Chapter 5: Optimal Coordinates

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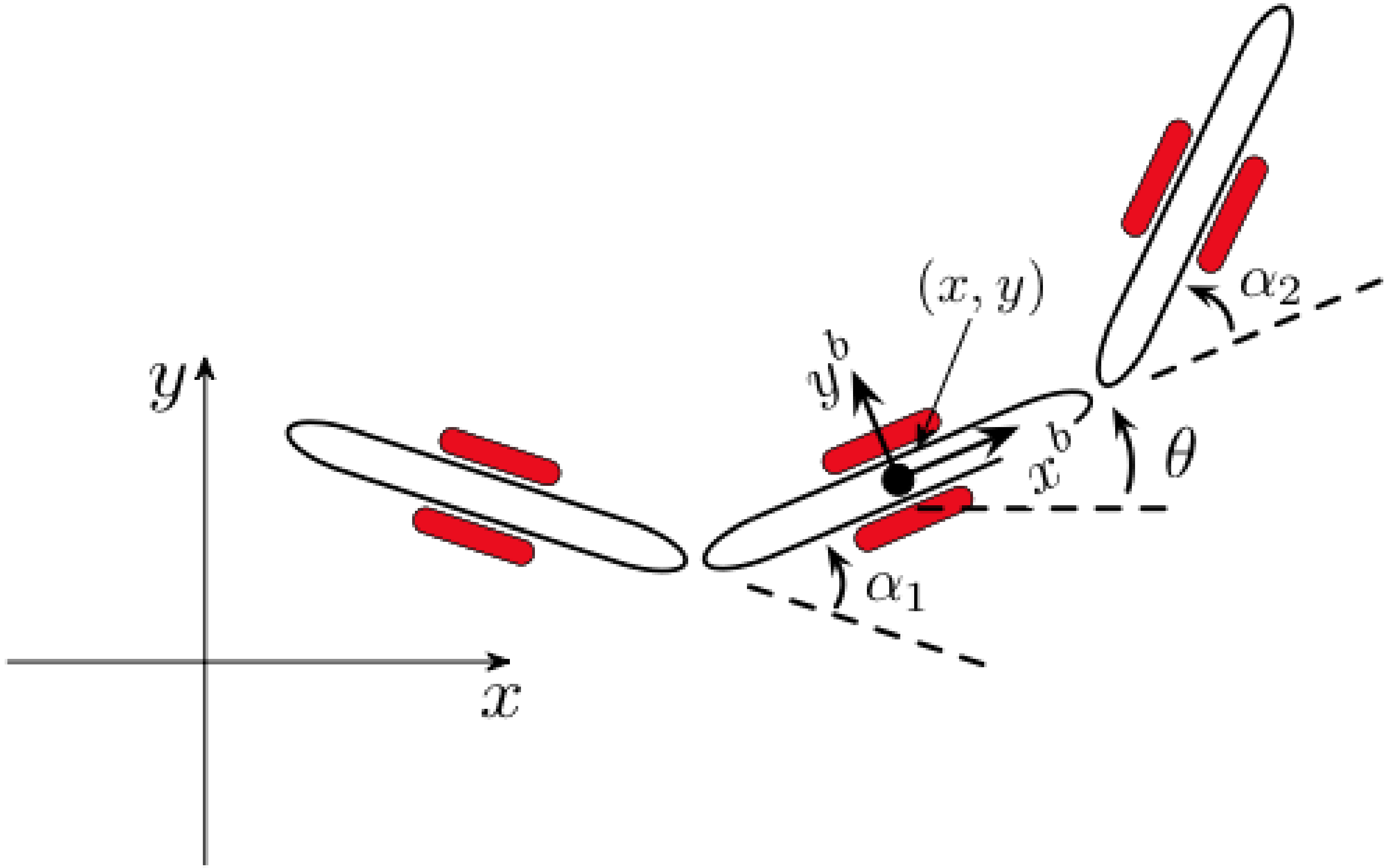
Take Lie Bracket from Micro to Macro

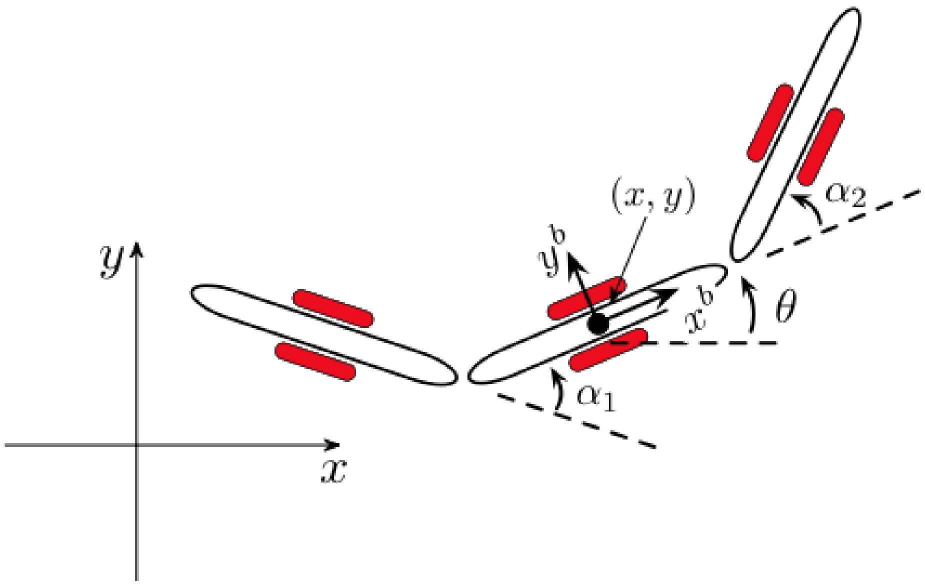
Coordinates Matter

taking the right coordinates allows us to do above

Three-link kinematic snake

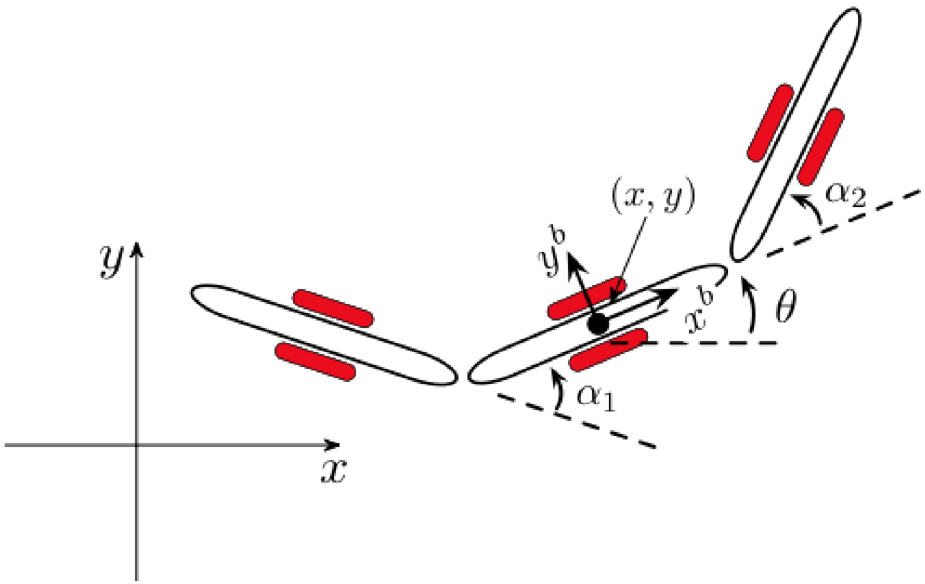


Connection vector fields



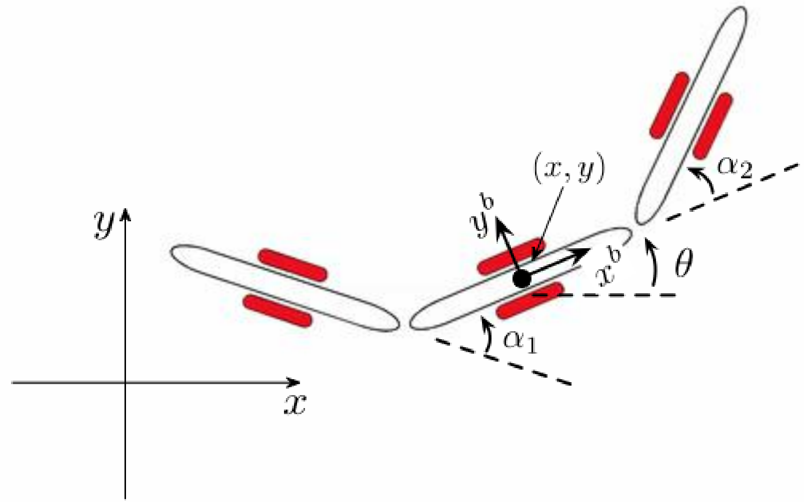
$$\overbrace{\xi}^{\text{body velocity}} = -\underbrace{\mathbf{A}(\alpha)}_{\text{local connection}}\overbrace{\dot{\alpha}}^{\text{shape velocity}}$$

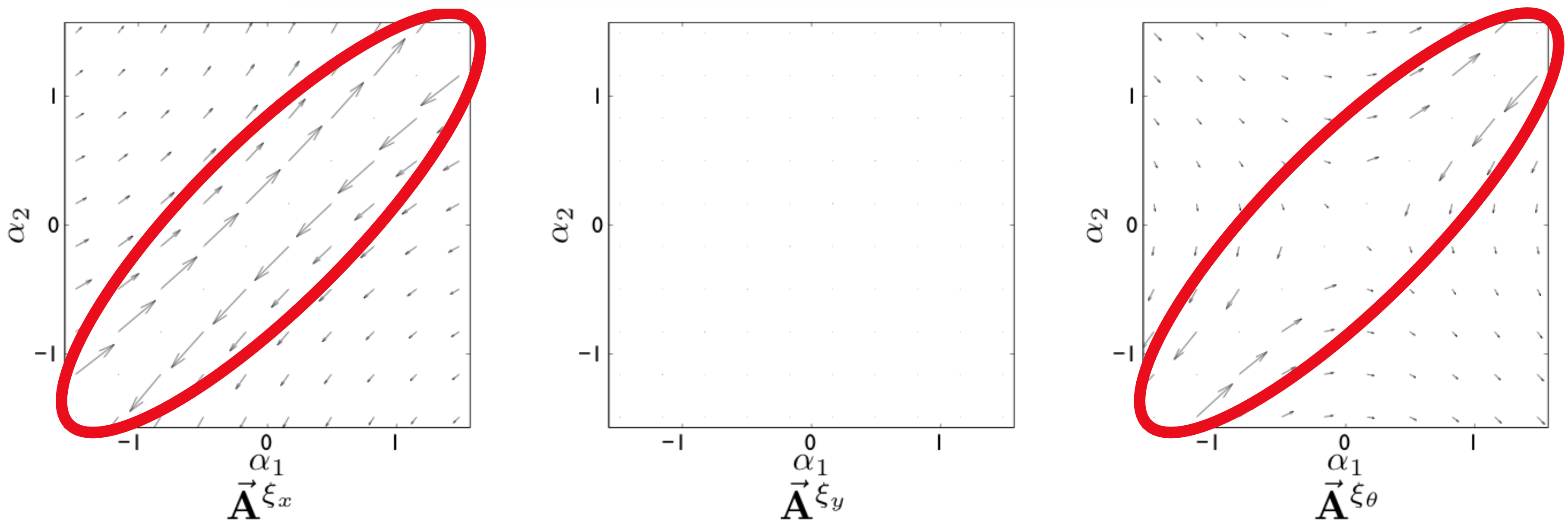
Connection vector fields



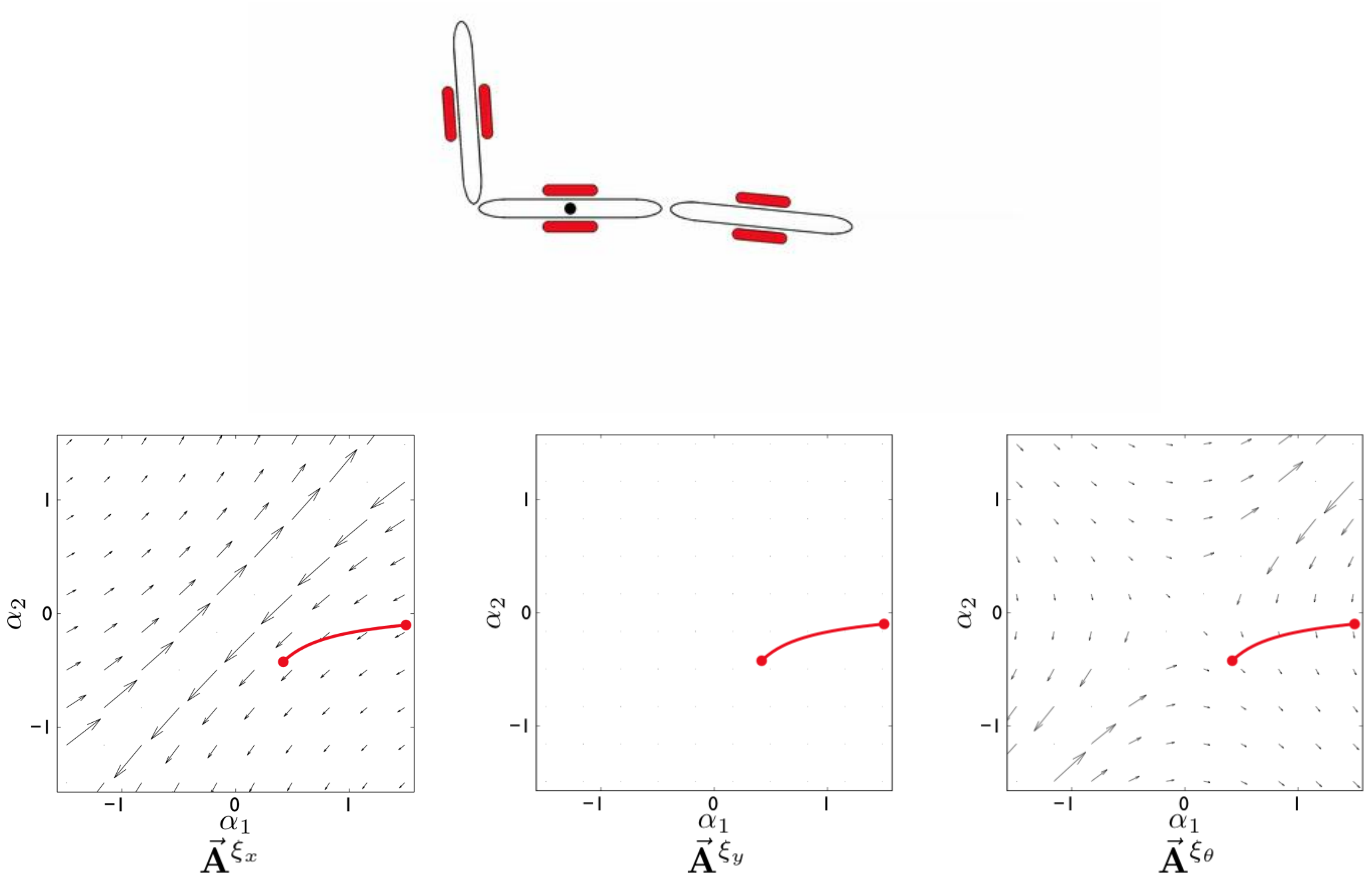
$$\begin{bmatrix} \xi_x \\ \xi_y \\ \xi_\theta \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{A}}^{\xi_x} \\ \vec{\mathbf{A}}^{\xi_y} \\ \vec{\mathbf{A}}^{\xi_\theta} \end{bmatrix} \dot{\alpha}$$

