

## Lecture 15

### Symmetric matrices, quadratic forms, matrix norm, and SVD

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices
- norm of a matrix
- singular value decomposition

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### Eigenvalues of symmetric matrices

suppose  $A \in \mathbf{R}^{n \times n}$  is symmetric, i.e.,  $A = A^T$

**fact:** the eigenvalues of  $A$  are real

to see this, suppose  $Av = \lambda v$ ,  $v \neq 0$ ,  $v \in \mathbf{C}^n$

then

$$\bar{v}^T Av = \bar{v}^T (\lambda v) = \lambda \bar{v}^T v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\bar{v}^T Av = \overline{(Av)}^T v = \overline{(\lambda v)}^T v = \bar{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have  $\lambda = \bar{\lambda}$ , i.e.,  $\lambda \in \mathbf{R}$  (hence, can assume  $v \in \mathbf{R}^n$ )

Symmetric matrices, quadratic forms, matrix norm, and SVD

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## Eigenvectors of symmetric matrices

**fact:** there is a set of orthonormal eigenvectors of  $A$ , i.e.,  $q_1, \dots, q_n$  s.t.  
 $Aq_i = \lambda_i q_i$ ,  $q_i^T q_j = \delta_{ij}$

in matrix form: there is an orthogonal  $Q$  s.t.

$$Q^{-1}AQ = Q^T A Q = \Lambda$$

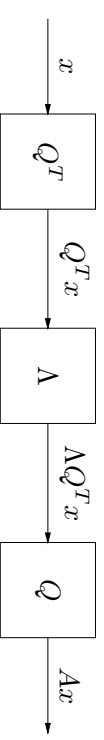
hence we can express  $A$  as

$$A = Q \Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T$$

in particular,  $q_i$  are both left and right eigenvectors

## Interpretations

$$A = Q \Lambda Q^T$$



linear mapping  $y = Ax$  can be decomposed as

- resolve into  $q_i$  coordinates
- scale coordinates by  $\lambda_i$
- reconstitute with basis  $q_i$

or, geometrically,

- rotate by  $Q^T$
- diagonal real scale ('dilation') by  $\Lambda$
- rotate back by  $Q$

decomposition

$$A = \sum_{i=1}^n \lambda_i q_i q_i^T$$

expresses  $A$  as linear combination of 1-dimensional projections

**example:**

$$\begin{aligned} A &= \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \\ &= \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left( \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)^T \end{aligned}$$

