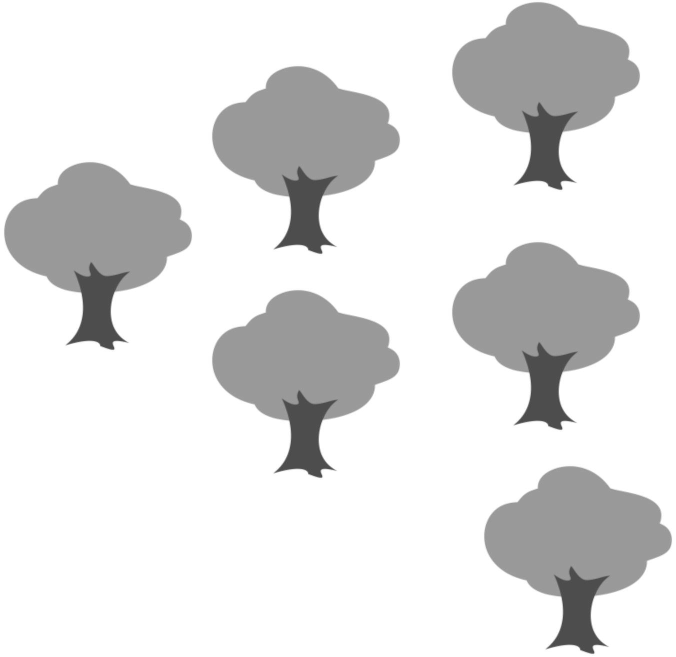
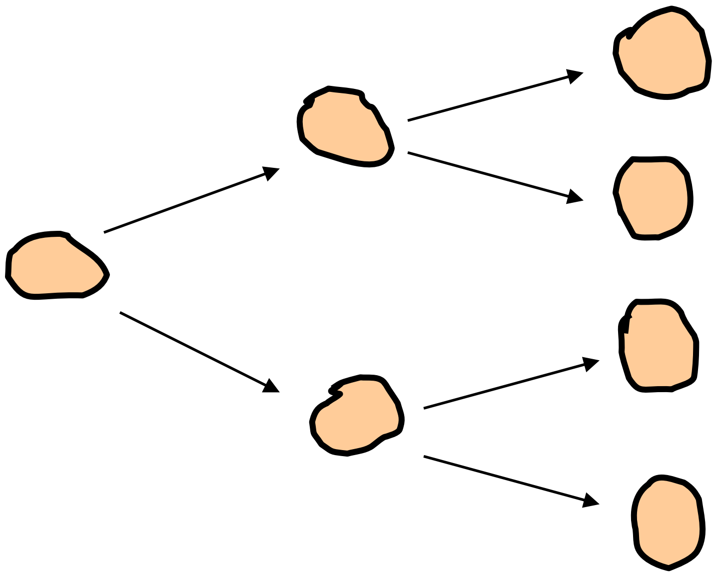


Modeling population growth

Simple growth model



The model:

$$\frac{dx}{dt} = r.x$$

x = population at time t

r = rate constant for growth.

Assumption: no death

The problem Statement

The model: $$\frac{dx}{dt} = rx$$

Questions:

a) At *time* = *0*, the population is x_0.

What will be the population at time t ?

b) How the population changes with time ?

Integration to get the answers

$$\frac{dx}{dt} = r.x \longrightarrow x = f(t)$$

$$\int_{x_0}^{x} \frac{dx}{x} = r.\int_0^t dt$$

$$\Rightarrow [\ln x]_{x_0}^{x} = r.[t]_0^t$$

$$\Rightarrow \ln x - \ln x_0 = r.(t-0)$$

$$\Rightarrow \ln \frac{x}{x_0} = r.t$$

$$\Rightarrow \frac{x}{x_0} = e^{r.t}$$

$$\Rightarrow x = x_0.e^{r.t}$$

Question 1

Starting with initial population of x_0, what will be the population at time t?

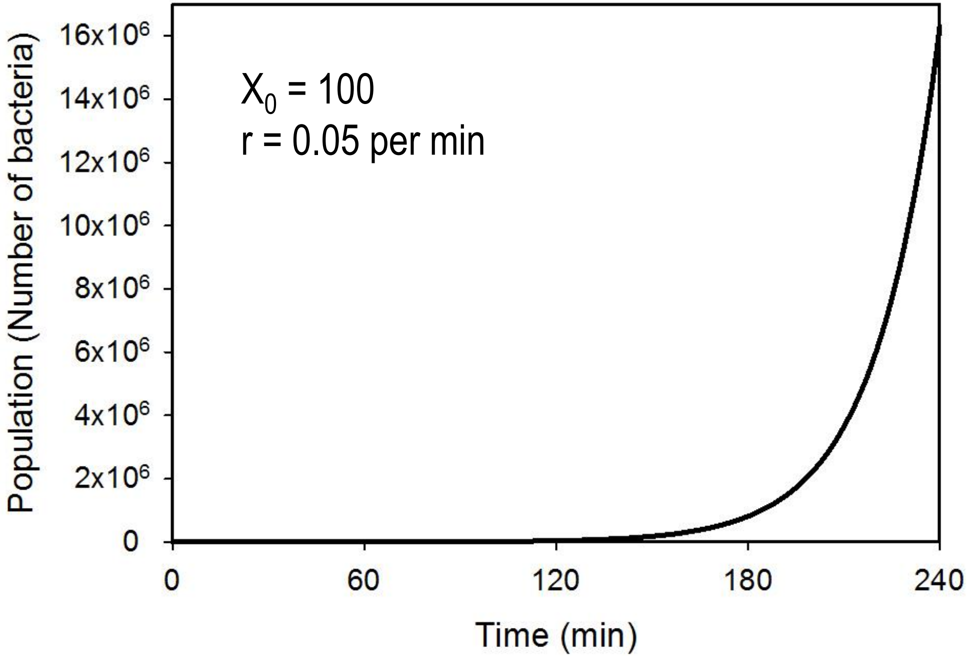
$$\frac{dx}{dt} = r.x \xrightarrow{\text{Integration, with initial conditions}} x = x_0.e^{r.t}$$

