# 2D Discrete Fourier Transform (DFT)

# Outline

- Circular and linear convolutions
- 2D DFT
- 2D DCT
- Properties
- Other formulations
- Examples

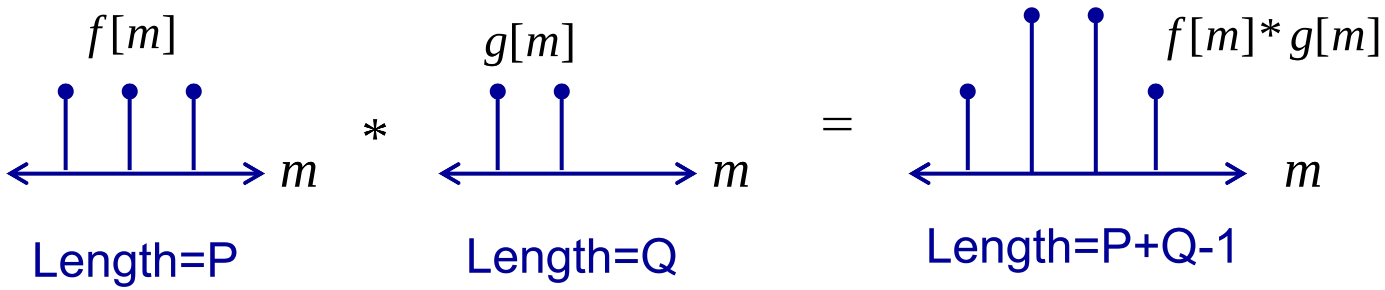
# Circular convolution

- Finite length signals ($N_0$ samples) → circular or periodic convolution
  - the summation is over 1 period
  - the result is a $N_0$ period sequence

$$\boxed{c[k] = f[k] \otimes g[k] = \sum_{n=0}^{N_0-1} f[n]g[k-n]}$$

- The circular convolution is equivalent to the linear convolution of the zero-padded equal length sequences

$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$

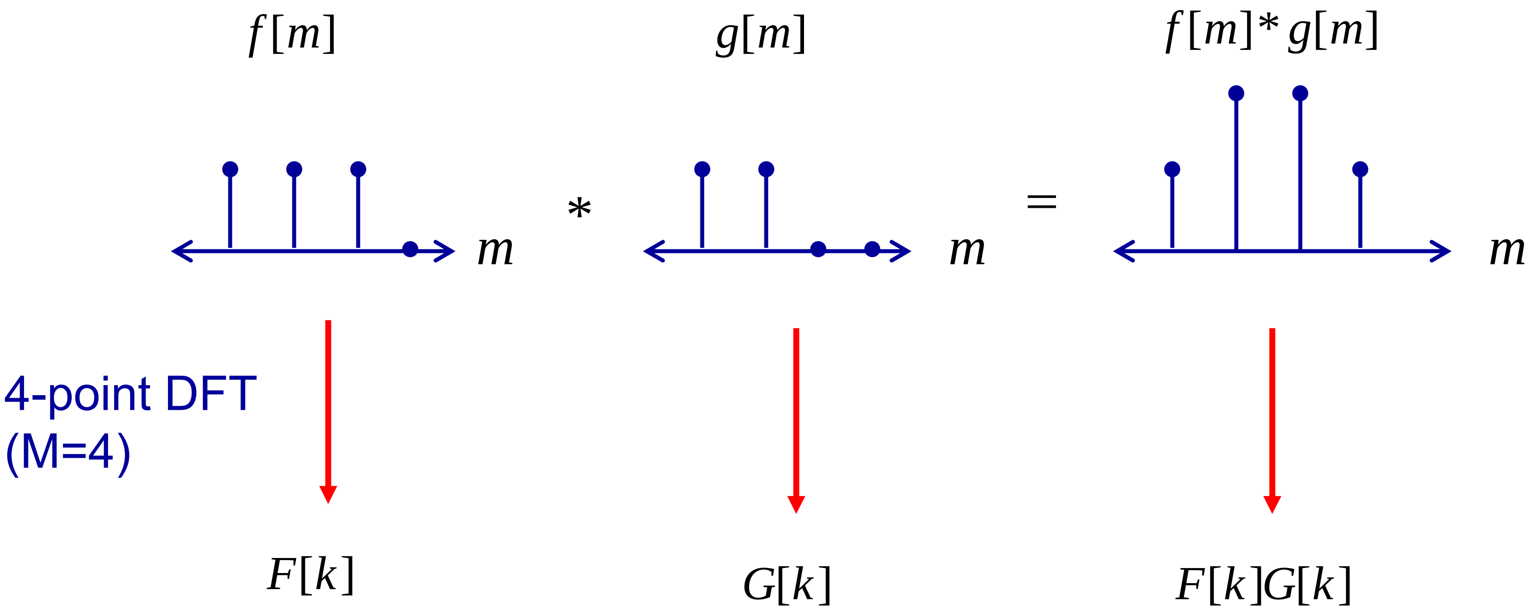



For the convolution property to hold, M must be *greater than or equal* to P+Q-1.

# Convolution

$$f[m] * g[m] \Leftrightarrow F[k]G[k]$$

- Zero padding



# In words

- Given 2 sequences of length N and M, let y[k] be their linear convolution

$$y[k] = f[k] * h[k] = \sum_{n=-\infty}^{+\infty} f[n]h[k-n]$$

- y[k] is also equal to the circular convolution of the two suitably zero padded sequences making them consist of the same number of samples

$$c[k] = f[k] \otimes h[k] = \sum_{n=0}^{N_0-1} f[n]h[k-n]$$

$$N_0 = N_f + N_h - 1: \text{ length of the zero-padded seq}$$

- In this way, the linear convolution between two sequences having a different length (filtering) can be computed by the DFT (which rests on the circular convolution)
  - The procedure is the following
    - Pad f[n] with $N_h$-1 zeros and h[n] with $N_f$-1 zeros
    - Find Y[r] as the product of F[r] and H[r] (which are the DFTs of the corresponding zero-padded signals)
    - Find the inverse DFT of Y[r]
- **Allows to perform linear filtering using DFT**

# 2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size $MxN$

or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.
- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a sampled version of the 2D DTFT

# 2D Discrete Fourier Transform (2D DFT)

- 2D Fourier (discrete time) Transform (DTFT) [Gonzalez]

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

a-periodic signal
periodic transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

periodized signal
periodic and sampled transform

2D DFT can be regarded as a sampled version of 2D DTFT.

