Digital Image Processing

Lecture #8

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Announcement

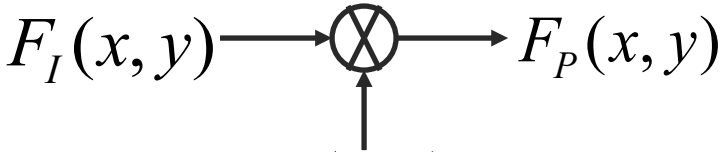
- Class Information
  - The following schedule

03/23	Lecture 5	05/11	proposal
03/30	Lecture 6 & 7	05/18	Lecture 10
04/06	Lecture 8	05/25	Lecture 11
04/13	RealSense	06/01	Lecture 12
04/20	midterm	06/08	Demo
04/27	RealSense	06/15	Demo
05/04	Lecture 9	06/22	Final Package Due

Image Sampling & Transforms

Image Sampling & Transforms

- **Image sampling**
  - **A/D conversion**
    - Usually deals with arrays of numbers obtained by spatially sampling points of a physical image
    - $F_I(x,y)$: continuous, infinite-extent, ideal image field
    - $F_P(x,y)$: the sampled image



transform function

$$S(x,y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\Delta y)$$

Dirac delta function

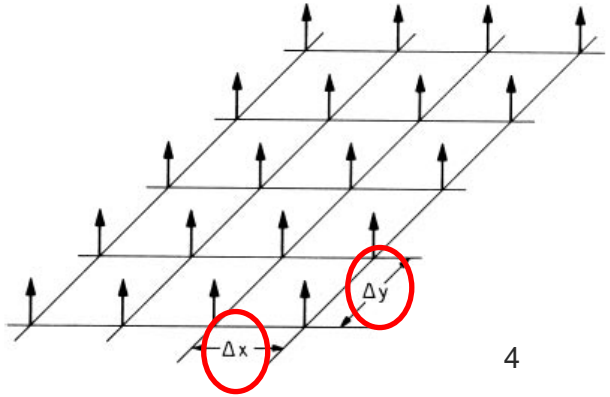


Image Sampling & Transforms

- **Image sampling**
  - **Sampling function** $S(x,y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(x - j\Delta x, y - k\Delta y)$
    - **An infinite array of the Dirac delta functions arranged in a grid of spacing,** $(\Delta x, \Delta y)$
  - **Space domain**

$$F_P(x,y) = F_I(x,y)S(x,y) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} F_I(j\Delta x, k\Delta y)\delta(x - j\Delta x, y - k\Delta y)$$

  - **Transform domain (Fourier transform)**

(Convolution Theorem)

$$\mathcal{F}_P(\omega_x,\omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_P(x,y)\exp\{-i(\omega_x x + \omega_y y)\}dxdy = \frac{1}{4\pi^2}\mathcal{F}_I(\omega_x,\omega_y) \otimes S(\omega_x,\omega_y)$$

Sampling frequency

where $S(\omega_x,\omega_y) = \frac{4\pi^2}{\Delta x \Delta y} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \delta(\omega_x - j\omega_{xs}, \omega_y - k\omega_{ys})$

and $\omega_{xs} = 2\pi/\Delta x \quad and \quad \omega_{ys} = 2\pi/\Delta y$

Image Sampling & Transforms

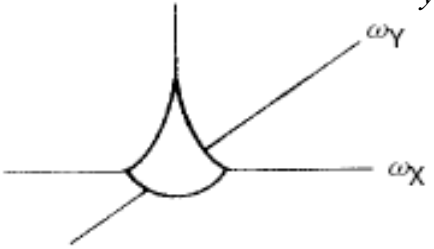
- **Image sampling**
  - **Transform domain (Fourier transform)**
    - **Assume the spectrum of the ideal image is band-limited**

$$\mathcal{F}_P(\omega_x,\omega_y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_P(x,y)\exp\{-i(\omega_x x+\omega_y y)\}dxdy=\frac{1}{4\pi^2}\mathcal{F}_I(\omega_x,\omega_y)\otimes S(\omega_x,\omega_y)$$

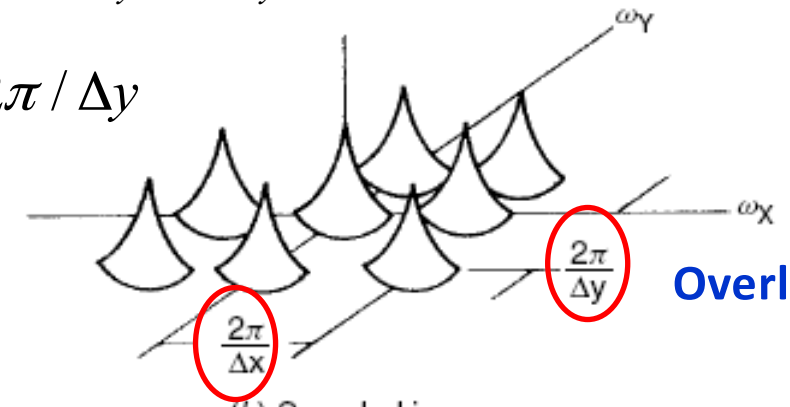
$$=\frac{1}{\Delta x\Delta y}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{F}_I(\omega_x-\alpha,\omega_y-\beta)\times\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\delta(\alpha-j\omega_{xs},\beta-k\omega_{ys})d\alpha d\beta$$

$$=\frac{1}{\Delta x\Delta y}\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\mathcal{F}_I(\omega_x-j\omega_{xs},\omega_y-k\omega_{ys})$$

where $\omega_{xs}=2\pi/\Delta x$ *and* $\omega_{ys}=2\pi/\Delta y$



(a) Original image



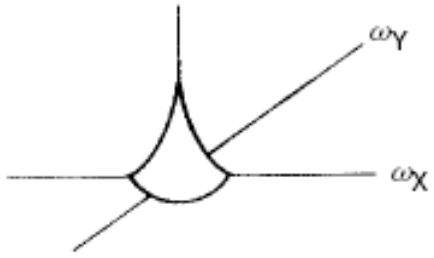
(b) Sampled image

Overlapping?

