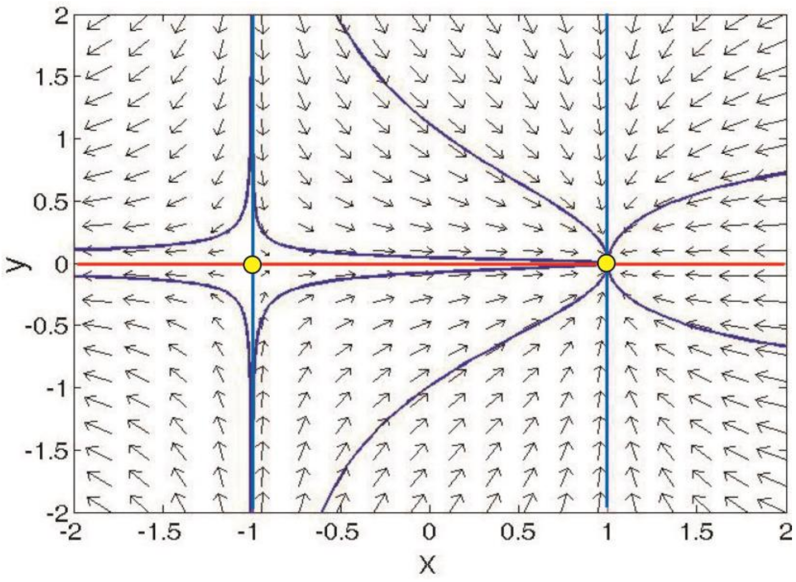


Concepts of Bifurcation

How parameters affect steady states

$$\frac{dx}{dt} = m - x^2$$

At steady state,

$$\frac{dx}{dt} = m - x^2 = 0$$

$$\Rightarrow x^2 = m$$

$$\Rightarrow x = \pm\sqrt{m}$$

When, $m > 0$

x has two steady states,

$$x = +\sqrt{m} \;\; \& \;\; x = -\sqrt{m}$$

When, $m = 0$

x has only one steady state,

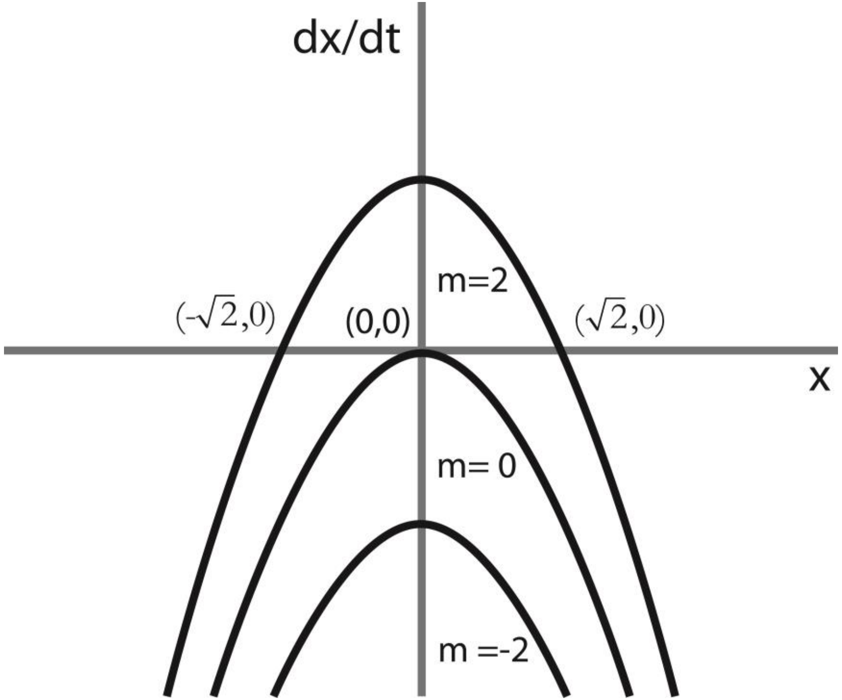