# 2D DFT: Periodicity

- A [M,N] point DFT is periodic with period [M,N]
  - Proof

$$F[k,l] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$\begin{aligned}
\boxed{F[k+M,l+N]} &= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k+M}{M}m+\frac{l+N}{N}n\right)} \\
&= \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\cancelto{1}{e^{-j2\pi\left(\frac{M}{M}m+\frac{N}{N}n\right)}} \\
&= \boxed{F[k,l]}
\end{aligned}$$

(In what follows: spatial coordinates=k,l, frequency coordinates: u,v)

# 2D DFT: Periodicity

- Periodicity

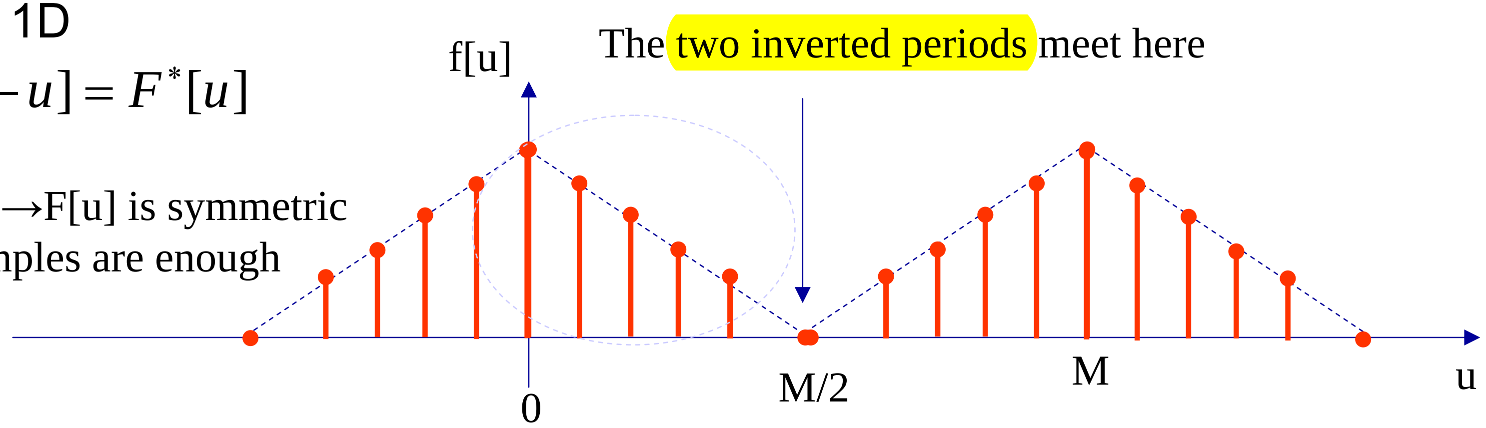
$$F[u,v]=F[u+mM,v]=F[u,v+nN]=F[u+mM,v+nN]$$

$$f[k,l]=f[k+mM,l]=f[k,l+nN]=f[k+mM,l+nN]$$

- This has important consequences on the implementation and energy compaction property
  - 1D

$$F[N-u]=F^*[u]$$

f[k] real→F[u] is symmetric
M/2 samples are enough



# Periodicity: 1D

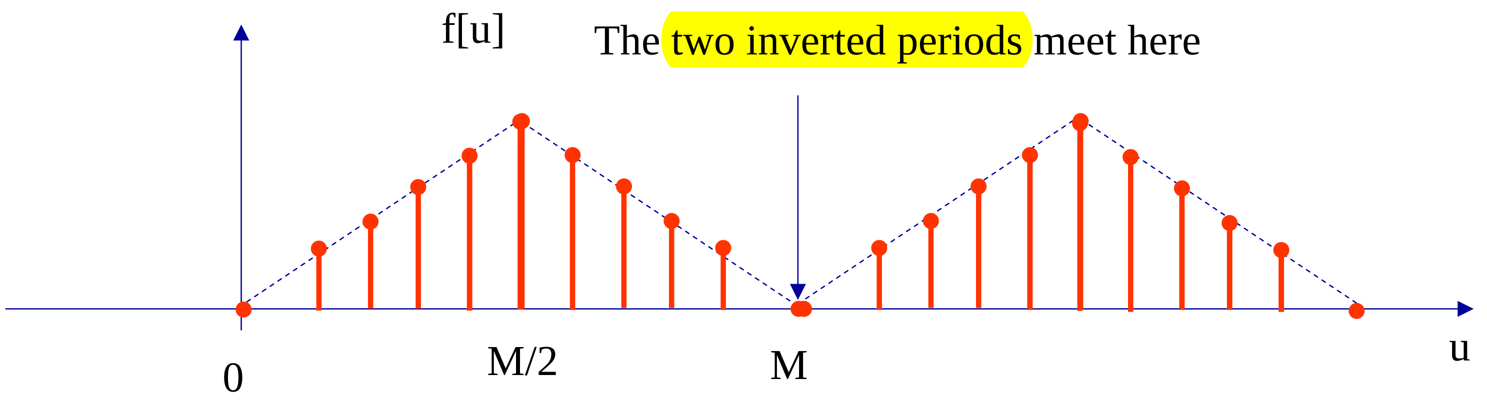
$$f[k] \leftrightarrow F[u]$$

$$f[k]e^{j2\pi\frac{u_0 k}{M}} \leftrightarrow F[u-u_0]$$

$$u_0 = \frac{M}{2} \rightarrow e^{j2\pi\frac{u_0 k}{M}} = e^{j2\pi\frac{Mk}{2M}} = e^{j\pi k} = (-1)^k$$

