0.1 Velocity Renormalisation

I. $\epsilon \approx 1$

For easier notation we will write the momentum cutoff as Λ_0 and Λ_0/s as Λ. Note that $\Lambda\partial_\Lambda f(\Lambda) = -s\partial_s f(\Lambda_0/s)$

In a momentum cutoff scheme (keeping only the fermionic momentum inside the shell) self energy

$$\Sigma(i\omega, \mathbf{k}) = -\frac{e^2}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\mathbf{q}\frac{\theta(\Lambda_0 > q > \Lambda)}{\epsilon_\Lambda(\omega, \mathbf{k}-\mathbf{q})|\mathbf{k}-\mathbf{q}|}\frac{i\omega + v_\Lambda(q)\sigma.\mathbf{q}}{\omega^2 + (v_\Lambda(q)q)^2} \tag{1}$$

First assume that $\epsilon(\omega, \mathbf{k}) = 1 + \frac{2\pi e^2}{k}\Pi(\omega, \mathbf{k}) \approx 1$.

$$\partial_\Lambda\Sigma(k) = -\frac{e^2}{4\pi}\int d^2\mathbf{q}\frac{\delta(q-\Lambda)}{|\mathbf{k}-\mathbf{q}|}\frac{\sigma.\mathbf{q}}{q} \tag{2}$$

taking $\mathbf{k} = k(1,0)$ and $\mathbf{q} = q(\cos(\phi), \sin(\phi)$ we have

$$\begin{aligned}
\partial_\Lambda\Sigma(k) &= -\frac{e^2}{4\pi}\int_0^{2\pi}\int_0^{\infty} q\,dq\,d\phi\frac{\delta(q-\Lambda)}{q\sqrt{k^2+q^2-2kq\cos(\phi)}}\left(\sigma_1 q\cos(\phi) + \sigma_2 q\sin(\phi)\right)\\
&= -\frac{e^2}{4\pi}\frac{\sigma_1 k}{k}\int_0^{2\pi} d\phi\frac{\Lambda\cos(\phi)}{\Lambda\sqrt{1-2(k/\Lambda)\cos(\phi)+(k/\Lambda)^2}}\\
\Lambda\partial_\Lambda\Sigma(k) &= -\frac{e^2}{4\pi}\sigma.\mathbf{k}\frac{\Lambda}{k}\int_0^{2\pi} d\phi\frac{\cos(\phi)}{\sqrt{1-2(k/\Lambda)\cos(\phi)+(k/\Lambda)^2}}
\end{aligned} \tag{3}$$

from the relation $G^{-1}(i\omega, k) = i\omega - v_F\sigma.\mathbf{k} - \Sigma(k)$ we recover $\partial_\Lambda\Sigma(k) = \sigma.\mathbf{k}\partial_\Lambda v$

We can also use a slightly different substitution which is easier to generalise. Before that, transform $\mathbf{q} \to -\mathbf{q}$. Then we use the transform $\mathbf{k} = k(1,0)$ and $\mathbf{q} = \frac{k}{2}(\cosh(\mu)\cos(\nu) - 1, \sinh(\mu)\sin(\nu))$. This gives

$$\frac{e^2}{8\pi}\int_0^{2\pi} d\nu\int_0^{\infty} d\mu\,\delta(\frac{|k|}{2}(\cosh(\mu) - \cos(\nu)) - \Lambda)k\left(\sigma_1(\cosh(\mu)\cos(\nu) - 1) + \sigma_2\sinh(\mu)\sin(\nu)\right) \tag{4}$$

The σ_2 term should vanish by symmetry. We first do the μ integral. Using the property of Dirac delta function $\delta(f(x)) = \frac{\delta(a)}{f'(a)}$ where a is zero of the function f.

