EE102A Notes

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1 Signals and Systems

Signal: A function of an independent variable. The dimension of a signal is determined by dimension of it's input variable (eg. f([a,b,c]) would be three dimensional as would f(a,b,c) where f is the signal in question).
Warning: the term dimension may mean different things in different contexts, such as communication.

System: A system preforms a mapping on signals to produce new signals. Eg. $g(f(t))$ is a system that maps f (a signal of time, t) from it's space to another. While this can be from one subspace to some other part of the same subspace (such as voltage to voltage) it can also be one subspace to a different subspace (such as a speaker which maps voltage to sound waves. Systems, like signals have dimensionality and depend on the dimensionality of the input in the same way that signals do.

1.1 Classification of Signals

Continuous Time (CT) Signal: Let time be denoted t. A CT signal $x : A \to Y,\ x(t)$ is defined for all $t \in A$ where A is open and connected.

Discrete Time (DT) Signal: A DT signal $x : X \to Y$, $x(t)$ is defined for all $n \in X$ where $(\forall n \in X)\, n \in \mathbb{Z}$. The input n may correspond to an unequally spaced sampling time such as stock markets (Eg. $f(n-1) =$ Thursdays' stock prices, $f(n) =$ Friday's stock prices, $f(n+1) =$ Monday's stock prices - notice the jump between Friday and Monday). However, often n and $n+1$ are equally spaced in time and $x[n]$ corresponds to samples of a CT signal at multitudes of a sampling interval T.

1.2 More Signal Classification

Even Signal (CT or DC): A signal $f : X \to Y$, $f(x)$ that satisfies $\forall x \in X . f(-x) = f(x)$
Odd Signal (CT or DC): A signal $f : X \to Y$, $f(x)$ that satisfies $\forall x \in X . f(-x) = -f(x)$
Further, for arbitrary signal $f : X \to Y$, f(x), f(x) can be represented as the sum of an odd and an even function.

$$f(x) = f_e(x) + f +_o (x)$$

$$f_e = \frac{1}{2}(f(x) + f(-x)) \qquad f_o = \frac{1}{2}(f(x) - f(-x))$$

Periodic CT Signal: A CT signal $f : X \to Y$, $f(x)$ is periodic if $\exists T . (\forall x \in X . f(x) = f(x+T))$. We let the period T_0 be the smallest T satisfying this.
Periodic DT Signal: A DT signal $g : X \to Y$, $g[x]$ is periodic if $\exists N \in \mathbb{Z} . (\forall x \in X . f(x) = f(x+N))$. We let the period N be the smallest N satisfying this.
Fundamental Frequency: Denoted ω_0, for CT signals: $\omega_0 = -\frac{2\pi}{T_0} \left(\frac{\text{rad}}{\text{s}}\right)$, for DT signals: $\omega_0 = -\frac{2\pi}{N} \left(\frac{\text{rad}}{\text{s}}\right)$

1.3 Energy and Power Signals

1.3.1 CT Signals

Consider voltage $V(t)$ across a resistor of resistance R which induces current $i(t)$. The power at time t is $p(t) = V(t)i(t) = \frac{V^2(t)}{R} = i^2(t)R$. Now, let $R = 1\Omega$. Now, $p(t) = V(t)i(t) = V^2(t) = i^2(t)$. So the energy over time t_e of a system with average power p_a is $t_e p_a$. As we take the limit $\lim t_e \to 0$ we get that the energy over t_e approaches $p(t)dt$. As such, the energy over time (t_0, t_p) is $E_p = \int_{t_0}^{t_p} p(t)dt$. Extending this, we get that the total energy of a system is

$$E = \int_{-\infty}^{\infty} p(t)dt = \int_{-\infty}^{\infty} V^2(t)dt$$

Leading to the average power:

$$P_A = \lim T \to \infty \frac{1}{2T} \int_{-T}^{T} V^2(t)dt$$

More generally for complex $x(t)$:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_A = \lim T \to \infty \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

(where $|x(t)|$ is the complex modulus of $x(t)$)
Additionally, for a periodic signal, we have:

$$P_A = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt$$

1.3.2 DT Signals

With the same principles, we extend to DT signals as such:

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x[N]|^2$$

With period N:

$$P = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} |x[n]|^2$$

1.4 Classifications

Energy Signal: A signal with energy E and power P where $0 \leq E < \infty$ and $P = 0$
Power Signal: A signal with energy E and power P where $0 < P < \infty$ and $E = \infty$
All finite-valued periodic signals are power signals. Further, there are some signals that aren't power or energy signals.

