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PHYS 234

QUANTUM PHYSICS I



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Abstract

These notes are intended as a resource for myself; past, present, or future students of this course, and anyone interested in the material. The goal is to provide an end-to-end resource that covers all material discussed in the course displayed in an organized manner. If you spot any errors or would like to contribute, please contact me directly.

Robert Hill is a low temperature experimentalist, but this course will be mostly theoretical.

Albert Einstein once said, "Quantum mechanics is certainly imposing. But an inner voice tells me that it is not yet the real thing. The theory says a lot, but does not really bring us any closer to the secret of the 'old one'. I, at any rate, am convinced that He does not throw dice."

Richard Feynman said "I think I can safely say that nobody understand quantum mechanics."

So we're in for miserable experience with this course then? Well not really, there are some good reasons to study Quantum Physics:

- It's Extremely interesting!
 - Physically
 - Mathematically
 - Philosophically
- It is the science behind future technology!
- Waterloo is Quantum Valley!

1 The Photoelectric and Compton Effects

1.1 Historical Background

In classical physics we always observed things as behaving like waves or as particles. For example, there is

- Particle-like behaviour of radiation
- Wave-like behaviour of matter
- Wave-particle duality that combines the two

Let's explore the two sides of the coin. First, **what is a particle?** Some words that describe it are *point*, *localised*, *mass*, *solid* and similarly **what is a wave?** It can be described with words like *interference*, *oscillation*, *delocalised*, and *medium*. One such thing that we have had trouble with describing is **light**. Is it a wave or a particle?

1.2 Einstein's Theory of Photoelectric Effect

Radiant energy (light) is quantized into concentrated bundles (photons)

$$E = hf$$

His Photoelectric Equation (1905) states that

$$K_{\max} = hf - \omega_0$$

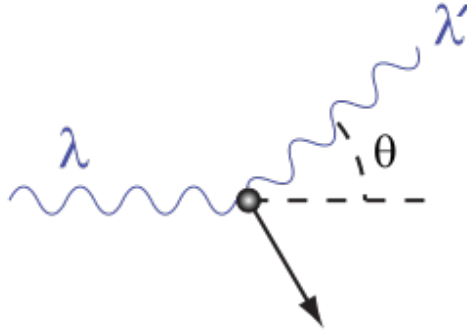


Figure 1.1: The Compton Effect. The scattered light has a different frequency; the frequency depends on the direction. A bigger deflection causes a bigger change in frequency.

1.3 Compton Effect

In the photoelectric effect, we treated light as being composed of individual light particles, called photons, that carry some energy. It then makes sense to think that the photons also have momentum.

Electromagnetic radiation is scattered by a target object. In classical theory, the charges in the target object will respond to the incoming wave and start to oscillate. All oscillating charges emit radiation at the frequency of oscillation, and this newly generated set of waves can also be detected at an angle θ with respect to the incoming wave. This classical model explains why the sky is blue and all that jazz. The scattering process itself, though, does not change the frequency of incoming and outgoing radiation.

However, an experimental problem occurred. In experiments with X-ray radiation on a graphene target, one observes that two separate frequencies at an angle θ result in different intensities. This effect is independent of the material, though intensities may vary.

Definition 1.1 (Compton Shift).

$$\Delta\lambda = \lambda_c(1 - \cos\theta)$$

Definition 1.2 (Compton wavelength).

$$\lambda_c = \frac{h}{m_0c}$$

2 De Broglie Wavelength and the Davisson-Germer Experiment

We have shown that wave phenomena can exhibit particle features. We can rewrite the momentum instead as $p = \frac{h}{\lambda}$ using a simple wave relationship. There is nothing in this reformed equation that has to do with light. This led to the following postulate.

2.1 The De Broglie Postulate (1924)

De Broglie's hypothesis was based on the grand symmetry of nature; if radiation has wave-particle duality, then so should matter.

Definition 2.1 (de Broglie Relation).

$$\lambda = \frac{h}{p}$$

2.2 The Davisson-Germer Experiment

We must first understand the Bragg Grating; it is an optical filter that reflects particular wavelengths and transmits all others. Note that reflection, however, is common to both waves and particles.

2.3 Final Words

The observation of both phenomena in one and the same experiment leads us also to the concept of delocalization, which goes beyond the simple concept of "being extended", because single quantum objects seem to be able to simultaneously explore regions in space-time that cannot be explored by a single object in any classical way.

3 Linear Algebra Review

We're going to begin by reviewing some mathematics that will be needed in the course. This is a physics course so we're going to be a little loosey-goosey.

3.1 Vector Spaces

Definition 3.1 (Vector Space). A **vector space** consists of a set of vectors : $(|\alpha\rangle, |\beta\rangle, |\gamma\rangle, \dots)$ which is closed under vector addition and scalar multiplication.

Vector Addition produces another vector, that is

$$|\alpha\rangle + |\beta\rangle = |\gamma\rangle$$

it is also commutative

$$|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$$

and associative

$$|\alpha\rangle + (|\beta\rangle + |\gamma\rangle) = (|\alpha\rangle + |\beta\rangle) + |\gamma\rangle$$

The null vector exists such that $|\alpha\rangle + |0\rangle = |\alpha\rangle$, and of course there is the inverse vector such that $|\alpha\rangle + |-\alpha\rangle = |0\rangle$.

Scalar Multiplication: The product of a scalar with a vector is another vector ($a|\alpha\rangle = |\gamma\rangle$). Note that scalar multiplication is distributive with respect to vector addition

$$a(|\alpha\rangle + |\beta\rangle) = a|\alpha\rangle + a|\beta\rangle$$

Scalar multiplication is distributive with respect to scalar addition too

$$(a + b)|\alpha\rangle = a|\alpha\rangle + b|\alpha\rangle$$

and it is associative with respect to the product of scalars.

$$a(b|\alpha\rangle) = (ab)|\alpha\rangle$$

then multiplication by zero and by ± 1 has

$$0|\alpha\rangle = |0\rangle, \quad 1|\alpha\rangle = |\alpha\rangle, \quad -1|\alpha\rangle = -|\alpha\rangle = |-\alpha\rangle$$

Linear Combinations of Vectors: To generate a linear combination of vectors

$$|\lambda\rangle = a|\alpha\rangle + b|\beta\rangle + c|\gamma\rangle$$

- (I) Any vector is linearly independent of a set of vectors if it cannot be written as a linear combination of them.
- (II) A set of vectors is linearly independent if each is linearly independent of the rest.
- (III) A collection of vectors is said to **span** the space if every vector can be written as a linear combination of them.
- (IV) A set of linearly independent vectors that span a space is called a **basis**.
- (V) The number of vectors in the basis is called the **dimension** of the space.

Co-ordinate Representation: With respect to a given basis, $|e_1\rangle, |e_2\rangle, |e_3\rangle, \dots, |e_n\rangle$, any given vector $|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \dots + a_n|e_n\rangle$ is uniquely defined by the ordered n -tuple of its components.

$$|\alpha\rangle \iff \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(the co-ordinate representation of $|\alpha\rangle$ with respect to the basis given by each $|e_1\rangle$.)

Coordinates depend on the chosen basis. In basis 1 $|\alpha\rangle = a_x|x\rangle + a_y|y\rangle \iff \begin{pmatrix} a_x \\ a_y \end{pmatrix}$ and then in basis 2 we see

$$|\alpha\rangle = a_{x'}|x'\rangle + a_{y'}|y'\rangle \iff \begin{pmatrix} a_{x'} \\ a_{y'} \end{pmatrix}.$$

Addition of vectors by adding corresponding components (when in the same basis) works as you might expect, too:

$$|\alpha\rangle + |\beta\rangle \iff (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$$

Also, scalar multiplication works by multiplying the scalar in each component

$$c|\alpha\rangle \iff (ca_1, ca_2, ca_3, \dots, ca_n)$$

so of course,

$$|0\rangle = (0, 0, 0, \dots, 0), \quad |-\alpha\rangle = (-a_1, -a_2, \dots, -a_n)$$

Inner Product: For every vector, $|\alpha\rangle$, in a vector space there exists a dual vector $\langle\alpha|$ in a corresponding dual vector space. Importantly, the dual vector to $c|\alpha\rangle$ is $C^*\langle\alpha|$ where $*$ denotes complex conjugation. So the inner product of $|\alpha\rangle$ and $|\beta\rangle$ is $\langle\alpha|\beta\rangle$ which is a scalar (complex number), hence $\langle\alpha|\beta\rangle$ is sometimes called **scalar product**.

- (I) $\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*$
- (II) $\langle\alpha|\alpha\rangle \geq 0$ (real and positive), so $\langle\alpha|\alpha\rangle = 0$ if $|\alpha\rangle = |0\rangle$.

(III) The norm of a vector $||\alpha|| = \sqrt{\langle\alpha|\alpha\rangle}$ generalized "length" of a vector.

(IV) Normalized $||\alpha|| = 1$.

(V) Orthogonal if $\langle\alpha|\beta\rangle = 0$, then $|\alpha\rangle$ is orthogonal to $|\beta\rangle$.