$x = 0$

When, $m < 0$

x has no real steady states

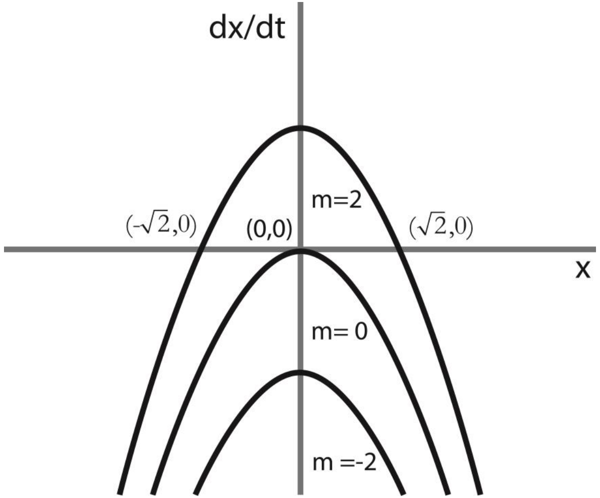
Effect of parameter on steady states

$$\frac{dx}{dt} = m - x^2$$

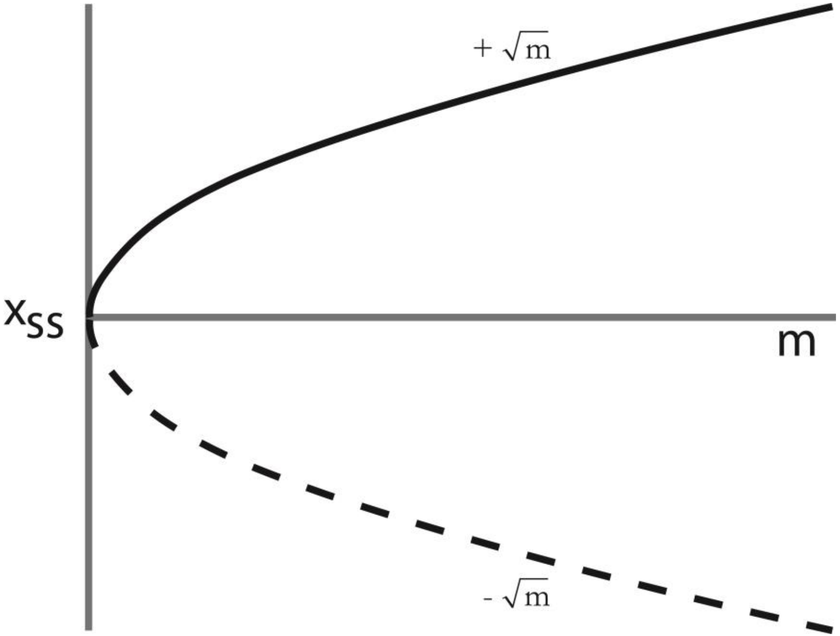


Effect of parameter on steady states

$$\frac{dx}{dt} = m - x^2$$



Bifurcation diagram



An example in a system of ODEs

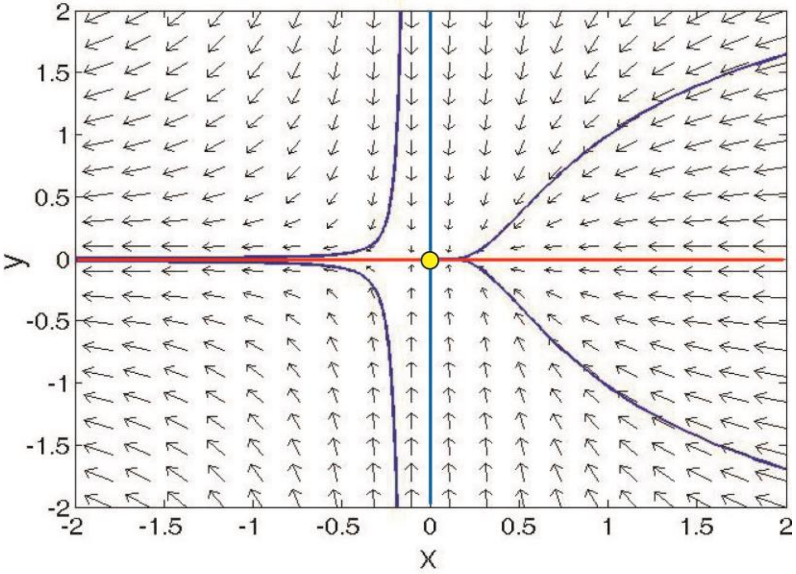
$$\frac{dx}{dt} = m - x^2 \qquad \frac{dy}{dt} = -y$$

