Contents

1 Constant Interaction

1.1 Free Hamiltonian

$$H_0 = \psi_k^\dagger h(k) \psi_k \tag{1}$$

where $\psi_k = (a_k, b_k)^T$ and $h(k) = \begin{pmatrix} 0 & \phi(k) \\ \phi^*(k) & 0; \end{pmatrix}$ with $\phi(k) = \sum_{\delta_j} e^{i\delta_j . k}$

The system becomes gapless when $\phi(k)$ vanishes. Let denote those two distinct points as K_1 and K_2.

We are interested in the physics near those two points, so we expand $h(k)$ near those two points to obtain for $k = K_1 + q$, $h(k) \approx v_F \sigma.q$ and for $k = K_2 + q$, $h(k) \approx -v_F \sigma^*.q$

For the ψ's we define

$$\Psi_{i,q} := \psi_{K_i+q} \tag{2}$$

Note that q can only take values near K_1 and K_2. But if we are interested in the soft mode limit it is irrelevant what value of energy we are considering at very high q. Hence we can treat the Hamiltonian as sum of two free theories

$$H_0 = \Psi_{1,q}^\dagger (v_F \sigma.q) \Psi_{1,q} + \Psi_{2,q}^\dagger (-v_F \sigma^*.q) \Psi_{2,q} \tag{3}$$

Then we have two Green's function for the two theories. These free propagators are

$$G_1^{(0)}(i\omega, k) = (i\omega - v_F \sigma.k)^{-1} = \frac{i\omega + v_F \sigma.k}{(i\omega)^2 - v_F^2 k^2} \tag{4}$$

$$G_2^{(0)}(i\omega, k) = (i\omega + v_F \sigma^*.k)^{-1} = \frac{i\omega - v_F \sigma^*.k}{(i\omega)^2 - v_F^2 k^2} \tag{5}$$

Now we can introduce the interactions. First consider interaction only inside each valley

$$H_{int} = \frac{V_1}{2!2!} \Psi_{1,k_4}^\dagger \Psi_{1,k_3}^\dagger \Psi_{1,k_2} \Psi_{1,k_1} + \frac{V_1}{2!2!} \Psi_{2,k_4}^\dagger \Psi_{2,k_3}^\dagger \Psi_{2,k_2} \Psi_{2,k_1} \tag{6}$$

here we have assumed same strength of interaction V_1 on both the valley. At this point it is enough to focus on one of the valley say the first one.

Note

- Integration sign over all the momentum as well as the overall delta function is omitted.

1.2 Renormalisation of two point function

For the first valley, at one loop, we have the diagram

This diagram 1 evaluates to

$$V_1 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} G_1^{(0)}(i\omega, k) \tag{7}$$

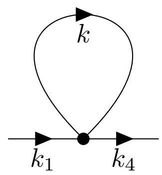


Figure 1: one loop diagram for two point function

We will first evaluate the ω integral. The poles are at $\pm i v_F |k|$. We can close the loop on either side to get contribution from only one of the pole. For example lets close the loop on the upper half complexplane to get.

$$V_1 \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{v_F|k| + v_F \sigma.k}{2v_F|k|} = V_1 \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{2} \left(1 + \frac{\sigma.k}{|k|}\right) \tag{8}$$

