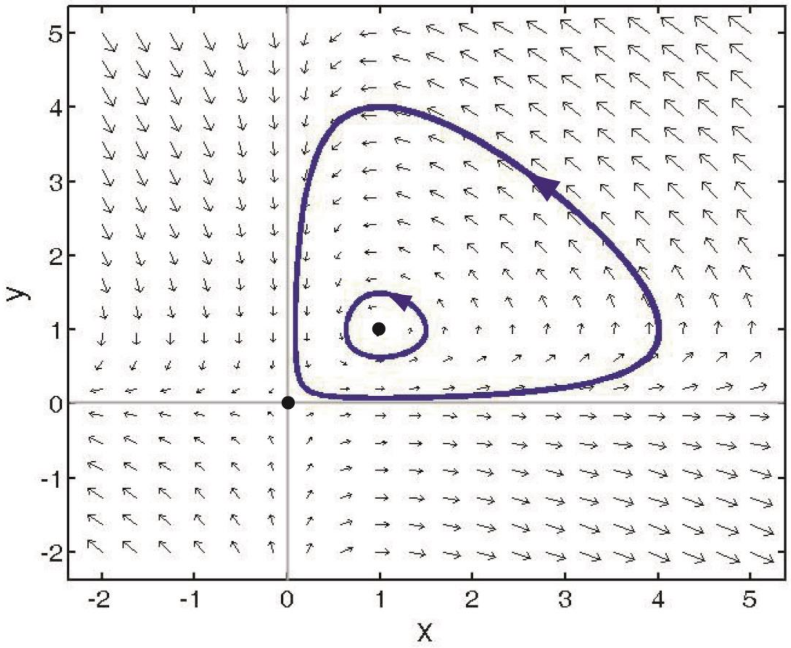


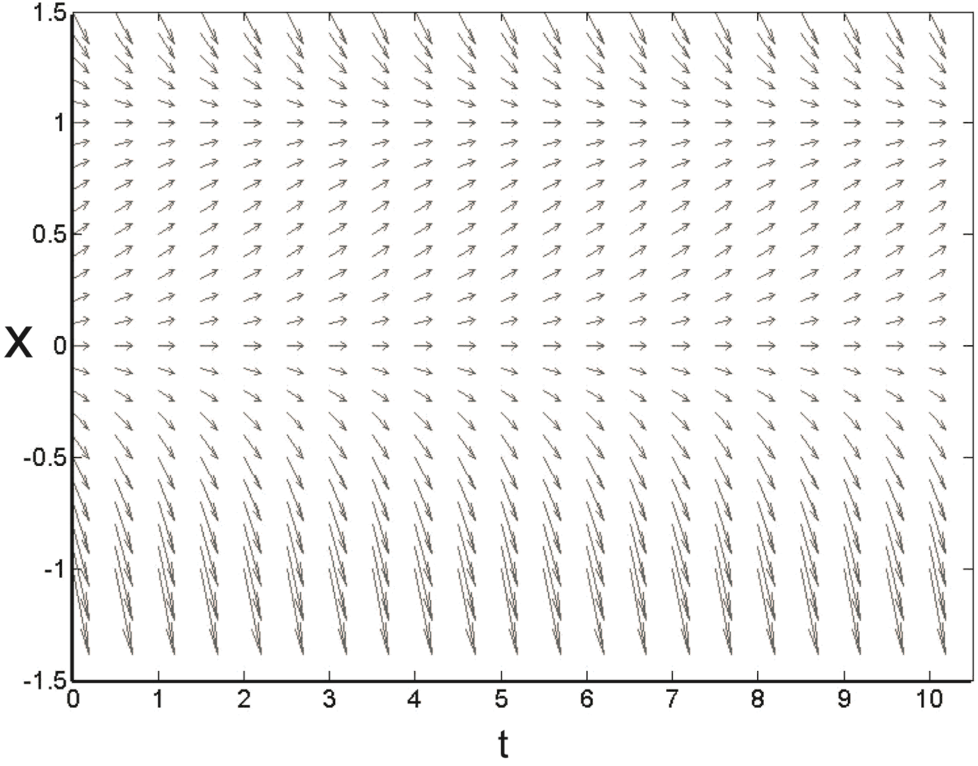

# Phase Plane Analysis-II

## Representing the dynamics for a single of ODEs

For a system with one dependent variable, we use direction field.