Image Sampling & Transforms

- **Image sampling**
  - **Transform domain (Fourier transform)**
    - **Assume the spectrum of the ideal image is limited and suppose that the cutoff frequency is $(\omega_{xc}, \omega_{yc})$**
      i.e. $\mathcal{F}_P(\omega_x, \omega_y)$ is non-zero only in the region bounded

      by $|\omega_x| \le \omega_{xc}$ $|\omega_y| \le \omega_{yc}$



(a) Original image

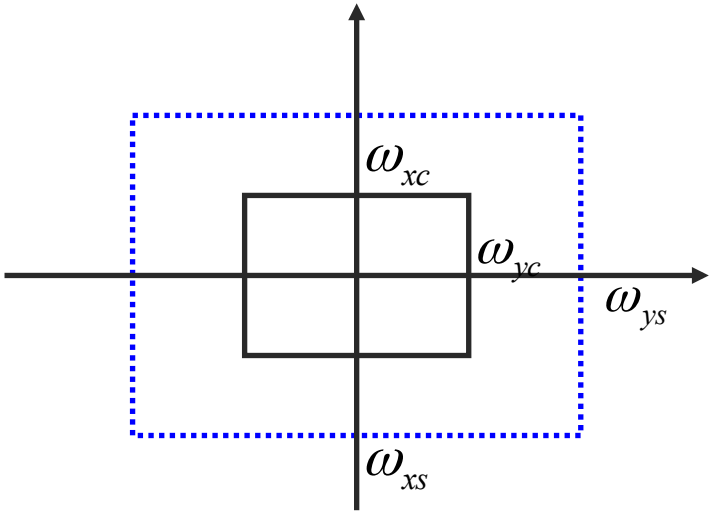


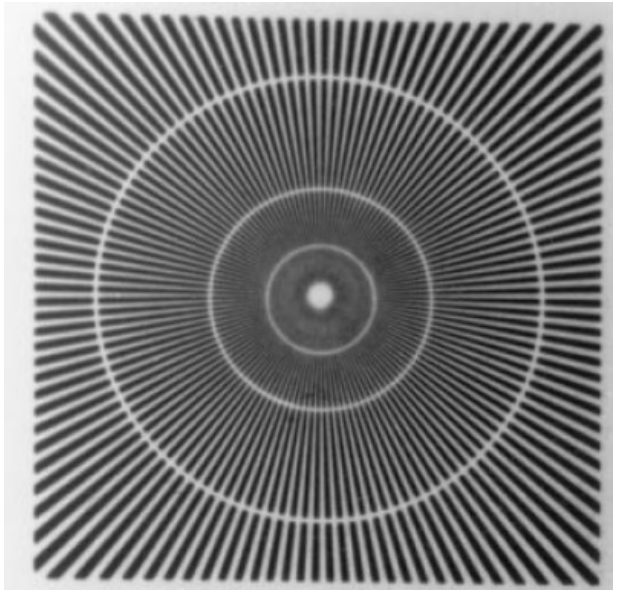
Image Sampling & Transforms

- **Image sampling**
  - **Transform domain (Fourier transform)**
    - If $\omega_{xc} \le \frac{\omega_{xs}}{2}$, $\omega_{yc} \le \frac{\omega_{ys}}{2}$,

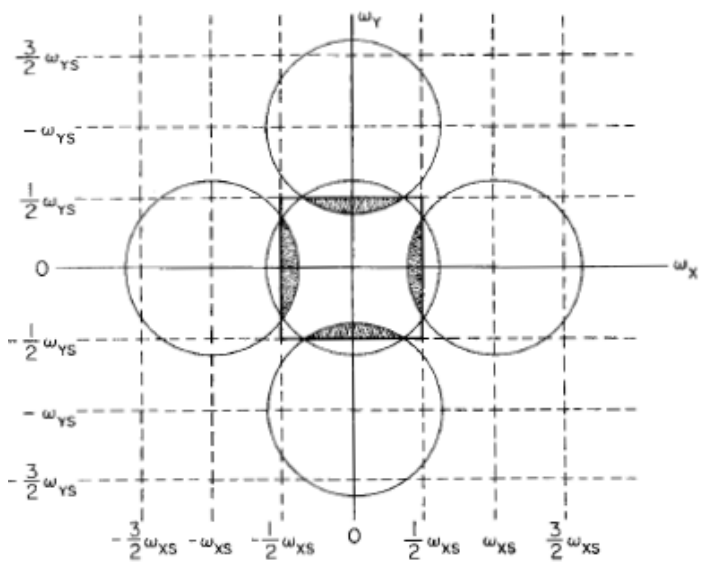
      there is no overlapping (no aliasing) between adjacent shifted waveforms of $\mathcal{F}_P(\omega_x, \omega_y)$
  - **Sampling Theorem**
    - To sample a band-limited signal, the sampling period must be no longer than one-half of the period of the finest details within the image to avoid aliasing
    - Generalization of the 1D Nyquist Theorem to the 2D case

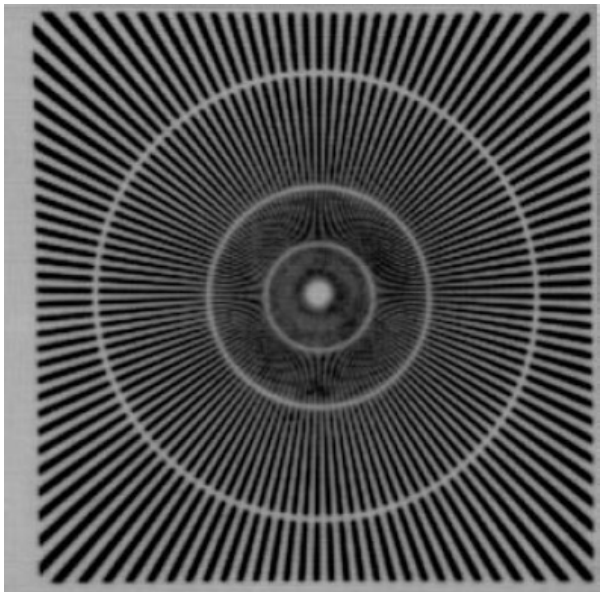
Image Sampling & Transforms

- **Sampling Theorem**
  - To sample a band-limited signal, the sampling period must be no longer than one-half of the period of the finest details within the image to avoid aliasing



Original image





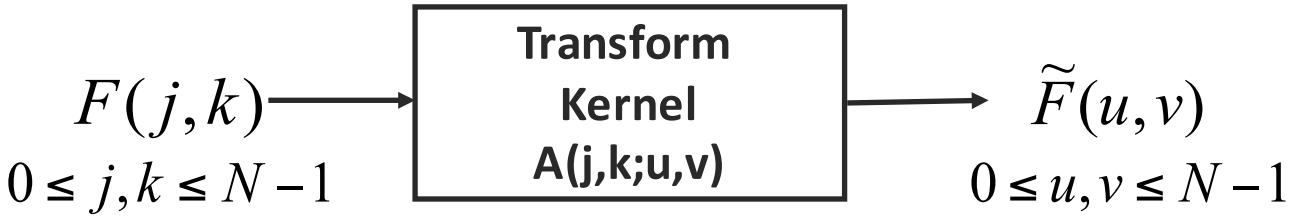
Sampled image

Image Sampling & Transforms

- Image Transform
  - Why image transform?
    - Feature extraction
    - Energy compaction
  - We are able to concentrate energy distribution over a small number of transform coefficients
    - Can be exploited for compression purpose
      - DCT – Discrete Cosine Transform
      - Block-based transform – 8x8

Image Sampling & Transforms

- **General 2D transform**
  - **Forward**



  - **Backward**

Image Sampling & Transforms

- **2D separable transform**
  - **General 2D transform**
    - **Forward**

$$\widetilde{F}(u,v) = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} F(j,k)A(j,k;u,v)$$

    - **Backward**

$$F(j,k) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \widetilde{F}(u,v)B(u,v;j,k)$$