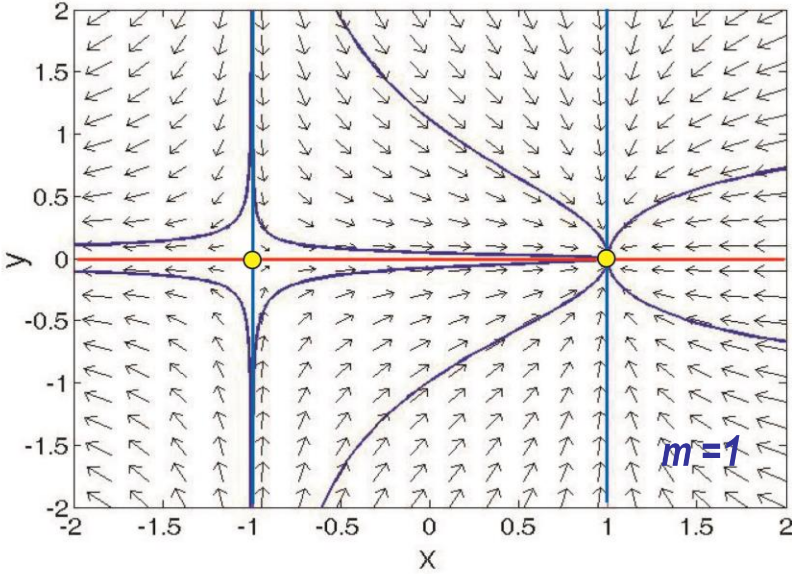
m < 0; No real steady state

m = 0; steady state at (0,0) → Saddle-node type

m > 0;
Steady state at $(-\sqrt{m}\ ,0)$ → Saddle point

Steady state at $(+\sqrt{m}\ ,0)$ → Stable node/sink





Bifurcation

If the variation of a parameter changes the **qualitative behavior** of the steady state(s), we call it bifurcation.

By qualitative behavior, we mean

a) **number of steady states**

b) **stability of the steady states**

Change in either of these two or both, will change the phase portrait. Therefore, bifurcation in a sense is change in the phase portrait of the system with change in a parameter.

Bifurcation in fish tank

The ODE

$$\frac{dx}{dt} = r(1-\frac{x}{k})x - d$$

At steady state $\frac{dx}{dt} = 0$

$$\therefore \quad r(1-\frac{x}{k})x - d = 0$$

$$\Rightarrow r.x - \frac{r.x^2}{k} - d = 0$$

$$\Rightarrow r.x^2 - k.r.x + k.d = 0$$

So at steady state,

$$x = \frac{kr \pm \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$$

Bifurcation in fish tank

The ODE

$$\frac{dx}{dt} = r(1 - \frac{x}{k})x - d$$

At steady state,

$$x = \frac{kr \pm \sqrt{k^2 r^2 - 4.r.k.d}}{2.r}$$

Case 1: d < 0 is not possible as d is rate of removal of fish

Case 2: For d = 0,

$$x = \frac{kr \pm \sqrt{k^2 r^2 - 4.r.k.0}}{2.r}$$

$$\Rightarrow x = \frac{kr \pm \sqrt{k^2 r^2}}{2.r}$$

$$\Rightarrow x = \frac{kr \pm kr}{2.r} = \frac{k \pm k}{2}$$

Steady state values of *x* are *k* and *0*

Bifurcation in fish tank

The ODE

$$\frac{dx}{dt} = r(1-\frac{x}{k})x - d$$

At steady state,

$$x = \frac{kr \pm \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$$

Case 3: When d > 0

Real solution exist iff $k^2r^2 \geq 4rkd$

$$or \quad d \leq \frac{kr}{4}$$

When $d = \frac{kr}{4}$

Steady state values of *x* is *k/2*

When $d < \frac{kr}{4}$

x has two steady states:

$$x = \frac{kr + \sqrt{k^2r^2 - 4.r.k.d}}{2.r} \qquad x = \frac{kr - \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$$