(VI) Orthogonal set $\langle a_i|a_j\rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Consider the orthonormal basis $|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle$, and

$$|\alpha\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle + \dots + a_n |e_n\rangle$$

$$|\beta\rangle = b_1 |e_1\rangle + b_2 |e_2\rangle + \dots + b_n |e_n\rangle$$

where we have the column vectors

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \quad |\beta\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

then with dual vectors

$$|\alpha\rangle = a_1^* |e_1\rangle + a_2^* |e_2\rangle + \dots + a_n^* |e_n\rangle$$

$$|\beta\rangle = b_1^* |e_1\rangle + b_2^* |e_2\rangle + \dots + b_n^* |e_n\rangle$$

so that we have row vectors

$$|\alpha\rangle = (a_1^*, a_2^*, \dots, a_n^*) \quad |\beta\rangle = (b_1^*, b_2^*, \dots, b_n^*)$$

Now we can see that these results interact in a kind of cool way, check this out:

$$\langle\alpha|\beta\rangle = (a_1^*, a_2^*, \dots, a_n^*) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$$

which is a complex number. The components of the linear expansion are inner products too:

$$|\alpha\rangle = a_1 |e_1\rangle + a_2 |e_2\rangle + \dots + a_n |e_n\rangle$$

Consider also that

$$\langle e_1|\alpha\rangle = \langle e_1|(a_1 |e_1\rangle + a_2 |e_2\rangle + \dots + a_n |e_n\rangle) = a_1 \langle e_1|e_1\rangle + a_2 \langle e_1|e_2\rangle + \dots + a_n \langle e_1|e_n\rangle = a_1$$

3.2 Matrices

Matrices represent linear transformations that take a vector in a vector space and map it to another vector.

$$|\alpha\rangle \longrightarrow |\alpha'\rangle = \hat{T} |\alpha\rangle$$

The transformation must be linear

$$\hat{T}(a |\alpha\rangle + b |\beta\rangle) = a \hat{T} |\alpha\rangle + b \hat{T} |\beta\rangle$$

Consider \hat{T} acting on n basic vectors, $|e_i\rangle$

$$\hat{T}|e_1\rangle = T_{11}|e_1\rangle + T_{21}|e_2\rangle + \dots + T_{n1}|e_n\rangle$$

That is, $|e_1\rangle$ is mapped to a new vector written as a linear combination of basis vectors, likewise

$$\begin{aligned}\hat{T}|e_2\rangle &= T_{12}|e_1\rangle + T_{22}|e_2\rangle + \dots + T_{n2}|e_n\rangle \\ &\vdots \\ \hat{T}|e_n\rangle &= T_{1n}|e_1\rangle + T_{2n}|e_2\rangle + \dots + T_{nn}|e_n\rangle\end{aligned}$$

which can be compactly expressed

$$\hat{T}|e_j\rangle = \sum_{i=1}^n T_{ij}|e_i\rangle \quad (j = 1, 2, \dots, n)$$

If $|\alpha\rangle$ is an arbitrary vector, expressed in terms of basis $|e_i\rangle$'s

$$|\alpha\rangle = a_1|e_1\rangle + a_2|e_2\rangle + \dots + a_n|e_n\rangle = \sum_{j=1}^n a_j|e_j\rangle$$

(and recall $a_i = \langle e_i|\alpha\rangle$). Then the effect of \hat{T} on $|\alpha\rangle$ is

$$\hat{T}|\alpha\rangle = \sum_{j=1}^n a_j \hat{T}|e_j\rangle = \sum_{j=1}^n \sum_{i=1}^n a_j T_{ij}|e_i\rangle = \sum_{i=1}^n \left(\sum_{j=1}^n T_{ij} a_j \right) |e_i\rangle$$

Hence \hat{T} takes a vector $|\alpha\rangle$, with components a_1, a_2, \dots, a_n and maps to a new vector α' with components $a'_i = \sum_{j=1}^n T_{ij} a_j$. So \hat{T} is characterized by n^2 elements, T_{ij} , which depend on the chosen basis. Express \hat{T} as a matrix.

$$\begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ T_{n1} & T_{n2} & \dots & T_{nn} \end{pmatrix}$$

where T_{ij} is a matrix element, the row is i and column j . Now if we want to express \hat{T} with respect to a particular set of basis vectors, the i -th element will define the values of matrix elements T_{ij}

$$\hat{T}|e_j\rangle = \sum_{i=1}^n |e_i\rangle \quad (j = 1, 2, \dots, n)$$

multiply on left by basis vector $\langle e_k|$,

$$\begin{aligned}\langle e_k|\hat{T}|e_j\rangle &= \langle e_k|\sum_{i=1}^n T_{ij}|e_i\rangle \\ &= \langle e_k|(T_{1j}|e_1\rangle + T_{2j}|e_2\rangle + \dots + T_{nj}|e_n\rangle) \\ &= (T_{1j} = \langle e_k|e_1\rangle + T_{2j}\langle e_k|e_2\rangle + \dots + T_{kj}\langle e_k|e_k\rangle + \dots + T_{nj}\langle e_k|e_n\rangle)\end{aligned}$$

where all terms except $\langle e_k | e_k \rangle$ go to 0. Now apply the orthonormal property of basis vectors $|e_i\rangle$ and thus

$$\langle e_k | \hat{T} | e_j \rangle = T_{kj} \quad \text{matrix element}$$

Once the basis is chosen, the i -th element will define the vector in coordinate representation and the linear transformation in matrix form.

Some matrix terminology,

Definition 3.2 (transpose). The interchange of rows and columns of the matrix. Transpose of column is row and vice versa. The transpose of a square matrix is to reflect elements in main diagonal.

Definition 3.3 (symmetric). A matrix is equal to its transpose (square matrices only).

Definition 3.4 (conjugate). The complex conjugate of every element.

Definition 3.5 (adjoint). The conjugate transpose of a matrix. Indicated by a dagger symbol \hat{T}^\dagger . A square matrix is **Hermitian** if matrix and adjoint are equal $\hat{T} = \hat{T}^\dagger$. Vector space and dual are related by adjoint.

$$|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a \implies \text{DUAL} = a^\dagger = (a_1^*, a_2^*, \dots, a_n^*)$$

The inner product $\langle \alpha | \beta \rangle = a^\dagger b$.

Definition 3.6 (product). Multiplication may not be commutative : $\hat{T}\hat{S} \neq \hat{S}\hat{T}$. The difference between orders is commutator

$$[\hat{S}, \hat{T}] = \hat{S}\hat{T} - \hat{T}\hat{S} \quad (\text{will be zero if } \hat{T} \text{ and } \hat{S} \text{ commute})$$

Definition 3.7 (eigenvalues, eigenvectors). Every linear transformation has special vectors that transform into scalar multiples of themselves.

$$\hat{T} |\alpha\rangle = \lambda |\alpha\rangle$$

where $|\alpha\rangle$ is the eigenvector, and λ is the eigenvalue.

Example 3.1. Find the eigenvalues and normalized eigenvectors of $\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$.

$$\begin{vmatrix} 5 - \lambda & -2 \\ -2 & 2 - \lambda \end{vmatrix} = 0 \quad (\text{Characteristic Equation})$$

This resolves to solving

$$\begin{aligned} (5 - \lambda)(2 - \lambda) - (-2)(-2) &= 0 \\ (\lambda - 1)(\lambda - 6) &= 0 \end{aligned}$$

Therefore $\lambda = 1$ or $\lambda = 6$ are eigenvalues. Eigenvectors

$$\bullet \lambda_1 = 1 \implies |\lambda_1\rangle = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \implies 5x_1 - 2y_1 = x_1 \text{ and } -2x_1 + 2y_1 = y_1$$

Rearranging these equations gives us that $2x_1 - y_1 = 0$. This means that any vector on the line $2x_1 - y_1 = 0$ is an eigenvector. So one possible vector is $|\lambda_1\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Now we can normalize by introducing a normalization constant,

$$|\lambda_1\rangle = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

then

$$\sqrt{\langle \lambda_1 | \lambda_1 \rangle} = 1 \implies \left(a(1, 2) \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)^{\frac{1}{2}} = a\sqrt{5} = 1 \implies a = \frac{1}{\sqrt{5}} \implies |\lambda_1\rangle = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

4 Introduction to the Formalism and Structure of Quantum Mechanics

We're going to cover a few topics including Angular Momentum and Spin, the Stern-Gerlach Experiment, Quantum State Vectors, Computing Probabilities, and Operators and Measure.