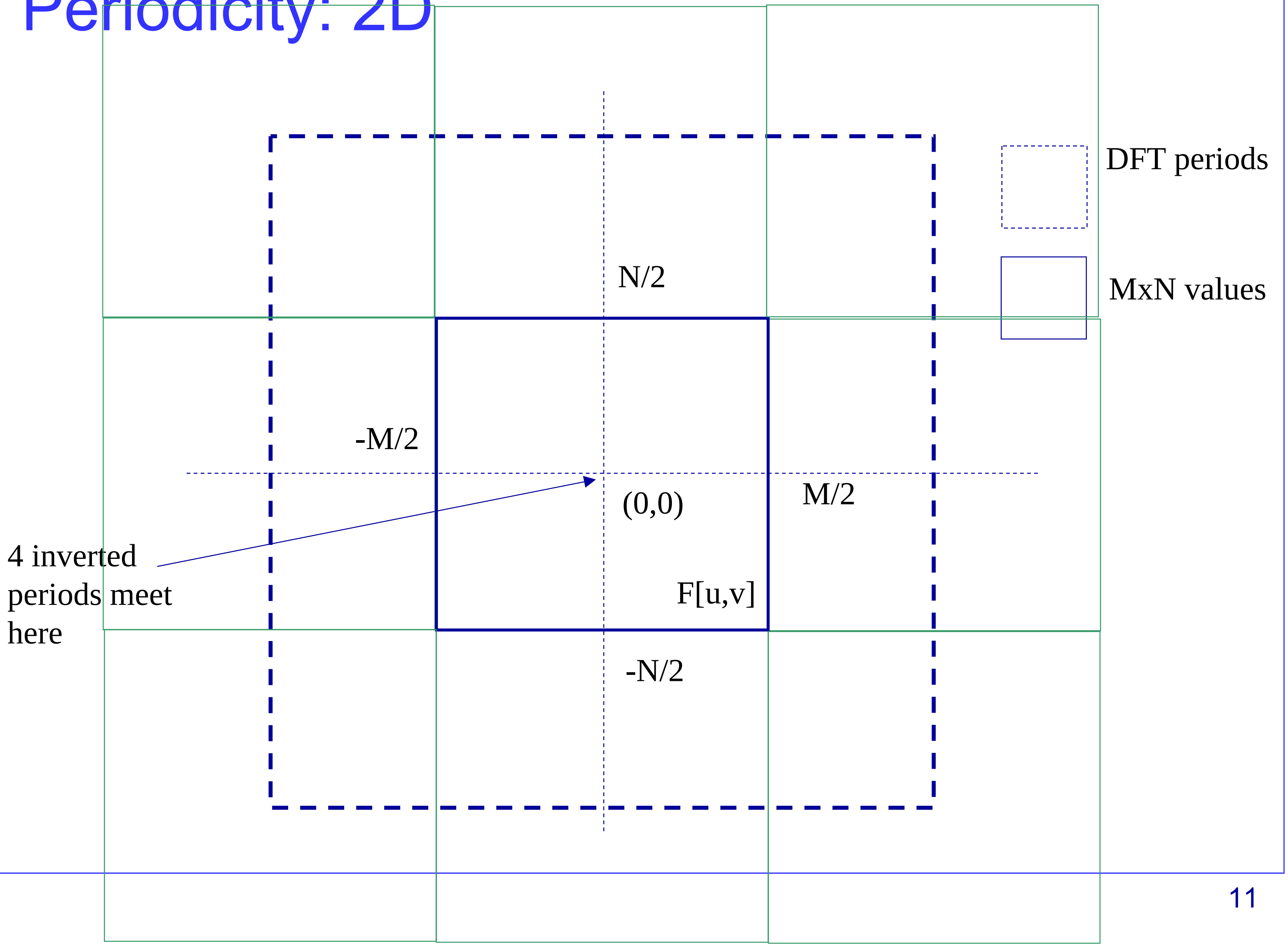
$$\boxed{(-1)^k f[k] \leftrightarrow F[u-\frac{M}{2}]}$$

changing the sign of every other sample puts F[0] at the center of the interval [0,M]




It is more practical to have one complete period positioned in [0, M-1]