$$\frac{e^2}{8\pi}\int_0^{2\pi} d\nu\int_0^{\infty} d\mu\frac{\delta(\mu - \cosh^{-1}(\frac{2\Lambda}{k} + \cos(\nu)))}{\frac{1}{2}\sqrt{(2\Lambda + k\cos(\nu))^2 - k^2}}k\sigma_1(\cosh(\mu)\cos(\nu) - 1) \tag{5}$$

$$\frac{e^2}{4\pi}\sigma.\mathbf{k}\int_0^{2\pi} d\nu\frac{(\frac{2\Lambda}{k} + \cos(\nu))\cos(\nu) - 1}{\sqrt{(2\Lambda + k\cos(\nu))^2 - (k)^2}} \tag{6}$$

Here we have to note that This makes sense if we have $\frac{2\Lambda}{k} + \cos(\nu) > 1$ as then only $\cosh^{-1}$ have real solutions.

II.

If we restrict both the electron and photon(scalar) momenta to lie in the shell then

$$\Sigma(i\omega, \mathbf{k}) = -\frac{e^2}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\mathbf{q}\frac{\theta(\Lambda_0 > q > \Lambda)\theta(\Lambda_0 > |\mathbf{k}+\mathbf{q}| > \Lambda)}{\epsilon_\Lambda(\omega, \mathbf{k}+\mathbf{q})|\mathbf{k}+\mathbf{q}|}\frac{i\omega + v_\Lambda(q)\sigma.\mathbf{q}}{\omega^2 + (v_\Lambda(q)q)^2} \tag{7}$$

Taking the derivative wrt Λ we have two terms first term being

$$\partial_\Lambda\Sigma(i\omega, \mathbf{k})_1 = -\frac{e^2}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\mathbf{q}\frac{\delta(q-\Lambda)\theta(\Lambda_0 > |\mathbf{k}+\mathbf{q}| > \Lambda)}{\epsilon_\Lambda(\omega, \mathbf{k}+\mathbf{q})|\mathbf{k}+\mathbf{q}|}\frac{i\omega + v_\Lambda(q)\sigma.\mathbf{q}}{\omega^2 + (v_\Lambda(q)q)^2} \tag{8}$$

and the other with $|\mathbf{q}+\mathbf{k}|$ inside the Dirac delta function and q inside Heaviside theta function.

Static Approximation

In static approximation we ignore the frequency dependence of ϵ. Then we can easily do the ω integral.

$$\partial_\Lambda \Sigma(i\omega, \mathbf{k})_1 = -\frac{e^2}{4\pi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int d^2\mathbf{q}\frac{\delta(q-\Lambda)\theta(\Lambda_0 > |\mathbf{k}+\mathbf{q}| > \Lambda)}{\epsilon_\Lambda(0, \mathbf{k}+\mathbf{q})|\mathbf{k}+\mathbf{q}|}\frac{\sigma.\mathbf{q}}{q} \tag{9}$$

We can repeat the substitution used earlier to get equation of the form

$$-\frac{e^2}{8\pi}\int_0^{2\pi} d\nu \int_0^{\infty} d\mu \frac{\delta(\mu - \cosh^{-1}(\frac{2\Lambda}{k}+\cos(\nu)))\theta(\Lambda_0 > \frac{k}{2}(\cosh(\mu)+\cos(\nu)) > \Lambda)}{\epsilon_\Lambda(\frac{k}{2}(\cosh(\mu)+\cos(\nu))\frac{1}{2}\sqrt{(2\Lambda+k\cos(\nu))^2-k^2}} k\sigma_1(\cosh(\mu)\cos(\nu)-1) \tag{10}$$

Carrying out the integration on μ we have

$$-\frac{e^2}{8\pi}\int_0^{2\pi} d\nu \frac{\theta(\Lambda_0 > \Lambda + k\cos(\nu) > \Lambda)}{\epsilon_\Lambda(\Lambda+k\cos(\nu))\frac{1}{2}\sqrt{(2\Lambda+k\cos(\nu))^2-k^2}} k\sigma_1((\frac{2\Lambda}{k}+\cos(\nu))\cos(\nu)-1) \tag{11}$$