Connection vector fields

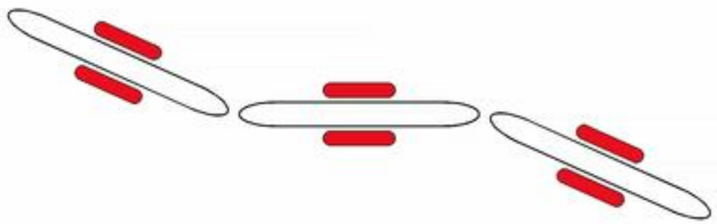


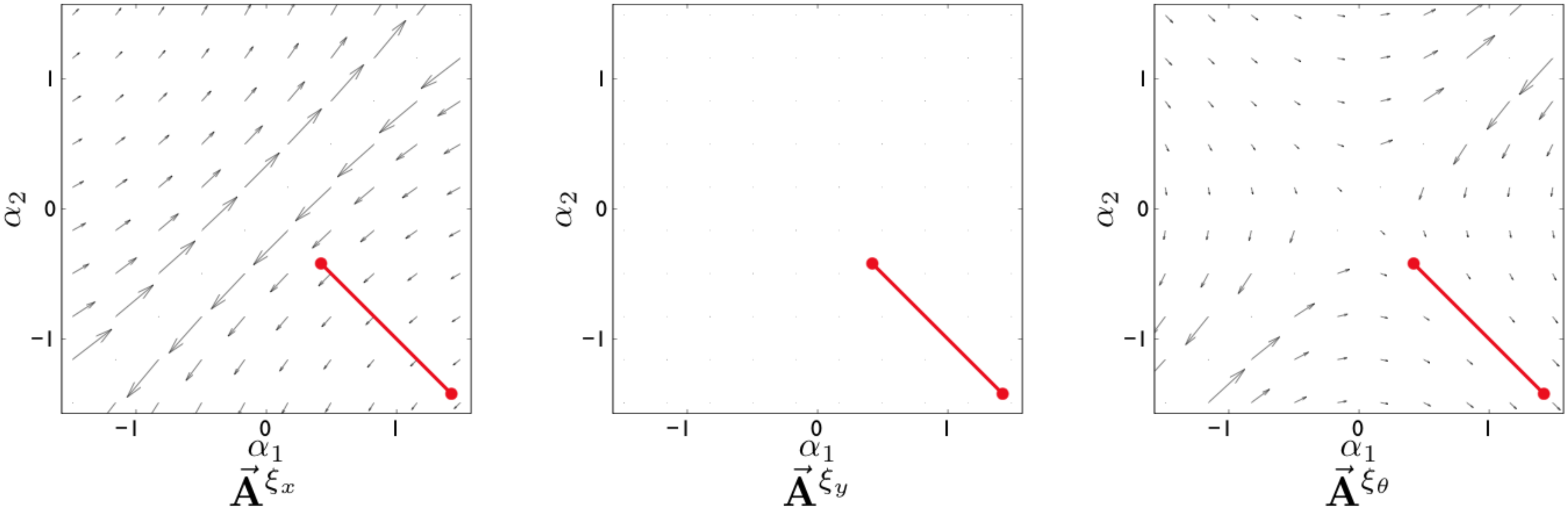


Connection vector fields

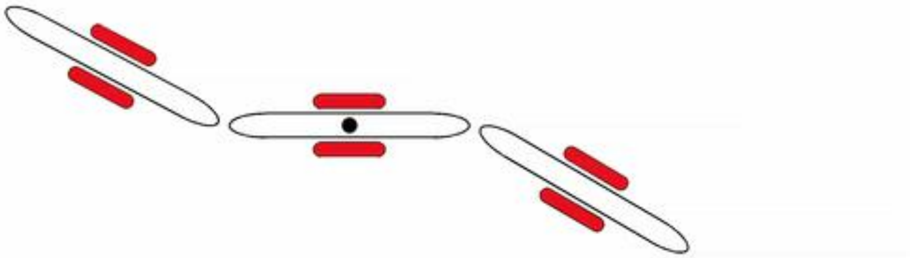


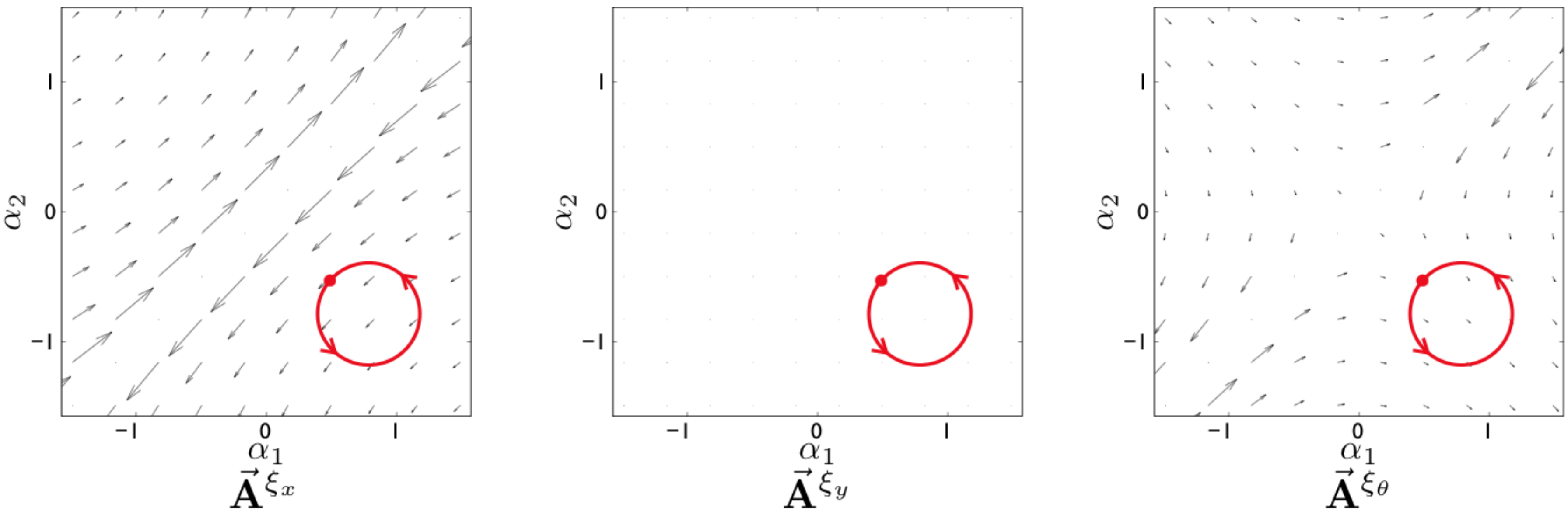
Connection vector fields





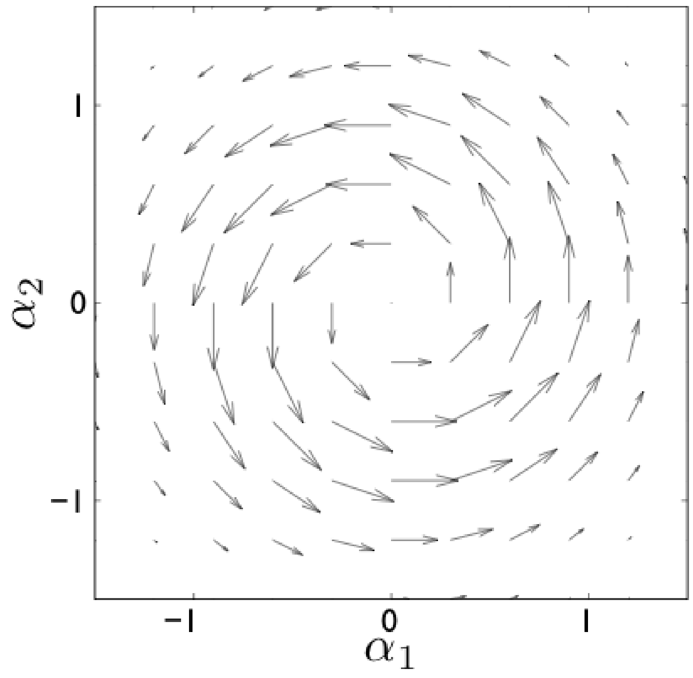
Gaits

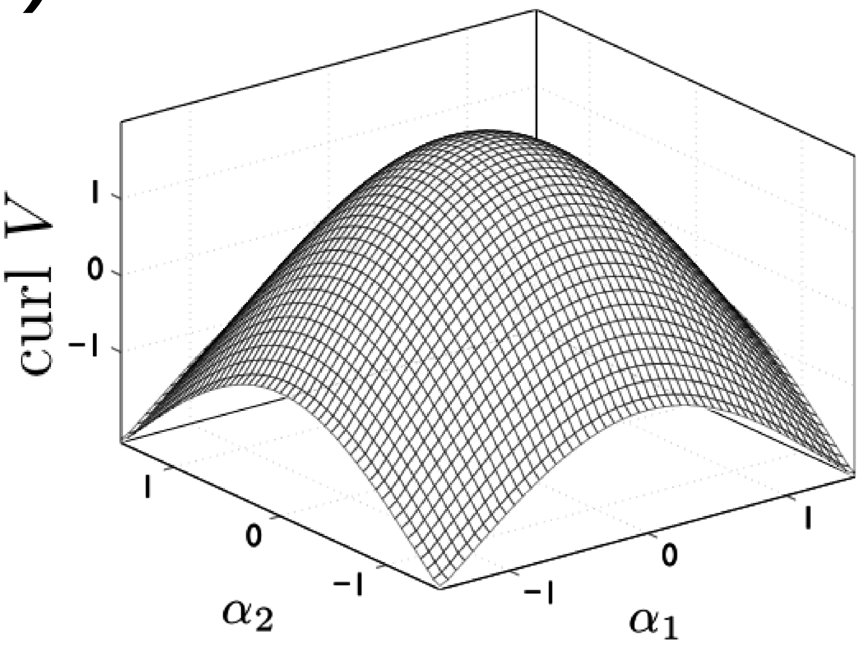




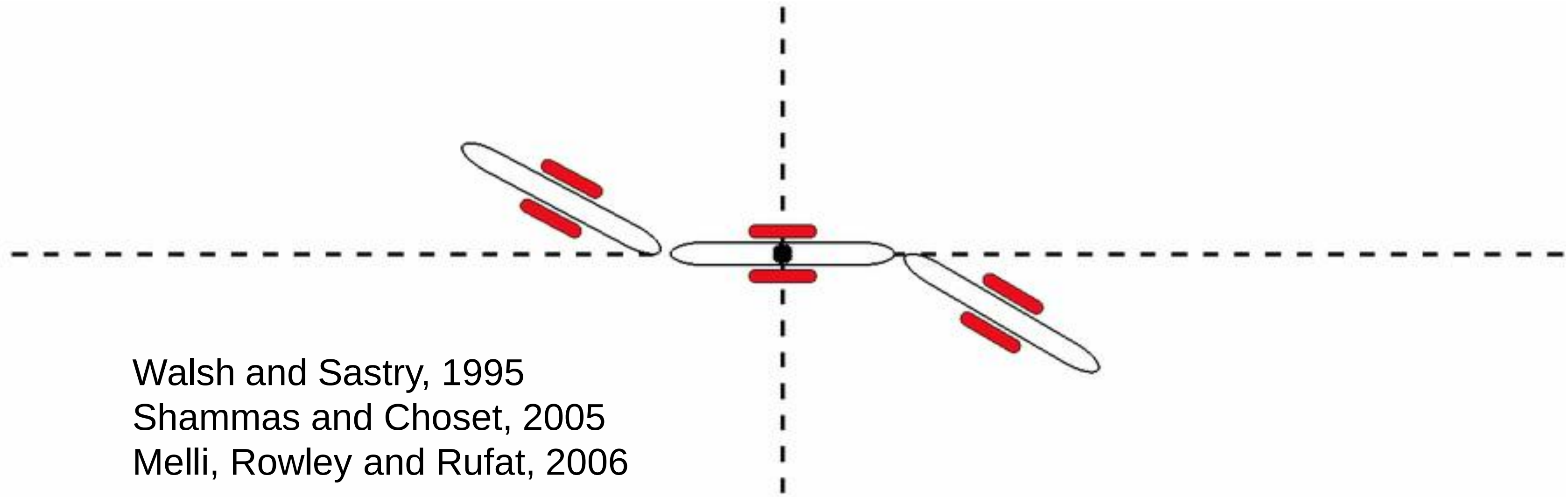
α_2
1
0
−1
−1
0
1
α_1
$\vec{\mathbf{A}}^{\xi_x}$
α_2
1
0
−1
−1
0
1
α_1
$\vec{\mathbf{A}}^{\xi_y}$
α_2
1
0
−1
−1
0
1
α_1
$\vec{\mathbf{A}}^{\xi_\theta}$

Stokes's theorem (Green's form)