Bifurcation in fish tank

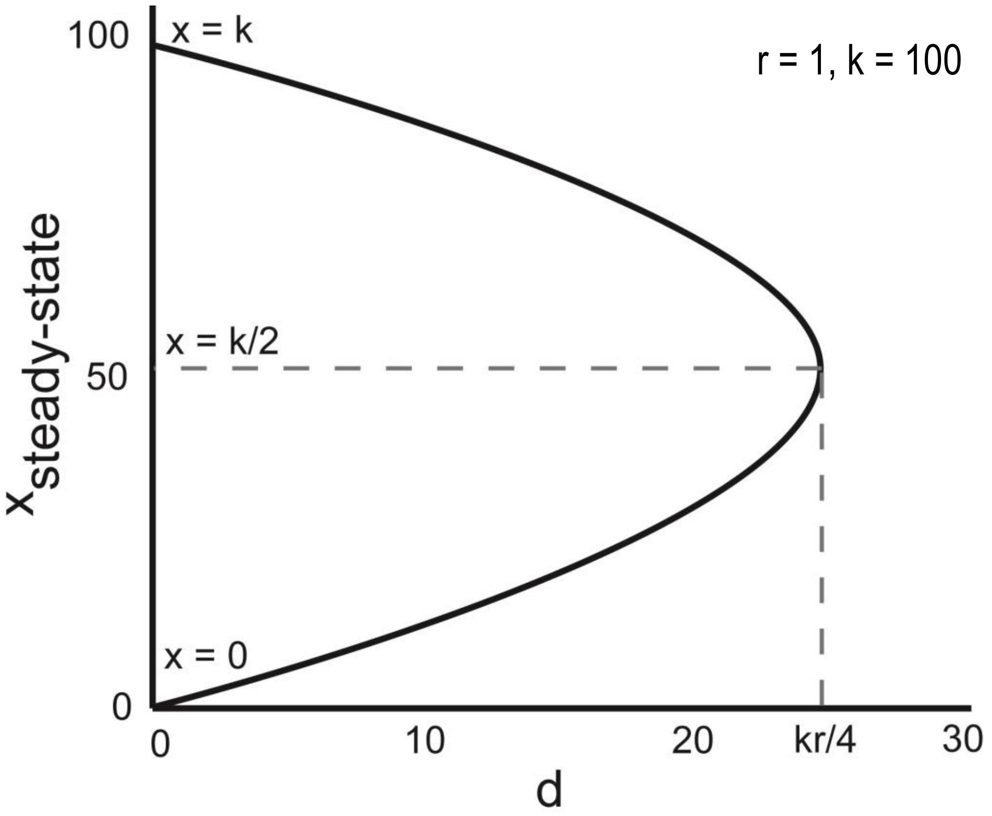
The ODE

$$\frac{dx}{dt} = r(1-\frac{x}{k})x - d$$

At steady state,

$$x = \frac{kr \pm \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$$

Bifurcation diagram



Bifurcation in fish tank

At steady state, $x = \dfrac{kr \pm \sqrt{k^2 r^2 - 4.r.k.d}}{2.r}$

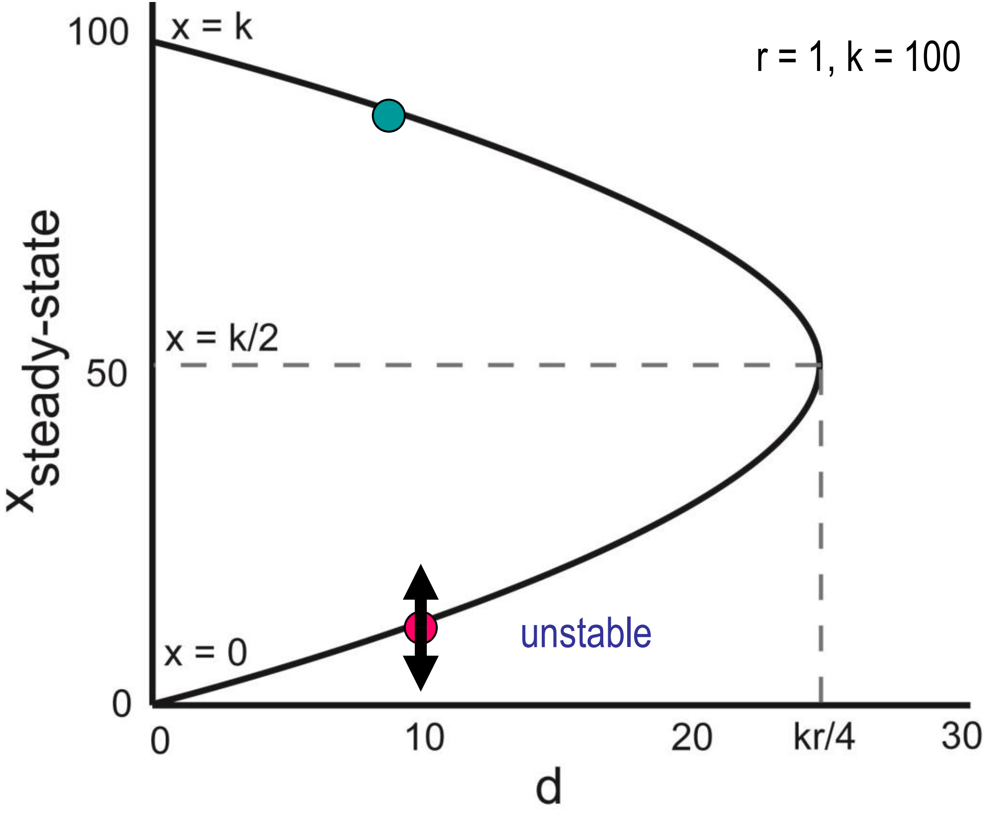
Let *d* = *10*

