Homework 1

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Problem 1

Proof. Let $\forall x \in (A \cap B)^c$. Under definition,

$$\begin{aligned}
&x \notin A \cap B \\
\Rightarrow &x \notin A \text{ or } x \notin B \\
\Rightarrow &x \in A^c \text{ or } x \in B^c \\
\Rightarrow &x \in A^c \cup B^c \\
\Rightarrow &(A \cap B)^c \subset A^c \cup B^c
\end{aligned}$$

Let $\forall y \in A^c \cup B^c$. Under Definition,

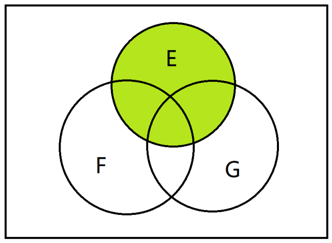
$$\begin{aligned}
&y \in A^c \text{ or } y \in B^c \\
\Rightarrow &y \notin A \text{ or } y \notin B \\
\Rightarrow &y \notin A \cap B \\
\Rightarrow &y \in (A \cap B)^c \\
\Rightarrow &A^c \cup B^c \subset (A \cap B)^c
\end{aligned}$$

Because $(A \cap B)^c \subset A^c \cup B^c$ and $A^c \cup B^c \subset (A \cap B)^c$,

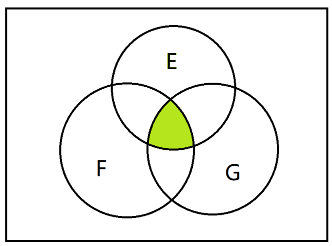
$$(A \cap B)^c = A^c \cup B^c$$

$\square$

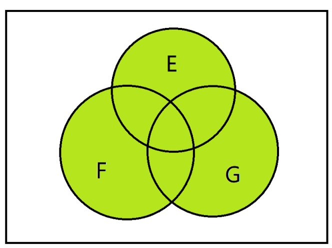
Problem 2



(a)



(b)



(c)

Problem 3

(a)

Proof. Because $\Omega = A \cup A^c$

$$B = (B \cap A) \cup (B \cap A^c)$$

Also, $(B \cap A) \cap (B \cap A^c) = \emptyset$

$$P(B) = P(B \cap A) + P(B \cap A^c)$$

Because $A \subset B$, $A = A \cap B$

$$P(B) = P(A) + P(B \cap A^c)$$

Considering $P(B \cap A^c) \geqslant 0$

$$P(A) \leqslant P(B)$$

$\square$

(b)

Proof. Because $(A \cap B) \cap (A \cap B^c) = \emptyset$

$$\begin{aligned} P(A) &= P((A \cap B) \cup (A \cap B^c)) \\ &= P(A \cap B) + P(A \cap B^c) \end{aligned}$$

Thus, $P(A \cap B^c) = P(A) - P(A \cap B)$ $\square$

1 Problem 4

Define $\Omega = \{(x_1, x_2, x_3, x_4, x_5) : x_i = 0, 1\}$, where x_i represents the ith result(0 = H,1 = T, * = both).
Obviously, there are 2^5 possible outcomes, and their probability are all equal.
(a)

$$\begin{aligned} P(HT***) &= \frac{1*1*2^3}{2^5} \\ &= \frac{1}{4} \end{aligned}$$

(b)

$$\begin{aligned} P(HHHHH) &= \frac{1}{2^5} \\ &= \frac{1}{32} \end{aligned}$$

(c)

$$\begin{aligned} P(\text{At most 2 heads}) &= P(0head) + P(1head) + P(2heads) \\ &= \frac{\binom{0}{5}}{2^5} + \frac{\binom{1}{5}}{2^5} + \frac{\binom{2}{5}}{2^5} \\ &= \frac{13}{16} \end{aligned}$$

(d)

$$\begin{aligned} P(\text{First Head in first 3 attempts}) &= 1 - P(TTT**) \\ &= 1 - \frac{1*1*1*2^2}{2^5} \\ &= \frac{7}{8} \end{aligned}$$

(e)

$$\begin{aligned}P(\text{At most 2 heads}) &= P(HTTTT)+P(HTTTH)+P(HTTHT)+P(HTHTT)+P(HTHTH)\\ &= \frac{5}{32}\end{aligned}$$

2 Problem 5

Let A, B, C represent a sum of 9, a sum of 10, other outcomes respectively, we have

$$P(A) = 8/81$$
$$P(B) = 9/81$$
$$P(C) = 64/81$$

Obviously,

$$\begin{aligned}P(\text{A is before B}) &= P(A)+P(CA)+P(CCA)+...\\ &= \frac{8}{81}+\frac{64}{81}\frac{8}{81}+(\frac{64}{81})^2\frac{8}{81}+...\\ &= \sum_{i=0}^{\infty}(\frac{64}{81})^i\frac{8}{81}\\ &= \frac{8}{17}\end{aligned}$$

3 Problem 6

Proof. for $n = 2$, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_1^c \cap A_2)$$

According to Problem 3.(c), we have

$$P(A_1^c \cap A_2) = P(A_2) - P(A_1 \cap A_2)$$

Thus,

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Suggest equation stands for $n = k$.

For $n = k + 1$, we have

$$
\begin{aligned}
& P(\bigcup_{i=1}^{k+1} A_i) \\
=& P(\bigcup_{i=1}^{k} A_i \cup A_{k+1}) \\
=& P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) - P(\bigcup_{i=1}^{k} A_i \cap A_{k+1}) \\
=& P(\bigcup_{i=1}^{k} A_i) + P(A_{k+1}) - P(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})) \\
=& \sum_{i=1}^{k} P(A_i) - ... + (-1)^{r+1} \sum_{i_1<i_2<...<i_r} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_r}) + P(A_{k+1}) \\
& - [\sum_{i=1}^{k} P(A_i \cap A_{k+1}) - ... + (-1)^{r+1} \sum_{i_1<i_2<...<i_r} P((A_{i_1} \cap A_{k+1}) \cap (A_{i_2} \cap A_{k+1}) \cap ... \cap (A_{i_r} \cap A_{k+1}))] \\
=& \sum_{i=1}^{k} P(A_i) - ... + (-1)^{r+1} \sum_{i_1<i_2<...<i_r} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_r}) + P(A_{k+1}) \\
& - [\sum_{i=1}^{k} P(A_i \cap A_{k+1}) - ... + (-1)^{r+1} \sum_{i_1<i_2<...<i_r} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_r}) \cap A_{k+1})] \\
=& \sum_{i=1}^{k+1} P(A_i) - \sum_{i_1<i_2}^{k+1} P(A_{i_1} \cap A_{i_2}) + ... + (-1)^{r+1} \sum_{i_1<i_2<...<i_r} P(A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_r}) + P(A_{k+1}) \\
& + (-1)^{k+2} P(A_1 \cap A_2 \cap ... \cap A_{k+1})
\end{aligned}
$$

Thus, for all $n \in N$, original equation stands. $\square$