# Periodicity: 2D

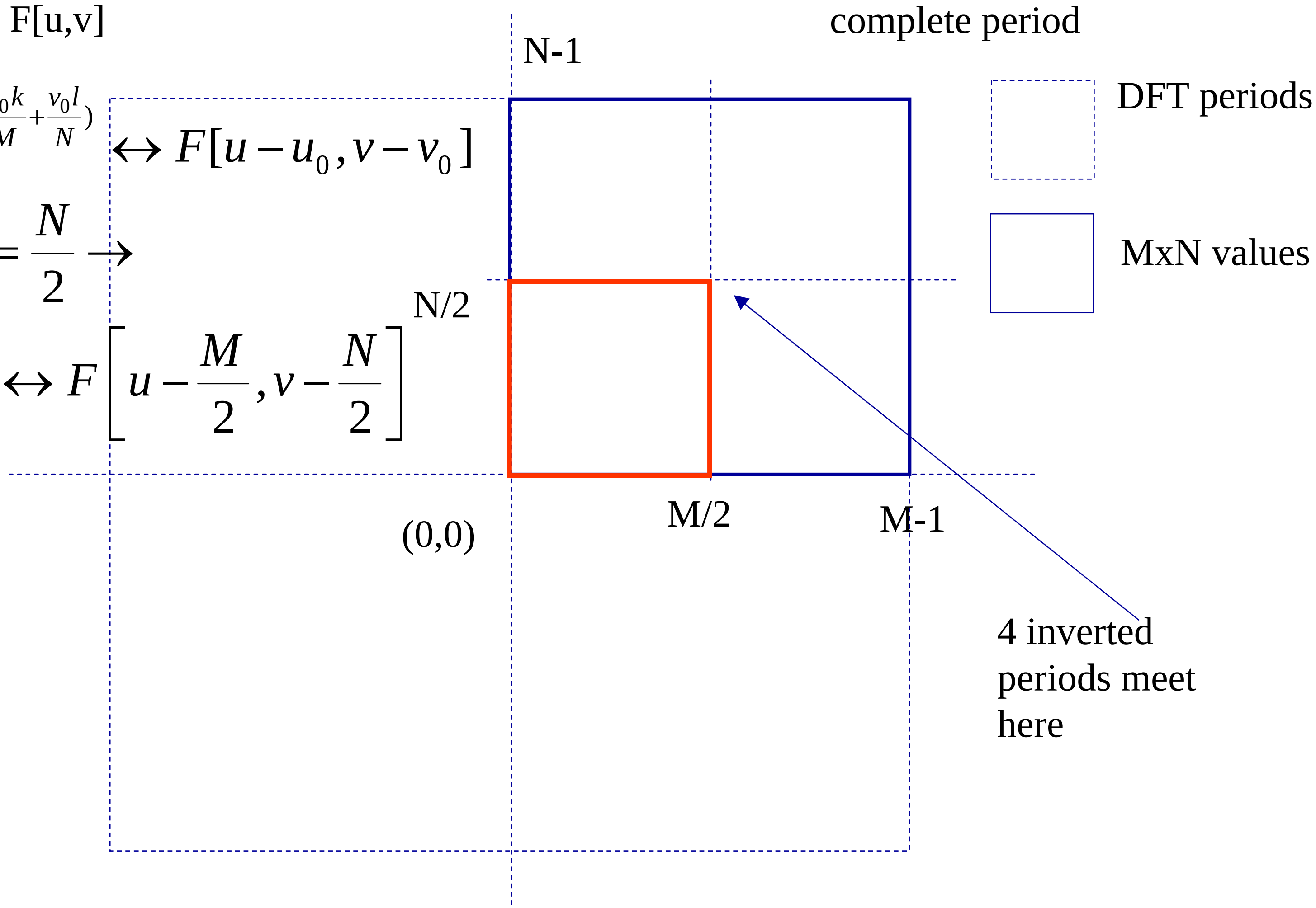


# Periodicity: 2D

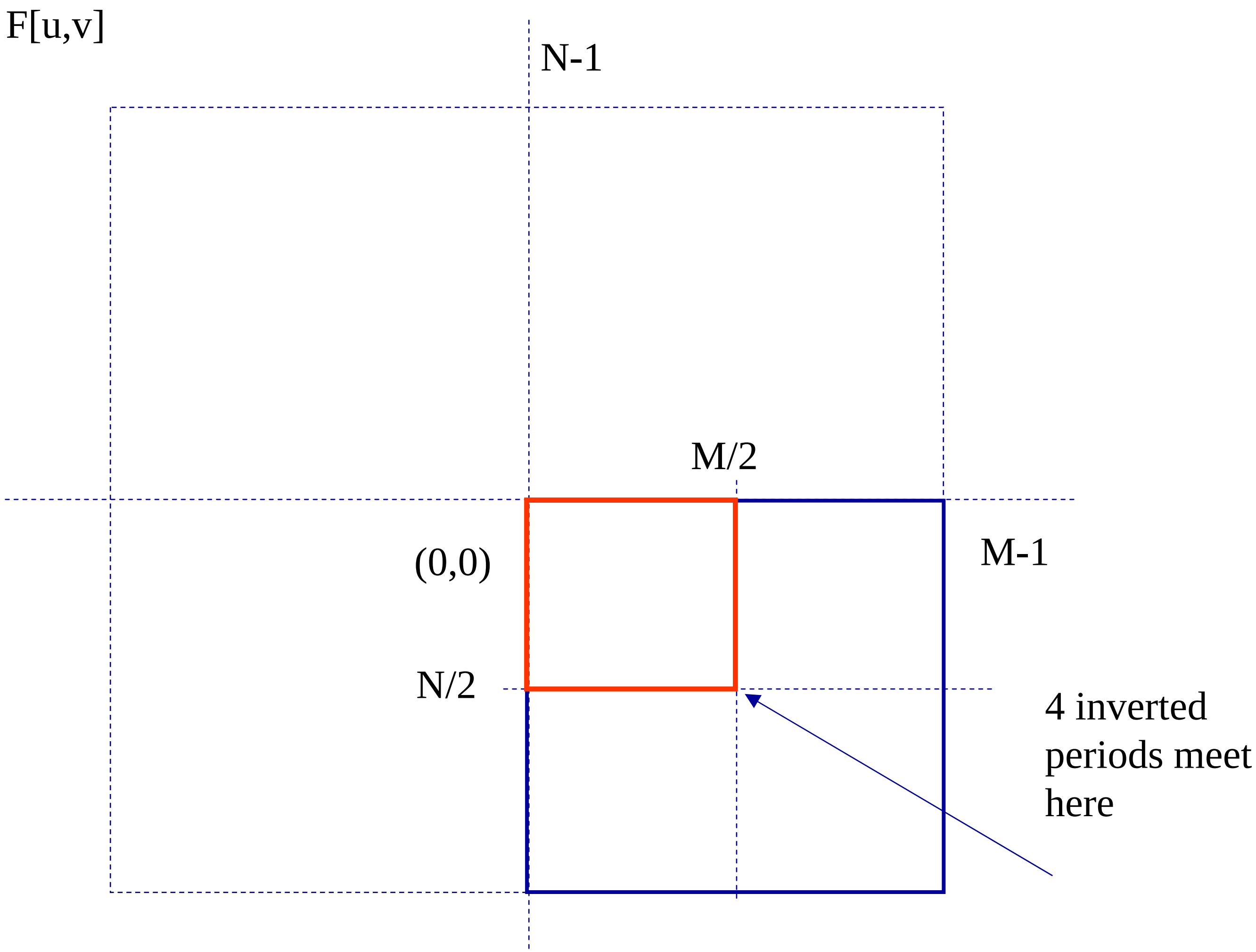
$$f[k,l]e^{j2\pi(\frac{u_0 k}{M}+\frac{v_0 l}{N})} \leftrightarrow F[u-u_0, v-v_0]$$

$$u_0 = \frac{M}{2}, v_0 = \frac{N}{2} \rightarrow$$

$$(-1)^{k+l} f[k] \leftrightarrow F\left[u-\frac{M}{2}, v-\frac{N}{2}\right]$$



# Periodicity: 2D



# Periodicity in spatial domain

- [M,N] point inverse DFT is periodic with period [M,N]

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

$$f[m+M, n+N] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}(m+M) + \frac{l}{N}(n+N)\right)}$$

$$= \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)} \cancelto{1}{e^{j2\pi\left(\frac{k}{M}M + \frac{l}{N}N\right)}}$$

$$= f[m,n]$$

# Angle and phase spectra

$$F[u,v] = |F[u,v]| e^{j\Phi[u,v]}$$

$$|F[u,v]| = \left[ \mathrm{Re}\{F[u,v]\}^2 + \mathrm{Im}\{F[u,v]\}^2 \right]^{1/2}$$

modulus (amplitude spectrum)

$$\Phi[u,v] = \arctan\left[ \frac{\mathrm{Im}\{F[u,v]\}}{\mathrm{Re}\{F[u,v]\}} \right]$$

phase

$$P[u,v] = |F[u,v]|^2$$

power spectrum

For a real function

$$F[-u,-v] = F^*[u,v]$$

conjugate symmetric with respect to the origin

$$|F[-u,-v]| = |F[u,v]|$$

$$\Phi[-u,-v] = -\Phi[u,v]$$

# Translation and rotation

$$f[k,l]e^{j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)} \leftrightarrow F\left[u-m, v-l\right]$$

$$f\left[k-m, l-n\right] \leftrightarrow F\left[u,v\right]^{-j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)}$$

$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \qquad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