Steady states are, *x* = *88.73* and *11.27*

$$\frac{dx}{dt} = r(1 - \frac{x}{k})x - d$$

x	dx/dt	sign	arrow
12	0.56	(+)ve	↑
11.27	0		→
10	-1	(-)ve	↓

Bifurcation diagram



Bifurcation in fish tank

At steady state, $x = \dfrac{kr \pm \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$

Let $d = 10$

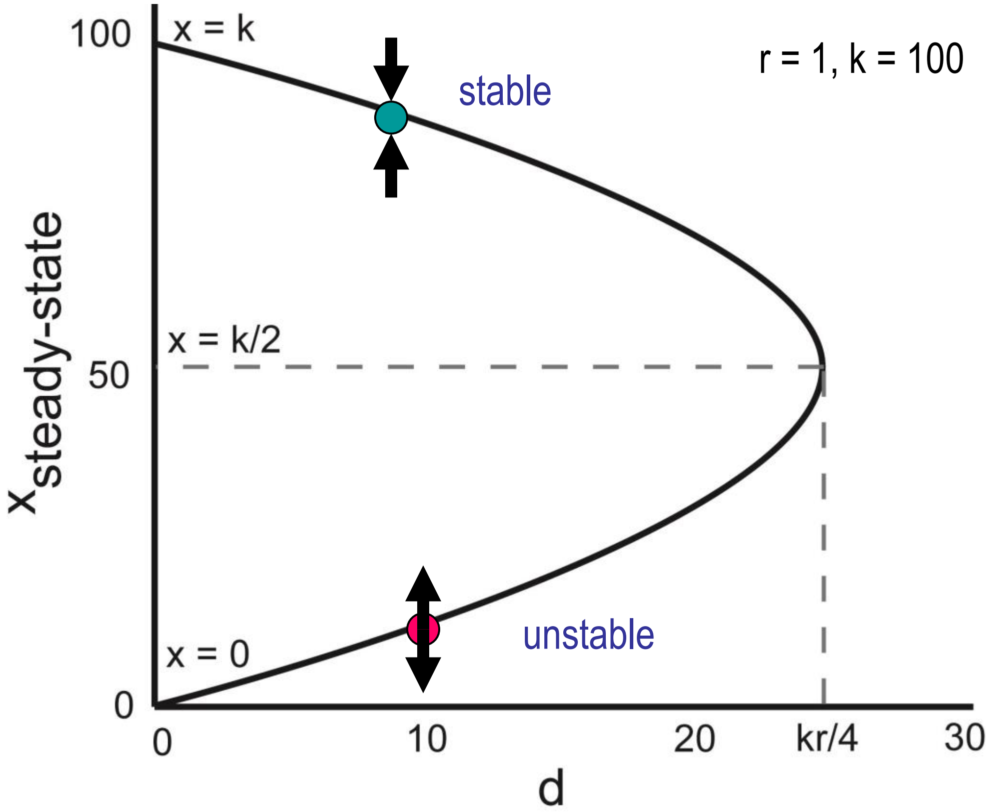
Steady states are, $x = 88.73$ and 11.27