The $\sigma.k$ part vanishes as it is an odd function of k. The other part gives

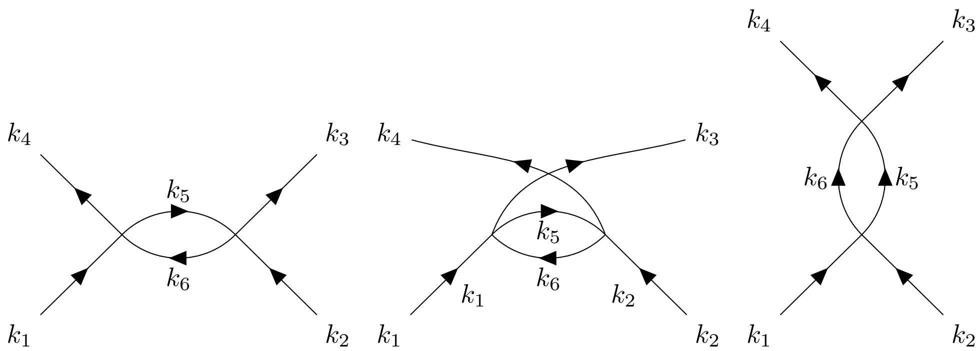
$$V_1 \int_{\frac{\Lambda}{s}}^{\Lambda} \frac{2\pi k dk}{(2\pi)^2} \frac{1}{2} = \frac{V_1 \Lambda^2}{8\pi} \left(1 - \frac{1}{s^2}\right) \tag{9}$$

Note

This introduces a term in the Hamiltonian of the form $\Psi_{1,q}^{\dagger} \left(\frac{V_1\Lambda^2}{8\pi}\left(1 - \frac{1}{s^2}\right)\right) \Psi_{1,q}$. It looks like a chemical potential term which can be tuned to remove the term.

1.3 Renormalisation of interaction

For the renormalisation of the interaction V_1 we have the following digrams (2)



(a) ZS

(b) ZS'

(c) BCS

Figure 2: oneloop diagrams with V_1 vertices

Each figure gives an integration of the form

$$V_1^2 \int_{-\infty}^{\infty} \frac{d\omega_5 d\omega_6}{(2\pi)^2} \int_{\frac{\Lambda}{s}<|k_5|,|k_6|<\Lambda} \frac{d^2k_5 d^2k_6}{(2\pi)^4} G_1^{(0)}(i\omega_5, k_5) G_1^{(0)}(i\omega_6, k_6) \tag{10}$$

Overall delta function for the momentum conservation at each vertex also fixes the relation between k_5 and k_6.

- **ZS:** $k_5 - k_6 = k_1 - k_4$
- **ZS$'$** : $k_5 - k_6 = k_1 - k_3$
- **BCS:** $k_5 + k_6 = k_1 + k_2$

For the calculation of β function we can set the external momenta and frequencies to be zero. This means that we have $k_5 = k_6$ and $\omega_5 = \omega_6$ for ZS and ZS' diagrams while $k_6 = -k_5$ and $\omega_6 = -\omega_5$ for BCS diagram. In all the cases we are supposed to calculate the same integral as in BCS diagram both frequency and momentum changes contributing an overall minus sign. To calculate the integral we will rewrite the propagator in a different form (called chiral representation)

$$G_1^{(0)}(i\omega, k) = \sum_{s=\pm 1} \frac{1}{2} \frac{1 + s\frac{\sigma.k}{|k|}}{i\omega - s v_F |k|} \tag{11}$$

So we get two kind of terms in 10. One with both the factor in the denominator having same sign of s's and the other having opposite signs. For the terms having same sign, the function have pole on one side of the complex plane and so we can always close the contour on the other side to get zero. For the case when we have opposite signs, we have the integrand

$$\frac{1}{4} \frac{(1 + \frac{\sigma.k}{|k|})(1 - \frac{\sigma.k}{|k|})}{(i\omega - v_F|k|)(i\omega + v_F|k|)} = \frac{1}{4} \frac{1 - \frac{(\sigma.k)(\sigma.k)}{k^2}}{(i\omega - v_F|k|)(i\omega + v_F|k|)} \tag{12}$$