4.1 Angular Momentum and Spin

Angular momentum and magnetic dipole moment (orbital). Consider an electron in a circular orbit, it has radius r , tangential velocity \vec{v} , and current going in the opposite direction of travel I , as well as dipole moment $\vec{\mu}_L$ and angular momentum \vec{L} . Now, to calculate the magnitude of the dipole moment,

$$\begin{aligned} |\vec{\mu}_L| &= IA \quad (\text{product of current and area}) \\ &= \frac{e}{T} \pi r^2 \quad (T = \text{period of electron}) \\ &= \frac{e}{\left(\frac{2\pi r}{v}\right)} \pi r^2 \\ &= \frac{e}{2} v r \\ &= \frac{e}{2m_e} m_e v r \\ &= \frac{e}{2m_e} |\vec{L}| \end{aligned}$$

The direction of $\vec{\mu}_L$ (follow the usual right hand rule)

$$\vec{\mu}_L = -\frac{e}{2m_e} \vec{L}$$

Next, we want to talk a little bit about **spin**. Spin is the intrinsic angular momentum \vec{S} which leads to an intrinsic dipole moment $\vec{\mu}_S$. This intrinsic property is a fundamental nature of particle and cannot be taken away (c.f., mass or charge). In analogy with orbital angular momentum,

$$\vec{\mu}_S = g \frac{q}{2m} \vec{S}$$

where g is the **gyromagnetic ratio**, q is the **charge**, and m is the **mass** of the particle.

For an electron, $g \approx 2$, $q = -e$, $m = m_e$, which means

$$\vec{\mu}_S = -\frac{e}{m_e} \vec{S}$$

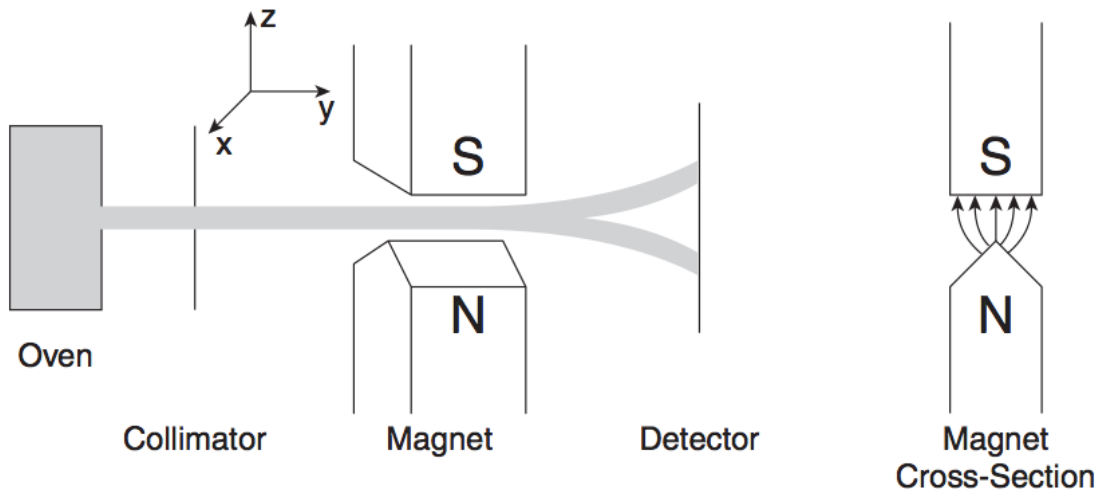


Figure 4.1: Stern-Gerlach experiment to measure the spin component of neutral particles along the z -axis. The magnet cross section at right shows the inhomogeneous field used in the experiment.

4.2 Stern Gerlach Experiments

This experiment was designed to measure the magnetic dipole moment of a particle (atom). A beam of atoms is passed through a magnetic field gradient and observations are made as to what happens to the trajectory.

So what are the physics in this experiment?

Potential energy of the magnetic dipole moment $\vec{\mu}$, in external field \vec{B}

$$E_{\text{magn}} = -\vec{\mu} \cdot \vec{B}$$

The force is negative of the gradient of the potential energy

$$\vec{F} = -\vec{\nabla}(-\vec{\mu} \cdot \vec{B})$$

In the Stern Gerlach experiment, the field gradient is in the z -direction, so only $\frac{dB_z}{dz} \neq 0$, so

$$\vec{F} = \mu_z \frac{dB_z}{dz} \hat{z} \quad (\hat{z} \text{ is unit vector in } z\text{-direction})$$

Atoms experience a force in the z -direction proportional to the z -component of magnetic dipole moment μ_z because we designed an experiment where only $\frac{dB_z}{dz} \neq 0$.

What is the classical expectation for silver atoms? ($47e^-$, 47 photons, 60/62 neutrons). Note that

$$-\mu_L \text{ or } \mu_S \propto \frac{1}{m}, \quad \text{so only consider electrons } (m_p \approx 2000m_e)$$

and there is only one non-closed (tell electron that contributes to angular momentum), it is in an s-shell ($\vec{L} = 0$), leaving only intrinsic angular momentum. For silver atoms,

$$\vec{\mu} = -g \frac{e}{2m_e} \vec{S} \quad (\text{with } g \approx 2)$$

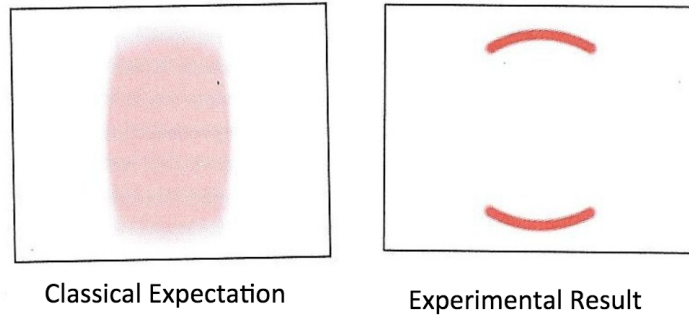


Figure 4.2: Space quantization as it appears in the experimental results of the Stern Gerlach experiment.

For a random gas of atoms, $\vec{\mu}$ is in all directions, so μ_z will have all possible values. So the force will range,

$$-\mu \frac{dB_z}{dz} \leq |\vec{F}| \leq +\mu \frac{dB_z}{dz}$$

which implies a circular beam spread in the z -direction.

It turns out that experimental results reveal that the beam is split into two. This is known as **space quantization**. This indicates that S_z has two possible values,

$$S_z = \pm \frac{\hbar}{2} \quad \left(\hbar = \frac{h}{2\pi} \right)$$

Splitting is associated with the field gradient $\frac{dB_z}{dz}$ since it can change direction and splitting tracks the direction of field gradient. The weird thing here is that **there is no bias to atom deflection**. There is a 50% deflection up rate and 50% deflection down rate. An individual atom is deflected in a probabilistic way. So there is no way of determining precisely what happens to an individual atom.

No what we'd like to do is strip down the experiment to the essentials and introduce language for additional study. First, there are **two possible outcomes**

$$S_z = +\frac{\hbar}{2} \text{ "spin up"} \quad S_z = -\frac{\hbar}{2} \text{ "spin down"}$$

Definition 4.1 (observable). The Quantum Mechanics term for the quantity being measured (S_z in this case.)

Definition 4.2 (analyser). Stern Gerlach device is some form of an analyser $(x, y, z, \theta, \hat{n})$

These are being called the essence of quantum mechanics, and there are a number of experiments involved, we're going to analyze each experiment, one by one.

Stern-Gerlach Experiment 1

In this experiment, no atoms are deflected down at the second analyzer. Also, each analyzer plays a different role; the first analyzer **prepared** the beam in a specific quantum state ($|+\rangle$) and so it is a **state preparation device**. The second analyzer **measures** the prepared beam.

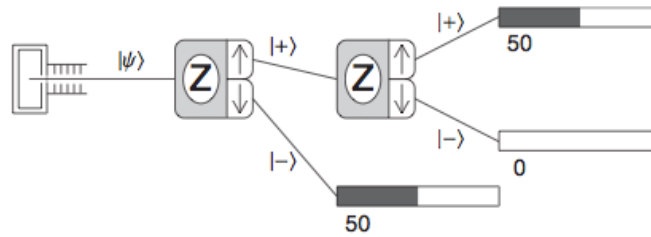


Figure 4.3: Experiment 1

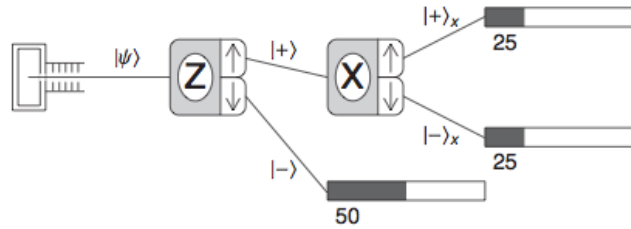


Figure 4.4: Experiment 2

Stern-Gerlach Experiment 2

The X analyzer has field gradient in the x -direction, (90° with respect to the z -direction). Also, atoms leaving spin-up / spin-down part of the X -analyzer have

$$S_x = +\frac{\hbar}{2} \quad / \quad S_x = -\frac{\hbar}{2}$$

For input beam $S_z = +\frac{\hbar}{2} [|+\rangle]$, then 50% are measured to have $S_x = +\frac{\hbar}{2}$. There is the same result for any two different X , Y , or Z analyser combinations.

Stern-Gerlach Experiment 3

Classically, we expect to be able to measure X , Y , and Z components and figure out the total spin direction. Experiment 3 shows this is not possible in quantum mechanics, as information is reset or lost when new measurements are made. This is a very generic and key feature of quantum mechanics and comes down to really the statement that **measurement disturbs the system**. By making a measurement, we change the information of the system. Every time we make a measurement, the system isn't what it was before you made the measurement.