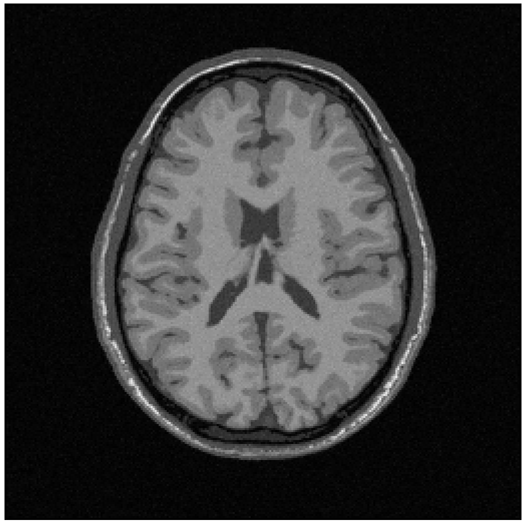
$$f\left[r, \vartheta+\vartheta_0\right] \leftrightarrow F\left[\omega, \varphi+\vartheta_0\right]$$

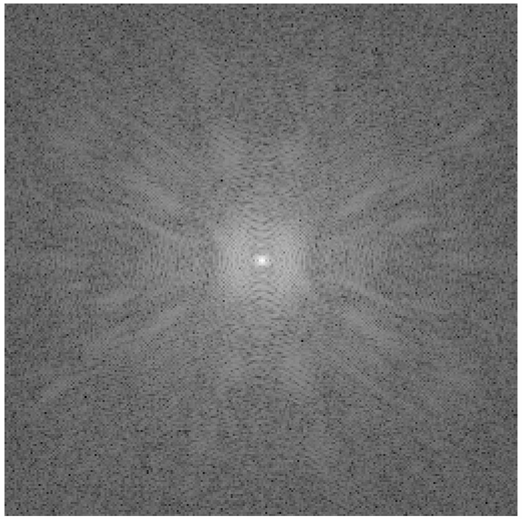
Rotations in spatial domain correspond equal rotations in Fourier domain

# mean value

$$F[0,0]=\frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}f[n,m]$$

DC coefficient





# Separability

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

- inverse transform

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

  - Because the transform kernels are separable and symmetric, the two dimensional transforms can be computed as sequential row and column one-dimensional transforms.
  - The basis functions of the transform are complex exponentials that may be decomposed into sine and cosine components.

# 2D DFT: summary

**TABLE 4.1**
Summary of some important properties of the 2-D Fourier transform.

| Property | Expression(s) |
|---|---|
| Fourier transform | $F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$ |
| Inverse Fourier transform | $f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$ |
| Polar representation | $F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$ |
| Spectrum | $\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \mathrm{Real}(F)$ and $I = \mathrm{Imag}(F)$ |
| Phase angle | $\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$ |
| Power spectrum | $P(u,v) = \lvert F(u,v)\rvert^2$ |
| Average value | $\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$ |
| Translation | $f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$<br>$f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$<br>When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then<br>$f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$<br>$f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$ |

# 2D DFT: summary

| | |
|---|---|
| Conjugate symmetry | $F(u,v) = F^*(-u,-v)$<br>$\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$ |
| Differentiation | $\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$<br>$(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$ |
| Laplacian | $\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$ |
| Distributivity | $\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$<br>$\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$ |
| Scaling | $af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert}F(u/a,v/b)$ |
| Rotation | $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$<br>$f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\varphi+\theta_0)$ |
| Periodicity | $F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$<br>$f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$ |
| Separability | See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result. |

# 2D DFT: summary

| Property | Expression(s) |
|---|---|
| Computation of the inverse Fourier transform using a forward transform algorithm | $\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by $MN$ gives the desired inverse. |
| Convolution† | $f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$ |
| Correlation† | $f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$ |
| Convolution theorem† | $f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$ |
| Correlation theorem† | $f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$ |

# 2D DFT: summary

Some useful FT pairs:

| | |
|---|---|
| *Impulse* | $\delta(x, y) \Leftrightarrow 1$ |
| *Gaussian* | $A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$ |
| *Rectangle* | $\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$ |
| *Cosine* | $\cos(2\pi u_0x + 2\pi v_0y) \Leftrightarrow \frac{1}{2}[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$ |
| *Sine* | $\sin(2\pi u_0x + 2\pi v_0y) \Leftrightarrow j\frac{1}{2}[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$ |

† Assumes that functions have been extended by zero padding.

# other formulations

# 2D Discrete Fourier Transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $l = 0,1,\ldots,N-1$

$k = 0,1,\ldots,M-1$

- Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

# 2D Discrete Fourier Transform

- It is also possible to define DFT as follows

$$F[k,l]=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

where $$k = 0,1,\ldots, M-1$$
$$l = 0,1,\ldots, N-1$$

- Inverse DFT

$$f[m,n]=\frac{1}{\sqrt{MN}}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

# 2D Discrete Fourier Transform

- Or, as follows

$$F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0,1,\ldots, M-1$ and $l = 0,1,\ldots, N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

# 2D DFT

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$\mathcal{F}(u, v) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} F(j, k) \exp\left\{\frac{-2\pi i}{N}(uj + vk)\right\}$$

- inverse transform

$$F(j, k) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \mathcal{F}(u, v) \exp\left\{\frac{2\pi i}{N}(uj + vk)\right\}$$

# 2D DCT

Discrete Cosine Transform

# 2D DCT

- based on most common form for 1D DCT

$$C(u) = \alpha(u)\sum_{x=0}^{N-1} f(x)\cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

u,x=0,1,…, N-1

$$f(x) = \sum_{u=0}^{N-1} \alpha(u)C(u)\cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