Take grid points at equal distance and draw an arrow at each grid point,

Slope of the arrow = $\dfrac{dx}{dt}$

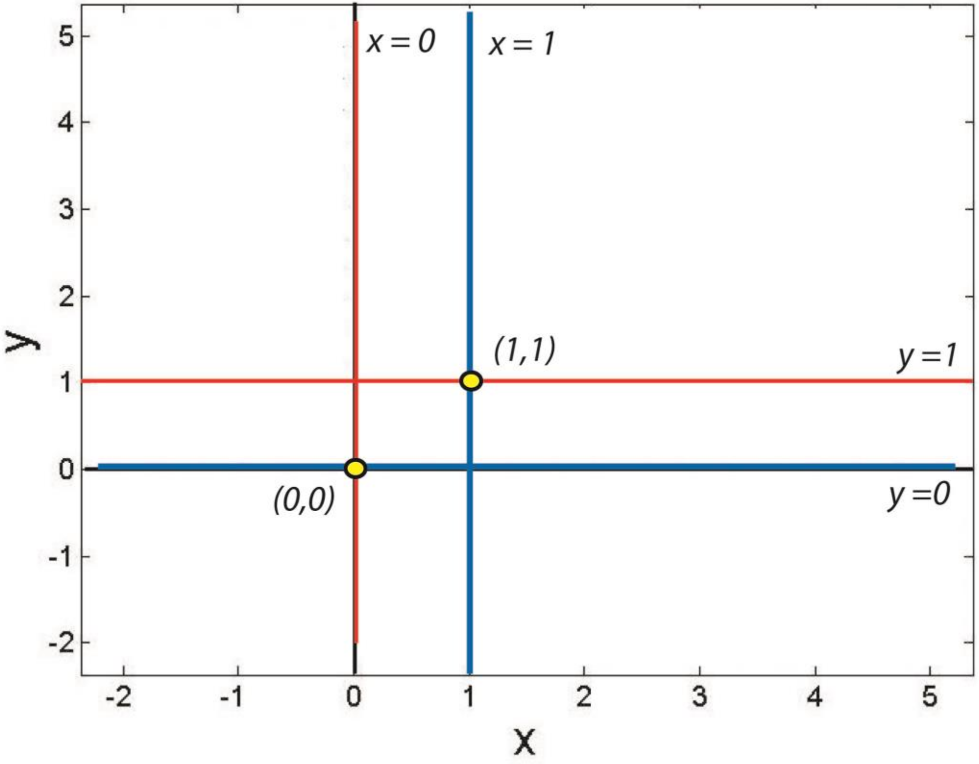


# Representing the dynamics for a system of ODEs

For a system with ODES, we have phase plane ( for example: *y vs x*)

Make a grid and take a grid point.

