frac{P(A \cap B)}{P(A \cap B) + P(A^c \cap B)} \\
=&\frac{P(AB)}{P(B)} \\
=&P(A|B)
\end{aligned}
$$

□