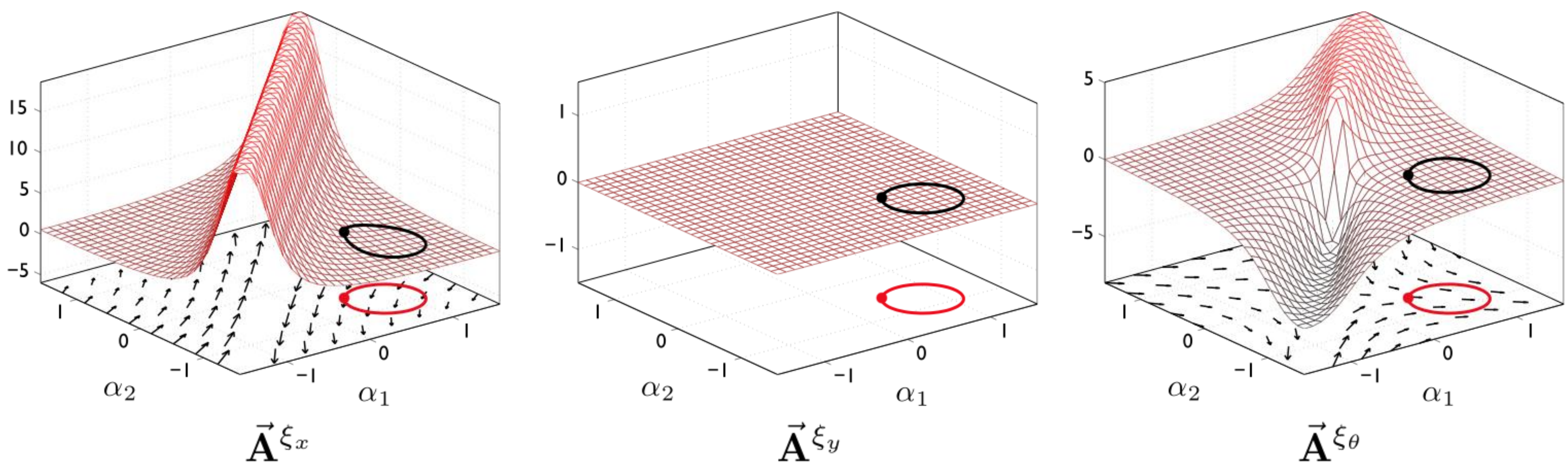




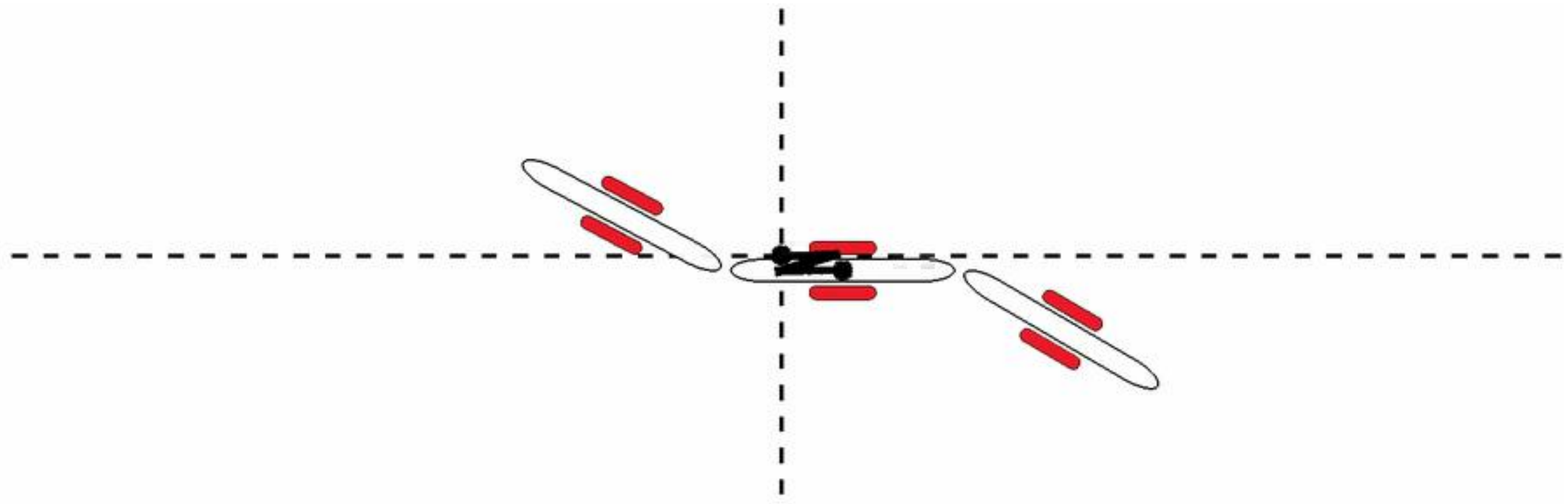
Gaits and Stokes’ s theorem

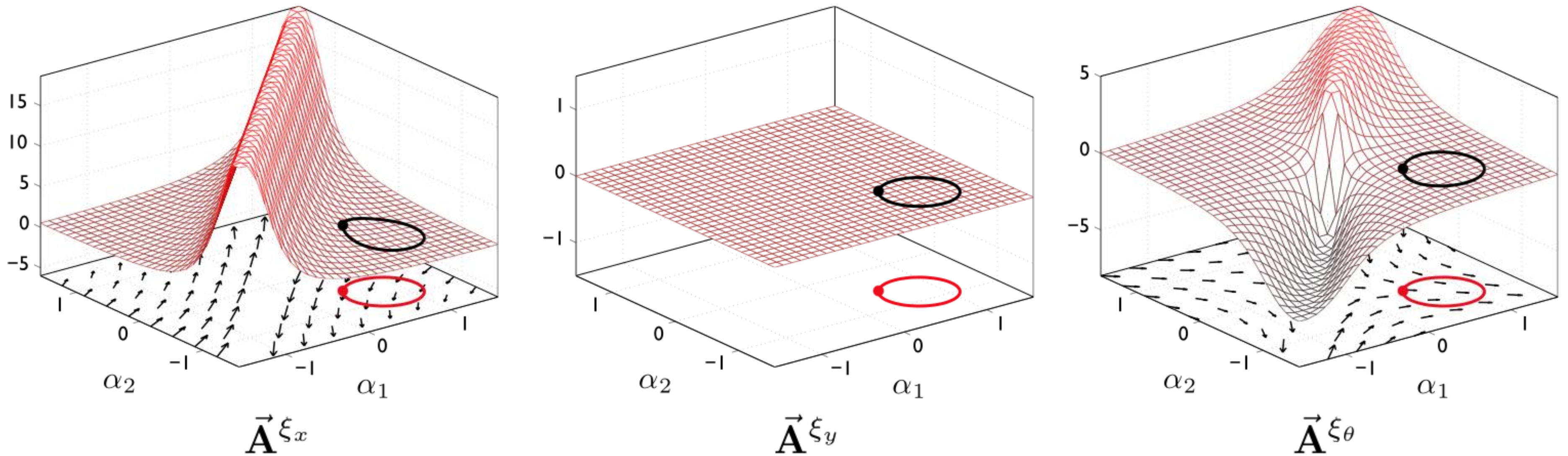


Walsh and Sastry, 1995
Shammas and Choset, 2005
Melli, Rowley and Rufat, 2006

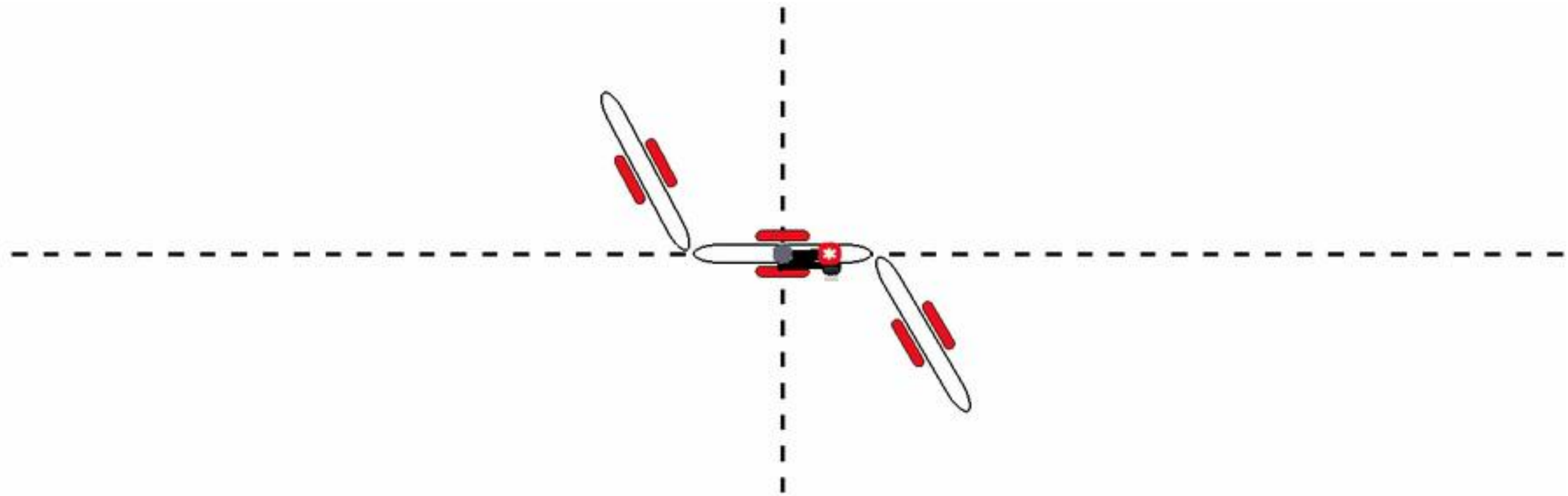


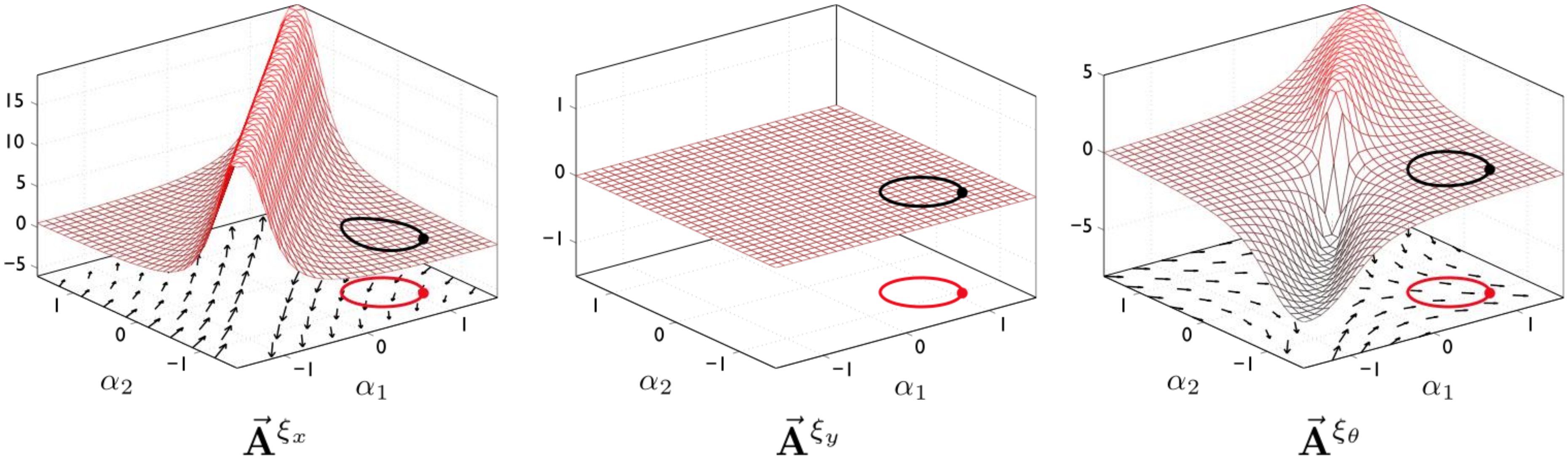
Gaits and Stokes's theorem



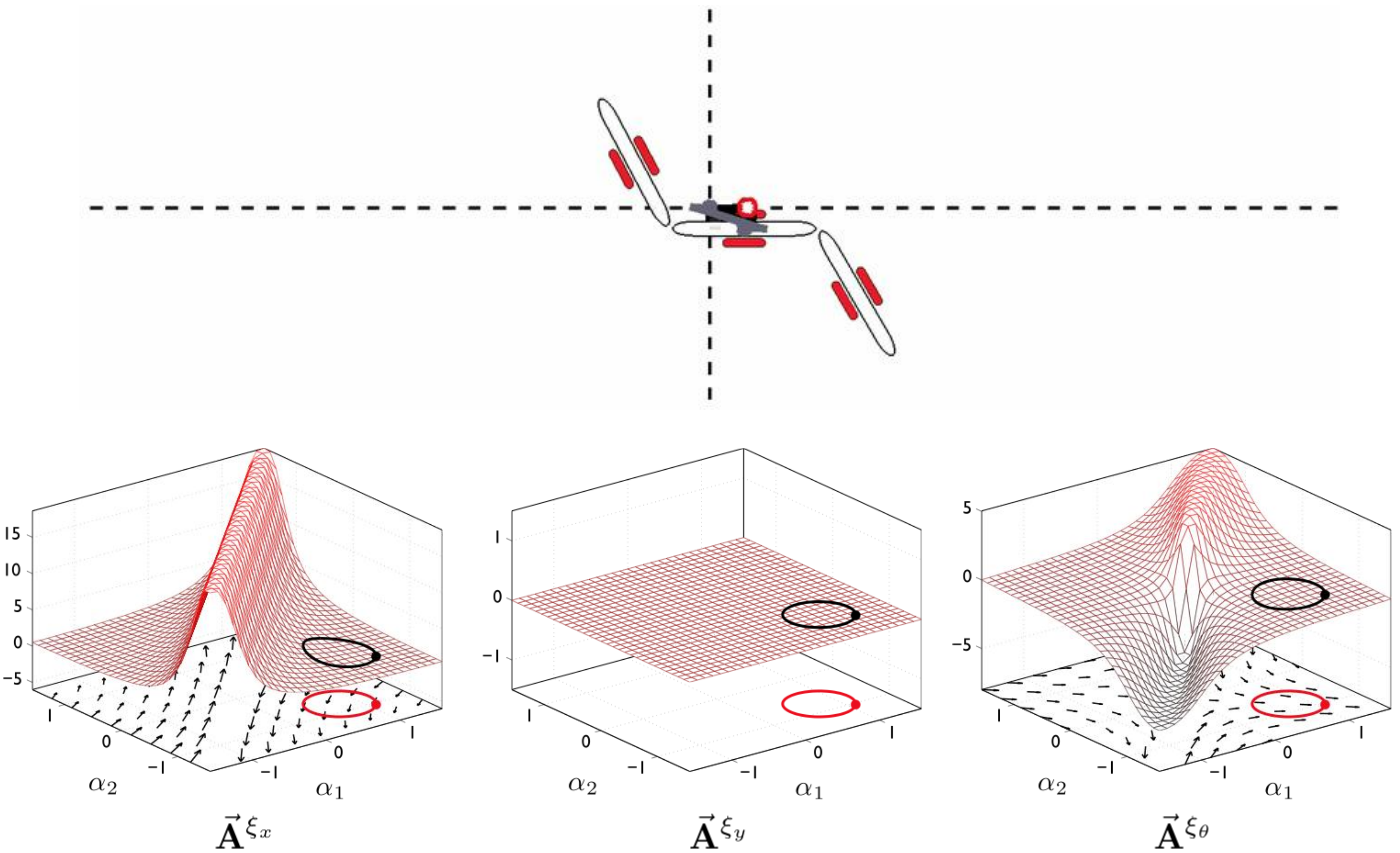


Gaits and Stokes’s theorem

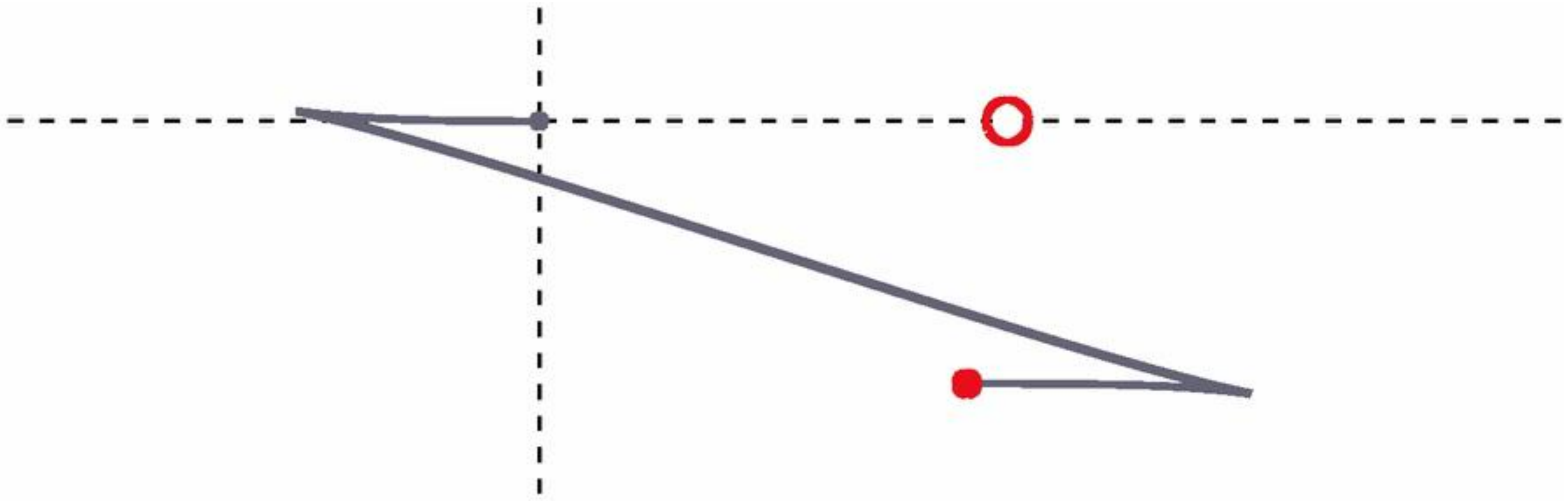


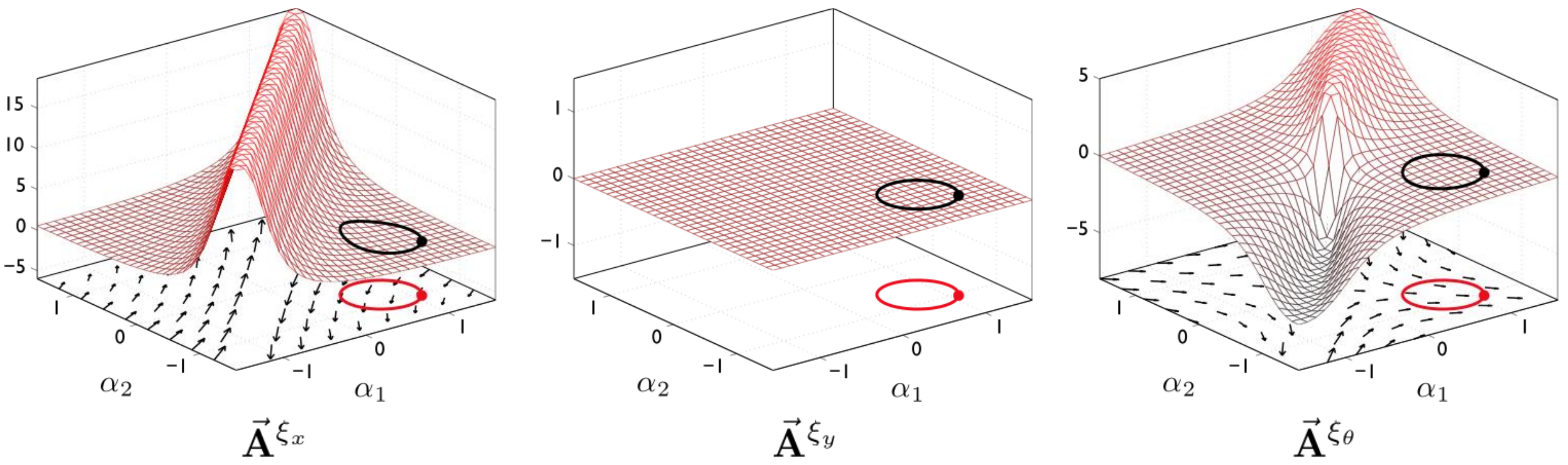


Gaits and Stokes' s theorem



Gaits and Stokes’s theorem





Displacement and the Lie Bracket

Displacement

$$\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \int_0^t \overbrace{\begin{bmatrix} \cos\theta(\tau) & -\sin\theta(\tau) & 0 \\ \sin\theta(\tau) & \cos\theta(\tau) & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{Rotate into world, } T_e L_g} \begin{bmatrix} \xi_x(\tau) \\ \xi_y(\tau) \\ \xi_\theta(\tau) \end{bmatrix} d\tau$$

This is a flow along a time-varying (left-invariant) vector field

Series representation over a gait

$$z(\phi) = \iint_\phi \overbrace{\underbrace{-\mathrm{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{[\mathbf{A}_1, \mathbf{A}_2]\, dr + \text{higher-order terms}}_{\text{noncommutativity}}}^{\text{Full Lie bracket}}$$

$$\exp(z) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

This is the “average” velocity vector for the system (with normalized time) – exponentiating this vector (flowing along the field for unit time) arrives at the same point as calculated by the displacement equation

Interpreting the series (I)

$$z(\phi) = \iint_{\phi} \overbrace{\underbrace{-\text{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{[\mathbf{A}_1, \mathbf{A}_2]\, dr + \text{higher-order terms}}_{\text{noncommutativity}}}^{\text{Full Lie bracket}}$$

Integral of the curl (exterior derivative) corresponds (via Stokes' s theorem) to the *body velocity integral* (BVI)

Body Velocity Integral

$$\zeta(t) = \begin{bmatrix} \zeta_x(t) \\ \zeta_y(t) \\ \zeta_\theta(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \xi_x(\tau) \\ \xi_y(\tau) \\ \xi_\theta(\tau) \end{bmatrix} d\tau$$

If we took only this term, it would be like saying "I know my average forward, lateral, and rotational velocities, so a first guess at my displacement would be the point reached by moving at that average velocity for the same amount of time."

Type Casting

- As a reminder, elements of SE(2) (the group) are of a different mathematical *type* than elements of se(2) (the tangent space at the identity)
- This is an especially important distinction when integrating:

T_eL_g means we interpret this calculation as a flow along a vector field – the output is a group element in SE(2)

$$\begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} = \int_0^t \overbrace{\begin{bmatrix} \cos\theta(\tau) & -\sin\theta(\tau) & 0 \\ \sin\theta(\tau) & \cos\theta(\tau) & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\text{Rotate into world, } T_eL_g} \begin{bmatrix} \xi_x(\tau) \\ \xi_y(\tau) \\ \xi_\theta(\tau) \end{bmatrix} \, d\tau$$

Here, we are just summing up the body velocities over time. The output is just the average body velocity, multiplied by the total time, and is this a vector in se(2)

$$\zeta(t) = \begin{bmatrix} \zeta_x(t) \\ \zeta_y(t) \\ \zeta_\theta(t) \end{bmatrix} = \int_0^t \begin{bmatrix} \xi_x(\tau) \\ \xi_y(\tau) \\ \xi_\theta(\tau) \end{bmatrix} \, d\tau$$

Displacements and BVIs

The BVI and cBVI (discussed below) are both elements of se(2). Informally, we often compare them with “displacements,” taking advantage of the fact that each element of se(2) maps to an element of SE(2) via the exponential map. This casual definition is helped by the property of the exponential map that if the rotational component of an se(2) element is zero, the exponential map is an identity map.

One thing to be careful of, however, is saying anything like “the BVI is the displacement in the body frame.”

• First, elements of SE(2) interpreted as displacements are themselves *always* with respect to the starting frame, so the above quote is misleading.
• Second, the above quote implies that the BVI resulting from an input can be turned into a world frame displacement simply by applying a transformation out of the body coordinates, which is false.

Interpreting the series (II)

$$z(\phi) = \iint_\phi \overbrace{\underbrace{-\mathrm{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{\left[\mathbf{A}_1, \mathbf{A}_2\right]}_{\text{noncommutativity}}}^{\text{Full Lie bracket}} \, dr + \text{higher-order terms}$$

Incorporating the second term gives us the *corrected body velocity integral* (cBVI)

The cBVI does not correspond to a time integral in the way that the BVI does – the Lie bracket term only has meaning over complete cycles.

Including the second term gives us a correcting factor to our BVI. It is like saying “If I know that I turn positively before moving forward, I know that my motion is biased to the left; to properly compute my average velocity, then, I should include a corresponding lateral component.”

Interpreting the series (III)

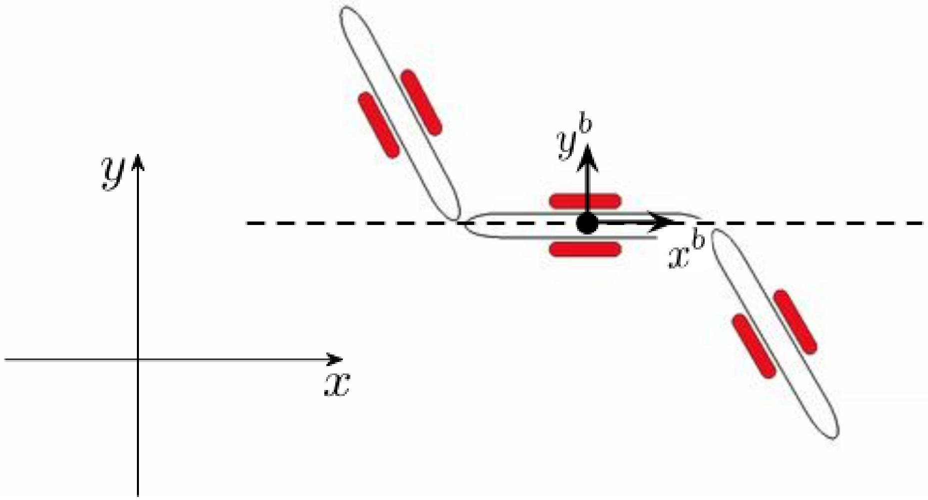
$$z(\phi) = \iint_{\phi} \overbrace{\underbrace{-\mathrm{curl}\mathbf{A}}_{\text{nonconservativity}} + \underbrace{[\mathbf{A}_1, \mathbf{A}_2]}_{}}^{\text{Full Lie bracket}} dr \underbrace{+ \text{ higher-order terms}}_{}$$

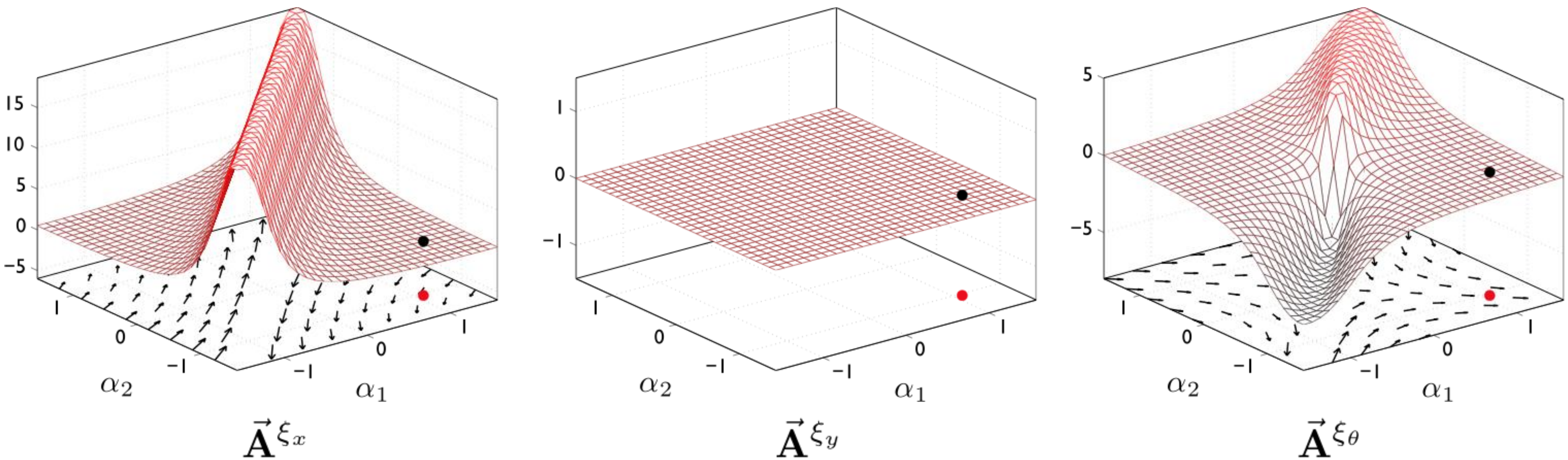
$$\underbrace{[\mathbf{A}_1, \mathbf{A}_2] \, dr + \text{higher-order terms}}_{\text{noncommutativity}}$$

The first two terms correspond to properties of the gait cycle (the closed curve it defines in the shape space) : the curl measures the change of the constraints over the shape space, and the local Lie bracket captures the effects of the *cyclic* segment ordering around its perimeter.