Energy Signal Example: A battery that dies after some time

Power Signal Example: An infinitely powered wall outlet

Example: Let CT signal $x(t)$ be defined on $[t_0, t_1]$ with amplitude a: As such, the energy is

$$E = \int_{t_0}^{t_1} a^2 dt = a^2(t_1 - t_0)$$

Example 2 Now, let $x(t) = a\sin(\omega_0 t)$ (Note: $\sin^2(x) = \frac{1}{2}(1 - cos(2x))$. We get that the energy is:

$$E = \int_{-\infty}^{\infty} a^2 \sin^2(x)dt = \frac{a^2}{2}\left(\int_{-\infty}^{\infty} 1dt - \int_{-\infty}^{\infty} \cos(2w_0 t)dt\right) = \infty$$

Linear System: In order for a system H to be linear, it must satisfy the following conditions

1. $H(ax(t)) \to aH(x(t))$
2. $H(x_1(t) + x_2(t)) = H(x_1(t)) + H(x_2(t))$

Time-invarient System: For a system H to be time-invarient, its shifted response must equal it's responce to a shifted signal.

$$H\Big(x(t-\delta)\Big)(t) = H\Big(x(t)\Big)(t-\delta)$$

Example 1: $H(x(t)) = x(-t)$ (Time **varient**)
Let x_2 be a shifted version of x: $x_2(t) = x(t-\delta)$.
The responce of system H to signal x_2 is accordingly:

$$H(x_2(t)) = x_2(-t) = x(-t-\delta)$$

For the system to be time-invarient, $H(x(t-\delta))$ must equal $H(x_2(t))$.
We see that $H(x(t-\delta)) = x(-(t-\delta)) = x(-t+\delta)$.
However, $x(-t+\delta) \neq x(t-\delta)$, so the system is Time Varient.

2 Unit and Impulse Functions

2.1 Discrete Time

In discrete time, the unit function ($u[n]$) and the impulse function ($\delta[n]$) are easily defined:

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

The discrete time impulse function presents a "sifting feature" in that when

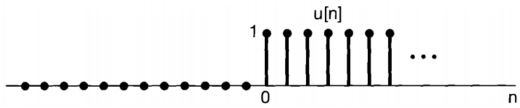

Figure 1: $u[n]$

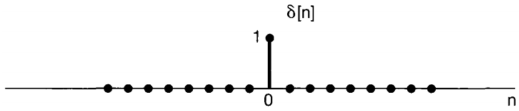

Figure 2: $\delta[n]$

integrated (or rather, summed) it can "sift" out a specific point.

$$\sum_{t=-\infty}^{\infty} x(t)\delta(t) = x(0)$$

And more generally,

$$\sum_{t=-\infty}^{\infty} x(t)\delta(t-k) = x(k)$$

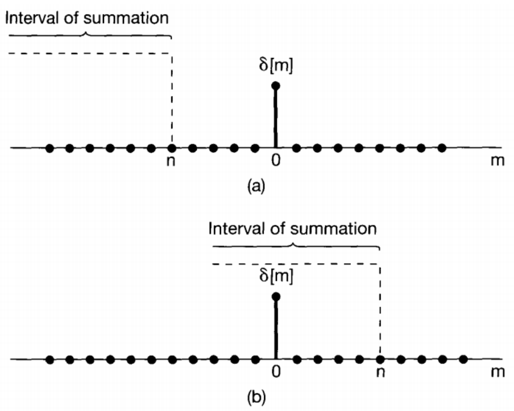

Figure 3: $\delta[n] : Sifting$

2.2 Continuous Time

In continuous time, things are a bit more hairy. The unit step function is defined as such:

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0, \end{cases}$$

Further, a second, more useful definition is as the derivative of the ramp function:

$$u(t) = \frac{d}{dt} r(t)$$

The impulse function is defined in terms of the unit step function:

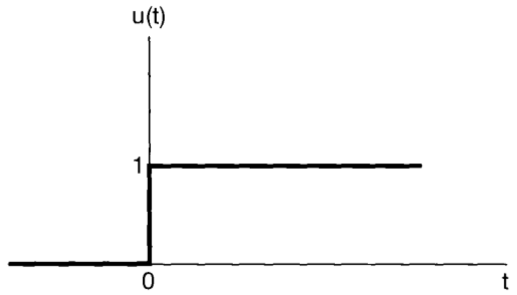

Figure 4: $u(t)$

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau \quad \rightarrow \quad \frac{d}{dt} u(t) = \delta(t)$$

Further, this gives us that the impulse function is the second derivative of the ramp function:

$$\delta(t) = \frac{d}{dt} u(t) = \frac{d^2}{d^2 t} r(t)$$

Much like DT impulse function, the CT impulse function also has sifting properties:

$$x(t)\delta(t) = x(0)\delta(t) \qquad x(t)\delta(x-k) = x(k)\delta(x-k)$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-k)dt = x(k)$$

3 System Classifications (Again)

Memory: A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at that same time. For example, the system specified by the relationship:

$$y[n] = 2x[n] + x[n]^2$$

is memoryless, as the value of $y[n]$ at any particular time n depends only on the value of $x[n]$ at that time. An example of a system with memmory is a summer or accumulator:

$$\sum_{k=-\infty}^{n} x[k]$$

Invertible System: A system is invertible if for every $H(x(t))$ there exists a function W such that $W(H(x(t))) = x(t)$ (aka H is invertible)