The delta function restricts the values of ν such that $\cos(\nu) > 1 - \frac{2\Lambda}{k}$ and the theta function here restricts that $\Lambda_0 > \Lambda + k\cos(\nu) > \Lambda$ i.e. $\frac{\Lambda_0 - \Lambda}{k} > \cos(\nu) > 0$. Similar calculations for the other part gives

$$-\frac{e^2}{8\pi}\int_0^{2\pi} d\nu \frac{\theta(\Lambda_0 > \Lambda - k\cos(\nu) > \Lambda)}{\epsilon_\Lambda(\Lambda)\frac{1}{2}\sqrt{(2\Lambda-k\cos(\nu))^2-k^2}} k\sigma_1((\frac{2\Lambda}{k}-\cos(\nu))\cos(\nu)-1) \tag{12}$$

with the constraints that $-\cos(\nu) > 1 - \frac{2\Lambda}{k}$ and $\frac{\Lambda_0-\Lambda}{k} > -\cos(\nu) > 0$.

We can recast the last two equation in a more convinient way. Theta function in the equation 11 restricts $\cos(\nu)$ to be positive while it is restricted to be negative in equation 12. So we can have integral running from just $-\pi/2$ to $\pi/2$ in the first equation while in the second equation we replace $\cos(\nu)$ with $-\cos(\nu)$ and have the same integral limits. Also as only $\cos(\nu)$ is going inside the integral, we can have the integral limit from 0 to $\pi/2$ and multiply with 2. Then we have

$$-\frac{e^2}{2\pi}\sigma.\mathbf{k}\int_0^{\pi/2} d\nu \frac{\theta(\Lambda_0 - \Lambda - k\cos(\nu))}{\sqrt{(2\Lambda+k\cos(\nu))^2-k^2}}\left(\frac{(\frac{2\Lambda}{k}+\cos(\nu))\cos(\nu)-1}{\epsilon_\Lambda(\Lambda+k\cos(\nu))} - \frac{(\frac{2\Lambda}{k}+\cos(\nu))\cos(\nu)+1}{\epsilon_\Lambda(\Lambda)}\right) \tag{13}$$

0.2 Polarisation Function

Polarisation function is given by

$$\Pi_\Lambda(Q) = -\int \frac{d^2\mathbf{k}}{(2\pi)^2}\hat{\theta}(q)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)\frac{\xi_\Lambda(k)+\xi_\Lambda(|\mathbf{k}+\mathbf{q}|)}{(\xi_\Lambda(k)+\xi_\Lambda(|\mathbf{k}+\mathbf{q}|))^2+\hat{\omega}^2}\left(1-\frac{\mathbf{k}.(\mathbf{k}+\mathbf{q})}{k|\mathbf{k}+\mathbf{q}|}\right) \tag{14}$$

where $\hat{\theta}(k) = \theta(\Lambda < k < \Lambda_0)$ and $\xi_\Lambda(k) = v_\Lambda(k)k$

In a static approximation we ignore the ω dependence to get the differential equation

$$\partial_\Lambda \Pi_\Lambda(q)_1 = \int \frac{d^2\mathbf{k}}{(2\pi)^2}\delta(k-\Lambda)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)\frac{1}{v_\Lambda(k)k + v_\Lambda(|\mathbf{k}+\mathbf{q}|)|\mathbf{k}+\mathbf{q}|}\left(1-\frac{\mathbf{k}.(\mathbf{k}+\mathbf{q})}{k|\mathbf{k}+\mathbf{q}|}\right) \tag{15}$$

We can use the same substitution as earlier to get

$$-\frac{1}{4\pi^2}\int_0^{2\pi} d\nu \frac{\theta(\Lambda < \Lambda + q\cos(\nu) < \Lambda_0)}{v_\Lambda(\Lambda)\Lambda + (\Lambda+q\cos(\nu))v_\Lambda(\Lambda+q\cos(\nu))}\frac{q^2\sin^2(\nu)}{\sqrt{(2\Lambda+q\cos(\nu))^2-q^2}} \tag{16}$$