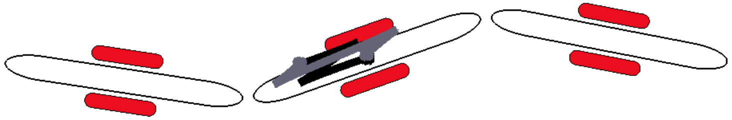
The higher-order terms correspond to the starting point on the gait, and so do not have a geometric (area-based) formulation. If we were to consider them, we would be making statements like "Parallel parking alternates forward, negative turning, backwards, positive turning. If I change the starting point, and move with negative turning, backwards, positive turning, forward translation, then the negative turning will have a bigger influence on my net motion."

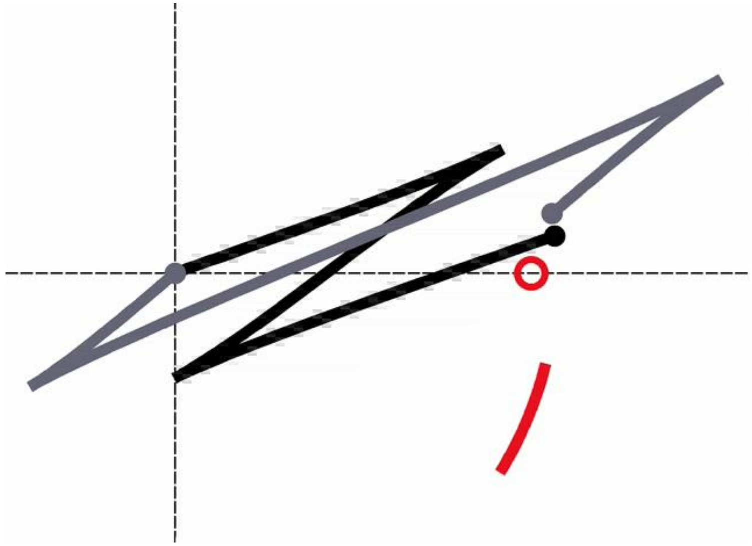
Mean Orientation

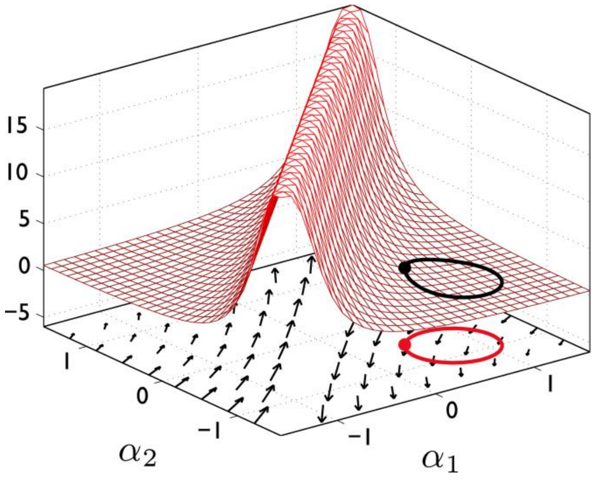




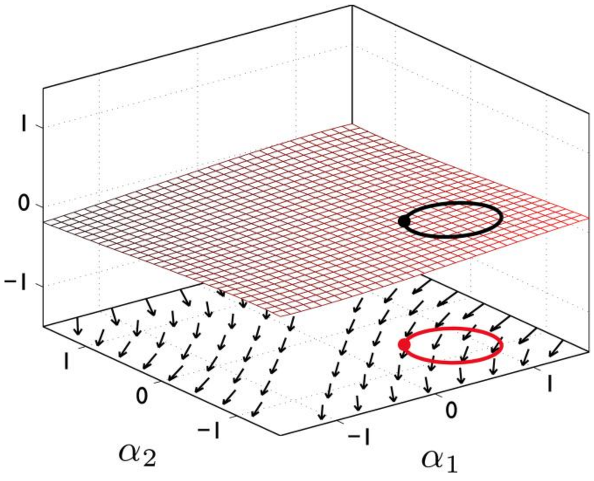
Mean Orientation



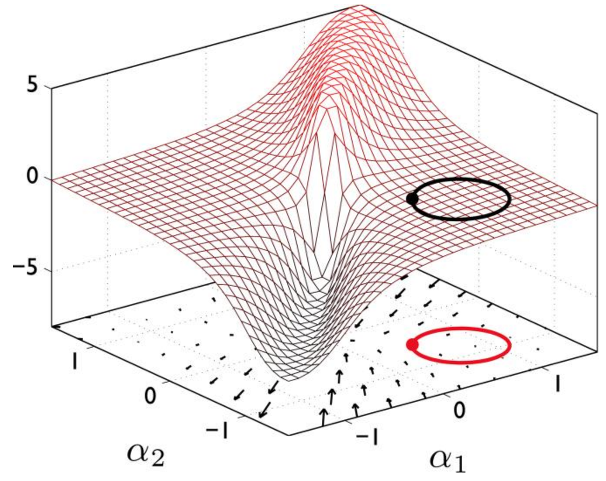




$\vec{\mathbf{A}}^{\xi_x}$

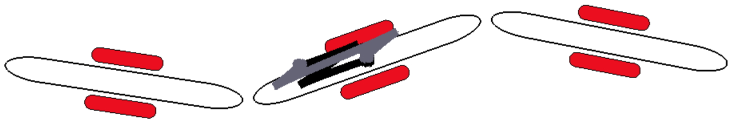


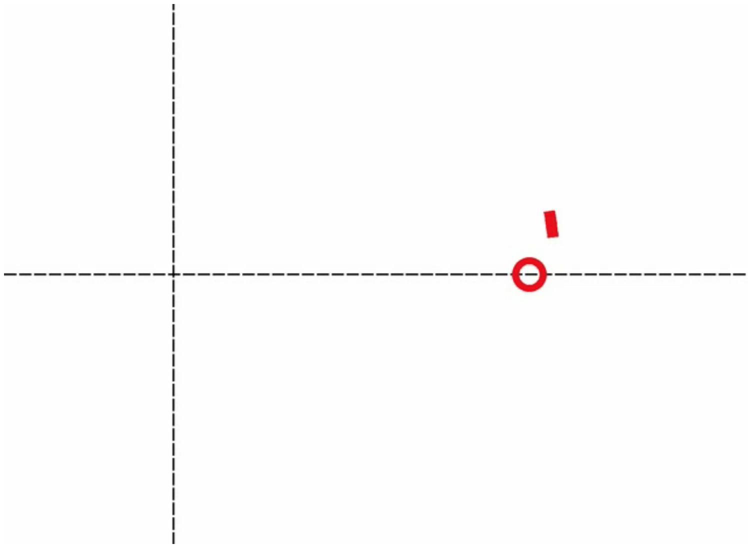
$\vec{\mathbf{A}}^{\xi_y}$

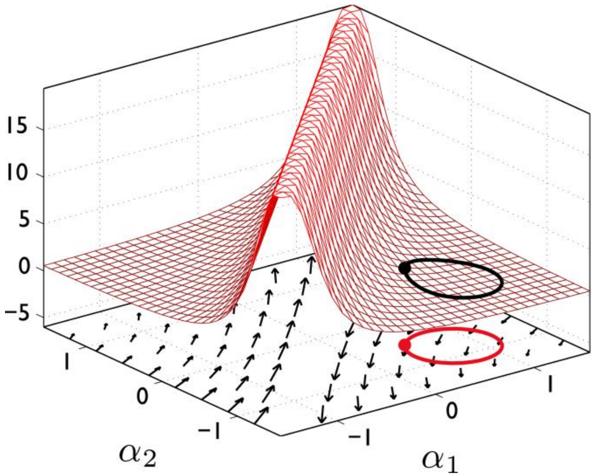


$\vec{\mathbf{A}}^{\xi_\theta}$

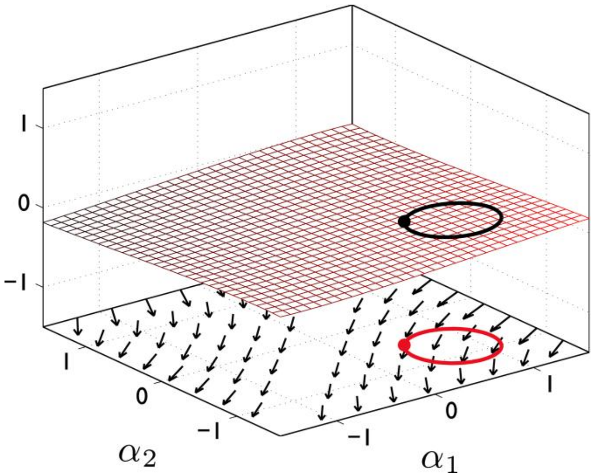
Mean Orientation



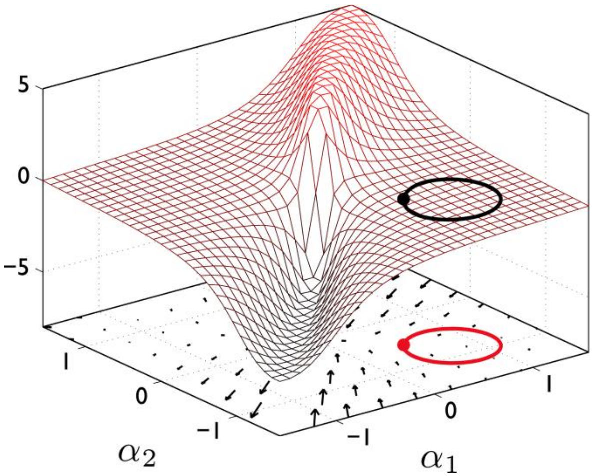




$\vec{\mathbf{A}}^{\xi_x}$



$\vec{\mathbf{A}}^{\xi_y}$

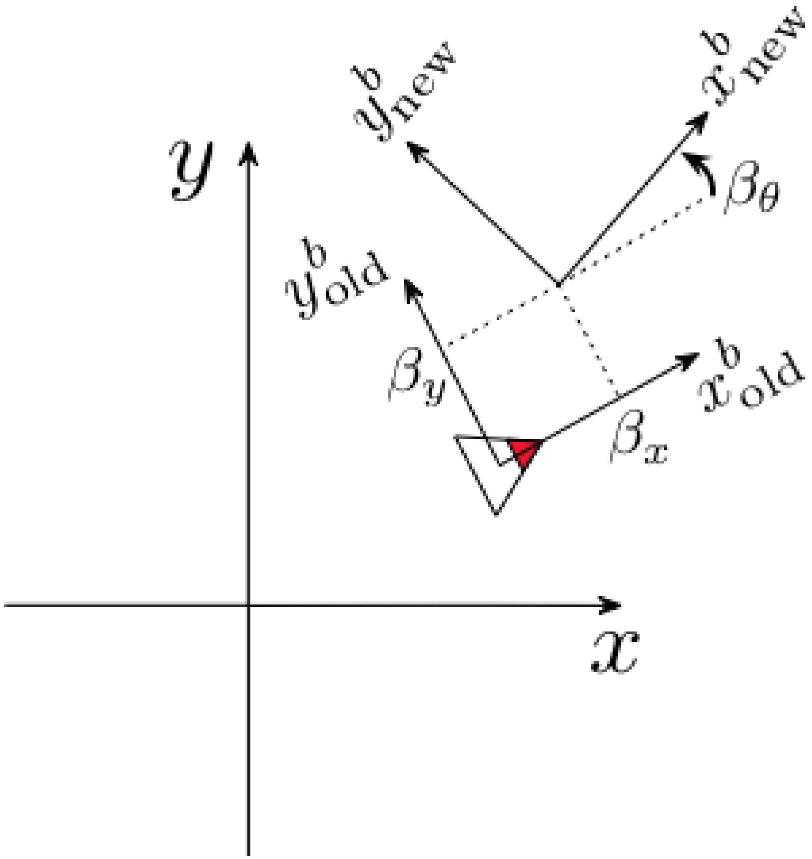


$\vec{\mathbf{A}}^{\xi_\theta}$

Available coordinate choices?

Good coordinate choice?

Available coordinate choices

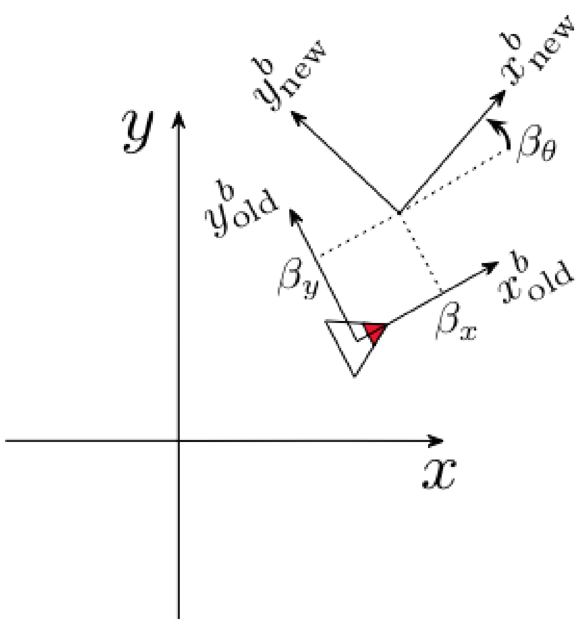


Valid body frame if (and only if)

$$\beta = f(\alpha)$$

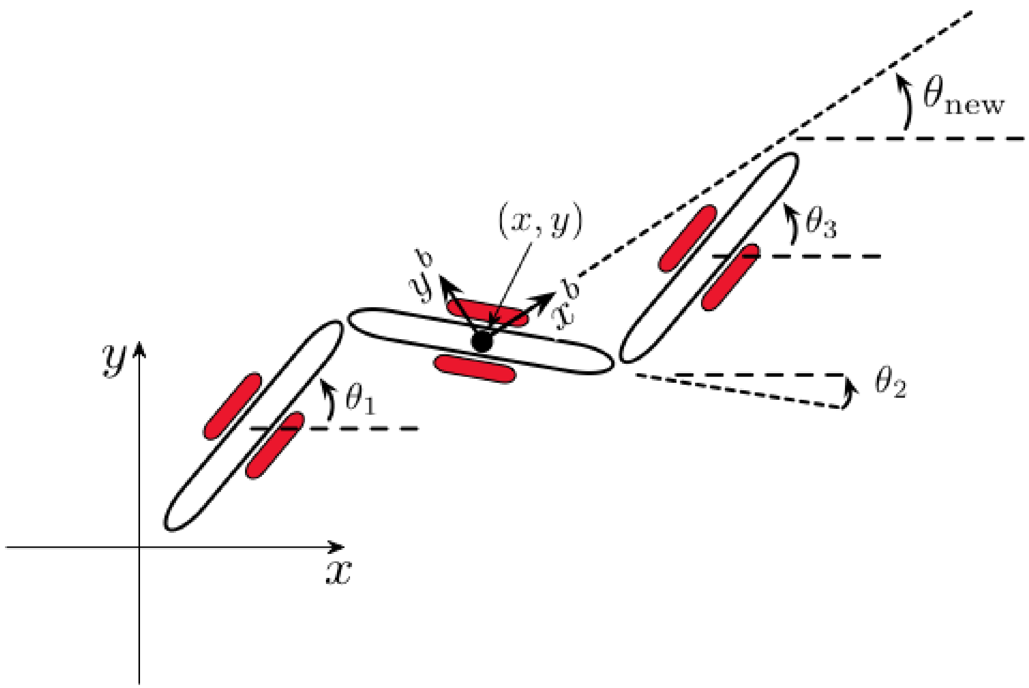
Effect of changing coordinates

$$\overbrace{\xi^{\text{new}}}^{\text{new body velocity}} = \begin{bmatrix} \cos\beta_\theta & \sin\beta_\theta & 0 \\ -\sin\beta_\theta & \cos\beta_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot (\overbrace{\xi}^{\text{original body velocity}} + \overbrace{\dot{\beta}}^{\text{relative velocities}} + \overbrace{B_\times \xi_\theta}^{\text{cross product}})$$



New velocity for mean orientation

$$\xi^{\text{new}} = \cancel{\begin{bmatrix} \cos\beta_\theta & \sin\beta_\theta & 0 \\ -\sin\beta_\theta & \cos\beta_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}} \cdot (\xi + \overset{(x,y)}{\cancel{\dot{\beta}}} + \cancel{B_x}\, \xi_\theta)$$