Slope of an arrow at a grid point will represent $\dfrac{dy}{dx}$



# Representing the dynamics for a system of ODEs

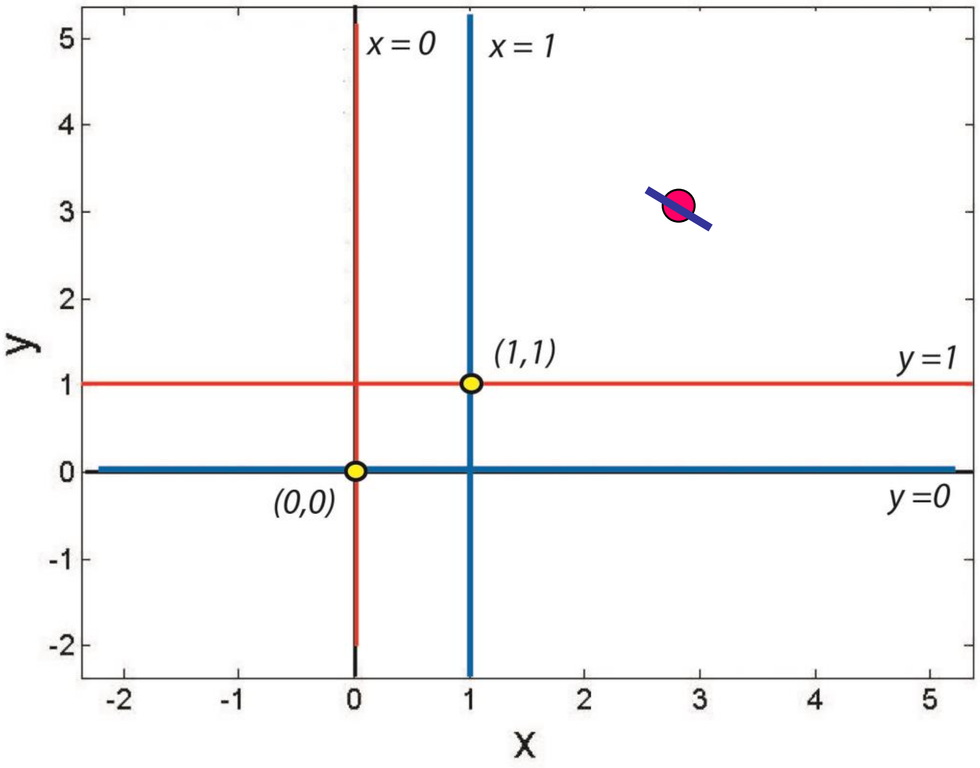
Calculating *dy/dx*

$$\frac{dx}{dt} = x - x.y \; ; \qquad \frac{dy}{dt} = x.y - y$$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} \qquad = \frac{xy - y}{x - xy}$$

For point (3,3)

$$\frac{dy}{dx} = \frac{xy - y}{x - xy} = \frac{3\times 3 - 3}{3 - 3\times 3} = -1$$



# Representing the dynamics for a system of ODEs

Decide the arrow head

$$\text{when } \frac{dy}{dt} > 0 \text{ and } \frac{dx}{dt} > 0 \quad : \nearrow$$

$$\text{when } \frac{dy}{dt} > 0 \text{ and } \frac{dx}{dt} < 0 \quad : \nwarrow$$

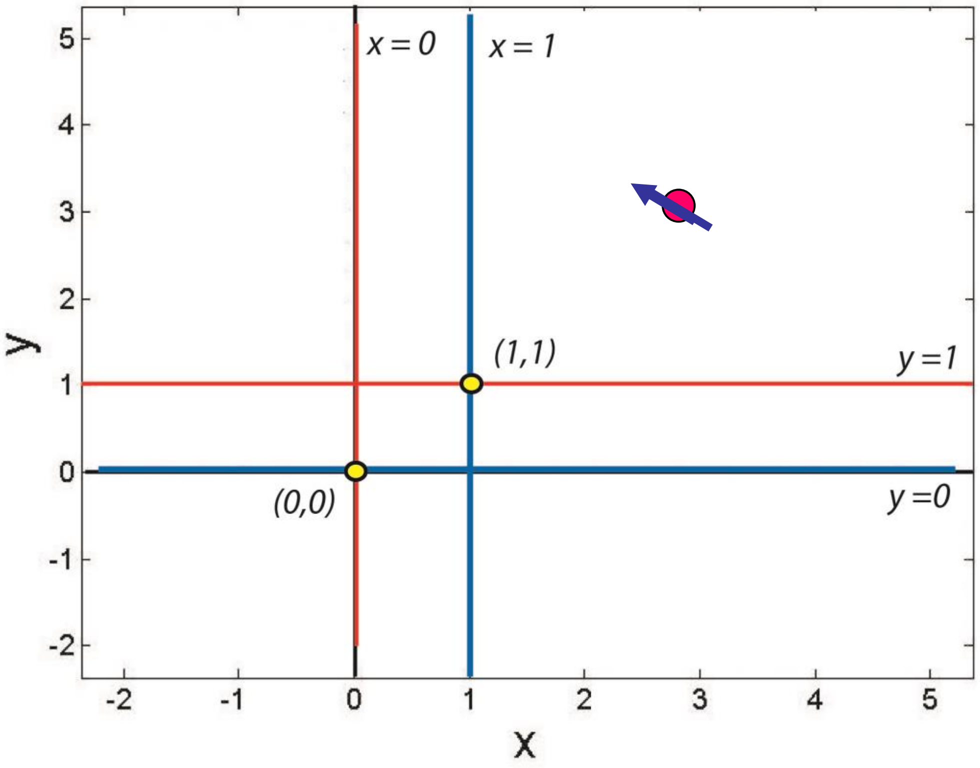
$$\text{when } \frac{dy}{dt} < 0 \text{ and } \frac{dx}{dt} < 0 \quad : \swarrow$$

$$\text{when } \frac{dy}{dt} < 0 \text{ and } \frac{dx}{dt} > 0 \quad : \searrow$$