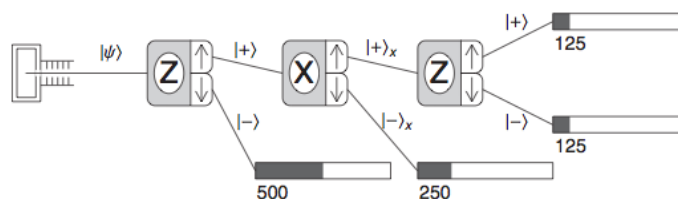


Figure 4.5: Experiment 3

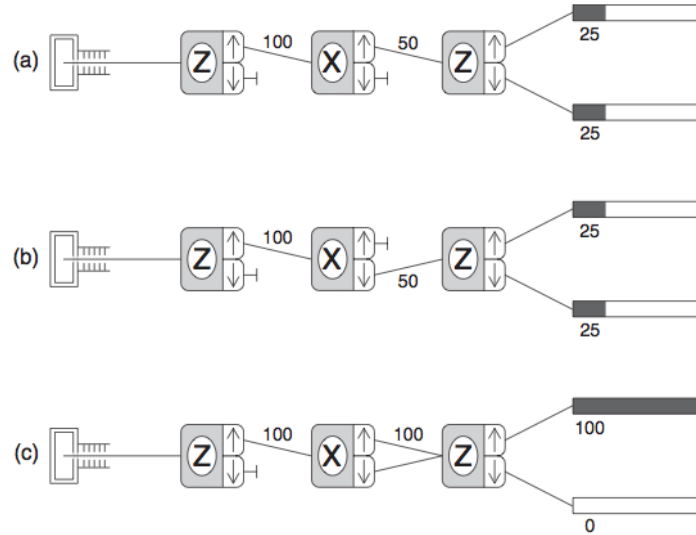


Figure 4.6: Experiment 4

This is a key feature of quantum mechanics. We cannot have simultaneous knowledge of more than one spin component; this is a fundamental incompatibility of knowing spin components along two or more directions. We can say that in quantum mechanics, S_x , S_y , S_z are **incompatible observables**. More specifically then, the state represented by $|+\rangle_z = |S_z = +\frac{\hbar}{2}\rangle$ or $|+\rangle_x = |S_x = +\frac{\hbar}{2}\rangle$ but **not** $|S_z = +\frac{\hbar}{2}, S_x = +\frac{\hbar}{2}\rangle$.

Stern-Gerlach Experiment 4

The first two demonstrate experiment 3 for both parts of the middle X analyzer independently. The third one gives a surprising result, allowing both parts into the Z analyzer simultaneously then it is as if the middle measurement had not occurred. This is reminiscent of interference; adding two outputs results in enhancement in one sector and reduction or cancellation in the other.

In summary,

- Experiment 1: State preparation. If we know the input state and choose an appropriate experiment, we remeasure the state with certainty.
- Experiment 2: Probabilistic nature of quantum measurement when the measurement is not matched to the input state.
- Experiment 3: Measurement disturbs the system leading to incompatible observables.
- Experiment 4: Quantum mechanical interference effects can be observed.

4.3 Quantum State Vectors

Definition 4.3 (Postulate 1). We label the input state with a left **ket**, ($|\Psi\rangle$), and label the output states with $|+\rangle$ for spin up and $|-\rangle$ for spin down.

Thus the ket, $|\Psi\rangle$ is part of a vector space called a **Hilbert Space**. The dimensionality of the Hilbert Space depends on the **observable**. For the S_z observable (z -component of angular momentum), it has two possible values,

$$S_z = \pm \frac{\hbar}{2}$$

Each value is associated with a state vector $|+\rangle, |-\rangle$ (note that no subscript will in general mean the z -direction, which is just our notation here), so the Hilbert Space is 2D.

- Much like $\hat{x}, \hat{y}, \hat{z}$ vectors span 3D geometric space, the kets $|+\rangle$ and $|-\rangle$ span the 2D Hilbert Space associated with observable S_z .
- They are **complete** (only 2 possible outcomes).
- They are **orthogonal**; the result is either spin up or spin down.
- They are also **normalised**; All Quantum state vectors can (or should be) normalized such that $\langle\Psi|\Psi\rangle = 1$.

Orthonormal properties are characterised mathematically as

$$\begin{aligned}\langle+|+\rangle &= 1 & \langle+|-\rangle &= 0 \\ \langle-|-\rangle &= 1 & \langle-|+\rangle &= 0\end{aligned}$$

Completeness ensures that $|+\rangle, |-\rangle$ can be used as a basis to express any general Quantum Mechanical state as a linear combination of them

$$\text{General State Vector} \quad |\Psi\rangle = a|+\rangle + b|-\rangle \quad \text{with } a, b \text{ complex scalars}$$

The coefficients are inner products, for example consider multiplying on the left by $|+\rangle$,

$$\begin{aligned}\langle+|\Psi\rangle &= \langle+|(a|+\rangle + b|-\rangle)\rangle \\ &= a\langle+|+\rangle + b\langle+|-\rangle \\ &= a(1) + b(0) \\ &= a\end{aligned}$$

Likewise, $\langle-|\Psi\rangle = b$. Therefore,

$$|\Psi\rangle = (\langle+|\Psi\rangle)|+\rangle + (\langle-|\Psi\rangle)|-\rangle$$

As an aside, the $\langle+|$ is the bra to the $|+\rangle$, so together $\langle+|+\rangle$ is a bra-ket. ☺.

The dual vector (bra) to the Quantum Mechanical ket $|\Psi\rangle$:

$$\begin{aligned}\langle\Psi| &= a^*\langle+| + b^*\langle-| \\ \langle\Psi|+\rangle &= a^*\langle+|+\rangle + b^*\langle-|+\rangle = a^*\end{aligned}$$

Hence,

$$\langle\Psi|+\rangle = a^* = (a)^* = (\langle+|\Psi\rangle)^* = \langle+|\Psi\rangle^*$$

Finally,

$$\langle\Psi| = (\langle\Psi|+\rangle)\langle+| + (\langle\Psi|-\rangle)\langle-|$$

In Quantum Mechanics we require that all kets (vectors) are normalized.

Example 4.1. Given a general quantum state vector expressed as a linear combination of the basis kets for the 2D Hilbert Space associated with the S_z observable:

$$|\Psi\rangle = a|+\rangle + b|-\rangle$$

Derive an expression for the coefficients, a and b , which when satisfied ensure that $|\Psi\rangle$ is normalized.

First we normalize $|\Psi\rangle$, so

$$\begin{aligned}\langle\Psi|\Psi\rangle &= 1 \\ &= (a^*\langle+| + b^*\langle-|)(a|+\rangle + b|-\rangle) \\ &= a^*a\langle+|+\rangle + a^*b\langle+|-\rangle + b^*a\langle-|+\rangle + b^*b\langle-|-\rangle \\ &= a^*a + b^*b \\ &= 1 \quad (\text{requires normalization}) \\ &= |a|^2 + |b|^2\end{aligned}$$

or since

$$\begin{aligned}a &= \langle+|\Psi\rangle & a^* &= \langle\Psi|+\rangle \\ b &= \langle-|\Psi\rangle & b^* &= \langle\Psi|-\rangle\end{aligned}$$

which implies

$$|\langle+|\Psi\rangle|^2 + |\langle-|\Psi\rangle|^2 = 1$$

Definition 4.4 (Postulate 4). The probability of obtaining the value $\pm\frac{\hbar}{2}$ in a measurement of the observable S_z on a system in the state $|\Psi\rangle$ is

$$P_{\pm} = |\langle\pm|\Psi\rangle|^2$$

where $|\pm\rangle$ is the basis ket of S_z corresponding to the result $\pm\frac{\hbar}{2}$.

- (i) $\langle+|\Psi\rangle$ is the **Probability Amplitude**, it must be "squared" (multiply by the complex conjugate) to get a probability.
- (ii) Convention is usually to put order of the inner product as $\langle\text{out}|\text{in}\rangle$ but since

$$P = |\langle\text{out}|\text{in}\rangle|^2 = \langle\text{out}|\text{in}\rangle \langle\text{in}|\text{out}\rangle$$

it doesn't really matter.

$|\text{in}\rangle$ is the input state, and $|\text{out}\rangle$ is the output state, whose probability for measurement we are calculating.

Example 4.2. A Z-analyzer is used to prepare atoms in the state "spin-up". The spin state if these atoms is then measured using a 2nd Z-analyzer. What is the probability that these atoms are measured as "spin-up" and "spin-down"?

$$P = |\langle\text{out}|\text{in}\rangle|^2 \quad \text{Postulate 4}$$

The first analyzer prepares the input state to the 2nd analyzer : $|\text{in}\rangle = |+\rangle$. The probability when $|\text{out}\rangle = |+\rangle$ implies that

$$P_+ = |\langle+|+\rangle|^2 = 1$$

then similiary for when $|\text{out}\rangle = |-\rangle$ we get

$$P_- = |\langle -|+\rangle|^2 = 0$$

This agrees perfectly with our observations for Stern Gerlach Experiment 1.