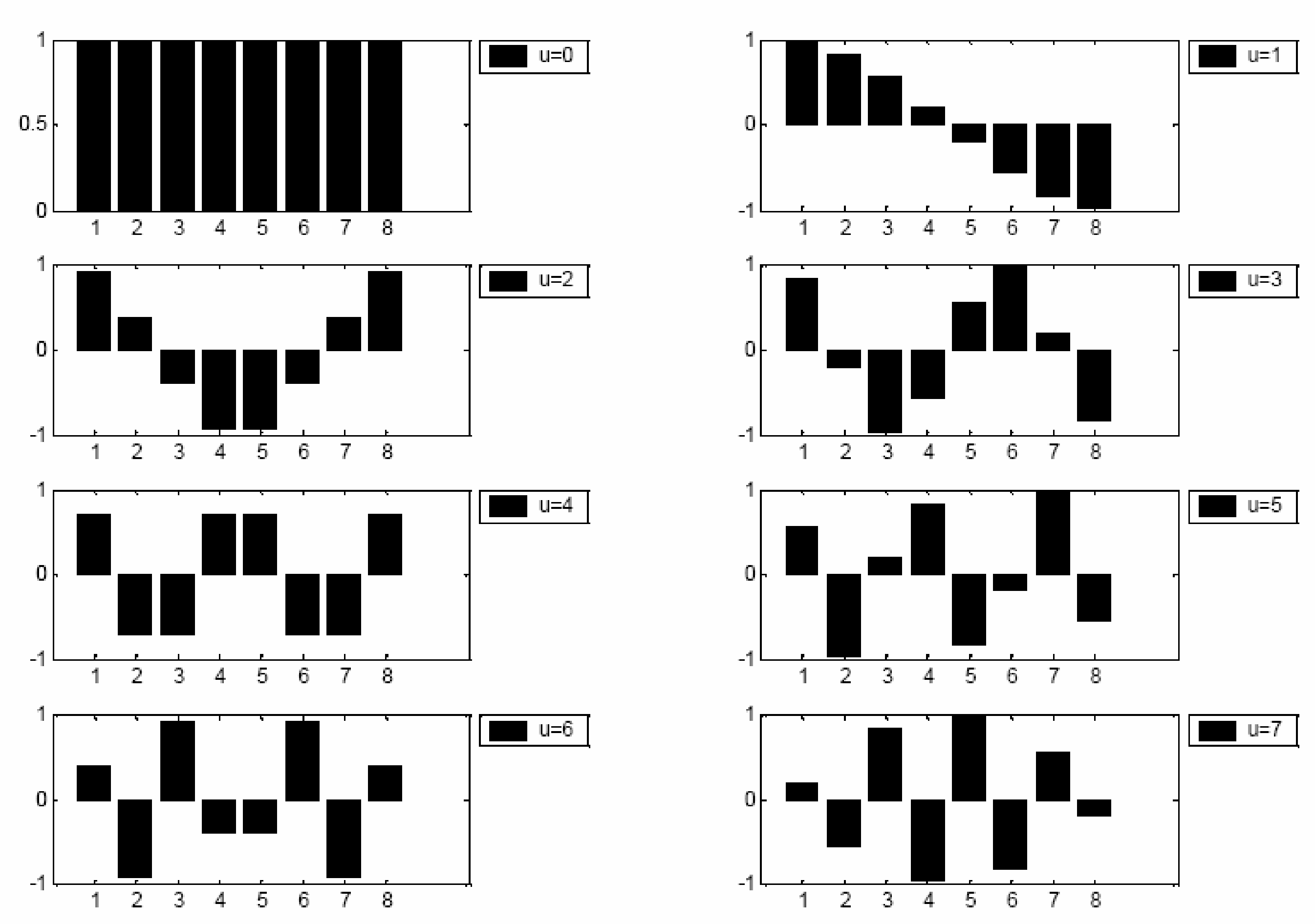
$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & for \quad u = 0 \\ \sqrt{\frac{2}{N}} & for \quad u \neq 0. \end{cases}$$

$$C(u=0) = \sqrt{\frac{1}{N}}\sum_{x=0}^{N-1} f(x).$$

"mean" value

# 1D basis functions

Figure 1




Cosine basis functions are orthogonal

# 2D DCT

- Corresponding 2D formulation

direct $$C(u,v)=\alpha(u)\alpha(v)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

*u,v=0,1,…., N-1*

$$\alpha(u)=\begin{cases}\sqrt{\dfrac{1}{N}} & for \quad u=0\\ \sqrt{\dfrac{2}{N}} & for \quad u\neq 0.\end{cases}$$

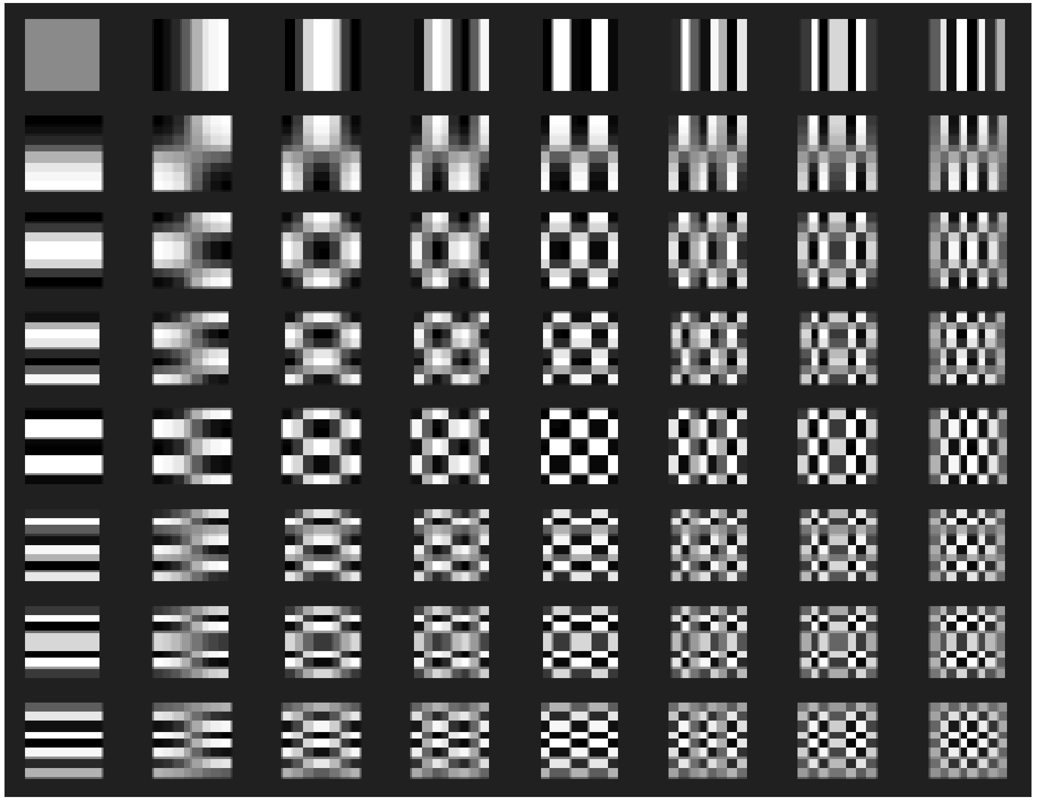
inverse $$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\alpha(u)\alpha(v)C(u,v)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