Similarly the other term which is given by

$$-\frac{1}{4\pi^2}\int_0^{2\pi} d\nu \frac{\theta(\Lambda < \Lambda - q\cos(\nu) < \Lambda_0)}{v_\Lambda(\Lambda)\Lambda + (\Lambda-q\cos(\nu))v_\Lambda(\Lambda-q\cos(\nu))}\frac{q^2\sin^2(\nu)}{\sqrt{(2\Lambda-q\cos(\nu))^2-q^2}} \tag{17}$$

Note that these two integrals along woth their constraints give the same result hence we have the equation

$$\boxed{\partial_\Lambda\epsilon_\Lambda(q) = \frac{2\pi e^2}{q}\partial_\Lambda\Pi_\Lambda(q) = -\frac{2e^2}{\pi}\int_0^{\frac{\pi}{2}} d\nu \frac{\theta(\Lambda < \Lambda + q\cos(\nu) < \Lambda_0)}{v_\Lambda(\Lambda)\Lambda + (\Lambda+q\cos(\nu))v_\Lambda(\Lambda+q\cos(\nu))}\frac{q\sin^2(\nu)}{\sqrt{(2\Lambda+q\cos(\nu))^2-q^2}}} \tag{18}$$

With an extra condition that $\cos(\nu) > 1 - \frac{2\Lambda}{q}$.

0.3 Frequency Dependence

Using eqn 14 we can also determine the frequency dependence of the dielectric function.

0.4 Finite Temperature calculations

Using Matsubara formulation it is very easy to get the finite temperature results as well. In this formalism, Frequancy integral from $-\infty$ to ∞ is replaced by sum over the Matsubara frequencies $\omega_n = 2n\frac{\pi}{\beta}$ for bosons and $\hat{\omega}_n = (2n+1)\frac{\pi}{\beta}$ for fermions. Hence we have the sum for self energy given by

$$-\frac{e^2}{2\pi}\frac{1}{\beta}\int_{\mathbf{q}}\sum_{n=-\infty}^{\infty}\frac{\hat{\theta}(q)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)}{|\mathbf{k}+\mathbf{q}|\epsilon_\Lambda(|\mathbf{k}+\mathbf{q}|)}\frac{i\hat{\omega}_n + v_\Lambda(q)\sigma.\mathbf{q}}{(i\hat{\omega}_n)^2-(v_\Lambda(q)q)^2} \tag{19}$$

where we have assumed the frequency independence of the dielctric function. Then the Matsubara sum is easy to do. First note that the function $f_\eta(z) = \frac{1}{e^{\eta\beta z}-\eta}$ have poles at the Matsubara frequencies with $\eta = +1$ for bosons and $\eta = -1$ for fermions. If then we take a function $F(z)$ which is obtained by substituting z in place of $i\hat{\omega}_n$ in eqn 19 without the $1/\beta$ factor. Then for a contour as shown in the figure 1 the integral $\int_C f_{-1}(z)F(z)dz$ evaluates to zero as we take the radius to infinity. Poles are shown in the figure. For the poles on the vertical axis we recover the sum in eqn 19 as the residue of $f_{-1}(z)$ is $-\frac{1}{\beta}$. As the whole integral over the contour vanishes the contribution from the other two poles must be equal and opposite to the matsubara sum. Hence we have