Quantum State Tomography is a way of determining the quantum state based on results of measurement and is the name of this method.

With regards to Stern Gerlach Experiment 2, let's apply this method:

The input state is $|+\rangle$ and we measure S_x with output states $|+\rangle_x$ with probability $\frac{1}{2}$ and $|-\rangle_x$ with probability $\frac{1}{2}$ (giving a sum probability of 1). Next we formulate mathematically using [Postulate 4](#) :

$$P = |\langle \text{out} | \text{in} \rangle|^2$$

So,

$$|\text{in}\rangle = |+\rangle \implies P_{+x} = |{}_x \langle + | + \rangle|^2 = \frac{1}{2} \quad (1)$$

$$P_{-x} = |{}_x \langle - | + \rangle|^2 = \frac{1}{2} \quad (2)$$

and the result should be the same if $|\text{in}\rangle = |-\rangle$ (Experiment 4b)

$$|\text{in}\rangle = |-\rangle \implies P_{+x} = |{}_x \langle + | - \rangle|^2 = \frac{1}{2} \quad (3)$$

$$P_{-x} = |{}_x \langle - | - \rangle|^2 = \frac{1}{2} \quad (4)$$

Since $|+\rangle$ and $|-\rangle$ are basis states that SPAN the 2D Hilbert Space then kets for outputs of other Stern Gerlach measurements can be expressed as linear combinations of them. For example, S_x output states

$$\begin{aligned} |+\rangle_x &= a |+\rangle + b |-\rangle \\ |-\rangle_x &= c |+\rangle + d |-\rangle \end{aligned}$$

for $a, b, c, d \in \mathbb{C}$. Now,

$$\begin{aligned} P_{+x} &= |{}_x \langle + | + \rangle|^2 = |(a^* \langle + | + b^* \langle - |) | + \rangle|^2 \\ &= |a^* \langle + | + b^* \langle - | + \rangle|^2 \\ &= |a|^2 \\ &= \frac{1}{2} \quad (\text{from experiment}) \end{aligned}$$

Similiary we can use (2), (3), and (4) to show that $|b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$.

The coefficients a, b, c, d are complex numbers implying that the amplitude of the phase is

$$re^{i\theta}$$

Note. The overall phase (global phase) of a quantum state vector is not physically meaningful - this means it doesn't affect the computation of probabilities (Assignment 3).

Only the relative phase between components of a ket (vector) is important, that is between a and b or c and d . This means that we chose one component to be real ($\theta = 0$) and one complex ($\theta \neq 0$).

Say $a = r_1 e^{i\theta_1}$ and $b = r_2 e^{i\theta_2}$, then only $\theta_2 - \theta_1$ matters for quantum mechanics so we rotate the vectors to make one real.

This means we can solve for a , b , c , and d in the following way

$$\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha} |-\rangle \\ |-\rangle_x &= \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\beta} |-\rangle \end{aligned}$$

We still need to determine the phases α and β . We can check,

$$P_{+x} = |\langle \text{out} | \text{in} \rangle|^2 = \left| \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\alpha} |-\rangle \right) |+\rangle \right|^2 = \frac{1}{2}$$

If we consider experiment 1 with an X -analyser, we would establish orthonormal properties of $|+\rangle_x$ and $|-\rangle_x$. That is,

$$\begin{aligned} {}_x \langle + | + \rangle_x &= {}_x \langle - | - \rangle_x = 1 \quad (\text{normalized}) \\ {}_x \langle + | - \rangle_x &= {}_x \langle - | + \rangle_x = 0 \quad (\text{orthogonal}) \end{aligned}$$

Also, orthogonality shows

$$\begin{aligned} {}_x \langle + | + \rangle_x &= \frac{1}{\sqrt{2}} (|+\rangle + e^{-i\beta} |-\rangle) \cdot \frac{1}{\sqrt{2}} (|+\rangle + e^{i\alpha} |-\rangle) = 0 \\ &= \frac{1}{2} \left(\langle + | + \rangle \xrightarrow{1} 1 + e^{-i\beta} \langle + | - \rangle \xrightarrow{0} 0 + e^{-i\alpha} \langle - | + \rangle \xrightarrow{0} 0 + e^{i(\alpha-\beta)} \langle - | - \rangle \xrightarrow{1} 1 \right) = 0 \\ &\implies \frac{1}{2} (1 + e^{i(\alpha-\beta)}) = 0 \\ &\implies e^{i\alpha} = -e^{i\beta} \end{aligned}$$

This is all the information we have to determine α and β . So, we choose $\alpha = 0$, so $e^{i\alpha}$ and $e^{i\beta} = -1$. Thus,

$$\begin{aligned} |+\rangle_x &= \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle) \\ |-\rangle_x &= \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle) \end{aligned}$$

Similar analysis leads to,

$$|+\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}(|+\rangle - i|-\rangle)$$

4.4 Matrix Notation

If an ordered set of vectors is chosen as a basis, then we can express a general state as a linear combination of them.

$$|\Psi\rangle = a|+\rangle + b|-\rangle$$

or as an ordered array of coefficients a, b in co-ordinate representation.

$$|+\rangle \xrightarrow[\text{basis}]{S_z} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \langle +|\Psi\rangle \\ \langle -|\Psi\rangle \end{pmatrix} \quad (\text{column vector})$$

$$|\Psi\rangle = |+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\Psi\rangle = |-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}|-\rangle \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Note. Basis vectors are unit vectors when expressed in co-ordinate representation with respect to their own basis. For example,

$$|\Psi\rangle = |+\rangle = 1|+\rangle + 0|-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

For the dual space,

$$|\Psi\rangle = a^* \langle +| + b^* \langle -| \xrightarrow[\text{basis}]{S_z} (a^*, b^*) \quad \text{row vector}$$

remember that the bra is the conjugate transpose of ket. So for the inner product,

$$\langle \Psi|\Psi\rangle = (a^* \ b^*) \begin{pmatrix} a \\ b \end{pmatrix} = |a|^2 + |b|^2$$

Let's revisit $|\pm\rangle_y$ in Matrix Notation, first consider experiment 2, but with X , replaced by Y (the probabilities measured will be the same).

Then, if $|\text{in}\rangle = |+\rangle$,

$$Prob_{+y} = |{}_y\langle +|+\rangle|^2 = \frac{1}{2}$$

$$Prob_{-y} = |{}_y\langle -|+\rangle|^2 = \frac{1}{2}$$

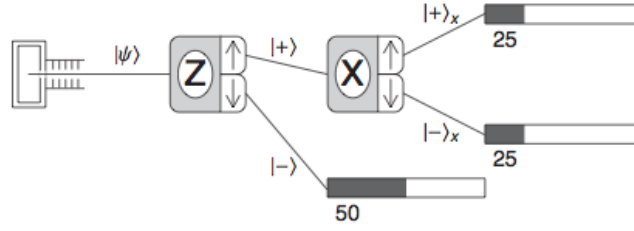


Figure 4.7: Experiment 2

otherwise if $|\text{in}\rangle = |-\rangle$,

$$\begin{aligned} \text{Prob}_{+y} &= |{}_y\langle + | - \rangle|^2 = \frac{1}{2} \\ \text{Prob}_{-y} &= |{}_y\langle - | - \rangle|^2 = \frac{1}{2} \end{aligned}$$

Then if we use $|+\rangle_y = r|+\rangle + s|-\rangle$ and $|-\rangle_y = t|+\rangle + u|-\rangle$ and orthonormality of $|\pm\rangle_y$ to get

$$\begin{aligned} |+\rangle_y &= \frac{1}{\sqrt{2}} (|+\rangle + e^{i\theta} |-\rangle) \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \\ |-\rangle_y &= \frac{1}{\sqrt{2}} (|+\rangle - e^{i\theta} |-\rangle) \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -e^{i\theta} \end{pmatrix} \end{aligned}$$

To determine the phase angle θ we need to consider Experiment 2 but with state preparation $|\pm\rangle_x$ (not $|\pm\rangle$, (z-analyzer))

$$|\text{in}\rangle = |+\rangle_x \quad \text{Prob}_{+y} = |{}_y\langle + | + \rangle_x|^2 = \frac{1}{2}$$

compute the inner product using Matrix notation

$$\begin{aligned} {}_y\langle + | + \rangle_x &= \frac{1}{\sqrt{2}} (1 \quad e^{-i\theta}) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} (1 + e^{-i\theta}) \\ |{}_y\langle + | + \rangle_x|^2 &= \frac{1}{2} (1 + e^{-i\theta}) \frac{1}{2} (1 + e^{i\theta}) = \frac{1}{4} (1 + e^{-i\theta} + e^{i\theta} + 1) \end{aligned}$$

then using Euler,

$$|{}_y\langle + | + \rangle_x|^2 = \frac{1}{2} (1 + \cos \theta) = \frac{1}{2} \quad \leftarrow \text{from experiment}$$

which implies $\cos \theta = 0$ and so $\theta = \pm \frac{\pi}{2}$. So the two possible outcomes correspond to RH or LH orientation of Y with respect to X and Z . Let's choose the RH solution $\theta = \frac{\pi}{2}$ then

$$|+\rangle_y \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle_y \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