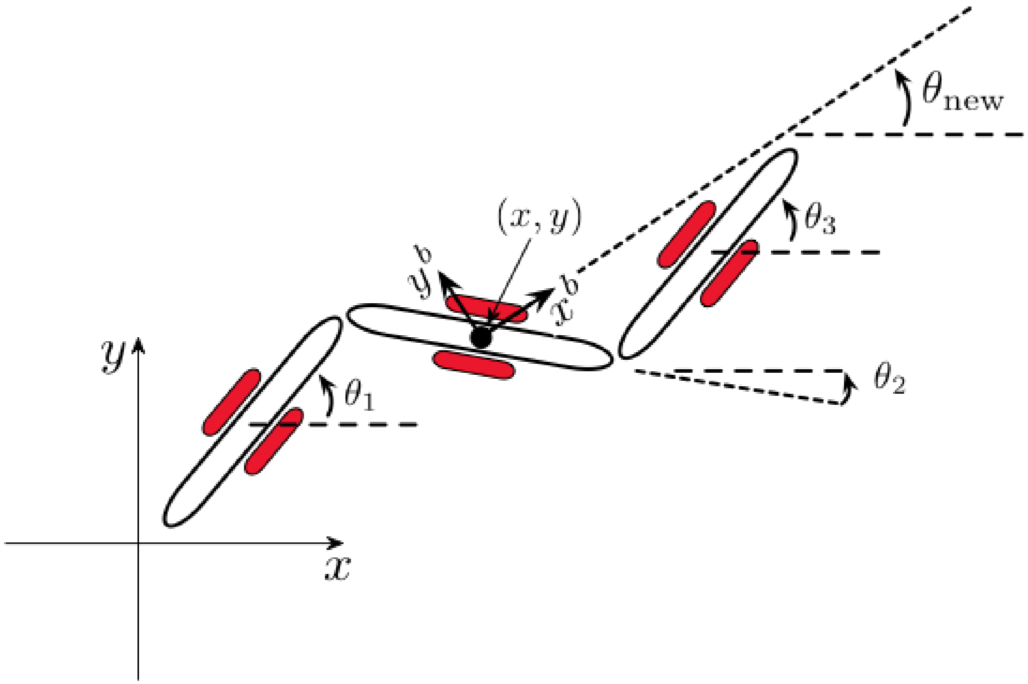
New rotational vector field

$$\xi_\theta^{\text{new}} = \xi_\theta + \dot{\beta}_\theta$$

$$\xi_\theta^{\text{new}} = \vec{\mathbf{A}}_{\text{new}}^{\xi_\theta} \cdot \overbrace{\dot{\alpha}}^{\text{shape velocity}}$$

$$\xi_\theta = \vec{\mathbf{A}}^{\xi_\theta} \cdot \dot{\alpha}$$

$$\dot{\beta}_\theta = \nabla_\alpha \beta_\theta \cdot \dot{\alpha}$$



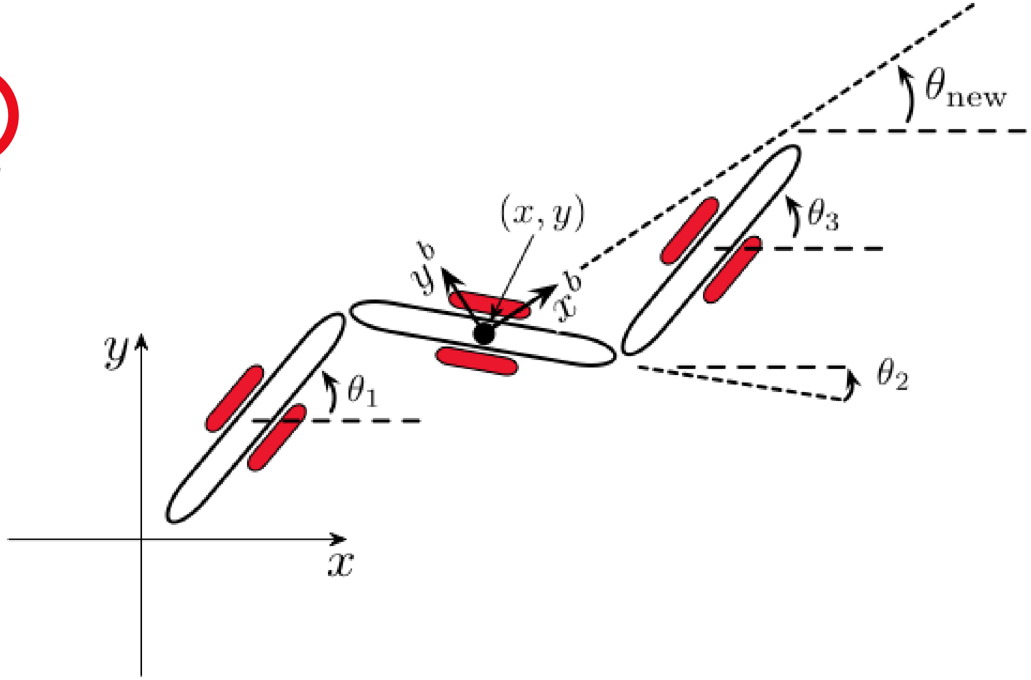
New rotational vector field

$$\vec{\mathbf{A}}^{\xi_\theta}_{\text{new}} = \vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta$$

$$\theta_{\text{new}} = \frac{\theta_1 + \theta_2 + \theta_3}{3}$$

$$\theta_1 = \theta_2 - \alpha_1$$

$$\theta_3 = \theta_2 + \underbrace{\alpha_2}_{\text{joint angles}}$$



New rotational vector field

$$\vec{\mathbf{A}}^{\xi_\theta}_{\mathrm{new}} = \vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta$$

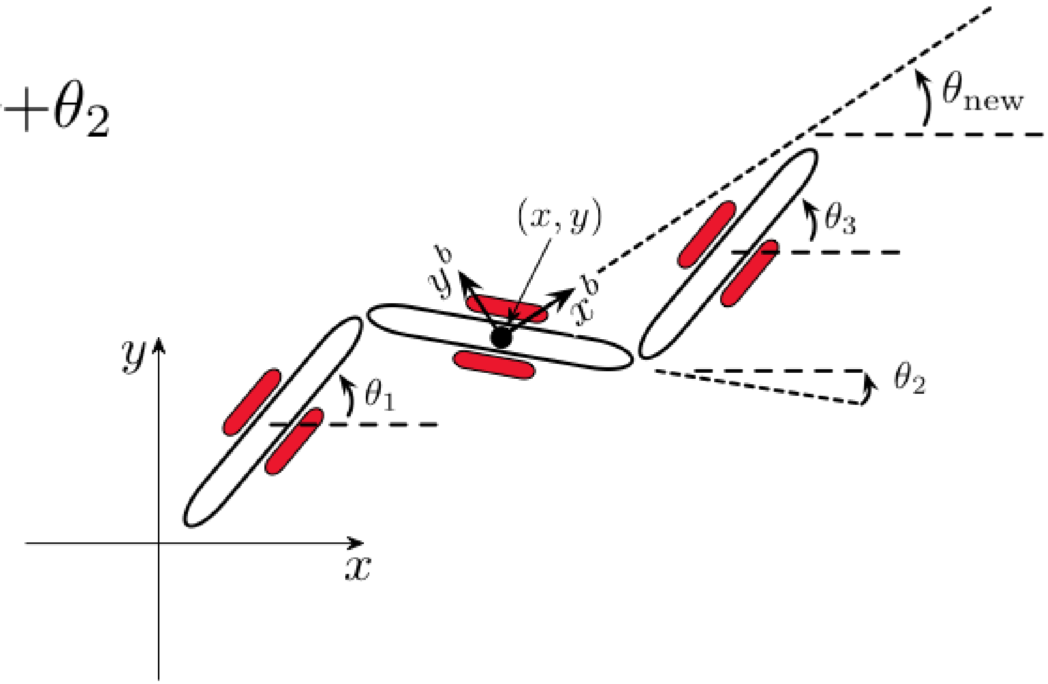
$$\theta_{\mathrm{new}} = \frac{-\alpha_1 + \alpha_2}{3} + \theta_2$$

change of orientation

$$\beta_\theta = \theta_{\mathrm{new}} - \theta_2$$

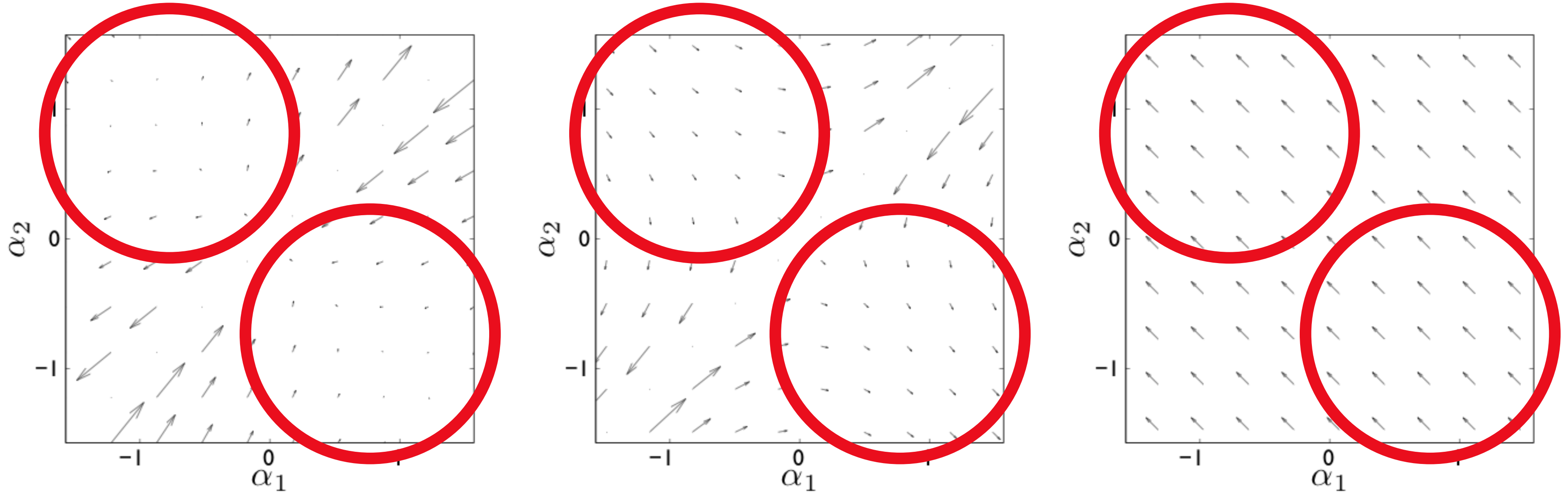
gradient with respect to shape

$$\nabla_\alpha \beta_\theta = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

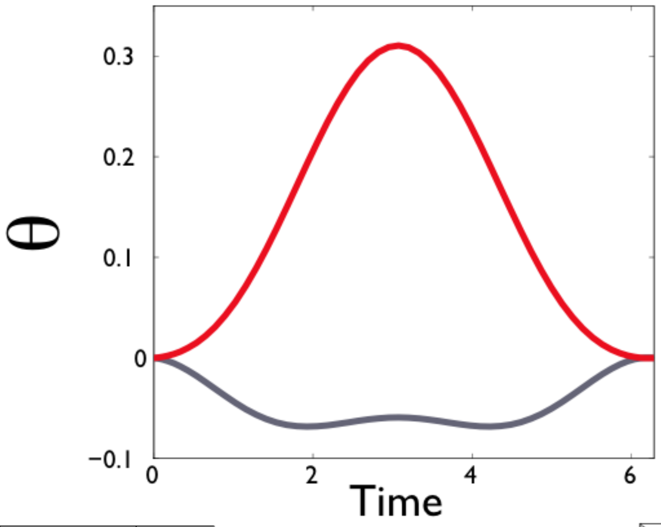


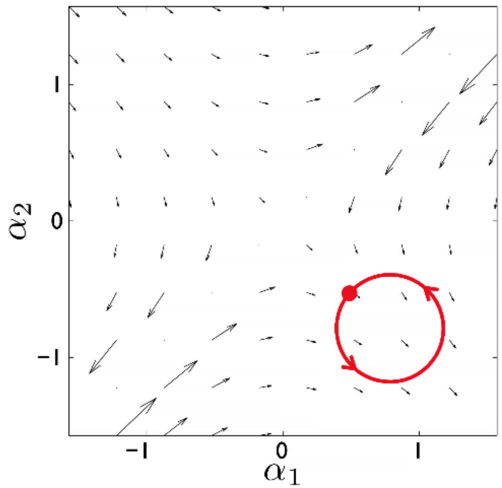
New rotational vector field

$$\vec{\mathbf{A}}^{\xi_\theta}_{\text{new}} = \vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta$$



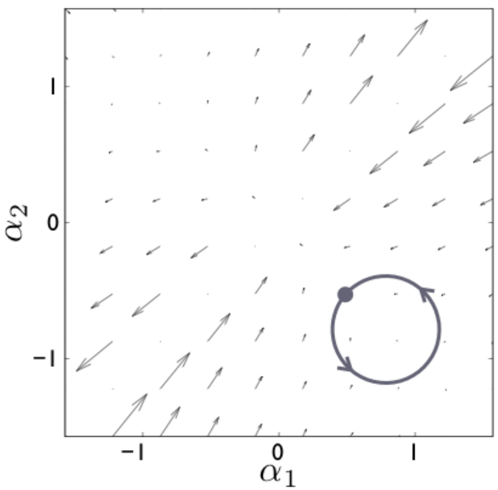
Orientation during gait





Original

$$\vec{\mathbf{A}}^{\xi_\theta}$$



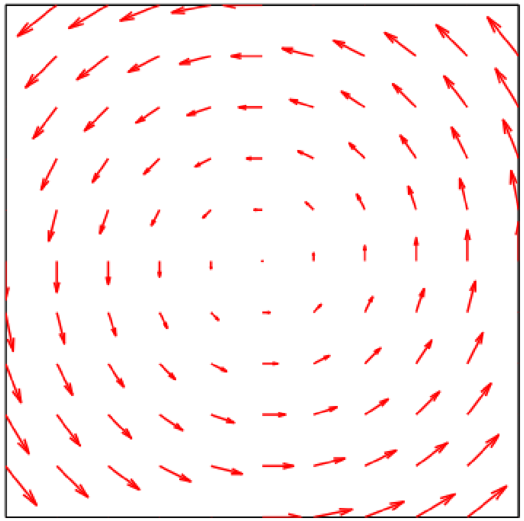
Mean

Optimal orientation

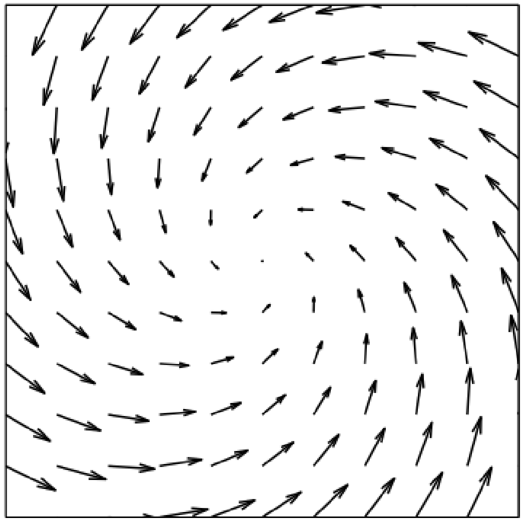
Find $\overbrace{\nabla_\alpha \beta_\theta}^{\text{change of orientation}}$ that minimizes

$$\iint_\Omega \| \underbrace{\vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta}_{\vec{\mathbf{A}}^{\xi_\theta}_{\text{new}}} \|^2 \, d\Omega$$

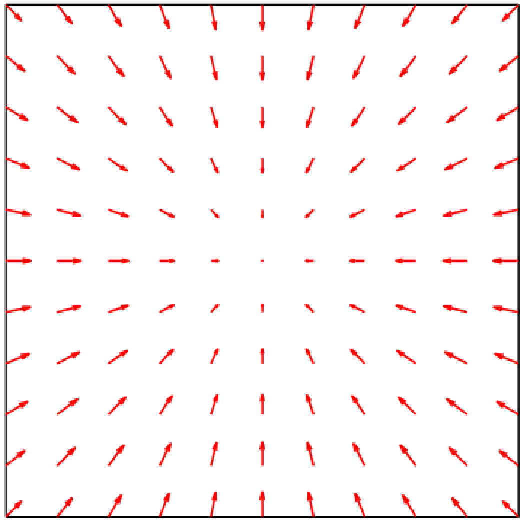
Hodge-Helmholtz Decomposition



Same Curl



Original



Closest Gradient

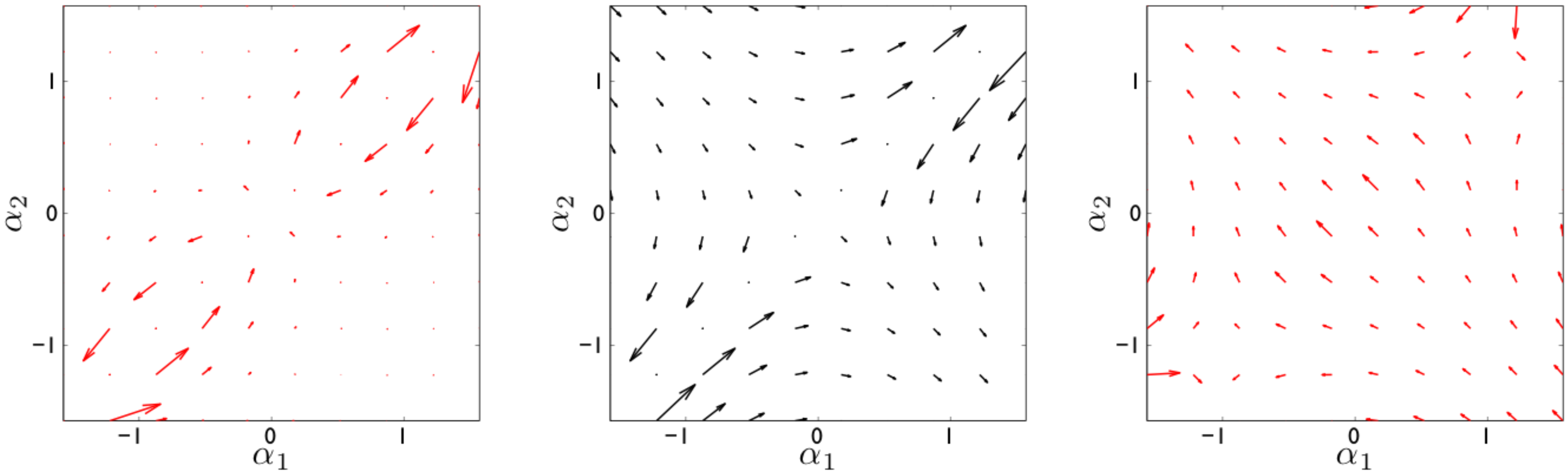
Original = Same Curl + Closest Gradient

Find $\nabla_\alpha \beta_\theta$ that minimizes

$$\iint_\Omega \| \vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta \|^2 \; d\Omega$$

