Stephen Boyd

Lecture 15
Symmetric matrices, quadratic forms, matrix norm, and SVD

- eigenvectors of symmetric matrices
- quadratic forms
- inequalities for quadratic forms
- positive semidefinite matrices
- norm of a matrix
- singular value decomposition

Eigenvalues of symmetric matrices

suppose $A \in \mathbf{R}^{n\times n}$ is symmetric, $i.e.$, $A = A^T$

fact: the eigenvalues of A are real

to see this, suppose $Av = \lambda v$, $v \neq 0$, $v \in \mathbf{C}^n$

then

$$\overline{v}^T Av = \overline{v}^T (Av) = \lambda \overline{v}^T v = \lambda \sum_{i=1}^n |v_i|^2$$

but also

$$\overline{v}^T Av = \overline{(Av)}^T v = \overline{(\lambda v)}^T v = \overline{\lambda} \sum_{i=1}^n |v_i|^2$$

so we have $\lambda = \overline{\lambda}$, $i.e.$, $\lambda \in \mathbf{R}$ (hence, can assume $v \in \mathbf{R}^n$)

Eigenvectors of symmetric matrices

fact: there is a set of orthonormal eigenvectors of A, *i.e.*, $q_1, \ldots, q_n$ s.t. $Aq_i = \lambda_i q_i$, $q_i^T q_j = \delta_{ij}$

in matrix form: there is an orthogonal Q s.t.

$$Q^{-1}AQ = Q^TAQ = \Lambda$$

hence we can express A as

$$A = Q\Lambda Q^T = \sum_{i=1}^n \lambda_i q_i q_i^T$$

in particular, q_i are both left and right eigenvectors

Interpretations

$A = Q\Lambda Q^T$

$$\xrightarrow{x} \boxed{Q^T} \xrightarrow{Q^Tx} \boxed{\Lambda} \xrightarrow{\Lambda Q^Tx} \boxed{Q} \xrightarrow{Ax}$$

linear mapping $y = Ax$ can be decomposed as

- resolve into q_i coordinates
- scale coordinates by λ_i
- reconstitute with basis q_i

or, geometrically,

- rotate by Q^T
- diagonal real scale ('dilation') by Λ
- rotate back by Q

decomposition

$$A = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

expresses A as linear combination of 1-dimensional projections

example:

$$\begin{aligned} A &= \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)^T \end{aligned}$$