    - **If A and B are separable, we have**

$$A(j,k;u,v) = A_X(j,u)A_Y(k,v)$$

$$B(u,v;j,k) = \underbrace{B_X(u,j)}_{\text{x-direction}}\underbrace{B_Y(v,k)}_{\text{y-direction}}$$

Image Sampling & Transforms

- **2D separable transform**
  - **The real advantage is the low computational cost of the separable transform**
    - **General 2D transform**

$$\tilde{F}(u,v) = \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} F(j,k)A(j,k;u,v)$$

$O(N^2)$ **per (u,v)** **&** $O(N^4)$ **for all (u,v)**

    - **Separable 2D transform**

$$\tilde{F}(u,v) = \sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} F(j,k)A_X(j,u)\right)A_Y(k,v)$$

$2N$ **per (u,v)** **&** $2N^3$ **for all (u,v)**

Image Sampling & Transforms

- **Unitary Transform**
  - **Consider 1D transform**

$$\tilde{F}(u) = \sum_{j=0}^{N-1} F(j)A(j,u)$$

$$\underline{f} = \begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(N-1) \end{bmatrix} \quad \underline{\tilde{f}} = \begin{bmatrix} \tilde{F}(0) \\ \tilde{F}(1) \\ \vdots \\ \tilde{F}(N-1) \end{bmatrix} \quad A = \begin{bmatrix} A(0,0) & \cdots & A(N-1,0) \\ \vdots & \ddots & \vdots \\ A(0,N-1) & \cdots & A(N-1,N-1) \end{bmatrix}$$

  - **Forward** $\underline{\tilde{f}} = A\underline{f}$
  - **Backward** $\underline{f} = B\underline{\tilde{f}}$

$$\Rightarrow B = A^{-1}$$

Gaussian elimination $\Rightarrow O(N^3)$

Image Sampling & Transforms

- **Unitary Transform**
  - The inverse is easy to compute

$$A^H A = AA^H = I \quad \Rightarrow A^{-1} = A^H$$

$$A^H = (A^*)^T = (A^T)^*$$

- **Karhunen Loeve Transform (KLT)**
  - Linear transform
  - Ideal for energy compaction
  - Principal component analysis (PCA)/Singular Value Decomposition (SVD)
  - Basis functions are image dependent
  - High computation cost for obtaining basis images

Image Sampling & Transforms

- **Discrete Fourier Transform (DFT)**
  - **Idea**
    - **Any function that periodically repeats itself can be expressed as a sum of sine and cosine of different frequencies.**

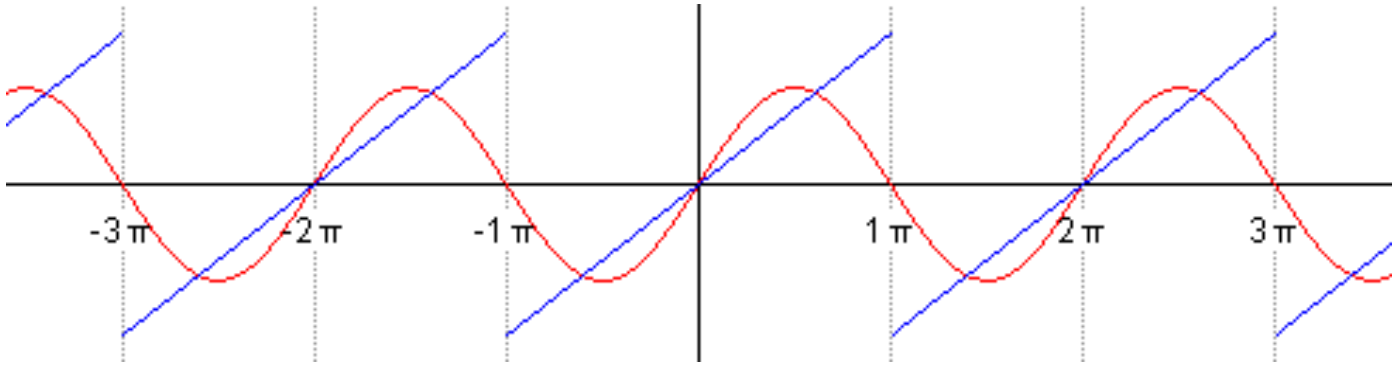


Image Sampling & Transforms

- **Discrete Fourier Transform (DFT)**
  - **Transform**

$$F(u,v) = \frac{1}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(j,k)\exp\{-\frac{2\pi i(uj+vk)}{n}\}$$

  - **Inverse DFT**

$$f(j,k) = \frac{1}{n}\sum_{u=0}^{n-1}\sum_{v=0}^{n-1} F(u,v)\exp\{\frac{2\pi i(uj+vk)}{n}\}$$

$i = \sqrt{-1}$ **and** $f(j,k)$ **is the input sequence**

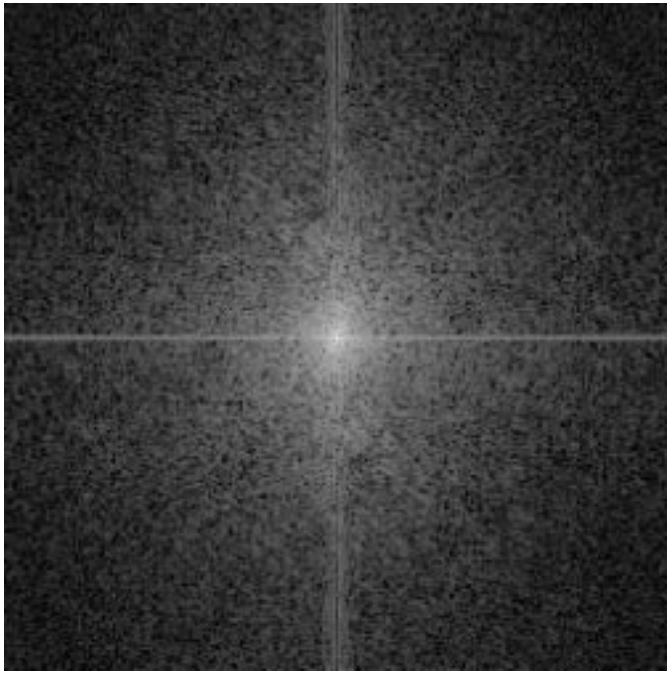
Image Sampling & Transforms

- **Discrete Fourier Transform (DFT)**
  - **Fast Fourier Transform (FFT)**
    - $O(n \log n)$
  - **Not so popular in image compression**
    - **performance is not good enough**
    - **computational cost for complex number is expensive**