$$\frac{dx}{dt} = r(1 - \frac{x}{k})x - d$$

x	dx/dt	sign	arrow
90	-1	(-)ve	↓
88.73	0		→
85	2.75	(+)ve	↑

Bifurcation diagram

r = 1, k = 100



Bifurcation in fish tank

The ODE

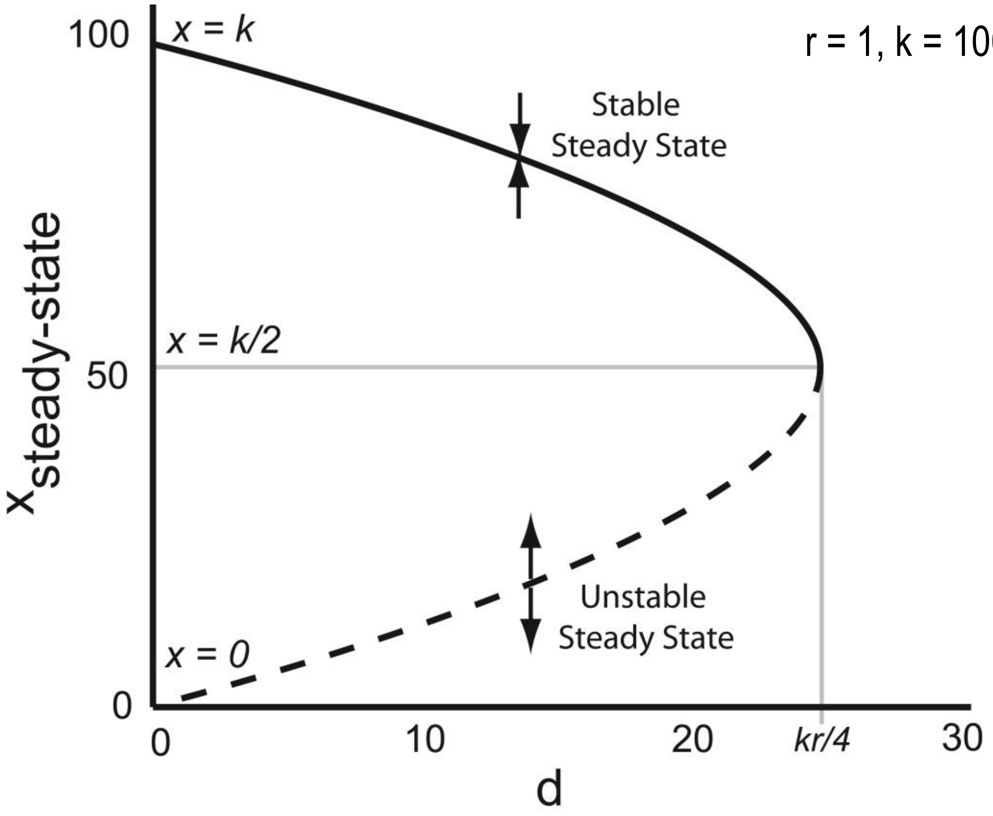
$$\frac{dx}{dt} = r(1-\frac{x}{k})x - d$$