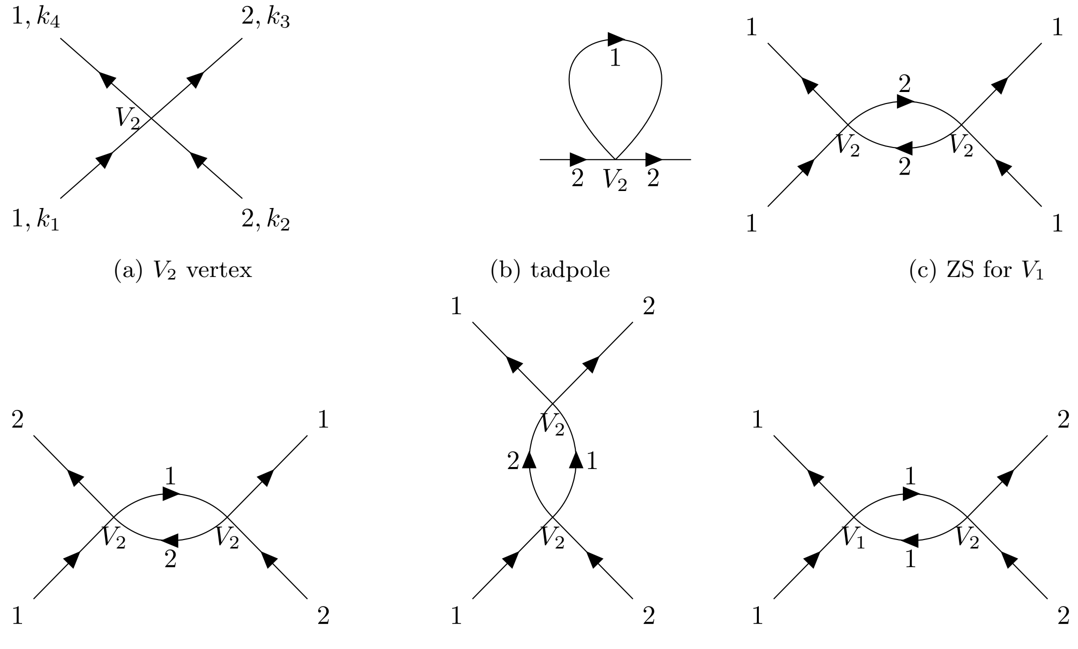
The numerotor vanishes as $(\sigma.k)(\sigma.k) = k^2$. So all the diagrams evaluates to be zero. Therefore we do not have any renormalisation for the interaction term at one loop.

1.4 Turning intervalley potential on

We can also have some intervalley potential which connects both the valley with an interaction strength V_2

$$H_{intval} = \frac{V_2}{2!2!} \Psi^\dagger_{1,k_4} \Psi^\dagger_{2,k_3} \Psi_{2,k_2} \Psi_{1,k_1} + \frac{V_2}{2!2!} \Psi^\dagger_{2,k_4} \Psi^\dagger_{1,k_3} \Psi_{1,k_2} \Psi_{2,k_1} \tag{13}$$

This kind of interaction allows the vertex of the type shown in the figure 3a. This allowed the one loop diagrams of the type shown in 3



(a) V_2 vertex (b) tadpole (c) ZS for V_1

(d) ZS' for V_2 (e) BCS for V_2 (f) ZS with V_1 and V_2 for V_2

Figure 3: one loop diagrams with new interaction

At one loop following cases arises,

- For two point function we get the same form of integral except that $G_1^{(0)}$ is replaced by $G_2^{(0)}$. The integrals evaluates to be the same, which gives contribution of $\frac{V_2}{8\pi}\left(1 - \frac{1}{s^2}\right)$

- We can have two V_2 vertex renormalising V_1 vertex as shown in the fig 3c. The integral corrosponding to the diagram is of the form eqn 10. This also vanishes.
- For the renormalisation of V_2 we have three kind of diagrams
  - The diagram consisting of one V_1 vertex and one V_2 vertex have the same form as eqn 10 hence vanishes.
  - The other two diagrams have both $G_1^{(0)}$ and $G_2^{(0)}$ in the integrand

$$V_2^2 \int_{-\infty}^{\infty} \frac{d\omega_5 d\omega_6}{(2\pi)^2} \int_{\frac{\Lambda}{s}<|k_5|,|k_6|<\Lambda} \frac{d^2k_5 d^2k_6}{(2\pi)^4} G_1^{(0)}(i\omega_5, k_5) G_2^{(0)}(i\omega_6, k_6) \tag{14}$$

relation between k_5 and k_6 turns out to be same as previous ones $k_6 = k_5$ and $k_6 = -k_5$. Lets focus on one of the integral (the other integral is also same with an overall sign). We can again use the similar chiral representation of $G_2^{(0)}$

$$G_2^{(0)}(i\omega, k) = \sum_{s=\pm1} \frac{1}{2} \frac{1 + s\frac{\sigma^*.k}{|k|}}{i\omega + s v_F |k|} \tag{15}$$