For point (3,3)

$$\frac{dy}{dt} = x.y - y = 3\times 3 - 3 = 6 > 0$$

$$\frac{dx}{dt} = x - x.y = 3 - 3\times 3 = -6 < 0$$

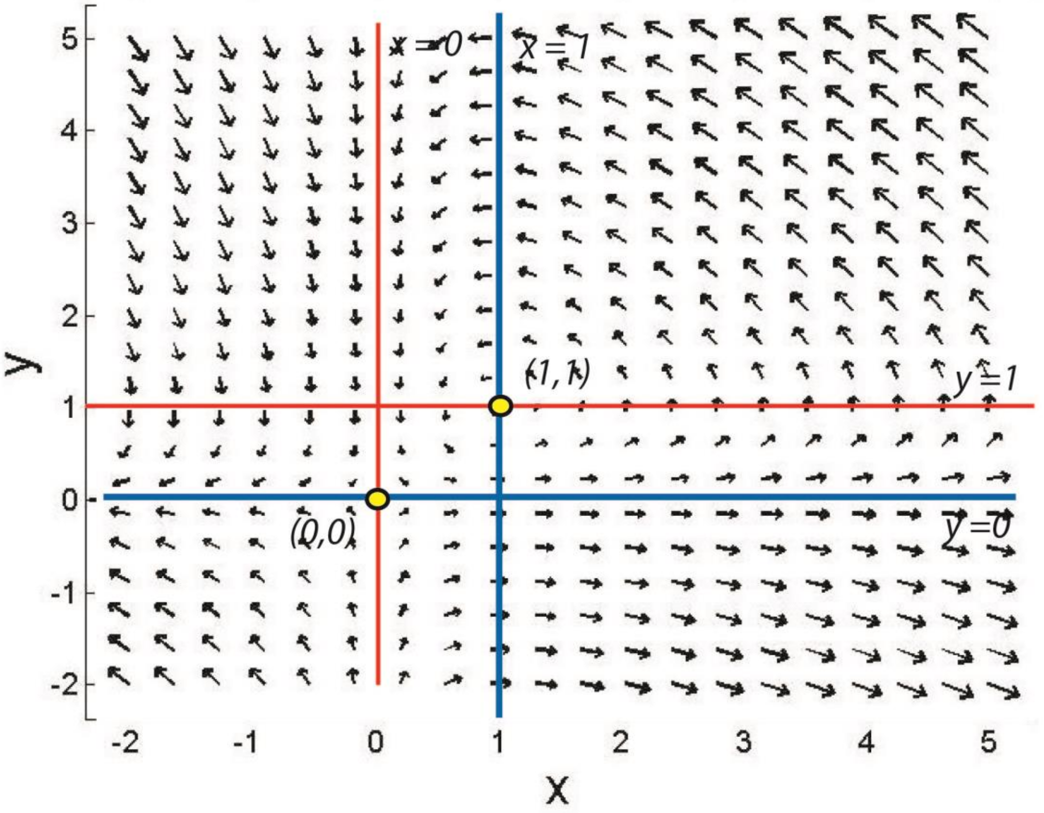


# Representing the dynamics for a system of ODEs

The Phase portrait

$$\frac{dx}{dt} = x - x.y \ ;$$