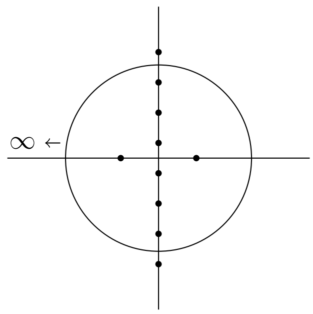


Figure 1: self energy contour

$$-\frac{e^2}{4\pi}\int_{\mathbf{q}}\frac{\hat{\theta}(q)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)}{|\mathbf{k}+\mathbf{q}|\epsilon_\Lambda(|\mathbf{k}+\mathbf{q}|)}\left(\frac{1}{e^{-\beta v_\Lambda(q)q}+1}\left(1+\frac{\sigma.\mathbf{q}}{q}\right)+\frac{1}{e^{\beta v_\Lambda(q)q}+1}\left(1-\frac{\sigma.\mathbf{q}}{q}\right)\right) \tag{20}$$

We can ignore the Identity part and focus on the part proportional to sigma matrices as the identity part will generate terms proportional to the chemical potential which can be removed by adding a counterterm. For the sigma part we have

$$\frac{e^2}{4\pi}\int_{\mathbf{q}}\frac{\hat{\theta}(q)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)}{|\mathbf{k}+\mathbf{q}|\epsilon_\Lambda(|\mathbf{k}+\mathbf{q}|)}\frac{\sigma.\mathbf{q}}{q}\tanh(\frac{\beta v_\Lambda(q)q}{2}) \tag{21}$$

Hence the differential equaition

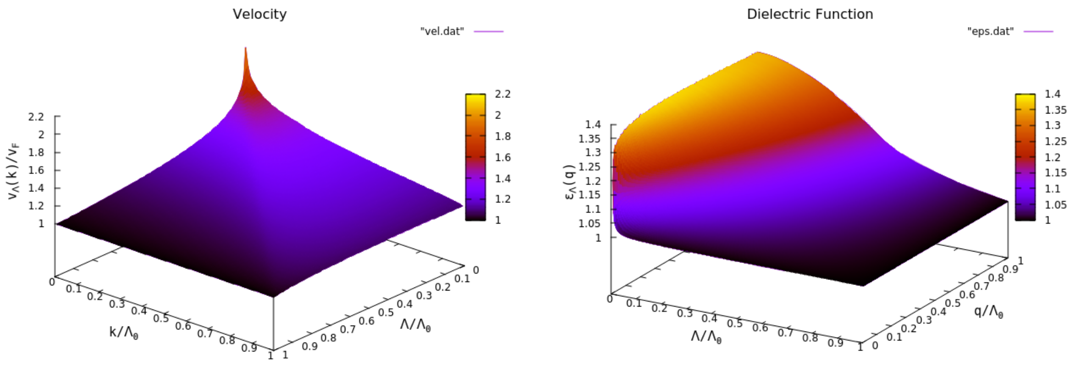
$$\partial_\Lambda v_\Lambda(k) = \frac{e^2}{2\pi}\int_{\mathbf{q}}\frac{\delta(q-\Lambda)\hat{\theta}(|\mathbf{k}+\mathbf{q}|)+\hat{\theta}(q)\delta(|\mathbf{k}+\mathbf{q}|-\Lambda)}{|\mathbf{k}+\mathbf{q}|\epsilon_\Lambda(|\mathbf{k}+\mathbf{q}|)}\frac{\sigma.\mathbf{q}}{q}\tanh(\frac{\beta v_\Lambda(q)q}{2}) \tag{22}$$

with the same substitution used earlier we get

$$\begin{aligned}\partial_\Lambda v_\Lambda(k) = \frac{e^2}{2\pi}\int_0^{\pi/2} d\nu\frac{\theta(\Lambda_0-\Lambda-k\cos(\nu))}{\sqrt{(2\Lambda+k\cos(\nu))^2-k^2}}&(\frac{(\frac{2\Lambda}{k}+\cos(\nu))\cos(\nu)-1}{\epsilon_\Lambda(\Lambda+k\cos(\nu))}\tanh(\frac{\beta v_\Lambda(\Lambda)\Lambda}{2})\\ &-\frac{(\frac{2\Lambda}{k}+\cos(\nu))\cos(\nu)+1}{\epsilon_\Lambda(\Lambda)}\tanh(\frac{\beta v_\Lambda(\Lambda+k\cos(\nu))(\Lambda+k\cos(\nu))}{2}))\end{aligned} \tag{23}$$