Image Sampling & Transforms

- Discrete Fourier Transform (DFT)
  - Example

original



log magnitude

Image Sampling & Transforms

- **Discrete Fourier Transform (DFT)**
  - **Application – Noise reduction**

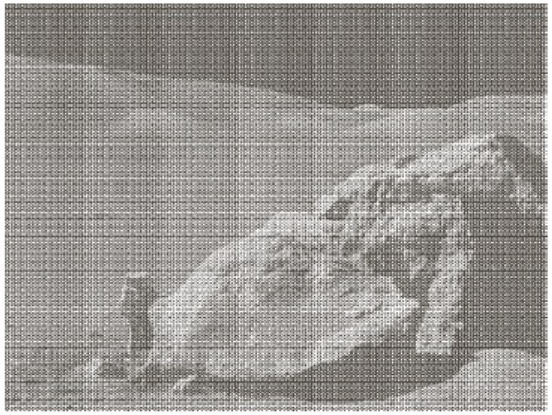


Image Sampling & Transforms

- **Discrete Fourier Transform (DFT)**
  - **Application– Low Pass Filter**

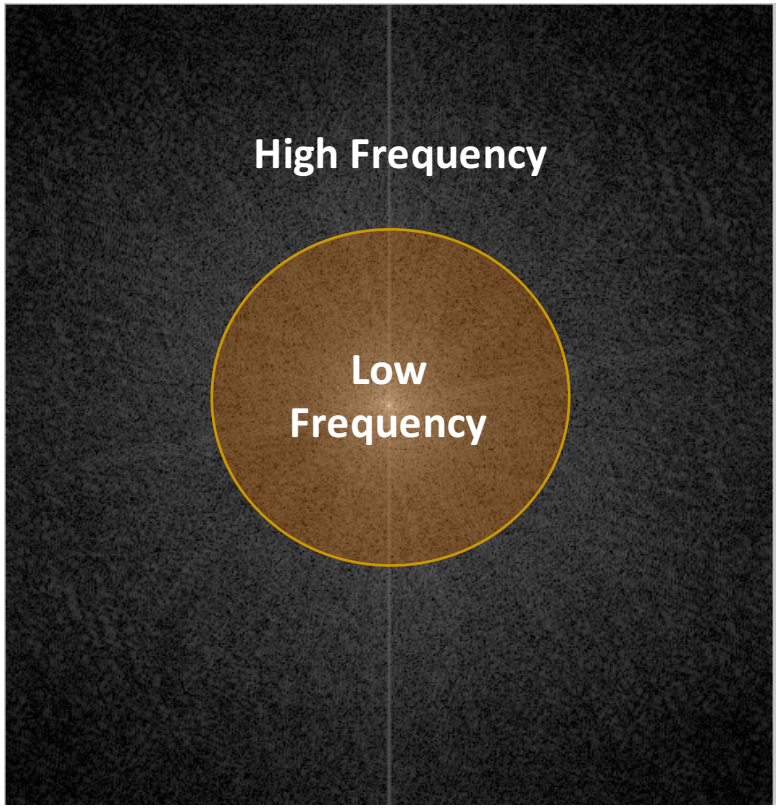


Image Sampling & Transforms

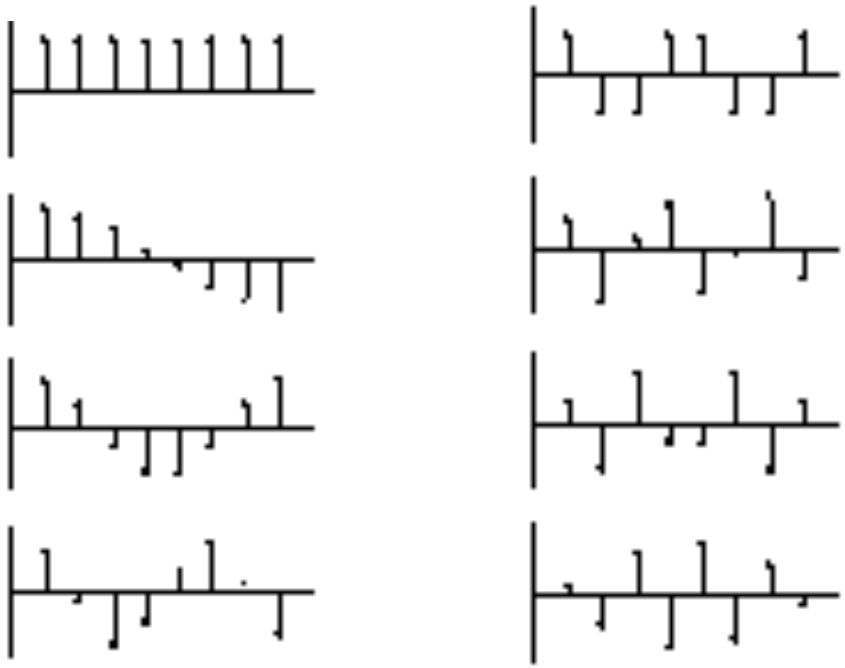
- **Discrete Fourier Transform (DFT)**
  - **Application– Low Pass Filter**

Original image

After low pass filter

Image Sampling & Transforms

- **Discrete Cosine Transform (DCT)**
  - **1D DCT basis functions**



All basis functions are cosine functions with different frequencies except the first basis function.

The signal is linear combination by these 8 basis functions. The result is the coefficient of linear combinations.

n=8

Image Sampling & Transforms

- **Discrete Cosine Transform (DCT)**
  - **Transform** f(j,k) : j is row, k is column

$$F(u,v) = \frac{C(u)C(v)}{4}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(j,k)\cos\left[\frac{(2j+1)u\pi}{2n}\right]\cos\left[\frac{(2k+1)v\pi}{2n}\right]$$

  - **Inverse DCT**

$$f(j,k) = \sum_{v=0}^{n-1}\sum_{u=0}^{n-1} C(u)C(v)F(u,v)\cos\left[\frac{(2j+1)u\pi}{2n}\right]\cos\left[\frac{(2k+1)v\pi}{2n}\right]$$

$$C(w) = \begin{cases} 1/\sqrt{2} & \text{if } w = 0 \\ 1 & \text{if } w = 1,2,\ldots,n\text{-}1 \end{cases}$$

Image Sampling & Transforms

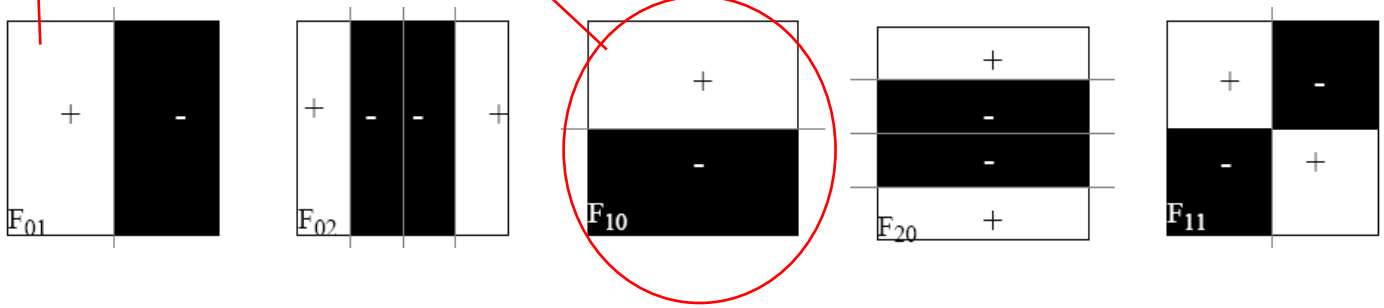
- **Discrete Cosine Transform (DCT)**
  - **Transform**