$$\frac{dy}{dt} = x.y - y$$



## Representing the dynamics for a system of ODEs

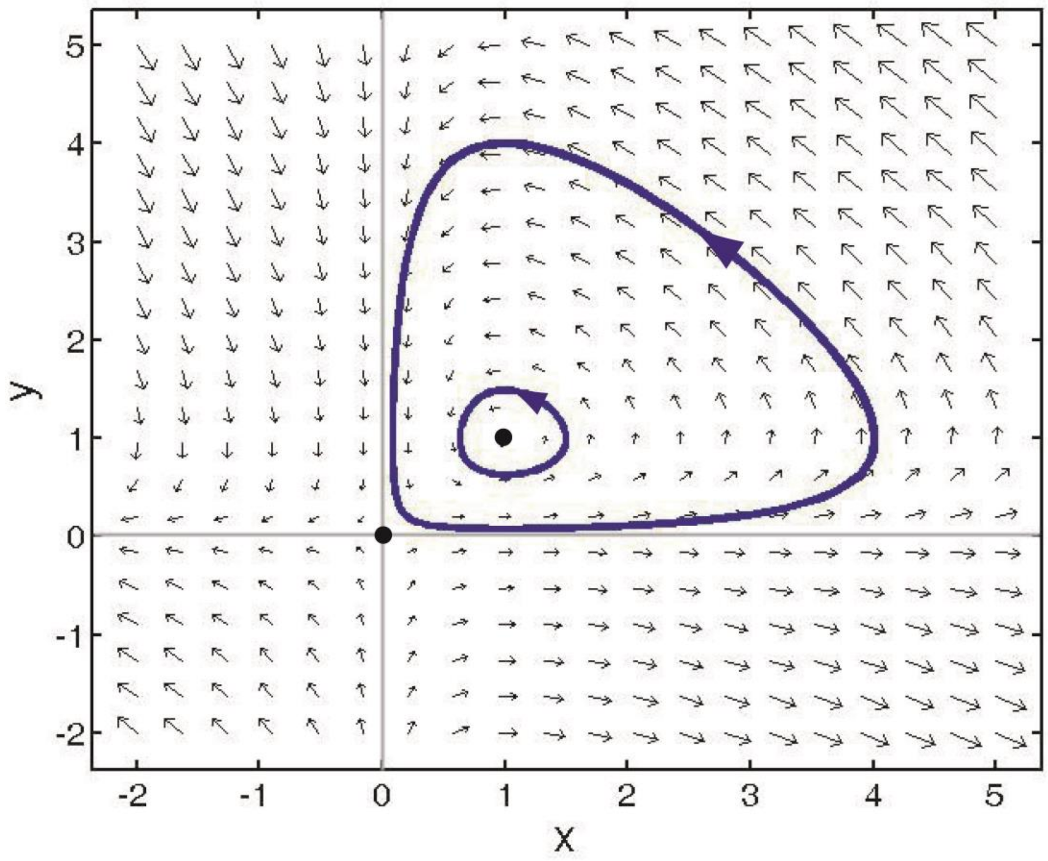
Following the arrows on the phase portrait, we get trajectory

Trajectory → Time evolution of the system

Properties of steady states :