4.5 General Quantum Systems

Consider a measurement of observable A , which yields results a_1, a_2, \dots, a_n (discrete and finite). The states associated with each outcome are described by kets, $|a_1\rangle, |a_2\rangle, \dots, |a_n\rangle$. These kets are orthogonal (in the sense

that any single measurement on a single particle yields only 1 outcome).

$$\text{Mathematically, } \langle a_i | a_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

The set of kets is complete (they span the Hilbert Space associated with observable A). They can be used to express any arbitrary state as a linear combination.

$$|\Psi\rangle = \alpha_1 |a_1\rangle + \alpha_2 |a_2\rangle + \cdots + \alpha_n |a_n\rangle$$

with $\alpha_i = \langle a_i | \Psi \rangle$. So, given $|\Psi\rangle$ as an input state to a measurement of observable A with results a_i and associates states $|a_i\rangle$, the probability for obtaining result a_i is given by

$$Prob_{(a_i)} = |\langle a_i | \Psi \rangle|^2 \quad [\text{Postulate 4}]$$

Example 4.3. Consider a quantum system with an observable A that has three possible measurement outcomes, with values a_1, a_2 and a_3 . The quantum state associated with each of these outcomes is described by the three orthonormal kets:

$$|a_1\rangle, |a_2\rangle, |a_3\rangle$$

A system is prepared in the state that is a superposition of the three basis kets:

$$|\Psi\rangle = 2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle$$

Calculate the probability for each of the possible outcomes if a measurement of A is made on the state.

So,

$$|\text{in}\rangle = |\Psi\rangle = 2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle \quad \text{with } |a_1\rangle, |a_2\rangle, |a_3\rangle \text{ orthogonal, normalised basis set}$$

First we normalize the state vector. Let $|\Psi\rangle = C(2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle)$, then

$$\begin{aligned} 1 &= \langle \Psi | \Psi \rangle \\ &= C^* (2\langle a_1| - 3\langle a_2| - 4i\langle a_3|) C (2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle) \\ &= |C|^2 (4\langle a_1|a_1\rangle + 9\langle a_2|a_2\rangle + 16\langle a_3|a_3\rangle) \\ |C|^2 &= \frac{1}{29} \end{aligned}$$

So, $C = \frac{1}{\sqrt{29}}$. Note that I skipped the full expansion, but remember that mismatched bra and ket pairs have inner product 0, and same pairs have inner product 1. Okay then, let's get the probabilities.

$$\begin{aligned}
 Prob_{a_1} &= |\langle \text{out} | \text{in} \rangle|^2 = Prob_{a_1} = |\langle a_1 | \Psi \rangle|^2 \\
 &= \left| \left\langle a_1 \left| \frac{1}{\sqrt{29}} (2|a_1\rangle - 3|a_2\rangle + 4i|a_3\rangle) \right. \right\rangle \right|^2 \\
 &= \left| \frac{1}{\sqrt{29}} \langle a_1 | (2\langle a_1 | a_1 \rangle - 3\langle a_1 | a_2 \rangle + 4i\langle a_1 | a_3 \rangle) \right\rangle \right|^2 \\
 &= \left| \frac{2}{\sqrt{29}} \right|^2 \\
 &= \frac{4}{29}
 \end{aligned}$$

Likewise, $Prob_{a_2} = \frac{9}{29}$, $Prob_{a_3} = \frac{16}{29}$ (check $Prob_{a_1} + Prob_{a_2} + Prob_{a_3} = 1$)

4.6 Quantum Mechanical Operators and Measurement

The goal is to be able to make prediction about measurements that haven't been done yet.

Operators, Eigenvalues, & Eigenvectors

Definition 4.5 (Postulate 2). A physical observable is represented mathematically by an operator A that acts on kets.

$$A|\Psi\rangle = \phi$$

where A is an operator that represents a physical observable.

For each operator there are "special" (eigen) kets that are not transformed by the operator except for being multiplied by a scalar constant, which has no measureable effect on the state (we will normalize anyway). So,

Eigenvector = unchanged ket (eigenstate, eigenket)

Eigenvalue = multiplicative constant

So, the Eigen-equation is

$$A|\Psi\rangle = a|\Psi\rangle$$

where A is an **operator**, a is an **eigenvalue**, and $|\Psi\rangle$ is an **eigenstate**.

Definition 4.6 (Postulate 3). The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator A . Eigenvalues are the outputs of measurements.

Example 4.4. Define operator S_z associated with measurement of the observable that in the z -component of intrinsic angular momentum.

$$\begin{aligned}
 S_z|+\rangle &= +\frac{\hbar}{2}|+\rangle \\
 S_z|-\rangle &= -\frac{\hbar}{2}|-\rangle
 \end{aligned}$$

We are using Abstract notation here. Note that the eigenvalues are $\pm\frac{\hbar}{2}$ and the eigenvectors of S_z are $|\pm\rangle$.

Hermitian Operators

Operators used in Quantum Mechanics are Hermitian operators. This means that the operator is equal to its adjoint (conjugate transpose), that is

$$A = A^\dagger$$

Why Hermitian?

- (i) Eigenvalues of Hermitian operators are real. (Quantum Mechanics interpretation is that they are results of measurements, to they must be real numbers (Energy, position, component of span))
- (ii) Eigenvectors of Hermitian operators form a complete set of basis vectors.
- (iii) Same operator for the dual space vectors. So,

$$A|\alpha\rangle = |\beta\rangle \quad \text{operator } A \text{ acts to right on ket } |\alpha\rangle$$

$$\langle\alpha|B = \langle\gamma| \quad \text{operator } B \text{ acts to left on bra } |\alpha\rangle$$

For $\langle\gamma| = |\beta\rangle$, then $B = A^\dagger$. So if $A = A^\dagger$ then the operator can act to the left or the right to give appropriate dual space relationship. Note that the dual space is transformed in the same way as original space.

$$A|\alpha\rangle = |\beta\rangle$$

$$\langle\alpha|A = \langle\beta|$$

Completeness Relationship

We have already seen operators, but didn't realize it.

$$|\Psi\rangle = a|+\rangle + b|-\rangle \quad \text{with } a = \langle+|\Psi\rangle, b = \langle-|\Psi\rangle$$

$$\begin{aligned} |\Psi\rangle &= \langle+|\Psi\rangle|+\rangle + \langle-|\Psi\rangle|-\rangle \\ &= |+\rangle\langle+|\Psi\rangle + |-\rangle\langle-|\Psi\rangle \\ &= (|+\rangle\langle+|)|\Psi\rangle + (|-\rangle\langle-|)|\Psi\rangle \\ &= (|+\rangle\langle+| + |-\rangle\langle-|)|\Psi\rangle \end{aligned}$$

Hence $|+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$, the identity operator. This is known as the completeness relation. Note that $|+\rangle\langle+|$ and $|-\rangle\langle-|$ are projection operators, and are an example of an outer product.

More generally for any orthonormal basis, $|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle$,

$$\sum_{i=1}^n |e_i\rangle\langle e_i| = \mathbb{1} \quad \text{identity}$$

Spectral Decomposition

Clearly there exists a very close relationship between an operator, its eigenvalues, and its eigenvectors. In general, any operator is related to its eigenvectors and eigenvalues by

$$\text{Operator } A = \sum_i a_i |a_i\rangle\langle a_i|$$

where a_i is an eigenvalue and $\langle a_i|$ and $|a_i\rangle$ are eigenvectors (eigenkets).

Example 4.5. Use the eigen-equations for the S_z operator to verify the spectral decomposition relationship.

$$S_z |+\rangle = +\frac{\hbar}{2} |+\rangle, \quad S_z |-\rangle = -\frac{\hbar}{2} |-\rangle \quad \text{are eigen-equations}$$

So we multiply by the appropriate bra,

$$S_z |+\rangle \langle +| = +\frac{\hbar}{2} |+\rangle \langle +|, \quad S_z |-\rangle \langle -| = -\frac{\hbar}{2} |-\rangle \langle -|$$

Add the two equations together and factorize

$$\begin{aligned} S_z |+\rangle \langle +| + S_z |-\rangle \langle -| &= +\frac{\hbar}{2} |+\rangle \langle +| + \frac{-\hbar}{2} |-\rangle \langle -| \\ S_z (|+\rangle \langle +| + |-\rangle \langle -|) &= +\frac{\hbar}{2} |+\rangle \langle +| + \frac{-\hbar}{2} |-\rangle \langle -| \\ S_z &= +\frac{\hbar}{2} |+\rangle \langle +| + \frac{-\hbar}{2} |-\rangle \langle -| \end{aligned}$$

4.7 More on Matrix Notation

Kets are 2D vectors (defined by Hilbert Space for Stern Gerlach experiments), and operators must be a 2×2 matrix. Matrix elements are defined (according to rules of linear algebra) as follows:

$$S_z \xrightarrow{\text{basis}} \begin{pmatrix} \langle +| S_z |+\rangle & \langle +| S_z |-\rangle \\ \langle -| S_z |+\rangle & \langle -| S_z |-\rangle \end{pmatrix} = \begin{pmatrix} \langle +| \frac{+\hbar}{2} |+\rangle & \langle +| \frac{-\hbar}{2} |-\rangle \\ \langle -| \frac{+\hbar}{2} |+\rangle & \langle -| \frac{-\hbar}{2} |-\rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We can check the math for our eigen-equations

$$S_z \xrightarrow{\text{basis}} \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad |+\rangle \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |-\rangle \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We want to see $S_z |+\rangle$,

$$S_z |+\rangle = +\frac{\hbar}{2} |+\rangle \xrightarrow{\text{basis}} +\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Note. Operators are always diagonal in their own basis and eigenvectors are always unit vectors in their own basis.

