Homework 4

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Problem 1

(a)
Because $X \sim Unif(0,1)$,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(X \ge 10^{-y}) \\ &= 1 - F_X(10^{-y}) \\ &= \begin{cases} 0 & y \le 0 \\ 1 - 10^{-y} & y > 0 \end{cases} \end{aligned}$$

Thus, the pdf of Y

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \begin{cases} ln10 \times 10^{-y} & y > 0 \\ 0 & y \le 0 \end{cases} \end{aligned}$$

(b)
Because $X \sim Unif(-\frac{\pi}{2}, \frac{\pi}{2})$,

$$f_X(x) = \begin{cases} \frac{1}{\pi} & X \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ 0 & Otherwise \end{cases}$$

Because $Y = tan(X)$ is strictly monotone given $X \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$\begin{aligned} & f_Y(y) \\ =& \frac{f_X(x)}{|J|} \\ =& \frac{1}{\pi(1+y^2)} \end{aligned}$$

Problem 2

Because $Y = \frac{1}{X}$ is strictly monotone given $X \in (5, \infty)$, for $Y \in (0, \frac{1}{5})$,

$$
\begin{aligned}
&f_Y(y) \\
=&\frac{f_X(x)}{|J|} \\
=&\frac{5/x^2}{|1/x^2|} \\
=&5
\end{aligned}
$$

for $Y \notin (0, \frac{1}{5})$, $f_Y(y) = 0$. Thus,

$$
\begin{aligned}
F_Y(y) &= \int_{-\infty}^{\infty} f_Y(y)dy \\
&= \begin{cases} 0 & y < 0 \\ 5y & 0 \leq y < \frac{1}{5} \\ 1 & y \geq \frac{1}{5} \end{cases}
\end{aligned}
$$

Problem 3

Proof.

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_{-\infty}^{0} f_X(x)(-\int_x^0 dy)dx + \int_0^{\infty} f_X(x)(\int_0^x dy)dx \\
&= -\int_{-\infty}^{0}\int_{-\infty}^{y} f_X(x)dxdy + \int_0^{\infty}\int_y^{\infty} f_X(x)dxdy \\
&= \int_0^{\infty}[\int_{-\infty}^{\infty} f_X(x)dx - \int_{-\infty}^{y} f_X(x)dx]dy - \int_{-\infty}^{0}[\int_{-\infty}^{y} f_X(x)dx]dy \\
&= \int_0^{\infty}[1 - F_X(y)]dy - \int_{-\infty}^{\infty} F_X(y)dy \\
&= \int_0^{\infty}[1 - F_X(x)]dx - \int_{-\infty}^{\infty} F_X(x)dx
\end{aligned}
$$

□

Problem 4

Because X is exponential distributed, $f_X(x) = \lambda e^{-\lambda x}, x > 0$
Denote $Y = X - a$, we have $f_Y(y) = \lambda e^{-\lambda(y+a)}, y > -a$

$$
\begin{aligned}
E[|Y|] &= \int_{-\infty}^{\infty} f_Y(y)|y|dy \\
&= \int_{0}^{\infty} f_Y(y)ydy - \int_{-a}^{0} f_Y(y)ydy \\
&= \int_{a}^{\infty} f_X(x)(x-a)dx - \int_{0}^{a} f_X(x)(a-x)dx
\end{aligned}
$$

Denote

$$
\begin{aligned}
F(x) &= \int f_X(x)(x-a)dx \\
&= -e^{-\lambda x}(x-a) - \int \lambda e^{-\lambda x} \\
&= e^{-\lambda x}(a - x - \frac{1}{\lambda}) + C
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&E[|Y|] \\
=&\int_{a}^{\infty} f_X(x)(x-a)dx - \int_{0}^{a} f_X(x)(a-x)dx \\
=&F(\infty) - 2F(a) + F(0) \\
=&\frac{1}{\lambda}(2e^{-\lambda a} + a\lambda - 1)
\end{aligned}
$$

Because

$$\frac{dE[|Y|]}{da} = 1 - 2e^{-\lambda a}$$

is strictly monotonically increasing.
$E[|X|]$ reaches its minimum when

$$\frac{dE[|Y|]}{da} = 0$$

Which is

$$
\begin{aligned}
1 - 2e^{-\lambda a} &= 0 \\
\rightarrow a &= \frac{\ln 2}{\lambda}
\end{aligned}
$$

Problem 5

Proof. for $P(X > km)$, let $n = (k-1)m + i, i = 1, 2, 3...$, and sum it up. We have

$$
\begin{aligned}
&P(X = m + (k-1)m + 1 | X > m) + P(X = m + (k-1)m + 2 | X > m) + ... \\
=&\frac{P(X = km + 1)}{P(X > m)} + \frac{P(X = km + 2)}{P(X > m)} + ... \\
=&\frac{P(X > km)}{P(X > m)} \\
=&P(X = m + (k-1)m + 1) + P(X = m + (k-1)m + 2) + ... \\
=&P(X > (k-1)m)
\end{aligned}
$$

It converges. Thus,

$$
\begin{aligned}
&P(X > km) \\
=&P(X > m)P(X > (k-1)m) \\
&... \\
=&P(X > m)^k
\end{aligned}
$$

Denote $P(X = 1) = p$ and let $m = 1$, we have

$$
\begin{aligned}
&P(X > i) \\
=&P(X > 1)^i \\
=&(1 - P(X = 1))^i \\
=&(1 - p)^i
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&P(X = i) \\
=&P(X > i - 1) - P(X > i) \\
=&(1 - p)^{i-1} - (1 - p)^i \\
=&(1 - (1 - p))(1 - p)^{i-1} \\
=&(1 - p)^{i-1}p
\end{aligned}
$$

□