Note that the denominator of both $G_1^{(0)}$ and $G_2^{(0)}$ are of the same form. So following the arguments as presented in evaluating eqn 10, we only have the cross terms. Both the cross terms are exactly same and equal to

$$V_2^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{4} \frac{1 - \frac{(\sigma.k)(\sigma^*.k)}{k^2}}{(i\omega - v_F|k|)(i\omega + v_F|k|)} \tag{16}$$

The numerator becomes $\left(1 - \frac{k_1^2 - k_2^2}{k_1^2 + k_2^2}\right) = \frac{2k_2^2}{k_1^2 + k_2^2}$ where k_1 and k_2 are the components of the two dimensional momentum vector k. We can replace k_2^2 with $\frac{1}{2}(k_1^2 + k_2^2)$, which will make the numerator unity. ω integral can be done by taking a contour closing on the upper half of the complex plane to get

$$V_2^2 \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \frac{1}{4} \frac{1}{2v_F|k|} = V_2^2 \int_{\frac{\Lambda}{s}}^{\Lambda} \frac{2\pi k dk}{(2\pi)^2} \frac{1}{4} \frac{1}{2v_F k} = \frac{V_2^2}{16\pi v_F}\left(\Lambda - \frac{\Lambda}{s}\right) \tag{17}$$

So these two terms together contribute $\frac{V_2^2 \Lambda}{8\pi v_F}\left(1 - \frac{1}{s}\right)$

1.5 Conclusion

- v_F does not renormalise as we change the cutoff momentum. To renormalise v_F we need the interaction to have momentum dependence at first order. Although at higher order we do get renormalisation of v_F (see 1.6)
- a chemical potential like term (mass like) arises due to renormalisation.

$$\mu' = s\left(\mu + \frac{(V_1 + V_2)\Lambda^2}{8\pi}\left(1 - \frac{1}{s^2}\right)\right) \tag{18}$$

- V_1 does not renormalise under $\Lambda \to \frac{\Lambda}{s}$ transformation.
- V_2 renormalises to

$$V_2' = V_2 - ZS' - \frac{1}{2}BCS = V_2 - \frac{1}{2}\frac{V_2^2\Lambda}{8\pi v_F}\left(1 - \frac{1}{s}\right) \tag{19}$$

As $ZS' = -BCS$.

1.6 Two point function second order

Let's focus on the second order correction in V_1 to two point function. This gives integral of the form

$$V_2^1 \int_{-\infty}^{\infty} \frac{d\omega_1 d\omega_2}{(2\pi)^2} \int_{\frac{\Lambda}{s}<|k_1|,|k_2|<\Lambda} \frac{d^2k_1 d^2k_2}{(2\pi)^4} G_1^{(0)}(i\omega_1, k_1) G_1^{(0)}(i\omega_2, k_2) G_1^{(0)}(-i(\omega_1+\omega_2), k-k_1-k_2) \quad (20)$$

where we have used the momentum conservation at each vertices to substitute for k_3 and ω_3. At first

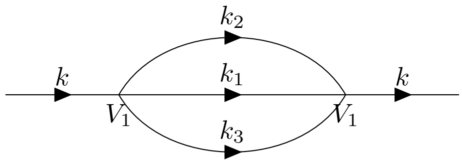


Figure 4: Two point function

we can look at one of the ω integrals

$$\sum_{s,s'=\pm 1} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \frac{1+s\frac{\sigma.k_1}{|k_1|}}{i\omega_1 - sv_F|k_1|} \frac{1+s'\frac{\sigma.(k-k_1-k_2)}{|k-k_1-k_2|}}{i(\omega_1+\omega_2) - s'v_F|k-k_1-k_2|} \quad (21)$$