4.8 Expectation Values

Quantum Mechanics allows us to compute probabilities for outcomes of measurements. Another useful quantity is the mean average value of many repeated measurements on the same initial system. This is known as the expectation value in Quantum Mechanics. Consider observable X with i eigenvalues x_i (measured elements).

$$\text{Expectation Value } \langle x \rangle = \sum_i x_i \text{prob}(x_i)$$

For z -component of angular momentum, characterised by S_z operator with eigenvalues $+\frac{\hbar}{2}$,

$$\langle S_z \rangle = +\frac{\hbar}{2} \text{prob}_+ + \frac{-\hbar}{2} \text{prob}_- \quad (1)$$

Given an input state $|\Psi\rangle$,

$$\begin{aligned}\langle S_z \rangle &= +\frac{\hbar}{2} |\langle +|\Psi \rangle|^2 + \frac{-\hbar}{2} |\langle -|\Psi \rangle|^2 \\ &= +\frac{\hbar}{2} (\langle \Psi|+ \rangle \langle +|\Psi \rangle) + \frac{-\hbar}{2} (\langle \Psi|- \rangle \langle -|\Psi \rangle) \\ &= \langle \Psi| \left(\underbrace{+\frac{\hbar}{2} |+\rangle \langle +| + \frac{-\hbar}{2} |-\rangle \langle -|}_{\text{spectral decomposition for } S_z} \right) |\Psi \rangle\end{aligned}$$

Then,

$$\langle S_z \rangle = \langle \Psi | S_z | \Psi \rangle \quad (2)$$

The expression (2) is completely general: given an operator Λ representing a Quantum Mechanical observable with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, and eigenkets $|\lambda_1\rangle, |\lambda_2\rangle, \dots, |\lambda_n\rangle$ then Λ can be expressed as

$$\Lambda = \sum_{i=1}^n \lambda_i |\lambda_i\rangle \langle \lambda_i| \quad \text{spectral decomposition}$$

If a measurement of Λ is made on an arbitrary input state $|+\rangle$, the expectation value of many repeated identical measurements will be

$$\langle \Lambda \rangle = \langle \Psi | \Lambda | \Psi \rangle \quad \text{Expectation value}$$

Definition 4.7 (trace). We are going to introduce a linear operation that assigns a number to each matrix.

$$\text{Tr} \{ |1\rangle \langle 2| \} = \langle 2|1 \rangle$$

Example 4.6.

$$|1\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad |2\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \langle 2|1 \rangle = (b_1^* \quad b_2^*) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = a_1 b_1^* + a_2 b_2^*$$

Also

$$|1\rangle \langle 2| = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (b_1^* \quad b_2^*) = \begin{pmatrix} a_1 b_1^* & a_1 b_2^* \\ a_2 b_1^* & a_2 b_2^* \end{pmatrix}$$

so

$$\text{Tr} \{ |1\rangle \langle 2| \} = a_1 b_2^* + a_2 b_1^*$$

Consider an arbitrary operator X and completeness relation for a set of basis kets

$$|+\rangle \langle +| + |-\rangle \langle -| = \mathbb{1}$$

then

$$\begin{aligned}\text{Tr} \{ X \} &= \text{Tr} \{ \mathbb{1} X \} = \text{Tr} \{ |+\rangle \langle +| + |-\rangle \langle -| X \} \\ &= \text{Tr} \{ |+\rangle (\langle +| X) + |-\rangle (\langle -| X) \} \\ &= \text{Tr} \{ |+\rangle (\langle +| X) \} + \text{Tr} \{ |-\rangle (\langle -| X) \} \\ &= \langle +| X |+\rangle + \langle -| X |-\rangle\end{aligned}$$

which is the sum of the diagonal elements of X when written in co-ordinate representation with respect to basis $|+\rangle$

and $|-\rangle$.

$$X = \begin{pmatrix} \langle + | X | + \rangle & \langle + | X | - \rangle \\ \langle - | X | + \rangle & \langle - | X | - \rangle \end{pmatrix}$$

Properties of Trace

- (i) $\text{Tr}\{X + Y\} = \text{Tr}\{X\} + \text{Tr}\{Y\}$
- (ii) $\text{Tr}\{\lambda X\} = \lambda \text{Tr}\{X\}$
- (iii) $\text{Tr}\{XY\} = \text{Tr}\{YX\}$
- (iv) $\text{Tr}\{XYZ\} = \text{Tr}\{YZX\} = \text{Tr}\{ZXY\}$

In regard to the expectation value for a Quantum Mechanical measurement for operator X and input state Ψ ,

$$\langle X \rangle = \langle \Psi | X | \Psi \rangle = \langle \Psi | (X | \Psi \rangle) = \text{Tr}\{(X | \Psi \rangle) \langle \Psi | \} = \text{Tr}\{X | \Psi \rangle \langle \Psi | \}$$

Separate the properties of measured system (atoms) from Observable X .

4.9 Stastical Operator / Density Matrix

Mixed state or Pure State (superposition state)?

So far we have discussed sources (input states) that are composed of a single type of atom - this may be a superposition state (pure state) like

$$|\Psi\rangle = a|+\rangle + b|-\rangle \quad \text{e.g.,} \quad |+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

How do we describe a source that is, say, a 50% mixture of atoms in $|+\rangle$ state and 50% in $|-\rangle$ state - at first glance this may appear similar to $|+\rangle_x$, but think about S_x .

Consider a mixture of $P_1|+\rangle_x$ atoms and $P_2|+\rangle_z$ atoms, ($P_1 + P_2 = 1$) and an arbitrary Stern Gerlach measurement S_n . The overall expectation value $\langle S_n \rangle$ is the weighted sum of individual expectation values.

$$\langle S_n \rangle = P_1 \langle S_n \rangle_x + P_2 \langle S_n \rangle_z$$

with

$$\langle S_n \rangle_x = {}_x \langle + | S_n | + \rangle_x = \text{Tr}\{S_n |+\rangle_x \langle +|\}$$

and

$$\langle S_n \rangle_z = {}_z \langle + | S_n | + \rangle_z = \text{Tr}\{S_n |+\rangle_z \langle +|\}$$

Thus

$$\begin{aligned} \langle S_n \rangle &= P_1 \text{Tr}\{S_n |+\rangle_x \langle +|\} + P_2 \text{Tr}\{S_n |+\rangle_z \langle +|\} \\ &= \text{Tr}\left\{ S_n \underbrace{(P_1 |+\rangle_x \langle +| + P_2 |+\rangle_z \langle +|)}_{\text{Statistical Operator}} \right\} \end{aligned}$$

Definition 4.8 (Statistical Operator (Density Matrix)).

$$\rho = P_1 |+\rangle_x \langle +| + P_2 |+\rangle_z \langle +|$$

the weighted sum of outer products of source atoms.

$$\langle S_n \rangle = \text{Tr} \{ S_n \rho \}$$

In general,

$$\rho = \sum_i P_i |x_i\rangle \langle x_i| \quad \text{with} \quad \sum_i P_i = 1$$

If there is only one type of atom, it is a pure state (e.g., $|\Psi\rangle$) and the statistical operator is $\rho = |\Psi\rangle \langle \Psi|$.

Example 4.7. (i) Use spectral decomposition and the expressions for $|+\rangle_x$, $|-\rangle_x$ in co-ordinate representation with respect to the S_z basis to find S_x as a matrix, also in the S_z basis.