# 2D basis functions

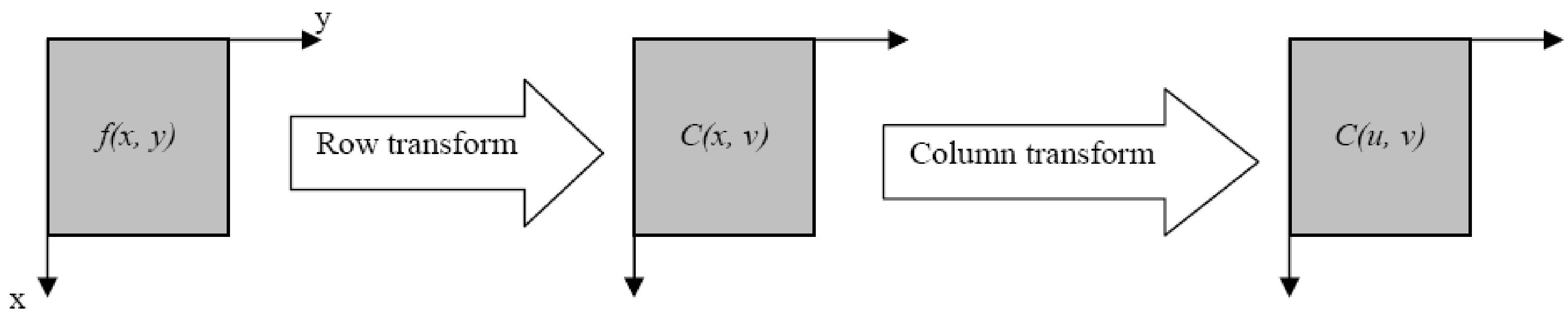
- The 2-D basis functions can be generated by multiplying the horizontally oriented 1-D basis functions (shown in Figure 1) with vertically oriented set of the same functions.
- The basis functions for $N = 8$ are shown in Figure 2.
  - The basis functions exhibit a progressive increase in frequency both in the vertical and horizontal direction.
  - The top left basis function assumes a constant value and is referred to as the *DC coefficient*.

# 2D DCT basis functions

Figure 2



# Separability




The inverse of a multi-dimensional DCT is just a separable product of the inverse(s) of the corresponding one-dimensional DCT , e.g. the one-dimensional inverses applied along one dimension at a time

# Separability

- Symmetry
  - Another look at the row and column operations reveals that these operations are functionally identical. Such a transformation is called a *symmetric transformation*.
  - A separable and symmetric transform can be expressed in the form

$$T = AfA$$

  - where A is a NxN symmetric transformation matrix which entries a(i,j) are given by

$$a(i,j) = \alpha(j) \sum_{j=0}^{N-1} \cos\left[\frac{\pi(2j+1)i}{2N}\right],$$

    - This is an extremely useful property since it implies that the transformation matrix can be pre computed offline and then applied to the image thereby providing orders of magnitude improvement in computation efficiency.

# Computational efficiency

- Computational efficiency
  - Inverse transform $f = A^{-1} T A^{-1}$.
  - DCT basis functions are orthogonal. Thus, the inverse transformation matrix of *A* is equal to its transpose i.e. $A^{-1} = A^T$. This property renders some reduction in the pre-computation complexity.