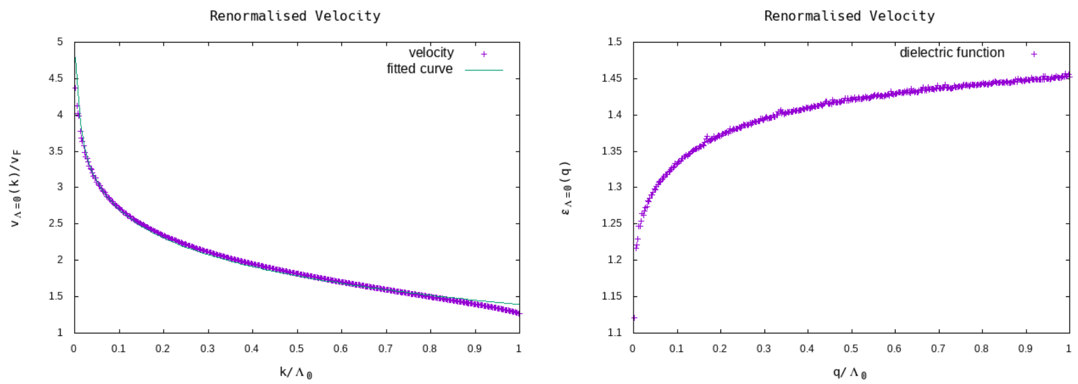
0.5 Numerical Solutions

- In the paper by Bauer, Sharma, Kopietz they used the two coupled differential equations which are the eqn 3 with frequency independent dielctric function given by eqn 18. These equations are numerically solved to get the following results.



(a) Velocity (b) Dielectric Function

Figure 2: Velocity and Dielectric function at $\Lambda \to 0$ with the approximation used by Bauer, Sharma, Kopietz

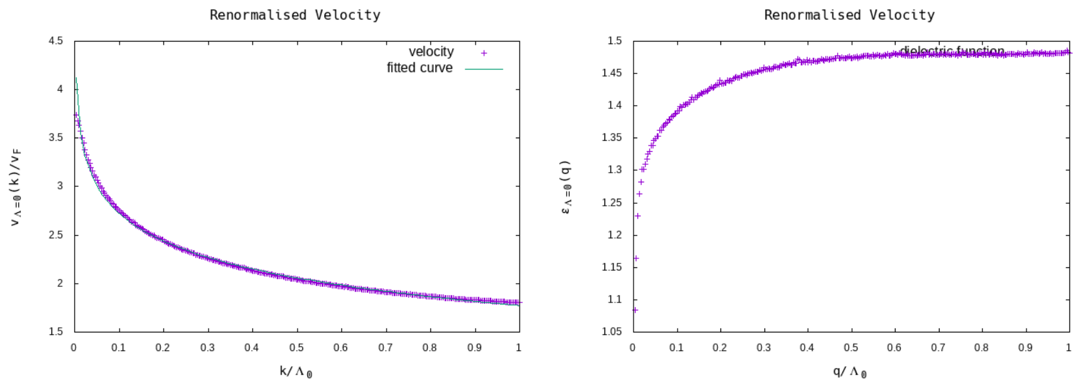


(a) Renormalised Velocity (b) Renormalised Dielectric Function

Figure 3: Velocity and Dielectric function at $\Lambda \to 0$ with the approximation used by Bauer, Sharma, Kopietz

Velocity in the limit $\Lambda \to 0$ shows logarithmic behaviour. Fitting the curve with the function of the form $a + b\log(\Lambda_0/k)$ gives $a = 1.393 \pm 0.003$ and $b = 0.568 \pm 0.002$. Leading order perturbation theory predicts b to be $\frac{\alpha}{4} = 0.55$.