The spectral decomposition for S_x is

$$S_x = +\frac{\hbar}{2} |+\rangle_x \langle +| + \frac{-\hbar}{2} |-\rangle_x \langle -|$$

and

$$|+\rangle_x \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |-\rangle_x \xrightarrow[\text{basis}]{S_z} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

then

$$\begin{aligned} S_x \xrightarrow[\text{basis}]{S_z} & \frac{+\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{-\hbar}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} \\ & = \frac{+\hbar}{2} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{-\hbar}{2} \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ & = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

(ii) A source of atoms is prepared as 20% $|+\rangle$ and 80% $|-\rangle_y$ and a measurement of the x -component of angular momentum is performed. What is the expectation value for this experiment?

$$\langle S_z \rangle = \text{Tr} \{ S_x \rho \}$$

and

$$\rho = \sum_i P_i |\lambda_i\rangle \langle \lambda_i| = \frac{20}{100} |+\rangle \langle -| + \frac{80}{100} |-\rangle_y \langle -|$$

which in the z -basis is

$$\begin{aligned} \rho \xrightarrow[\text{basis}]{S_z} & \frac{20}{100} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{80}{100} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \end{pmatrix} \\ & = \frac{1}{5} \begin{pmatrix} 3 & 2i \\ -2i & 2 \end{pmatrix} \end{aligned}$$

So,

$$\langle S_x \rangle = \text{Tr} \left\{ \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 3 & 2i \\ -2i & 2 \end{pmatrix} \right\} = \frac{\hbar}{10} \text{Tr} \left\{ \begin{pmatrix} 2i & 3 \\ 2 & -2i \end{pmatrix} \right\} = 0$$

5 Tutorials

5.1 Tutorial 1

1. Given two vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 6 \\ 4 \\ 1 \end{pmatrix}$$

- (a) Compute the inner product $\vec{a}^T \vec{b}$

$$1 \cdot 6 + 2 \cdot 4 + 4 \cdot 1 = 12$$

- (b) Compute the outer product $\vec{a} \vec{b}^T$

$$6 \cdot 1 + 4 \cdot 2 + 1 \cdot 4 = 12$$

2. Given two matrices

$$A = \begin{pmatrix} 1 & 9 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 \\ 5 & 6 \end{pmatrix}$$

- (a) Compute the product AB

$$\begin{pmatrix} 1 \cdot 0 + 9 \cdot 5 & 1 \cdot 2 + 9 \cdot 6 \\ 4 \cdot 0 + 3 \cdot 5 & 4 \cdot 2 + 3 \cdot 6 \end{pmatrix} = \begin{pmatrix} 45 & 56 \\ 15 & 20 \end{pmatrix}$$

- (b) Do these matrices commute?

Let's check

$$\begin{pmatrix} 0 \cdot 1 + 2 \cdot 4 & 0 \cdot 9 + 2 \cdot 3 \\ 5 \cdot 1 + 6 \cdot 4 & 5 \cdot 9 + 6 \cdot 3 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 29 & 63 \end{pmatrix}$$

No.

3. Given the matrix

$$C = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

- (a) Compute the eigenvalues of C The characteristic polynomial is

$$C(\lambda) = \det(C - \lambda I) = \det \begin{pmatrix} 7 - \lambda & 2 \\ 2 & 4 - \lambda \end{pmatrix} = (7 - \lambda)(4 - \lambda) - 4 = 24 - 11\lambda + \lambda^2 \implies \lambda = 8, 3$$

- (b) Compute the eigenvectors of C

$$C - 8I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \equiv \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \implies \vec{v} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad s, t \in \mathbb{R}$$

$$C - 3I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \implies \vec{v} = t \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad s, t \in \mathbb{R}$$

(c) Are the eigenvectors orthogonal?

Yes.

(d) Normalize the eigenvectors.

$$\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

4. Given the complex number

$$z = 3 + 3i$$

(a) Compute the complex conjugate of z (denoted \bar{z})

$$\bar{z} = 3 - 3i$$

(b) Compute the norm of z (denoted $|z|$)

$$|z| = \sqrt{z \cdot \bar{z}} = \sqrt{18}$$

(c) Express z as $z = re^{i\theta}$.

$$z = (\sqrt{18})e^{i \arctan(\frac{3}{3})} = \sqrt{18}e^{\frac{\pi i}{2}}$$

(d) Express z as $z = r(\cos\theta + i \sin\theta)$

$$z = \sqrt{18} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)$$

5.2 Tutorial 2

Tutorial 2 concerns coin flipping, the SPINS program, and connections to Quantum Theory. We played games.

5.3 Tutorial 3

We are given a matrix $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so first we find the eigenvalues:

$$(1 - \lambda)(-1 - \lambda) = 0 \implies \lambda = \pm 1$$

Then for $\lambda_1 = 1$ we get

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \implies \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and $\lambda_2 = -1$ which corresponds to $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Now we want to determine if this matrix is a reflection and if so on

what line; well actually this is pretty straightforward because you can see that for any vector, $M_2 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$ which means it is a vertical reflection (over the x -axis).

Also a little bit on Group Theory.

5.4 Tutorial 4

On the theory of ideal finite-dimensional Quantum Mechanics and for an application, a Numerical QST Exercise. Axioms for Finite-Dimensional Quantum Mechanics

- Axiom 0. Systems Exist
- Axiom 1. For each preparation device \mathcal{P} , there is an associated $|\Psi\rangle \in \mathbb{C}^d : \langle\Psi|\Psi\rangle = 1$.
- Axiom 2. For each transformation channel \mathcal{T} , there is an associated $u \in U(\mathbb{C}^d), |\Psi\rangle \rightarrow u|\Psi\rangle$
- Axiom 3. For each measurement device \mathcal{M} , there is an associated Hermitian A with eigenvalues λ_r and eigenkets $|\xi_r\rangle$ such that for input $|\phi\rangle$,

$$pr(\lambda_r) = |\langle\phi|\xi_r\rangle|^2$$

This is The Born Rule.

Note. $uu^\top = \mathbb{1}$ implies unitary, and $A = A^\top$ implies hermitian.

Example 5.1. Suppose that we have some preparation device with four possible states $\Psi_1, \Psi_2, \Psi_3, \Psi_4$, then we can describe Ψ_1 as

$$|\Psi_1\rangle = a|+_z\rangle + be^{i\theta}|-_z\rangle$$

for $a, b \in \mathbb{R}_+$, and $\theta \in [0, 2\pi]$. Then,

$$pr(+_z) = |\langle\Psi|+_z\rangle|^2 = a^2$$

and remember that $a^2 + b^2 = 1$. Also,

$$\begin{aligned} pr(+_x) &= |\langle\Psi|+_x\rangle|^2 \\ &= \left| \left(a \langle+_z| + be^{-i\theta} \langle-_z| \right) \left(\frac{|+_z\rangle + |-_z\rangle}{\sqrt{2}} \right) \right|^2 \\ &= \frac{1}{2} |a + be^{-i\theta}|^2 \\ &= \frac{1}{2} (a^2 + b^2 + ab(e^{-i\theta} + e^{i\theta})) \\ pr(+_x) &= \frac{1}{2} (1 + 2ab \cos \theta) \end{aligned}$$

Using this framework we can find a_r , b_r , and θ_r from experimental results (finding $pr(+_x)$, $pr(+_y)$, $pr(+_z)$ for each Ψ_r)

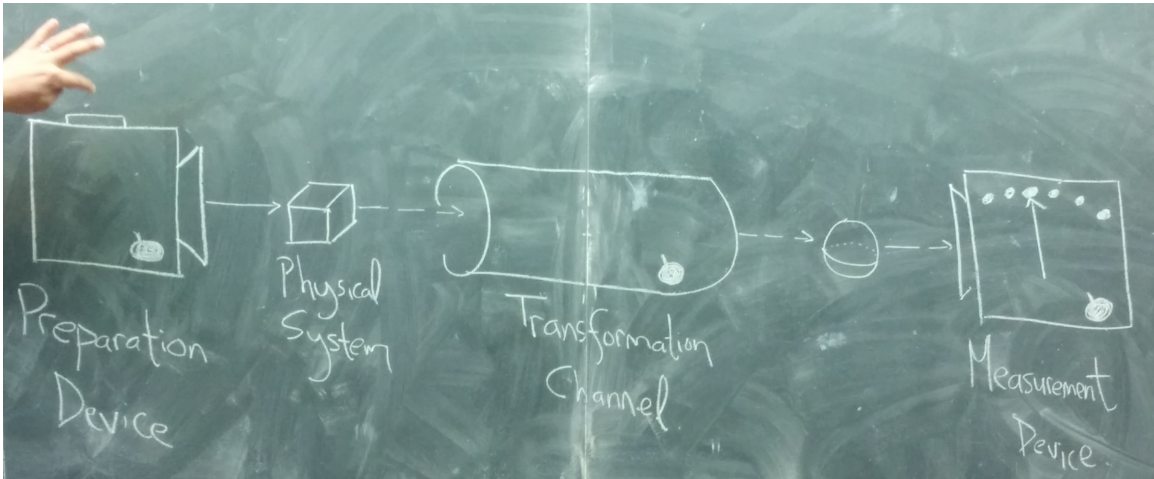


Figure 5.1: "Any physical experiment, classical, quantum, or other, can be viewed as the type of experiment described here." - L. Hardy (2001)

Example 5.2. Prove that $|\langle \Psi | \phi \rangle|^2 = |\langle \tau | \phi \rangle|^2$ where $|\tau\rangle = e^{i\theta} |\Psi\rangle$, for $|\Psi\rangle, |\phi\rangle \in \mathbb{C}^d$.

$$\begin{aligned} |\langle \tau | \phi \rangle|^2 &= |e^{-i\theta}|^2 |\langle \Psi | \phi \rangle|^2 \\ &= |\langle \Psi | \phi \rangle|^2 \\ \langle \tau | \phi \rangle &= \langle \Psi | \phi \rangle \end{aligned}$$