F(0,0) is to compute DC, the mean of the image

$$F(u,v) = \frac{C(u)C(v)}{4}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} f(j,k)cos\left[\frac{(2j+1)u\pi}{2n}\right]cos\left[\frac{(2k+1)v\pi}{2n}\right]$$

$$F_{10} = \frac{c_1 c_0}{4}\left[\cos\frac{\pi}{16}\left(\sum_{i=0}^{7} f(0,j) - \sum_{i=0}^{7} f(7,j)\right) + \cos\frac{3\pi}{16}\left(\sum_{i=0}^{7} f(1,j) - \sum_{i=0}^{7} f(6,j)\right) + \right.$$

$$\left. \cos\frac{5\pi}{16}\left(\sum_{i=0}^{7} f(2,j) - \sum_{i=0}^{7} f(5,j)\right) + \cos\frac{7\pi}{16}\left(\sum_{i=0}^{7} f(3,j) - \sum_{i=0}^{7} f(4,j)\right)\right]$$



F01 = F(0,1)
F02 = F(0,2)
F10 = F(1,0)
there are 8x8 F(u,v) for 8*8 basis functions

Image Sampling & Transforms

- **2D DCT basis functions (8x8)**

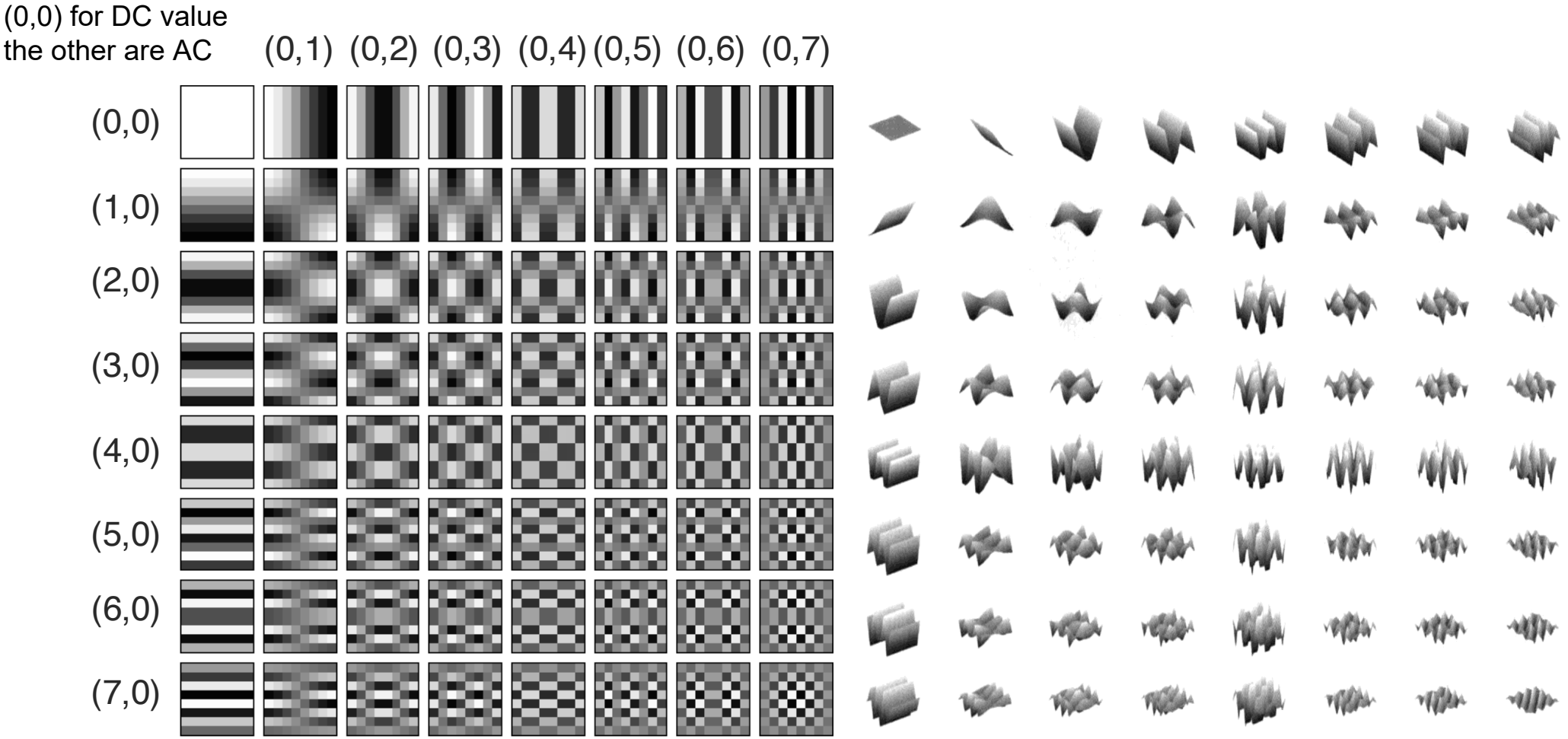


Image Sampling & Transforms

- The Meaning of DCT Coefficients
  - Represent the spatial frequency content within an 8x8 image block
  - The (0,0) coefficient is called DC coefficient
    - The average of the 64 image pixel values in the block
  - The rest of 63 coefficients are called AC coefficients
  - Move to the right of the block