Poles are at $\pm v_F|k_1|$ and at $-i\omega_2 \pm v_F|k-k_1-k_2|$. We can close the contour on either side to get residues from only two of them. This gives the integral to be

$$\left(1+\frac{\sigma.k_1}{|k_1|}\right) \sum_{s'=\pm 1} \frac{1+s'\frac{\sigma.(k-k_1-k_2)}{|k-k_1-k_2|}}{i\omega_2+v_F|k_1|-s'v_F|k-k_1-k_2|} + \left(1+\frac{\sigma.(k-k_1-k_2)}{|k-k_1-k_2|}\right) \sum_{s=\pm 1} \frac{1+s\frac{\sigma.k_1}{|k_1|}}{-i\omega_2+v_F|k-k_1-k_2|-sv_F|k_1|} \quad (22)$$

Note that the terms with $s = s' = 1$ cancels each other. For the other ω integral we first have to multiply the above result by $G_1^{(0)}(i\omega_2, k_2)$. All the terms in the integrands are of the form

$$\int_{\infty}^{\infty} \frac{d\omega_2}{2\pi} \frac{1}{(i\omega_2 - a)(i\omega_2 - b)} \quad (23)$$

where a and b are real numbers. The integrals trivially reduces to $\frac{\theta(b)-\theta(a)}{a-b}$. Thus we only have non zero contribution if a and b are of different signs. In this case a can take values from $\pm v_F|k_2|$ and b can take values from $\pm v_F(|k_1| + |k-k_1-k_2|)$. Closing the contour such a way to take only the contribution from the positive ones we get,

$$(1+\hat{\sigma}_{k_1})(1+\hat{\sigma}_{k_2})(1-\hat{\sigma}_{k-k_1-k_2}) \frac{1}{v_F(|k_1|+|k_2|+|k-k_2-k_2|)} + (1+\hat{\sigma}_{k-k_1-k_2})(1-\hat{\sigma}_{k_1}) \frac{1}{v_F(|k_1|-|k_2|+|k-k_1-k_2|)} \quad (24)$$

where $\hat{\sigma}_k = \frac{\sigma.k}{|k|}$.

2 Scalar Coupling

Rather than introducing a constant interaction which couples inter and intra valley fields, we can introduce a scalar field. The interaction is of the form

$$-ie(\Psi_{1,k}^{\dagger}\Psi_{1,k+q}\phi_q + \Psi_{2,k}^{\dagger}\Psi_{2,k+q}\phi_q) \quad (25)$$

To get Coulomb like interaction (i.e. the scalar field behaving like a photon field) we will assume the scalar field propagator to be $\frac{2\pi}{|q|}$. Feynman diagrams for the theory are shown in figure 5

2.1 Two point function

Tadpole Diagram

In the tadpole diagram 6a momentum consevation at the vertices gives $q = 0$. This term known as Hartree term usually gets cancelled by the background uniform charge distribution.

(a) electron and photon propagator (b) vertex

Figure 5: feynman diagrams for scalar coupling

(a) Tadpole Diagram (b) Electron Self Energy

Electron Self Energy

This diagram 6b translates to

$$(-ie)^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{\frac{\Lambda}{s}<|q|<\Lambda} \frac{d^2q}{(2\pi)^2} G_1^{(0)}(i\omega, k-q) \frac{2\pi}{|q|} \tag{26}$$

the ω integral is straightforward and gives following

$$-e^2 \int_{\frac{\Lambda}{s}<|q|<\Lambda} \frac{d^2q}{4\pi} \left(1+\frac{\sigma.(k-q)}{|k-q|}\right)\frac{1}{|q|} \tag{27}$$

the q integral is difficult to do and analytic result can be obtained in Mathematica. Although as we are not interested in the higher powers of external momentum k we can expand the integrand in powers of k and then get the integral easily. For example at order k we get $\frac{\sigma.ke^2}{2} log(s)$ which renormalises v_F.