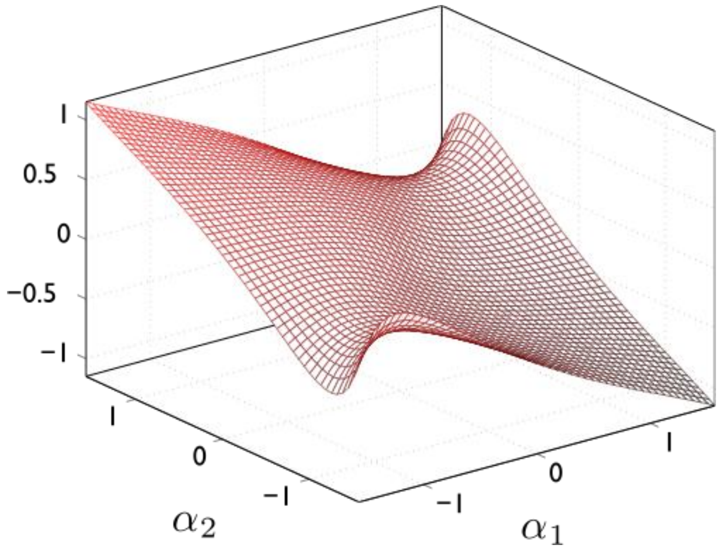
Same Curl

-Closest Gradient

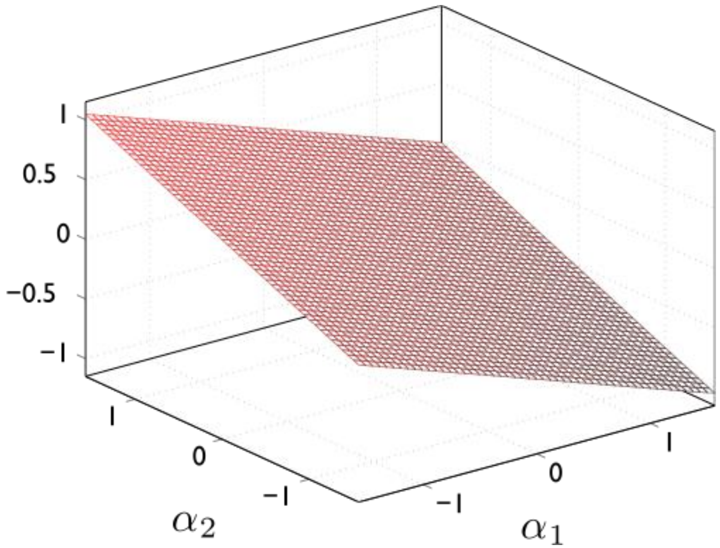


$$\vec{\mathbf{A}}^{\xi_\theta}_{\text{opt}} = \vec{\mathbf{A}}^{\xi_\theta} + \nabla_\alpha \beta_\theta$$

Optimal vs. Mean Orientation

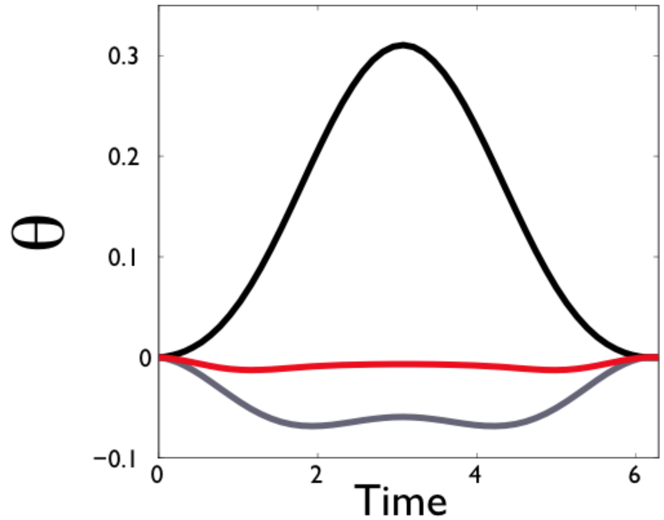


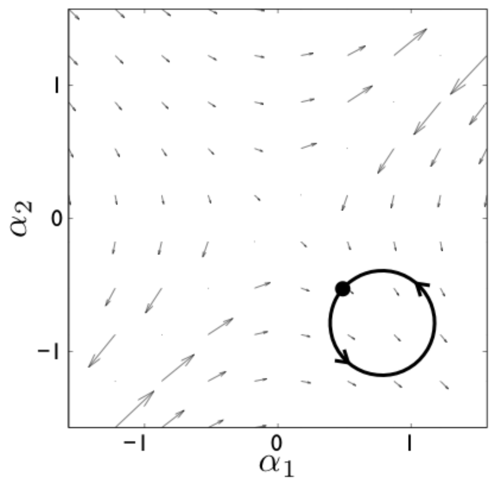
Optimal β_θ



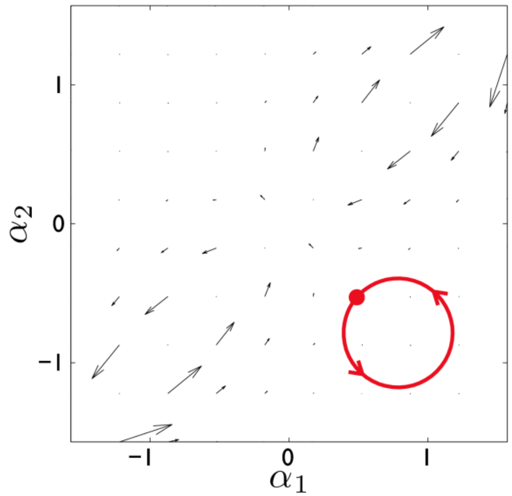
Mean β_θ

Orientation during gait

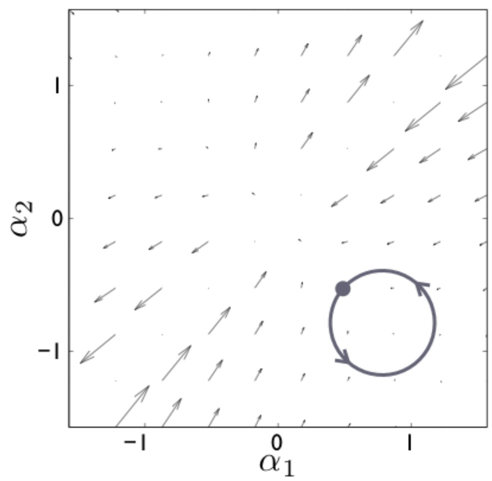




Original



Optimal



Mean

BVI–Displacement Error

$$\varepsilon_\zeta = \zeta - g = \int_0^t \begin{bmatrix} 1-\cos\theta & \sin\theta & 0 \\ -\sin\theta & 1-\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_x \\ \xi_y \\ \xi_\theta \end{bmatrix} d\tau$$

Optimal position

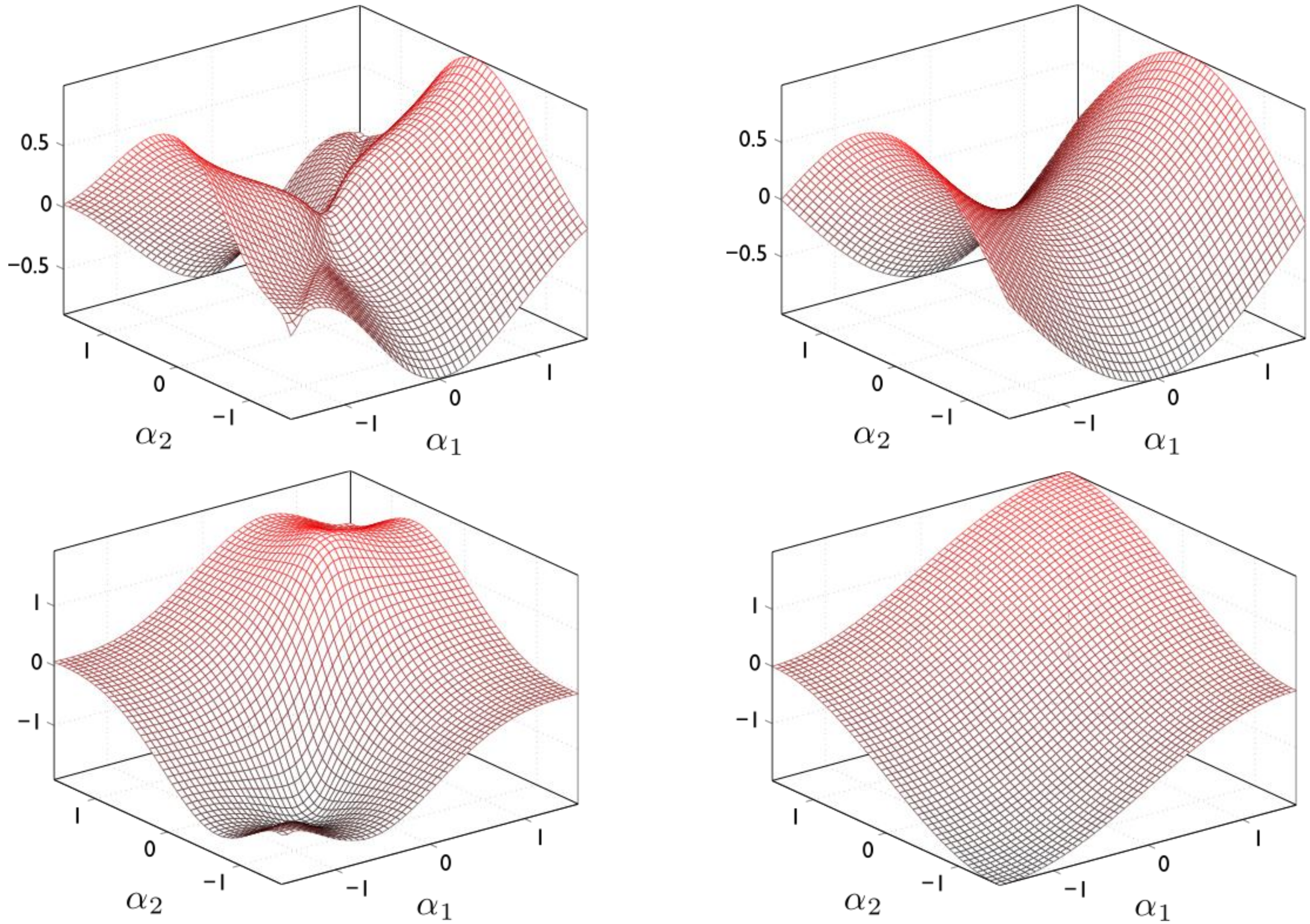
Find β_x and β_y that minimize

$$\iint_\Omega \|\vec{\mathbf{A}}^{\xi_x} + \nabla_\alpha \beta_x - \beta_y \vec{\mathbf{A}}^{\xi_\theta}\|^2 + \|\vec{\mathbf{A}}^{\xi_y} + \nabla_\alpha \beta_y + \beta_x \vec{\mathbf{A}}^{\xi_\theta}\|^2 \; d\Omega$$