    → the energy in higher horizontal frequencies
  - Move down the block

    → the energy in higher vertical frequencies

Image Sampling & Transforms

- Examples of 8x8 DCT Coefficients

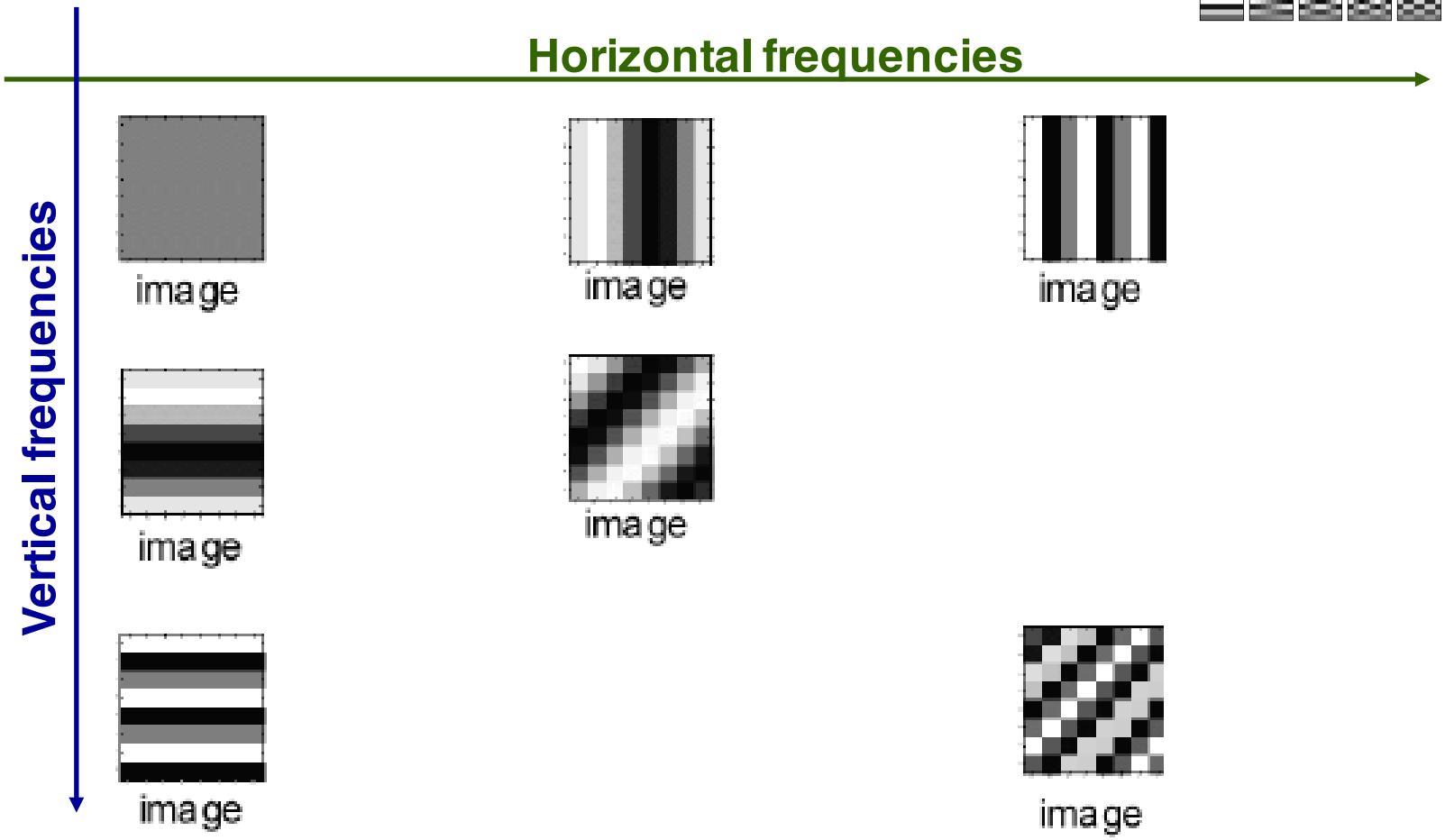


Image Sampling & Transforms

To operate on the compressed image directly without decompression and compression again.
It save time very much, but sometimes it loses the detail because the limited basis functions.

Discrete Cosine Transform (DCT)

Vertical-dominant	section 1	$\|F01\| > \|F10\|$	$F01 \le 0$	$F10 > 0$	
				$F10 < 0$	
	section 2		$F01 \ge 0$	$F10 > 0$	
				$F10 < 0$	
Horizontal-dominant	section 3	$\|F01\| < \|F10\|$	$F10 \le 0$	$F01 > 0$	
				$F01 < 0$	
	section 4		$F10 \ge 0$	$F01 > 0$	
				$F01 < 0$	

Vertical	$F10 = 0$	$F01 > 0$	
		$F01 < 0$	
Horizontal	$F01 = 0$	$F10 > 0$	
		$F10 < 0$	
Diagonal	$\|F01\| = \|F10\|$	$F01 < 0$, $F10 > 0$	
		$F01 > 0$, $F10 < 0$	
		$F01 > 0$, $F10 > 0$	
		$F01 < 0$, $F10 < 0$	

Bo Shen and Ishwar K. Sethi, "Direct feature extraction from compressed images,"1996

Image Sampling & Transforms

- **Discrete Cosine Transform (DCT)**
  - Use cosine function as its basis function
  - Performance similar to KLT
  - Fast algorithm available
  - Most popular in image compression
  - JPEG (Joint Photographic Experts Group)
  - The periodicity implied by DCT implies that less blocking artifact will be introduced than DFT

Image Sampling & Transforms

- **Hadamard transform**
  - Based on Hadamard matrix
    - A square array with elements of plus and minus 1's
    - Rows and columns are orthogonal
    - A normalized Hadamard matrix satisfies $HH^T = I$
    - The general form:

H^T=H^-1

$$H_{2N} = \frac{1}{\sqrt{2}} \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

$$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Sign Changes
0
3
1
2

$$H_8 = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Sign Changes
0
7
3
4
1
6
2
5

Image Sampling & Transforms

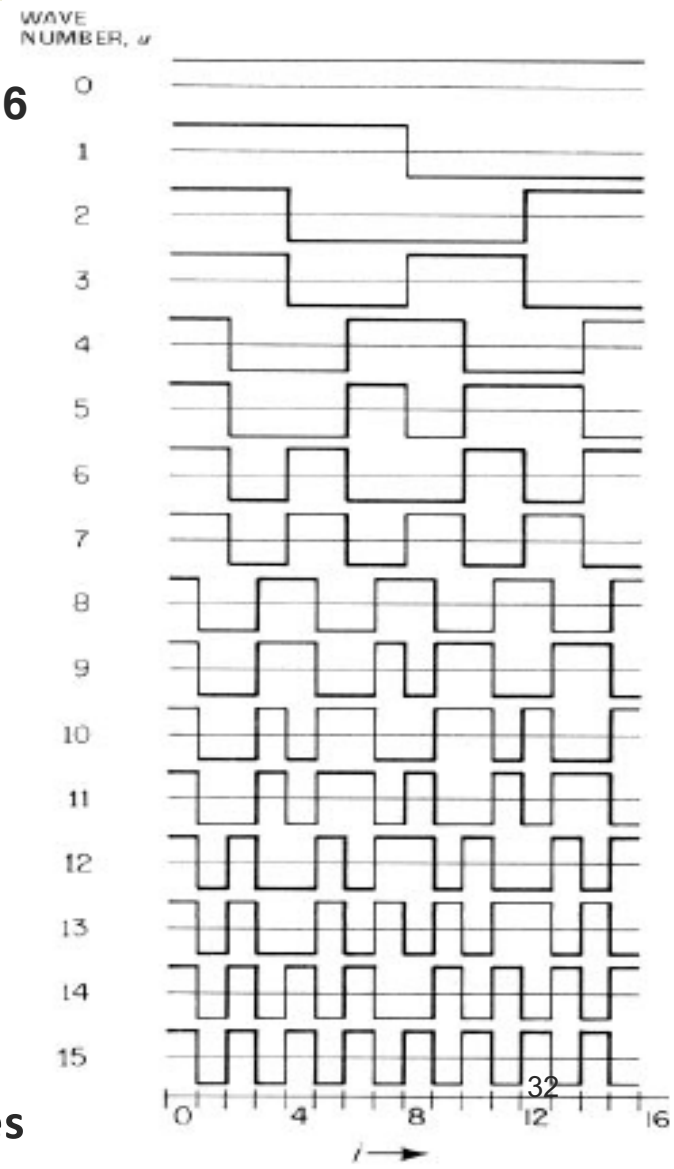
- **Hadamard transform**

e.g. N=8

$$\mathbf{H}_8 = \frac{1}{2\sqrt{2}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ \hline 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{array}\right]$$