At steady state,

$$x = \frac{kr \pm \sqrt{k^2r^2 - 4.r.k.d}}{2.r}$$

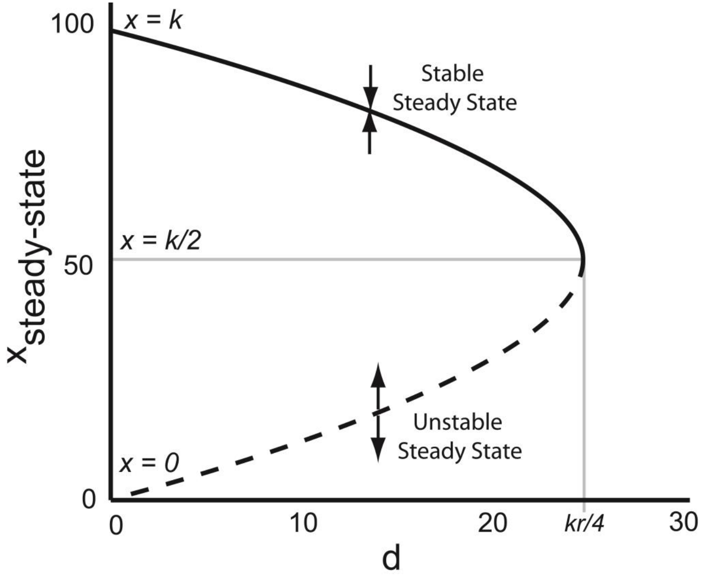
Bifurcation diagram

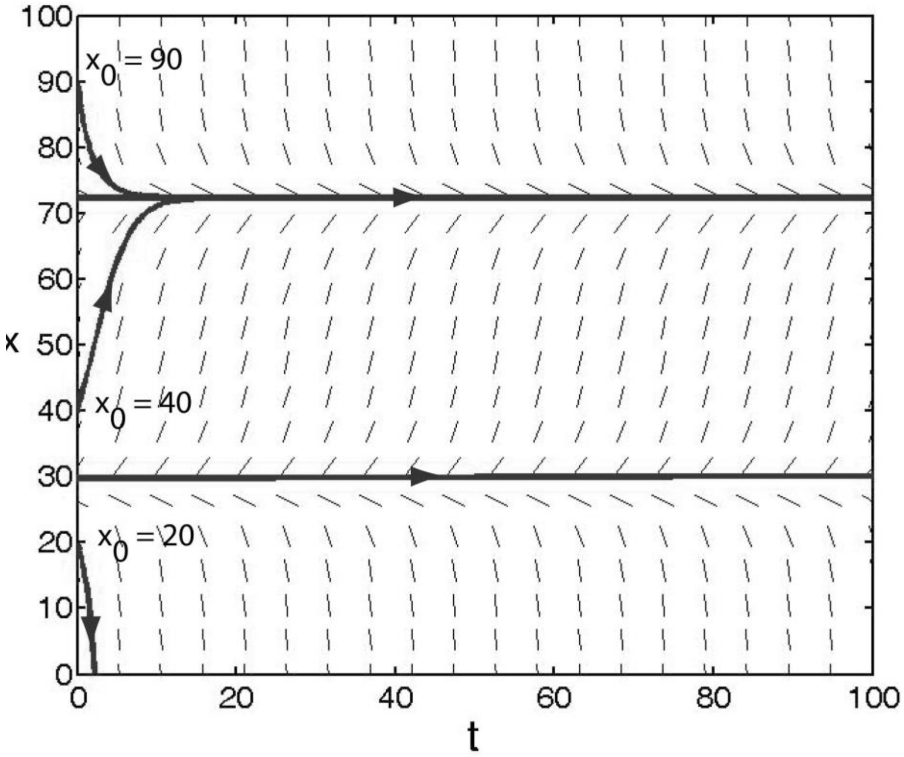
r = 1, k = 100



Effect of bifurcation on dynamics of fish population

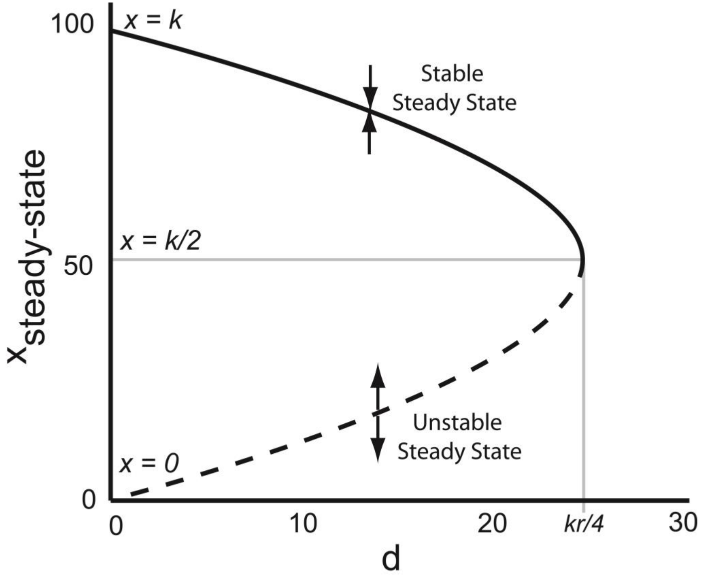
r = 1, k = 100, **d = 20**

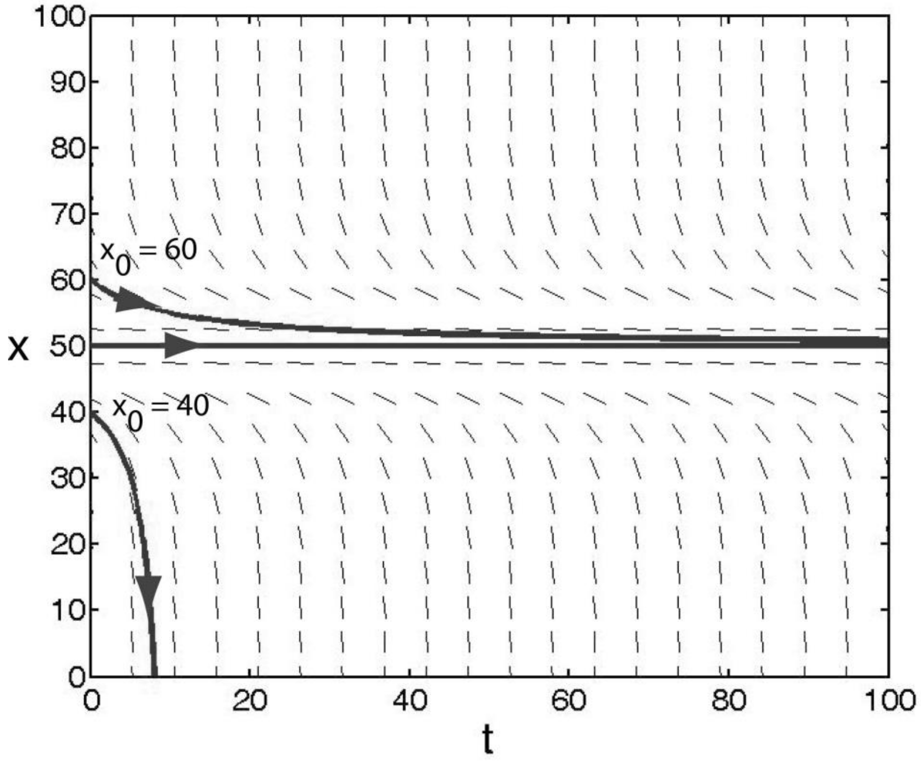




Effect of bifurcation on dynamics of fish population

r = 1, k = 100, **d = 25**





Key points:

1. Change in value of a parameter may affect: a) Number of steady states, b) Stability of steady state and c) both a and b.
2. Such change in qualitative behavior of the system due to change in parameter value is Bifurcation
3. Bifurcation diagram shows the effect of the bifurcation parameter on number and stability of steady states.