Causal System: A system mapping x to y is causal iff for any pair of inputs $x_1(t)$ and $x_2(t)$ satisfying:

$$x_1(t) = x_2(t), \forall t \leq t_0$$

the corresponding outputs satisfy

$$y_1(t) = y_2(t), \forall t \leq t_0$$

4 Convolution

The convolution is defined as follows:

$$\begin{aligned}(f * g)(t) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\,d\tau.\end{aligned}$$

It is commonly written as:

$$f(t) * g(t) \stackrel{\text{def}}{=} \underbrace{\int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau}_{(f*g)(t)},$$

A wonderful visual explanation of this can be found on wikipedia: `https://en.wikipedia.org/wiki/Convolution#Visual_explanation`

The Discrete Time convolution is as one would expect:

$$x * y = \sum_{k=-\infty}^{\infty} x(k)y(t-k) = \sum_{k=-\infty}^{\infty} x(t-k)y(k)$$

5 Impulse Response

Let H be an arbitary LTI system that takes signals x to outputs y like this:

$$H[x[t]] = y[t]$$

From before, we know that we can represent $x[t)$ as a sum of weighted signals:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[t-k] = ... + x[-1]\delta[t+1] + x[0]\delta[t] + x[1]\delta[t-1] + ...$$

So,

$$H[x[t]] = H[... + x[-1]\delta[t+1] + x[0]\delta[t] + x[1]\delta[t-1] + ...] = y[t]$$

Furhter, since H is linear, it follows that:

$$H[x[t]] = ... + H[x[-1]\delta[t+1]] + H[x[0]\delta[t]] + H[x[1]\delta[t-1]] + ... = y[t]$$

And, since $x[-1]$ and the like are constants:

$$H[x[t]] = ... + x[-1]H[\delta[t+1]] + x[0]H[\delta[t]] + x[1]H[\delta[t-1]] + ... = y[t]$$

Finnaly, since H is time-invarient:

$$H[x[t]] = ...+x[-1]H\Big[\delta[t]\Big][t+1]+x[0]H\Big[\delta[t]\Big][t]+x[1]H\Big[\delta[t]\Big][t-1]+... = y[t]$$

So, lets let $H\Big[\delta[t]\Big][t] = H_\delta[t]$ This is called the systems **impulse responce** (the system's responce to an impulse - go figure)

$$\sum_{k=-\infty}^{\infty} H_\delta[t-k]x[k]$$

Notice anything?

$$\sum_{k=-\infty}^{\infty} H_\delta[t-k]x[k] = H_\delta[t] * x[t]$$

By finding a LTI systems impulse responce, we know how it will react to any signal.

Some useful facts:

$$a(t) * (b(t) + c(t)) = a(t) * b(t) + a(t) * c(t)$$

If y_1 and y_2 are the outputs of two sytems such that:

$$y_1(t) = x(t) * h_1(t)$$

$$y_2(t) = x(t) * h_2(t)$$

Then the system $y(t) = y_1(t) + y_2(t)$ has impulse response $h_1(t) + h_2(t)$:

$$y(t) = y_1(t) + y_2(t) = x(t) * h_1(t) + x(t) * h_2(t) = x(t) * (h_1(t) + h_2(t))$$

The associative property also holds:

$$a(t) * (b(t) * c(t)) = (a(t) * b(t)) * c(t)$$

6 ADD TITLE HERE?

Rise time: the time taken by a signal to change from a specified low value to a specified high value

7 Fourier Series

$$s(x) = \overbrace{\frac{a_0}{2}}^{A_0} + \sum_{n=1}^{\infty} \left(a_n \sin(\phi_n) \cos\left(\tfrac{2\pi n x}{P}\right) + b_n \cos(\phi_n) \sin\left(\tfrac{2\pi n x}{P}\right) \right)$$

Where $s(x)$ is the fourier series approximation of a function with period P and:

$$a_n = \frac{2}{P} \int_{x_0}^{x_0+P} s(x) \cdot \cos\left(\tfrac{2\pi n x}{P}\right)\, dx$$

$$b_n = \frac{2}{P} \int_{x_0}^{x_0+P} s(x) \cdot \sin\left(\tfrac{2\pi n x}{P}\right)\, dx$$

However, there is also a complex exponential based version of the form:

$$s(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{2\pi n x}{P}}$$

where P is the period, and

$$c_n = \frac{1}{P} \int_{x_0}^{x_0+P} s(x) \cdot e^{-i\frac{2\pi n x}{P}} dx$$

Further, because there are apparently *only two letters in the alphabet*:

$$A_k = 2Re(c_n) \qquad B_k = -2Im(c_n)$$

Also helpful: $e^{ix} = \cos x + i \sin x$

For a rectangular pulse train we see that:

$$c_n = \begin{cases} \frac{1}{P}\int_{x_0}^{x_0+P} y(t)dt & n = 0 \\ \frac{T_p}{P} sinc\left(\frac{nT_p}{P}\right) & n \neq 0 \end{cases}$$

where

$$sinc(x) = \frac{sin(\pi x)}{\pi x}$$