Sign Changes
0
7
3
4
1
6
2
5

e.g. N=16



- **Applications**
  - data encryption and signal processing
  - data compression algorithms
    - HD Photo and MPEG-4 AVC
    - video compression
      - the sum of absolute transformed differences

Image Sampling & Transforms

- **Haar transform**
  - **Derived from the Haar matrix**

$$R_N = \begin{bmatrix} V_N \\ W_N \end{bmatrix}_{NxN}$$

$$\mathbf{V}_N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \end{bmatrix}$$

Scaling matrix

$$\mathbf{W}_N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & \vdots & & & & & & \vdots & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & -1 \end{bmatrix}$$

Wavelet matrix

e.g.

$$\mathbf{H}_4 = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Image Sampling & Transforms

- **Haar transform**
  - **1st-level Haar transform of a Nx1 vector f**

$$f_1 = R_N f = \left[a_1 \mid d_1\right]^T \quad where \quad \underbrace{a_1 = V_N f}_{\text{trend}};\ \underbrace{d_1 = W_N f}_{\text{fluctuation}}$$

global trend — local fluctuation

  - **2nd-level Haar transform of a Nx1 vector f**

$$f_2 = \left[a_2 \mid d_2 \mid d_1\right]^T \quad where \quad a_2 = V_{N/2} a_1;\ \ d_2 = W_{N/2} a_1$$

  - **The process continues until the full transformation**

$$\Rightarrow f_n = \left[a_n \mid d_n \mid d_{n-1} \mid \cdots \mid d_1\right]^T$$

Image Sampling & Transforms

- **Haar transform** wavelet transform
  - E.g. basis functions with N=16
  - Sample an input data sequence with finer and finer resolution increasing at the power of 2
  - Provide a transform domain in which type of differential energy is concentrated in localized regions

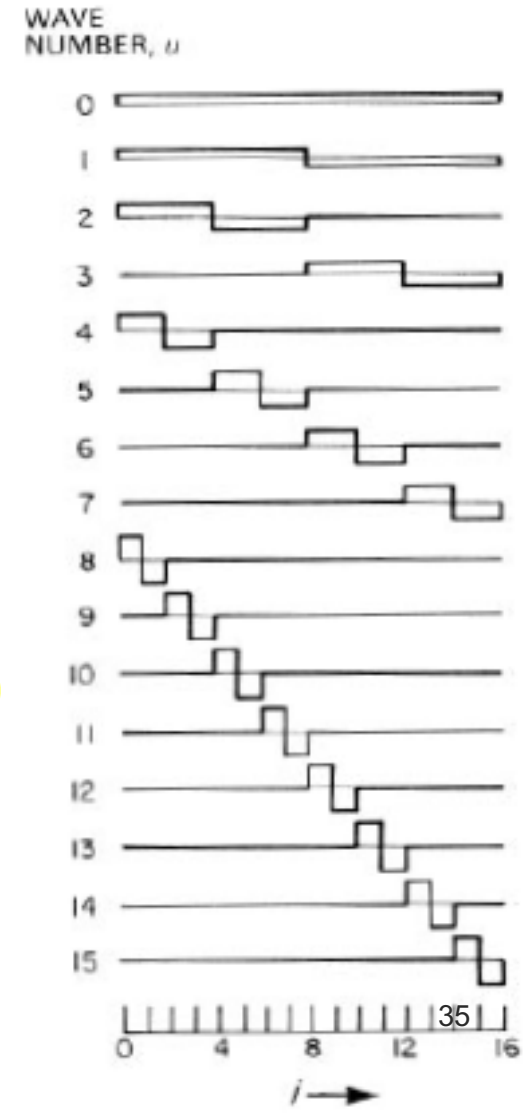
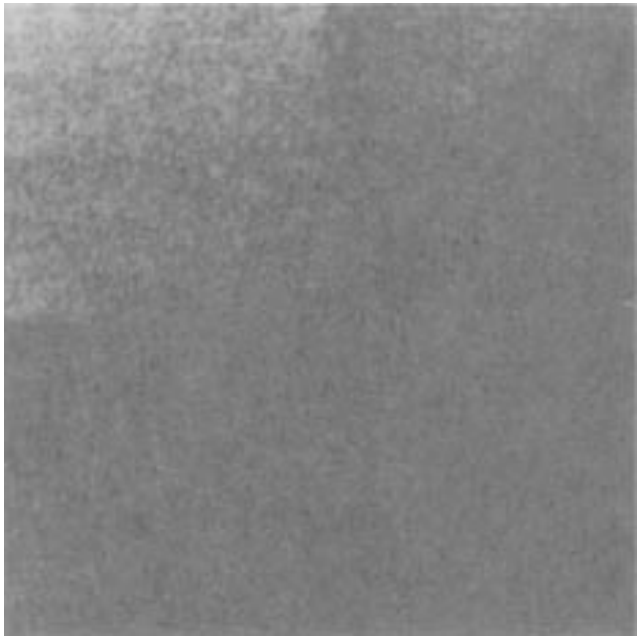


Image Sampling & Transforms

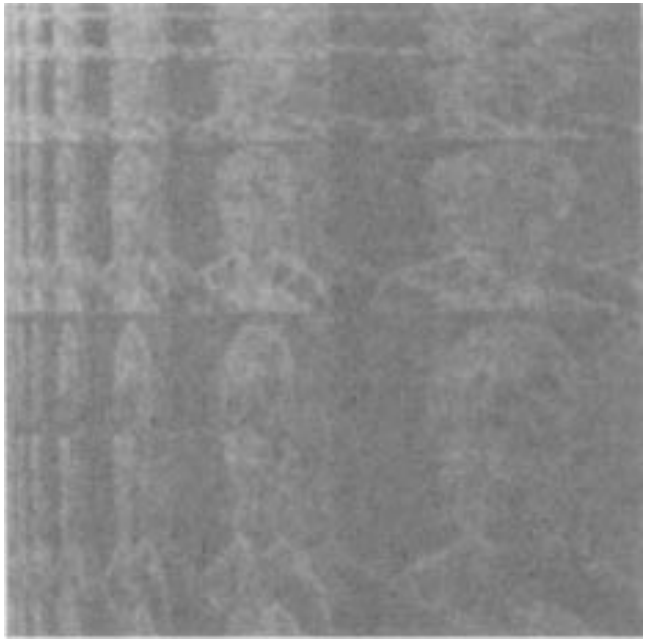
- **Hadamard transform and Haar transform**
  - **Example**

no spatial domain can be seen



Hadamard transform

sptial domain is still kept



Haar transform

Image Sampling & Transforms

- Hough transform
  - An edge is not a line…
  - How to detect lines in an image?