2.2 Photon Self Energy

With the vertex 5 one can form the diagram 7 which will give correction to the photon propagator. The correction is given by

$$e^2 \int_{\frac{\Lambda}{s}<|k|<\Lambda} \frac{d^2k}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_1^{(0)}(i\omega_1, k_1) G_1^{(0)}(i(\omega_1-\omega), k_1-k) \tag{28}$$

The ω integral can be evaluated as given in the appendix (see 34).

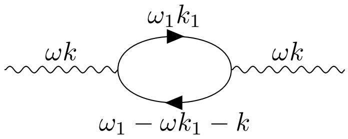


Figure 7: Photon Self Energy

2.3 Vertex Correction

Similarly one can calculate the vertex correction 8 as well

$$e^2 \int_{\frac{\Lambda}{s}<|q_1|<\Lambda} \frac{d^2q_1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_1^{(0)}(i\omega_1, k+q+q_1) G_1^{(0)}(i(\omega_1-\omega), k-q_1) \frac{2\pi}{|q_1|} \tag{29}$$

3 Appendix

Coulomb Interaction

The Coulomb interaction i.e. $\frac{1}{r}$ type of interaction in momentum space can be obtained by Fourier transforming

$$\int d^2\mathbf{r}\frac{e^{i\mathbf{k}.\mathbf{r}}}{r} = \int_0^\infty rdr\int_0^{2\pi} d\theta\frac{e^{ikr\cos(\theta)}}{r} = \int_0^\infty dr 2\pi \text{BesselJ}(0,kr) = \frac{2\pi}{k} \tag{30}$$

Integrals

This section summerises some of the frequent integrals used in the text.

we have

$$G_1^{(0)}(i\omega,k) = \sum_{s=\pm 1}\frac{1+s\frac{\sigma.k}{|k|}}{i\omega - sv_F|k|} \tag{31}$$

1.

$$I_1(k) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_1^{(0)}(i\omega,k) = \frac{1}{2}\left(1+\frac{\sigma.k}{|k|}\right)$$

2.

$$I_2(k_1,k_2) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_1^{(0)}(i\omega,k_1)G_1^{(0)}(i\omega,k_2)$$

The poles are at $\pm iv_F|k_1|$ and $\pm iv_F|k_2|$. We can close the contour on the upper half plane (of $i\omega$) to get only the contribution from $iv_F|k_1|$ and $iv_F|k_2|$. This gives

$$I_2(k_1,k_2) = \frac{1}{2v_F}\frac{1-\frac{\sigma.k_1}{|k_1|}\frac{\sigma.k_2}{|k_2|}}{|k_1|+|k_2|} = \frac{1}{2v_F}\frac{1-cos(\theta_{k_1,k_2})}{|k_1|+|k_2|} \tag{32}$$

3.

$$I_{2'}(k_1,k_2,\omega') = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G_1^{(0)}(i\omega,k_1)G_1^{(0)}(i\omega+i\omega',k_2) \tag{33}$$

This integral evaluates to

$$I_{2'}(k_1,k_2.\omega') = \left(\frac{\sigma.k_1}{|k_1|}-\frac{\sigma.k_2}{|k_2|}\right)\frac{i\omega'}{2(\omega'^2+v_F^2(|k_1|+|k_2|)^2)} + \frac{1}{2}\left(1-cos(\theta_{k_1,k_2})\right)\frac{v_F(|k_1|+|k_2|)}{\omega'^2+v_F^2(|k_1|+|k_2|)^2} \tag{34}$$

We can analytically continue the equation to real frequencies by the substitution $i\omega' \to \omega$.

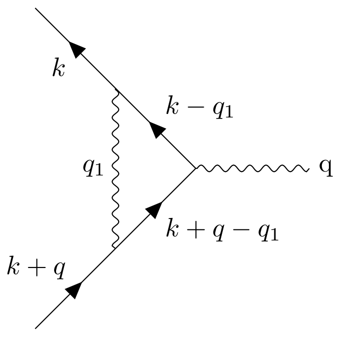


Figure 8: Vertex Correction