Optimal position

β_x

β_y

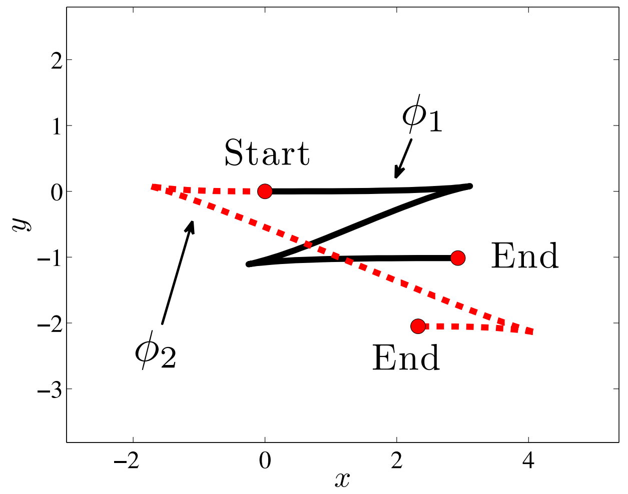


Optimal

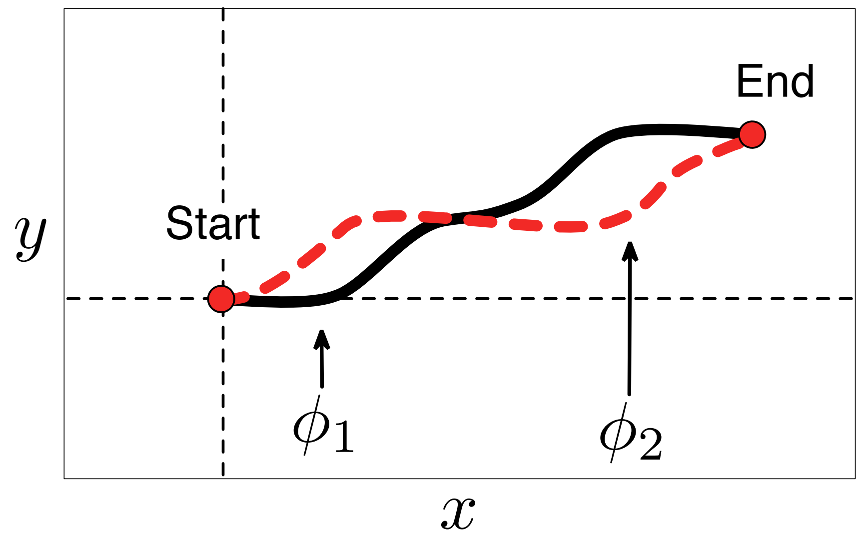
Center of mass

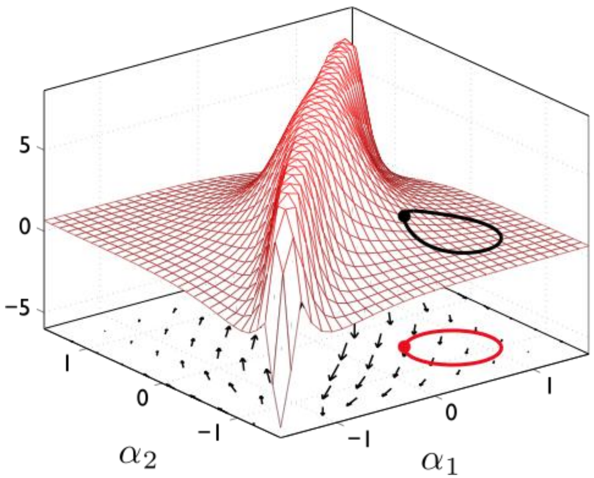
Optimal coordinates

Original

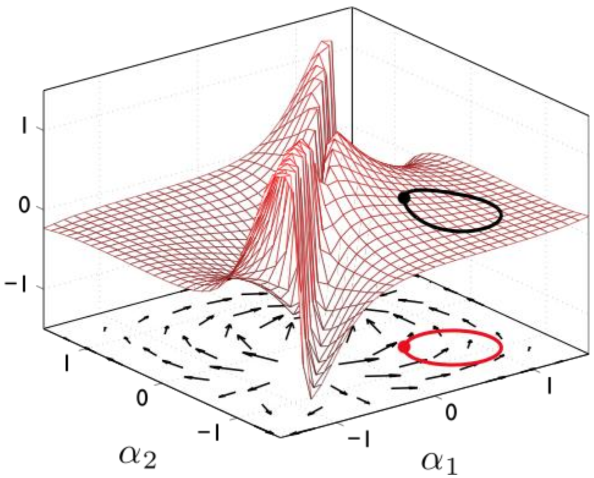


Optimized

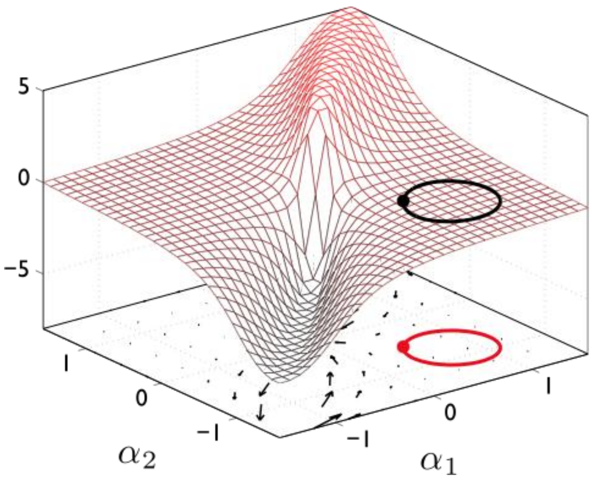




$\vec{\mathbf{A}}^{\xi_x}$

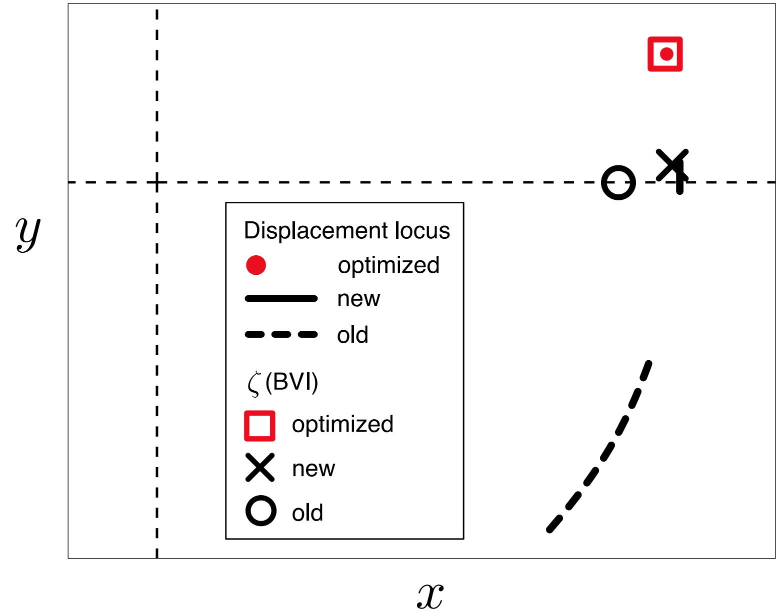


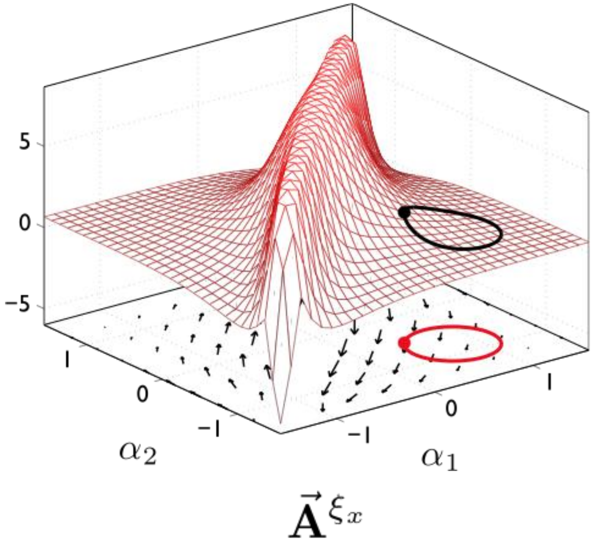
$\vec{\mathbf{A}}^{\xi_y}$

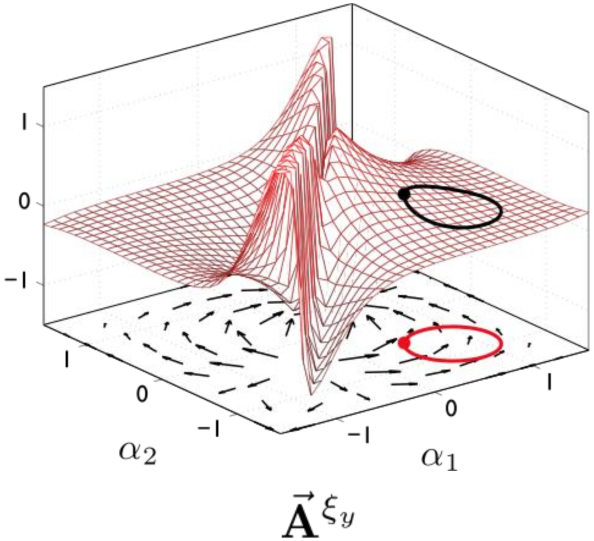


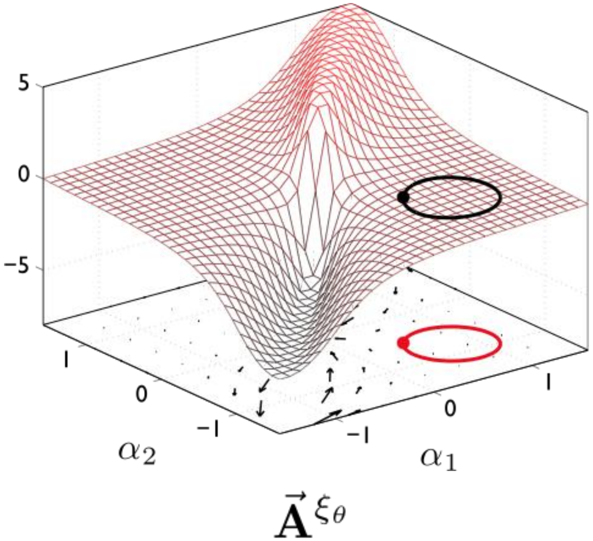
$\vec{\mathbf{A}}^{\xi_\theta}$

Optimal coordinates

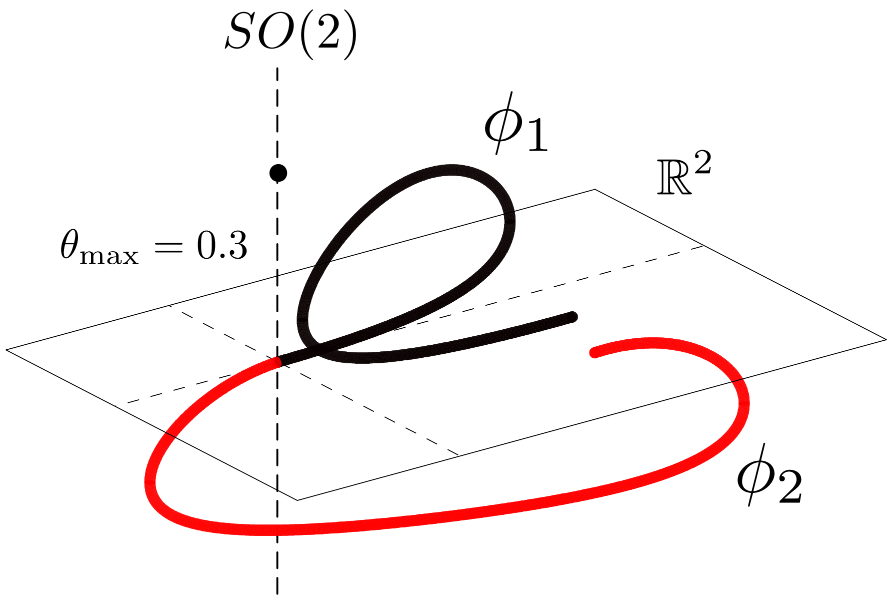




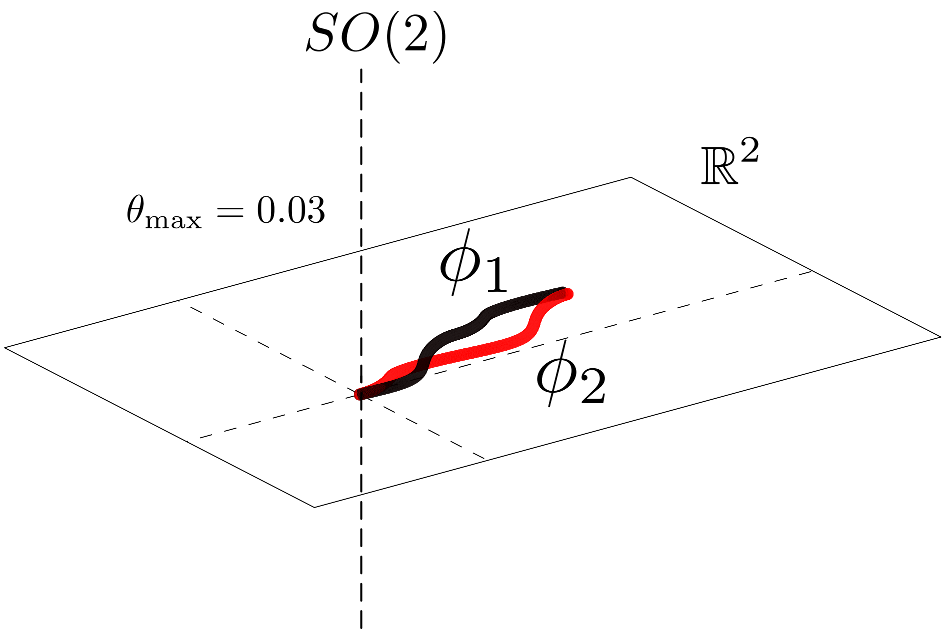




Images to replace optimal translation trajectory video



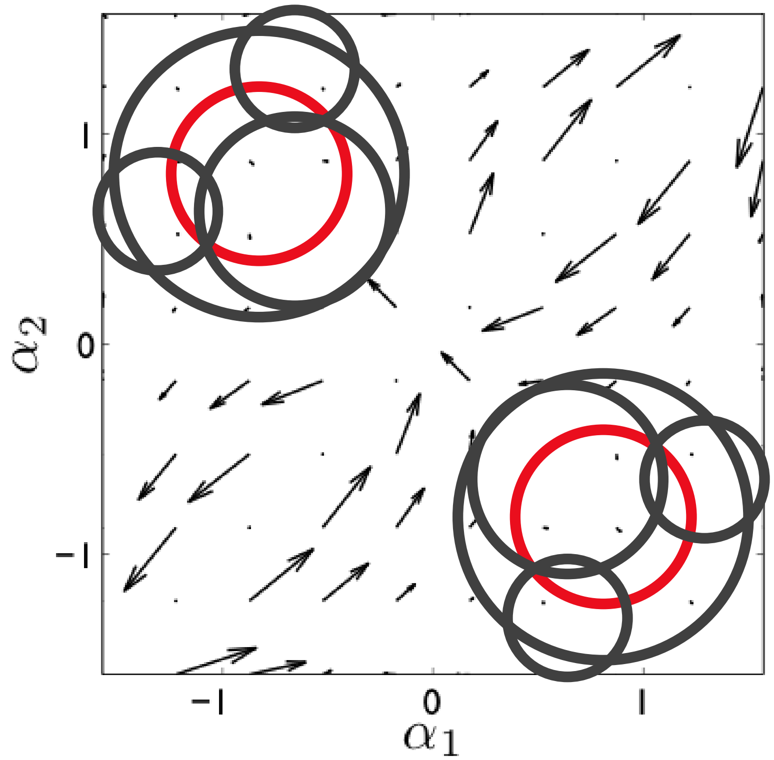
Original coordinates



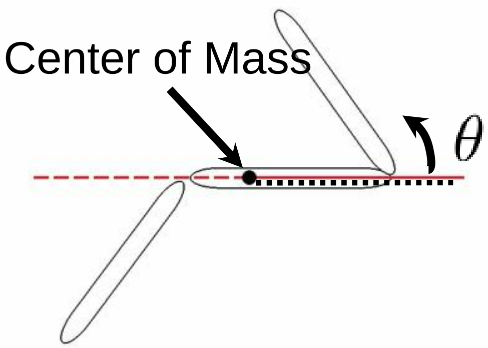
Optimal coordinates

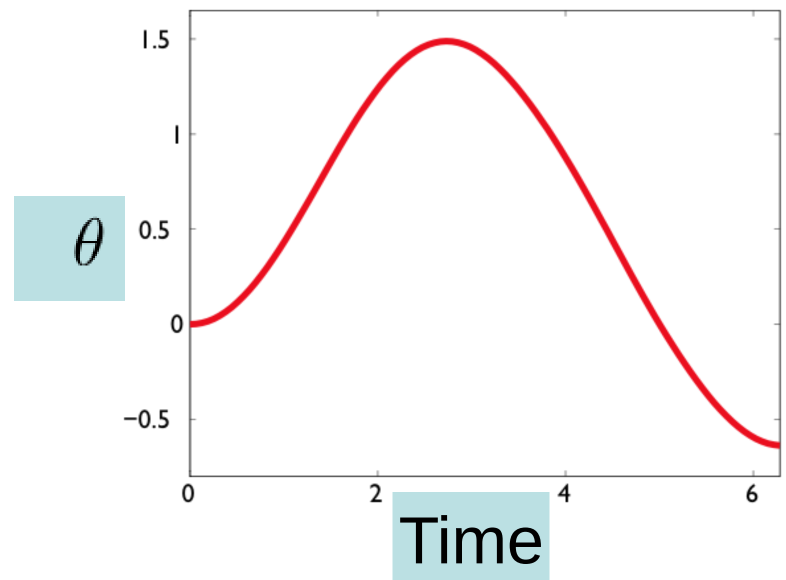
Generality

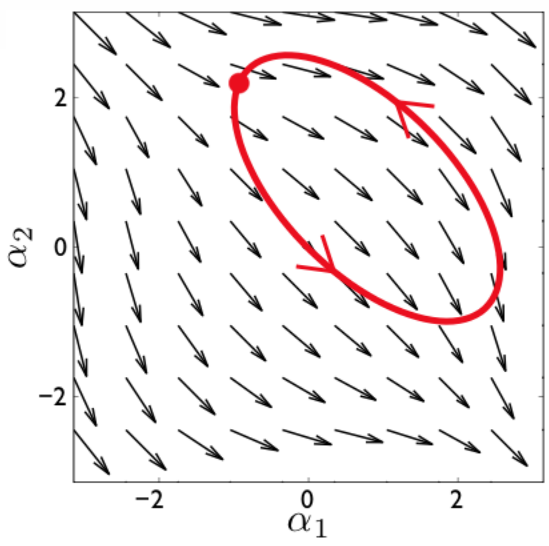
Gaits with BVI error less than 10%



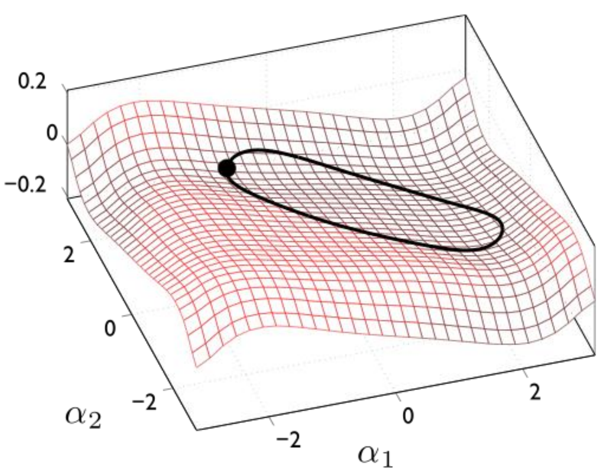
Floating snake



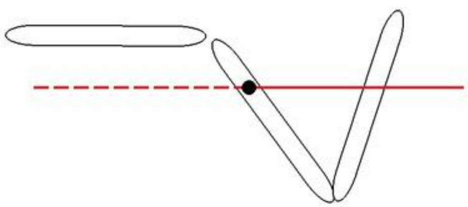


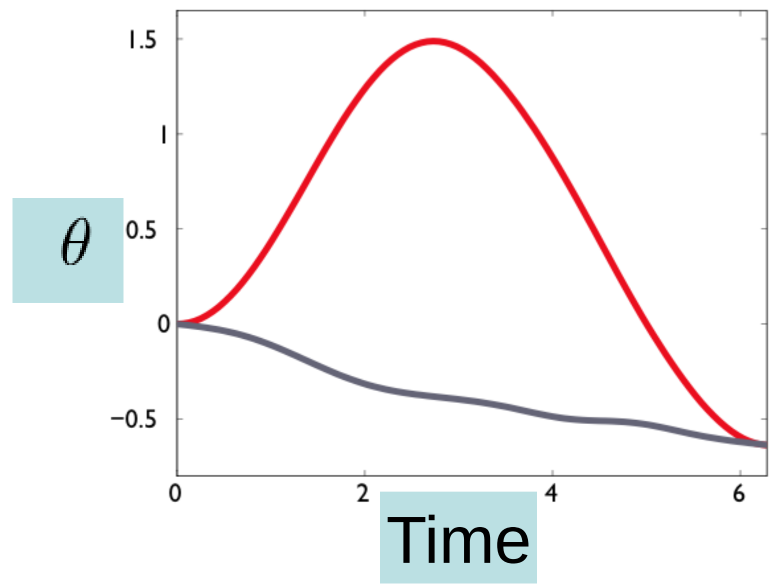


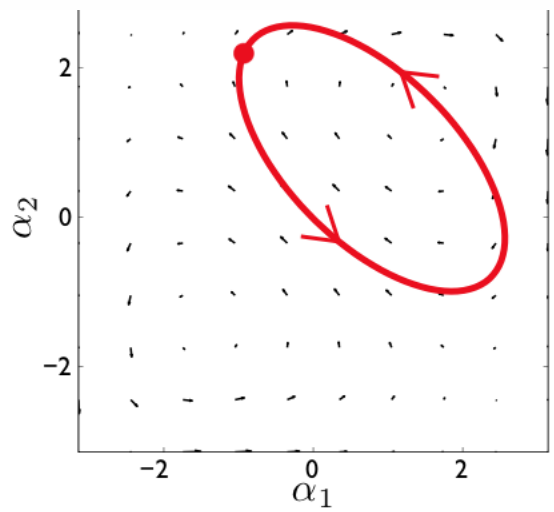
$$\vec{\mathbf{A}}^{\xi_\theta}$$



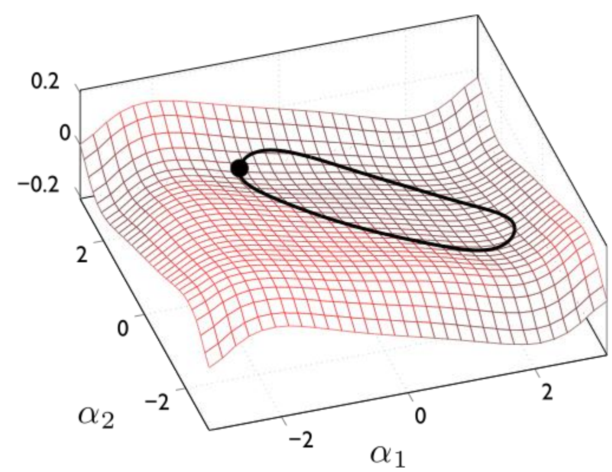
Floating snake







$$\vec{\mathbf{A}}^{\xi_\theta}$$



Conclusions

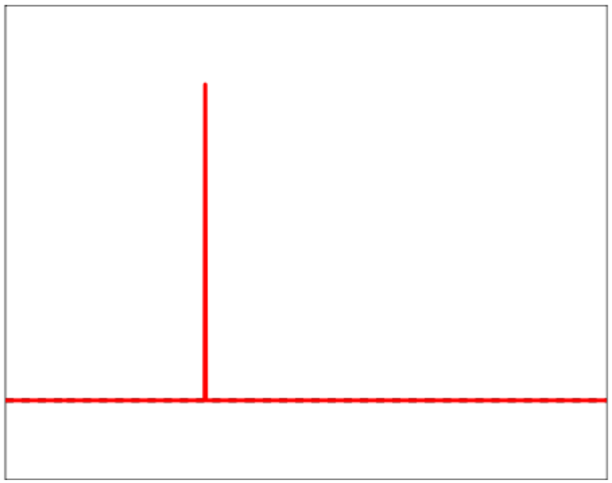
- The choice of position coordinates affects the system analysis
- In the “right” coordinates, the curl functions succinctly characterize locomotion performance
- Coordinate optimization strips out motion that will be “undone” over the course of a gait