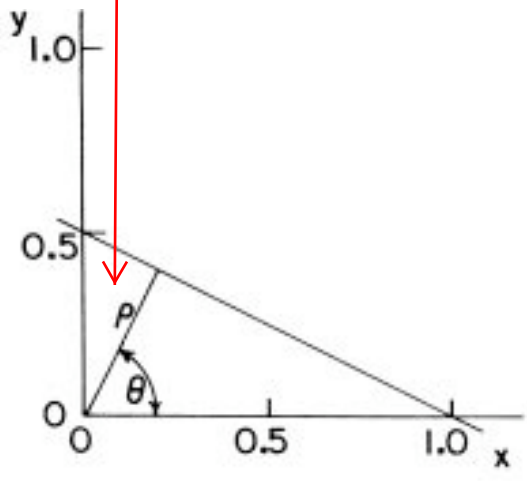
Image Sampling & Transforms

- Hough transform
  - How to detect lines in an image?
    - Search for the lines at every position with possible orientations
    - Voting scheme: Hough transform
  - Performed after edge detection
  - Can locate straight lines, circles, parabolas, ellipses, etc
    - As the curve can be specified in a parametric form
  - Advantages
    - Tolerate gaps between edges
    - Relatively unaffected by noise
    - Can deal with occlusion

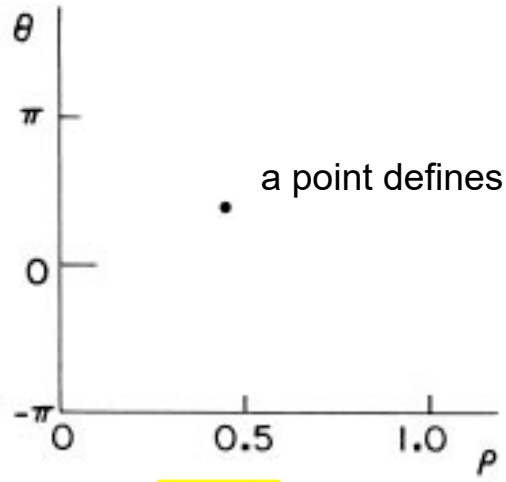
Image Sampling & Transforms

- **Hough transform**
  - **A straight line can be represented as** $y = mx + b$
    - Vertical lines? m is infinity, it's not a function of y
  - **Another representation is**

$$\rho = x\cos\theta + y\sin\theta, \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$



(a) Parametric line

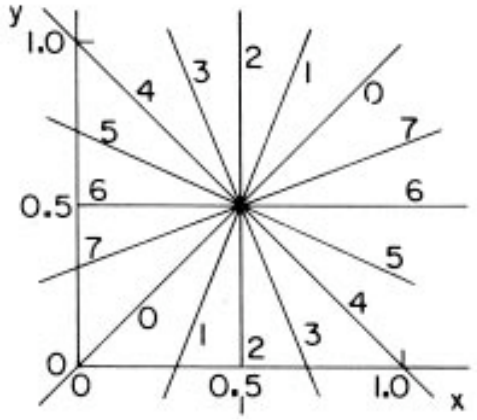


a point defines a line

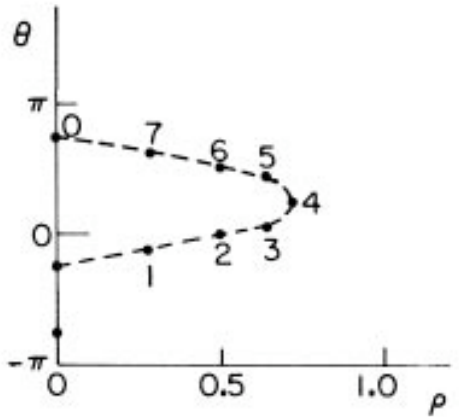
(b)Hough transform

Image Sampling & Transforms

- **Hough transform**



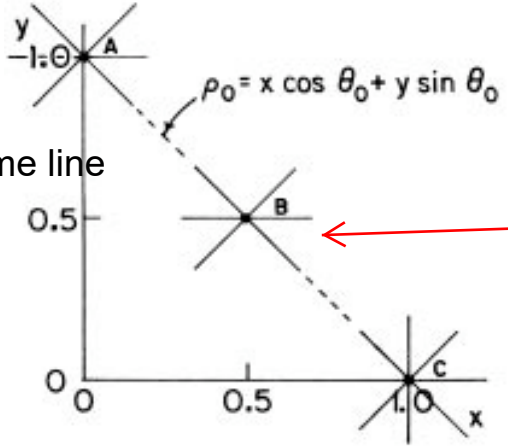
(c) Family of lines, common point

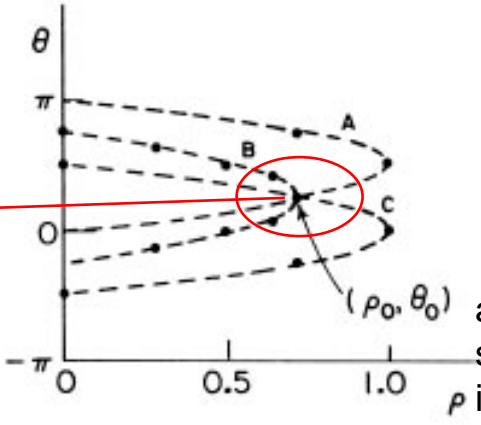


(d) Hough transform of (c)

colinear : points are in the same line

(e) Colinear points





(f) Hough transform of (e)

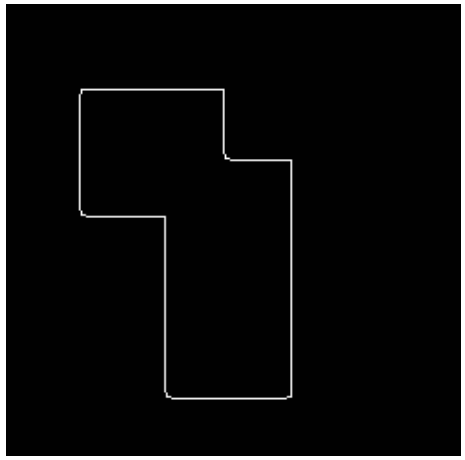
all the transform intersects the same point which means they are in the same line.

Image Sampling & Transforms

- **Hough transform for line detection**
  - Quantize the Hough Transform space
    - Identify the maximum and minimum values of ρ and θ
  - Generate an accumulator array A(ρ,θ)
  - For all edge points (x_i, y_i) in the image
    - Use gradient direction as θ
    - Compute ρ from the equation
    - Increment A(ρ,θ) by one
  - For all cells in A(ρ,θ)
    - Search for the maximum value
    - Calculate the equation of the line

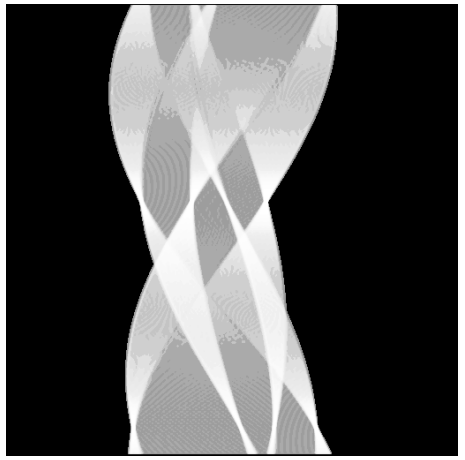
Image Sampling & Transforms

- Hough transform for line detection



H.E.

histogram equalization



Take a relative threshold

and convert back to image space