- Next we can use more general velocity renormalisation equation given by eqn 13which takes both the momenta inside the momentum shell for calculating velocity renormalisation 13 along with dielcetric function in eqn 18. This gives the plots below. The fit with $a + b\log\left(\frac{\Lambda_0}{k}\right)$ is better as can be seen from the graph below 4a although the fitted parameter have values $a = 1.773 \pm 0.002$ and $b = 0.413 \pm 0.002$. The second parameter differ significantly from the LO perturbation theory result.



(a) Renormalised Velocity (b) Renormalised Dielectric Function

Figure 4: Velocity and Dielectric function at $\Lambda \to 0$ with both the fermionic and bosonic momenta inside the shell to be integrated out

- Getting the frequency dependence of the dielectric function is easy from eqn 14 although as this can only be done numerically, knowledge of poles and branches are not known hence the integral in eqn 8 cannot be done hence we have to be satisfied with the velocity renormalisation in static approximation only. Here we used the previous approximation to get velocity data and used that to calculate dielectric function. We have used the dimensionless variable omega $= \frac{\omega}{v_F \Lambda_0}$.

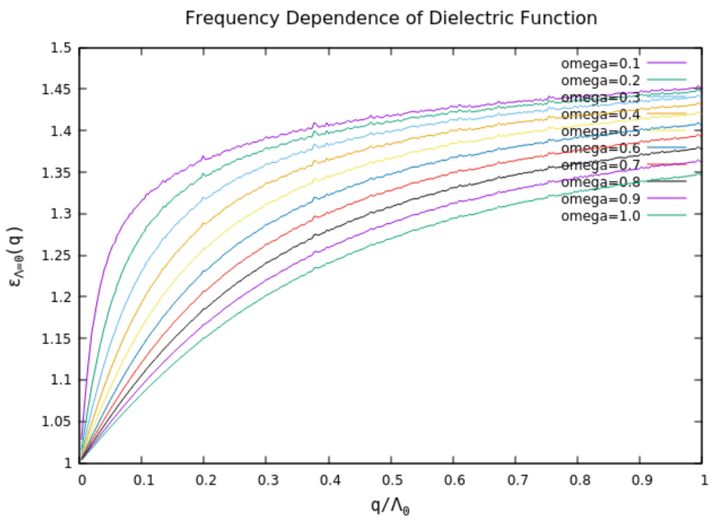


Figure 5: Frequency Dependence of Dielectric Function

- In a finite temperature case (eqn 22). The following graph is for the changes in velocity with temperature. Here we have ignored the temperature dependence of the dielectric function and used the previous formula in equation 18.

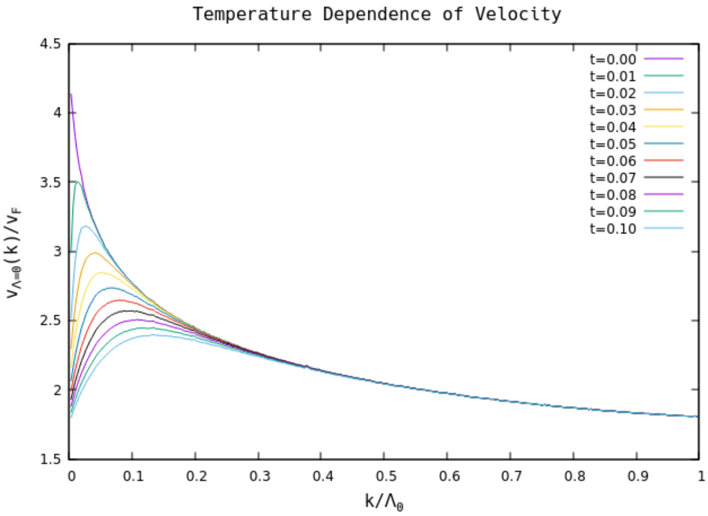


Figure 6: Temperature Dependence of velocity. Here t is in the units of $\frac{v_F \Lambda_0}{k_B}$