Future work

Modal Approach

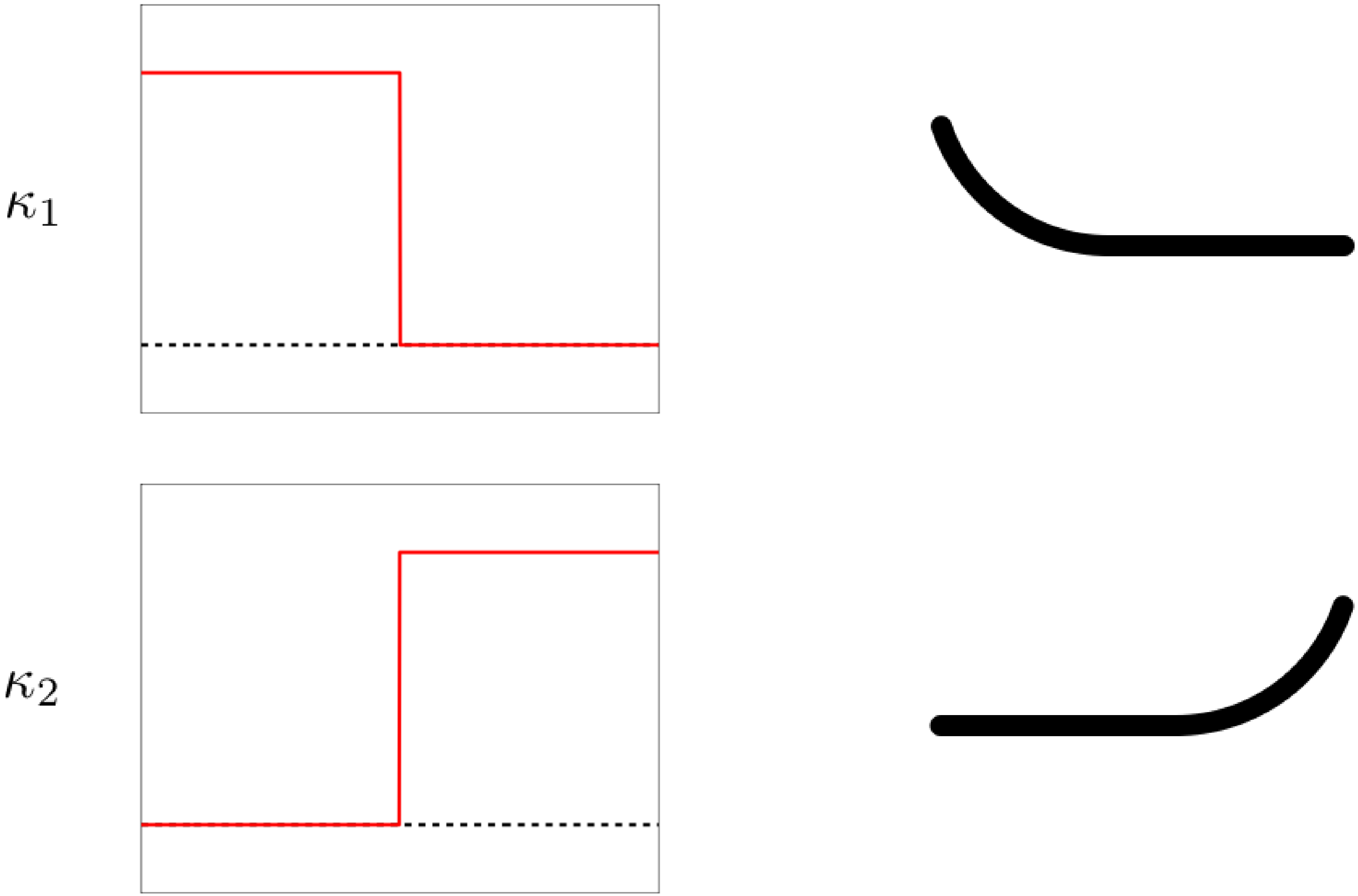
Discrete joints: Dirac delta curvature

κ_1



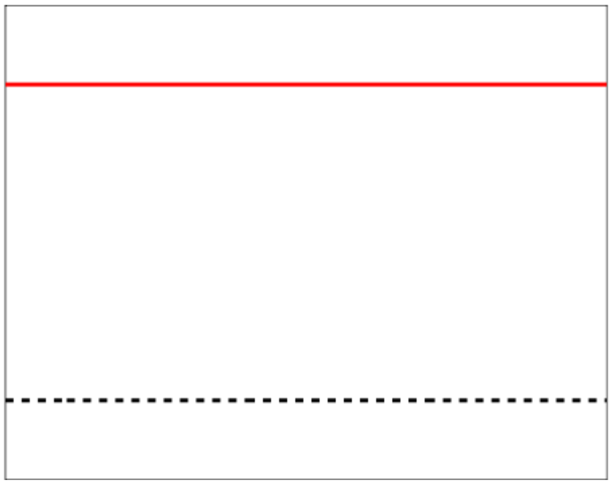
κ_2

Constant curvature

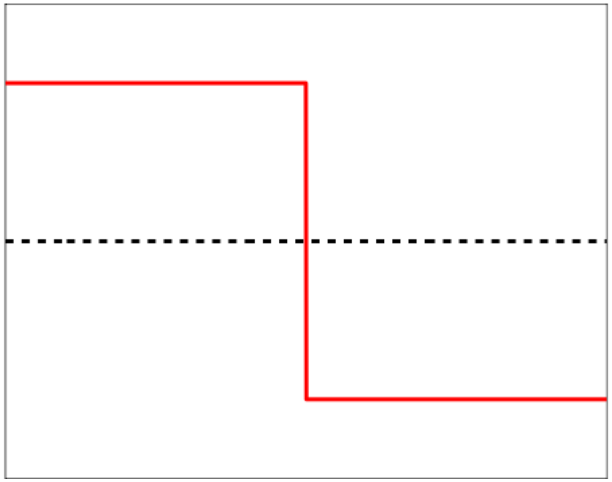


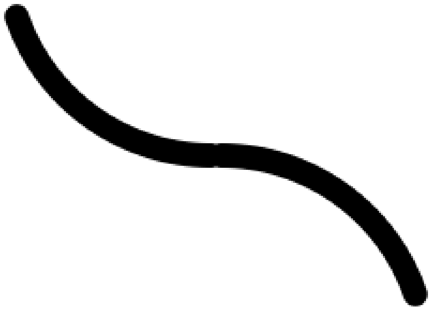
Alternate modes for constant curvature

κ_1

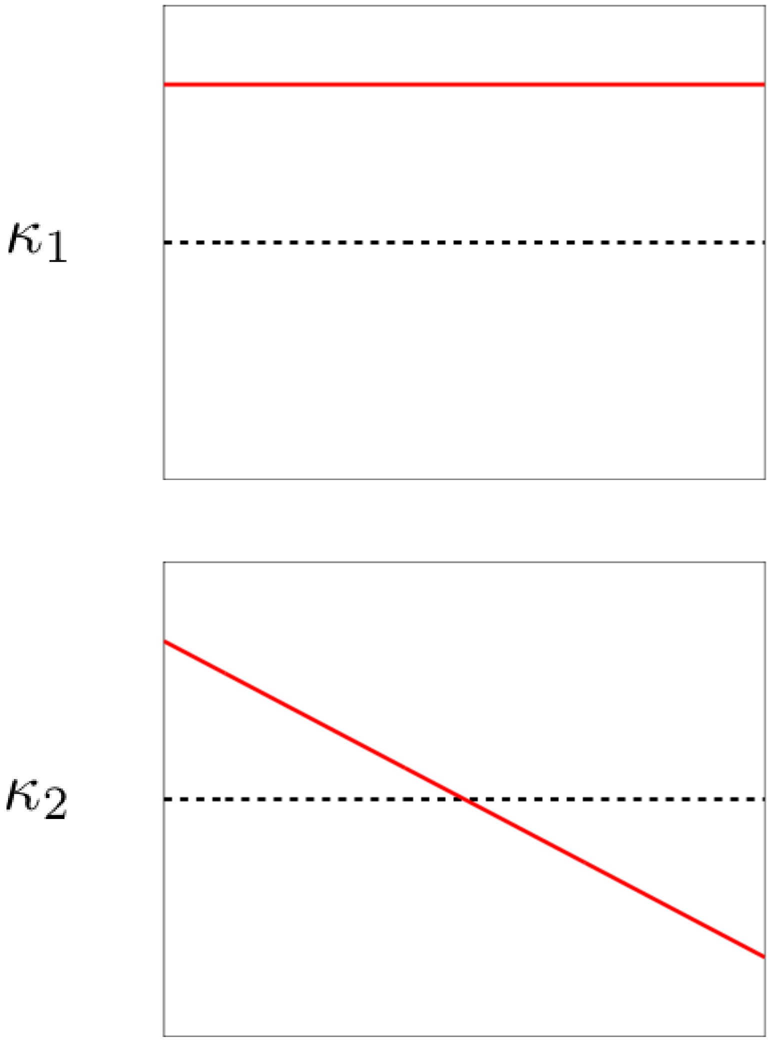


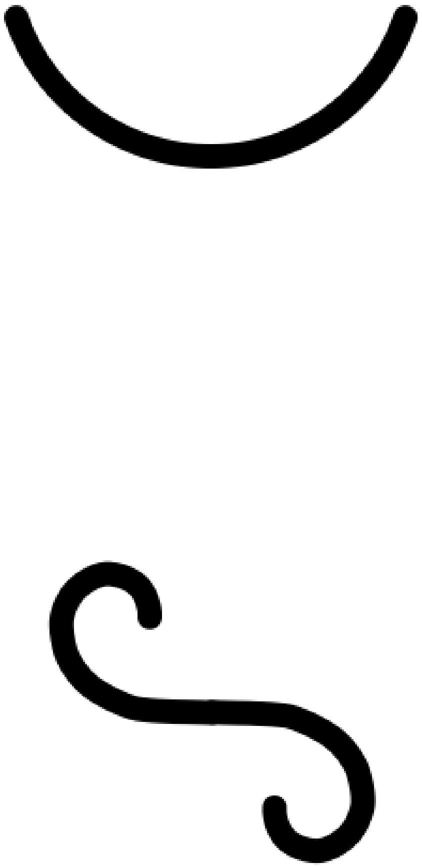
κ_2



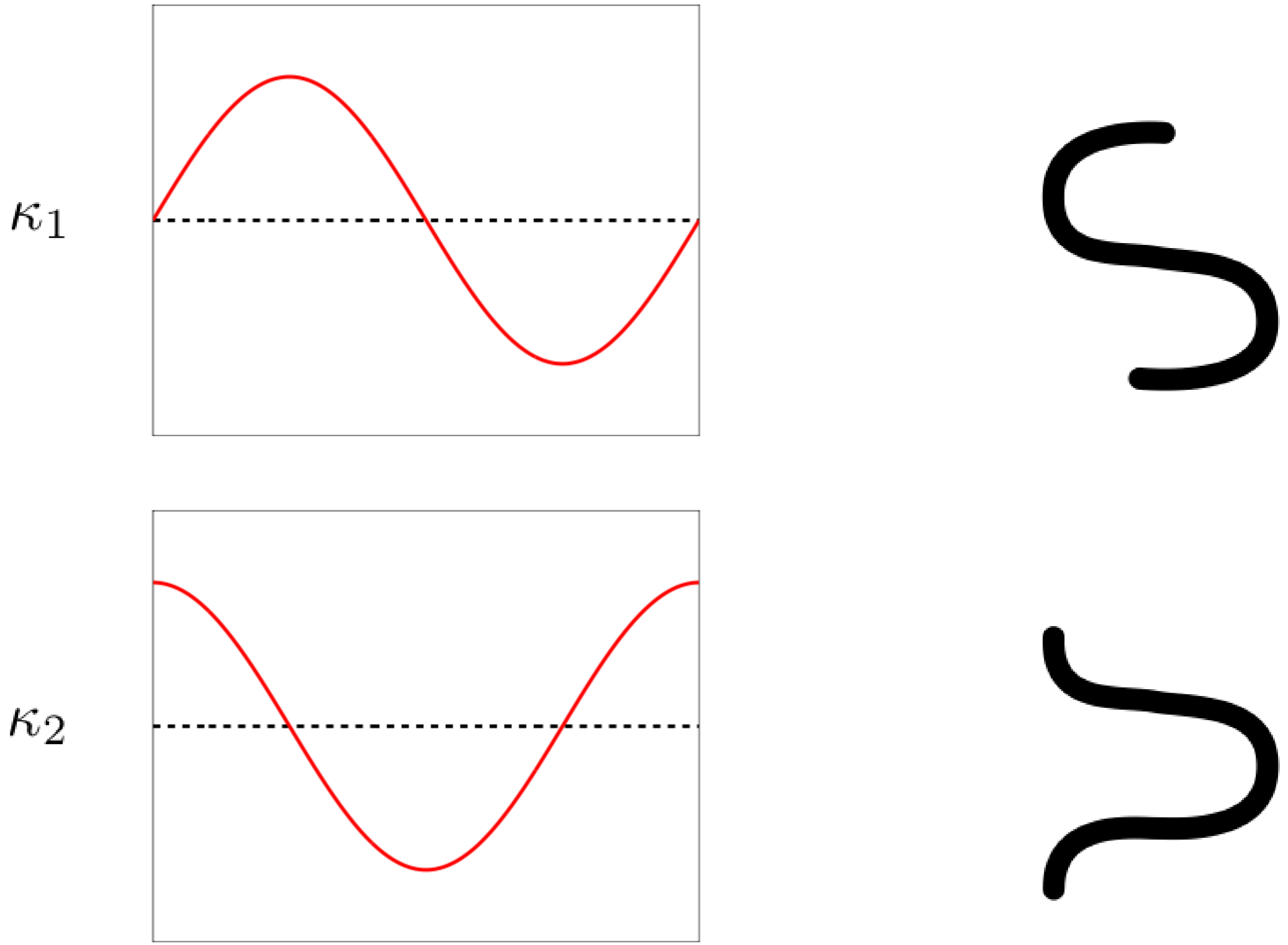


Polynomial curvature



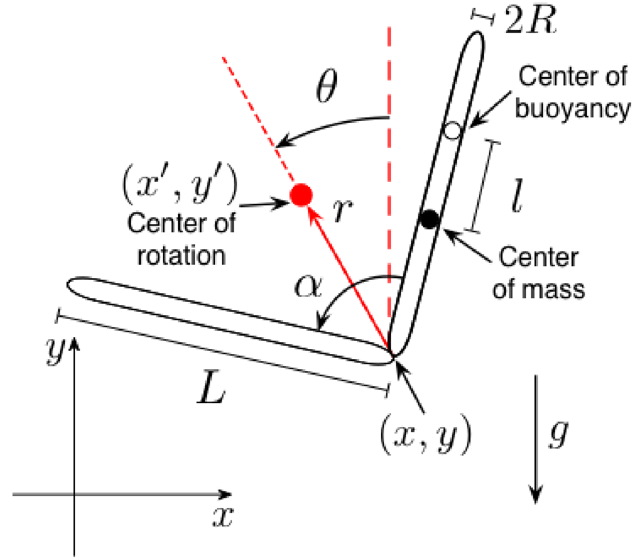


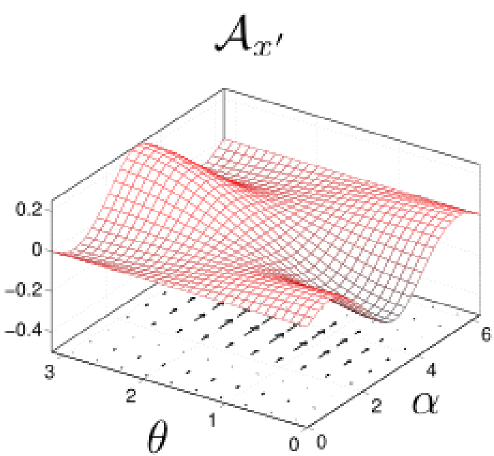
Sinusoidal curvature

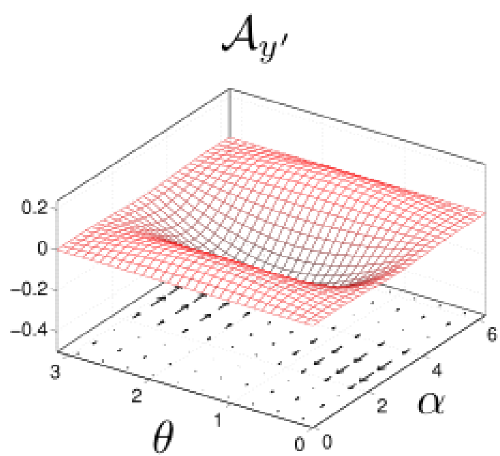


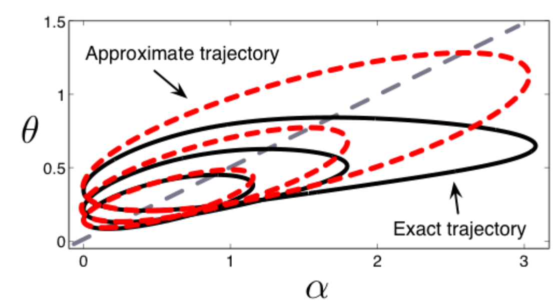
Buoyant scallop

With Burton and Hosoi at MIT





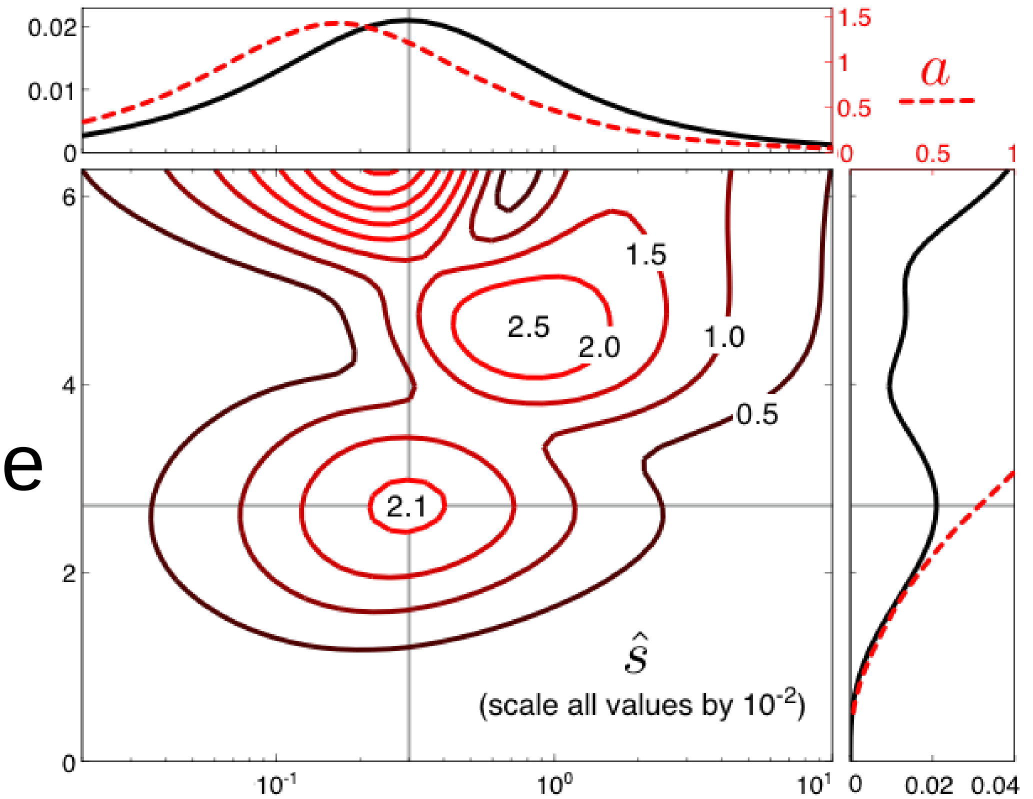




Buoyant scallop

mean speed

hinge amplitude



$$\frac{\text{gravity}}{\text{hinge speed}}$$

mean speed

Questions for me to worry about

- Higher dimensions
  - In three-dim base space, we can use the vectorfield analogy and note tha the curl is a three-dim vector field. In 2dim, this vectorfield is normal to our vectorfield and the scalar value we plot is its magnitude and by integrating its area, we are finding the flux of this field through the patch of the shape space bounded by the gait. If we go to three dimensions, the flux of the curl field bounded by ANY two surface bounded by the 1-d curve is the same (this has something to do with exact forms being closed), and the flux is equivalent to our area integral. If we go to higher dimensions, we have to abondon the ideas of vectorfields and curl but we have the same structure where we are building a two-surface with certain area integration properties and take its boundary as its gait.