Question 2

Overall dynamics of population

$$\frac{dx}{dt} = r.x$$

$$x = x_0 .e^{r.t}$$



Growth can not be infinite

Environment has limited resources.

It imposes constrain on growth.

There must be a maximum size of the population

The logistic growth model:

$$\frac{dx}{dt} = r.(1 - \frac{x}{k}).x$$

x = population at time *t*

r = rate constant for growth

k = carrying capacity

Integrate to get the function

$$\frac{dx}{dt} = r.(1-\frac{x}{k}).x$$

$$\Rightarrow k\frac{dx}{dt} = r.(k-x).x$$

$$\Rightarrow \frac{k}{(k-x).x}dx = r.dt$$

$$\Rightarrow \frac{dx}{x} + \frac{dx}{k-x} = r.dt$$

$$\int_{x_0}^{x}\frac{dx}{x} + \int_{x_0}^{x}\frac{dx}{k-x} = r.\int_0^t dt$$

$$\Rightarrow [\ln x]_{x_0}^{x} - [\ln(k-x)]_{x_0}^{x} = r.[t]_0^t$$

$$\Rightarrow \ln(x) - \ln(x_0) - \ln(k-x) + \ln(k-x_0) = r.t$$

$$\Rightarrow \ln(\frac{x.(k-x_0)}{x_0.(k-x)}) = r.t$$

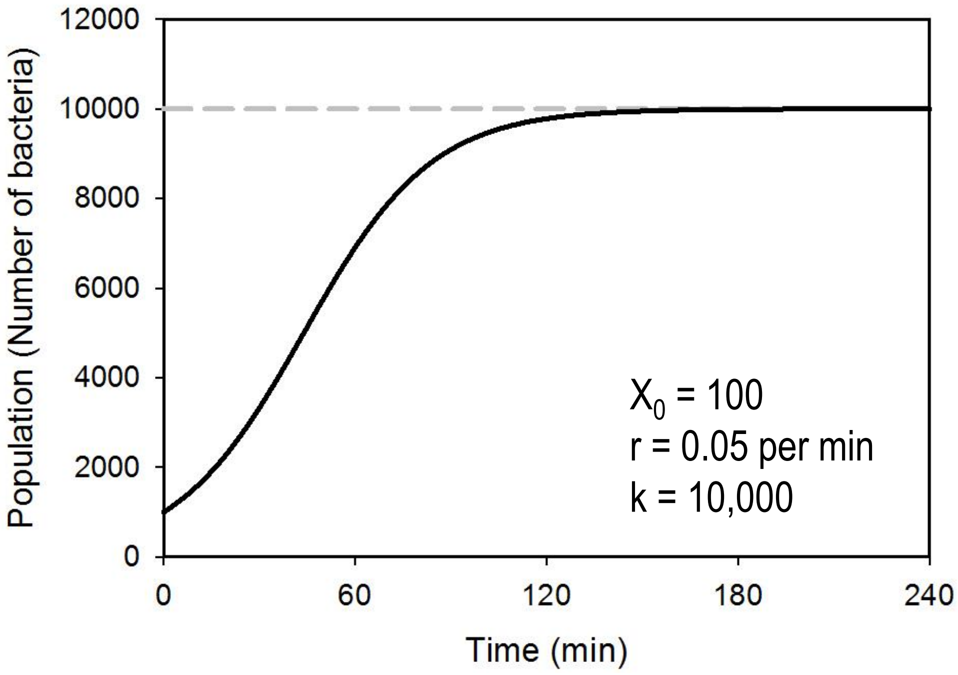
$$\Rightarrow \frac{x.(k-x_0)}{x_0.(k-x)} = e^{r.t}$$

$$\Rightarrow x = \frac{k}{1+(\frac{k}{x_0}-1)e^{-r.t}}$$

Population dynamics in logistic growth

$$\frac{dx}{dt} = r.(1-\frac{x}{k}).x$$

$$x = \frac{k}{1+(\frac{k}{x_0}-1)e^{-r.t}}$$



Key points:

1. Always start with simplest model
2. Represent the rate of the process by an ODE
3. Integrate the ODE to get the function describing dynamics of the process
4. Answer questions using this function
5. If required, do simplest modification to the model to add further complexity or to make it more realistic
6. Logistic equation has sigmoidal behavior and suitable for constrained growth processes