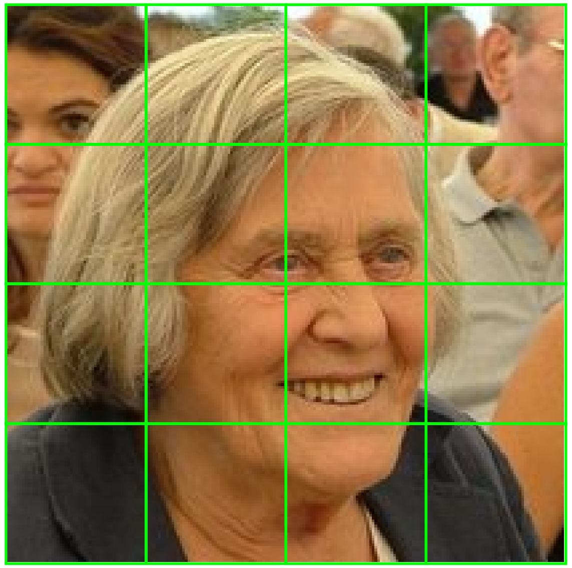
# Block-based implementation

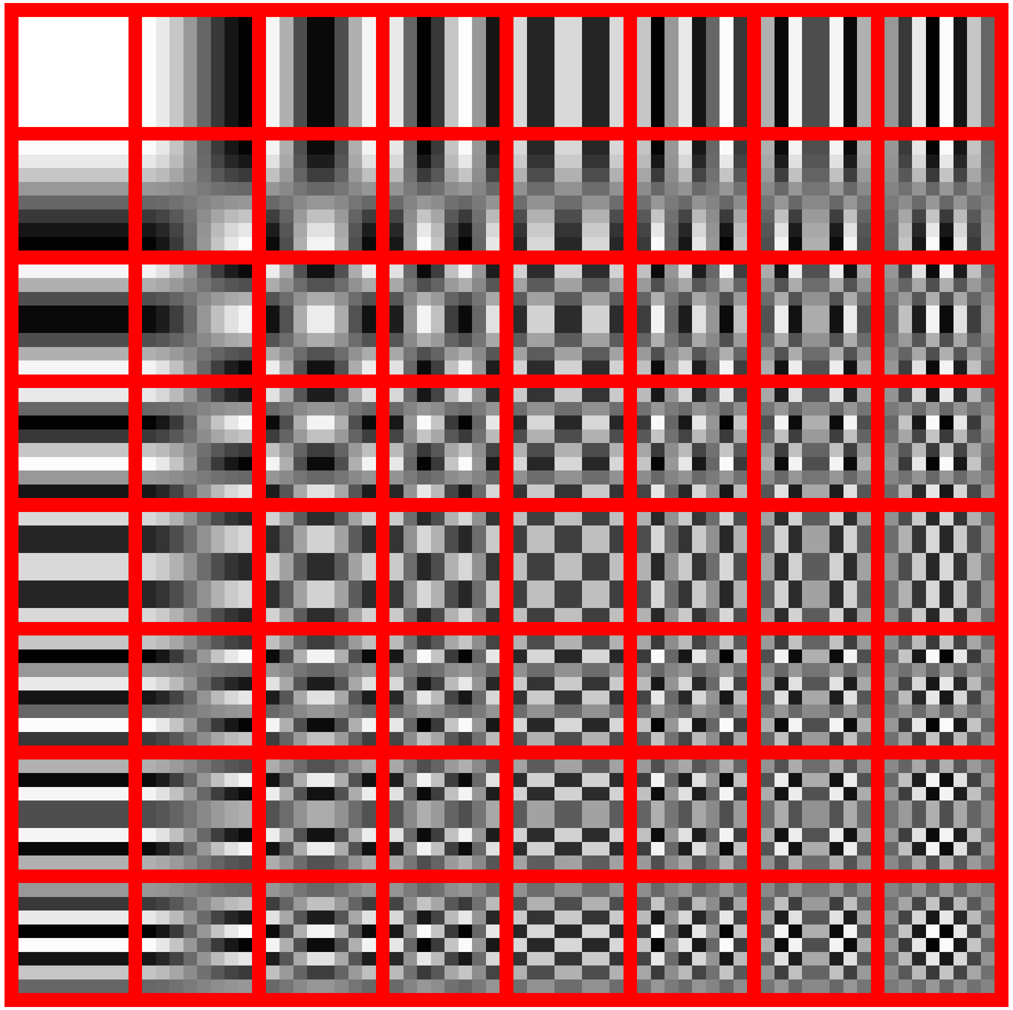
Basis function

*Block-based* transform

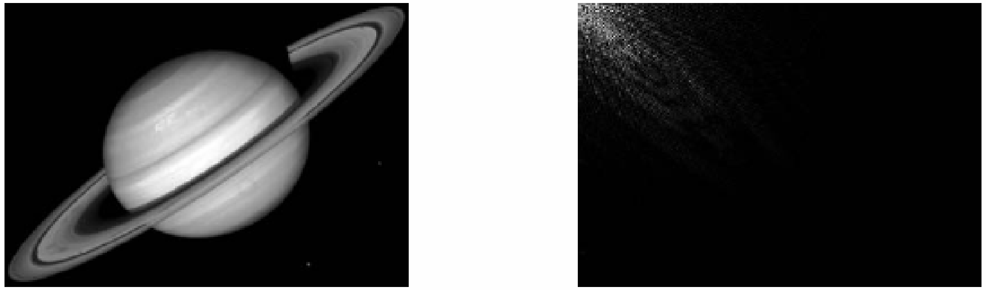
Block size
N=M=8

The source data (8x8) is transformed to a linear combination of these 64 frequency squares.



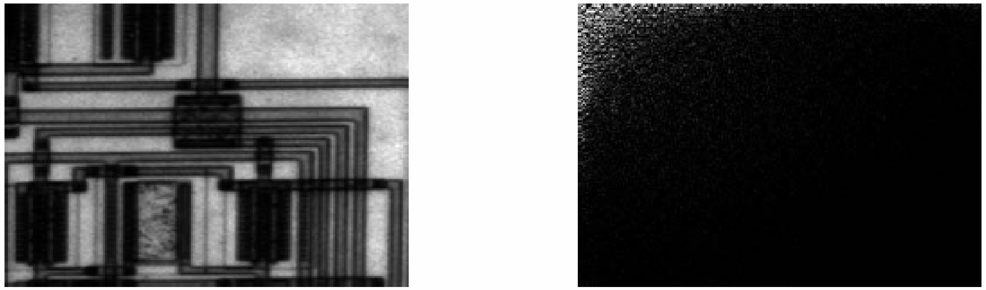


# Energy compaction

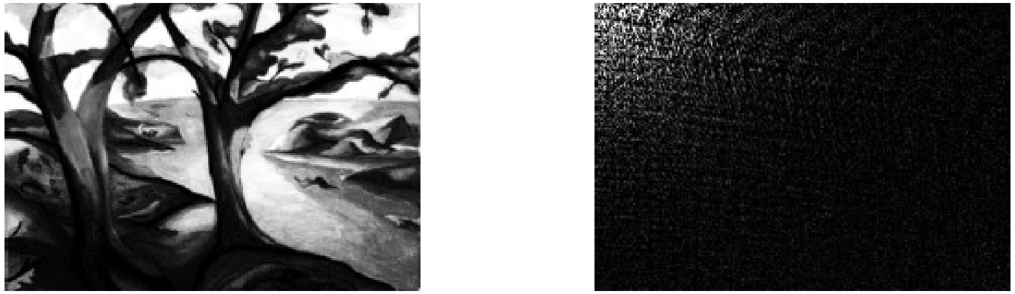


(a)

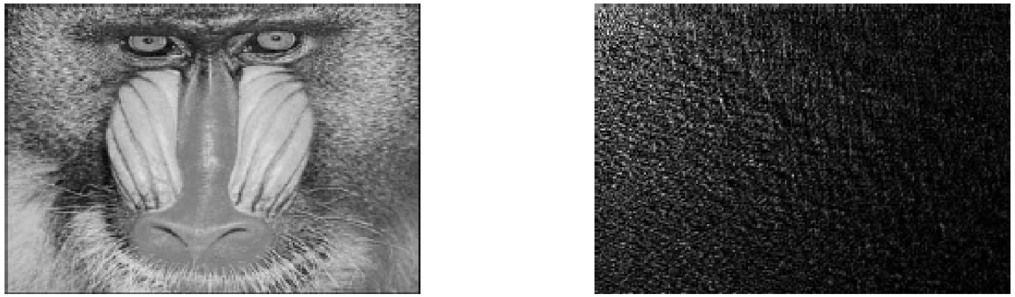
(b)



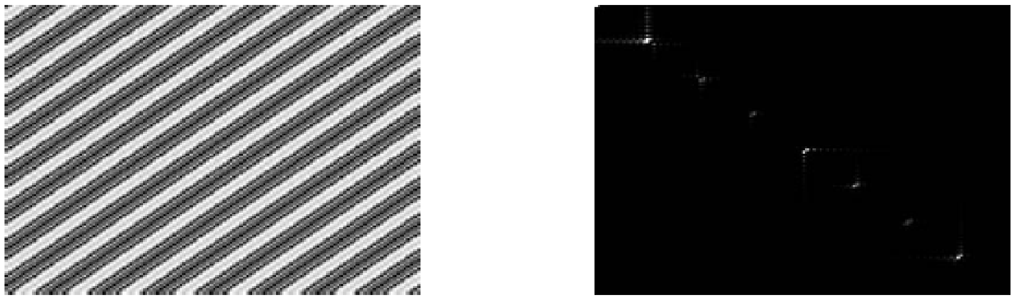
# Energy compaction



(d)



(e)



# Appendix

- Eulero's formula

$$A(j,k;u,v) = \exp\left\{\frac{-2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} - i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$

$$B(j,k;u,v) = \exp\left\{\frac{2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} + i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$