*(0, 0)* → Saddle point

*(1, 1)* → Center type

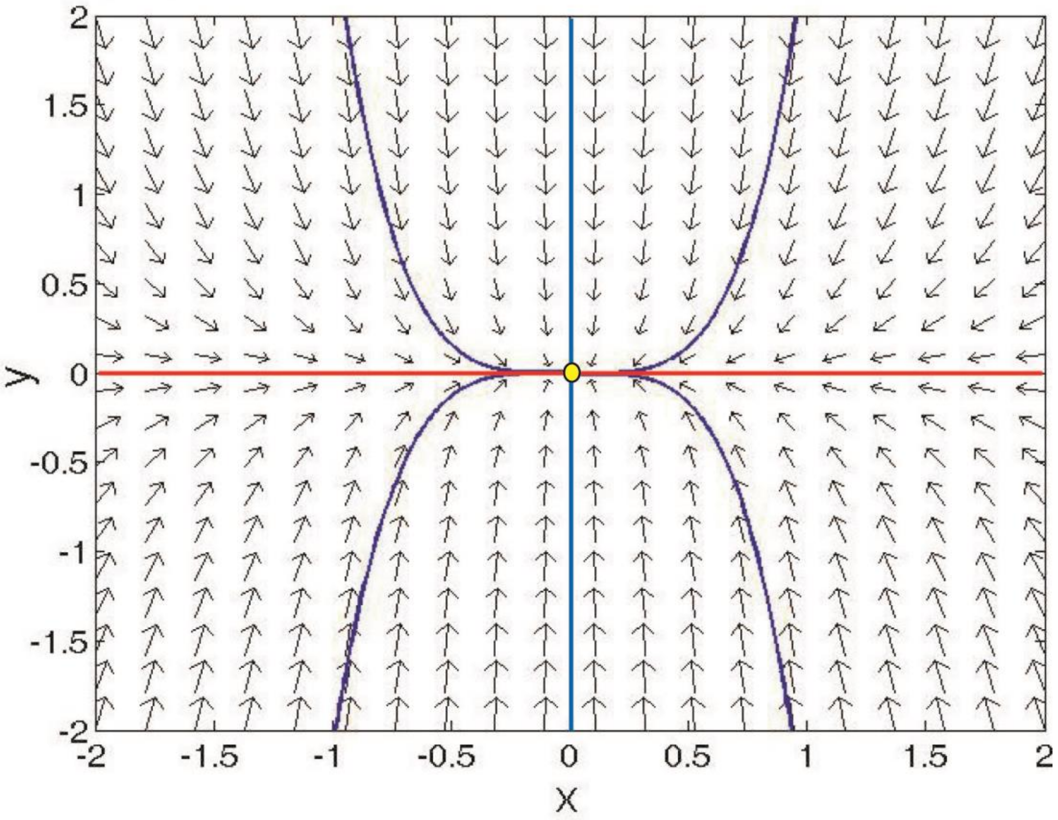


## Phase Portrait

Sink node or stable node

$$\frac{dx}{dt} = -x$$

$$\frac{dy}{dt} = -4.y$$

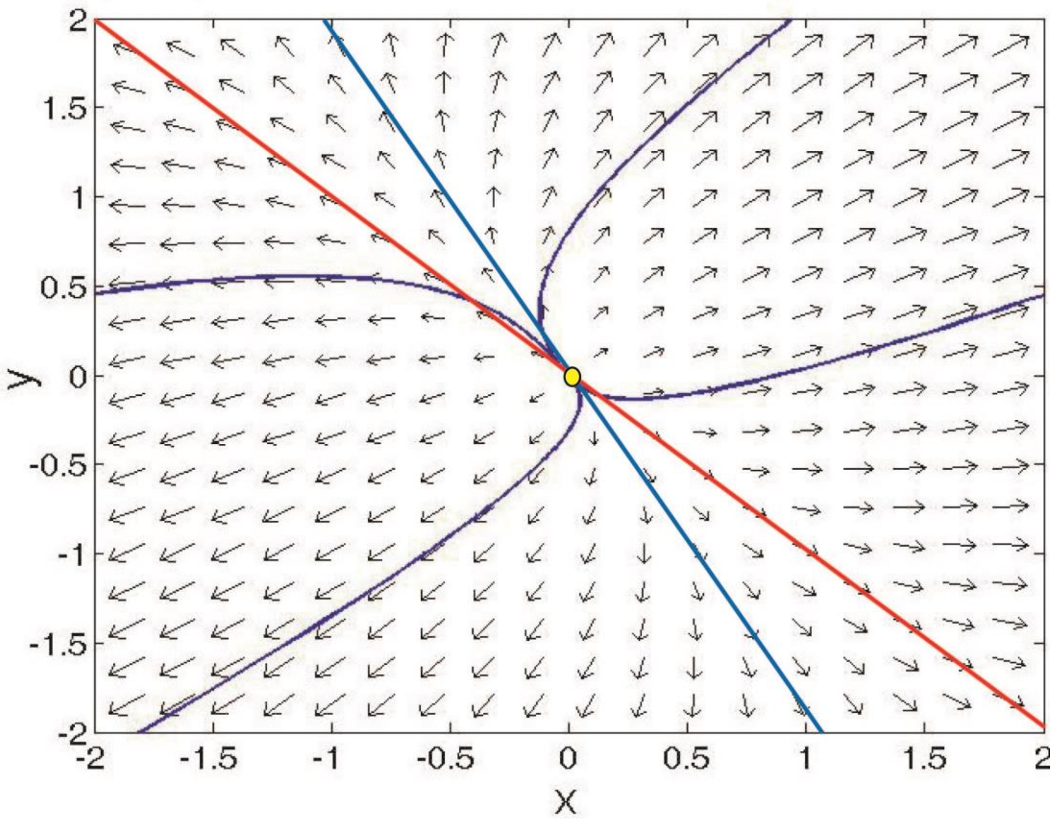


# Phase Portrait

Source node or unstable node

$$\frac{dx}{dt} = 2.x + y$$

$$\frac{dy}{dt} = 2.x + 2.y$$



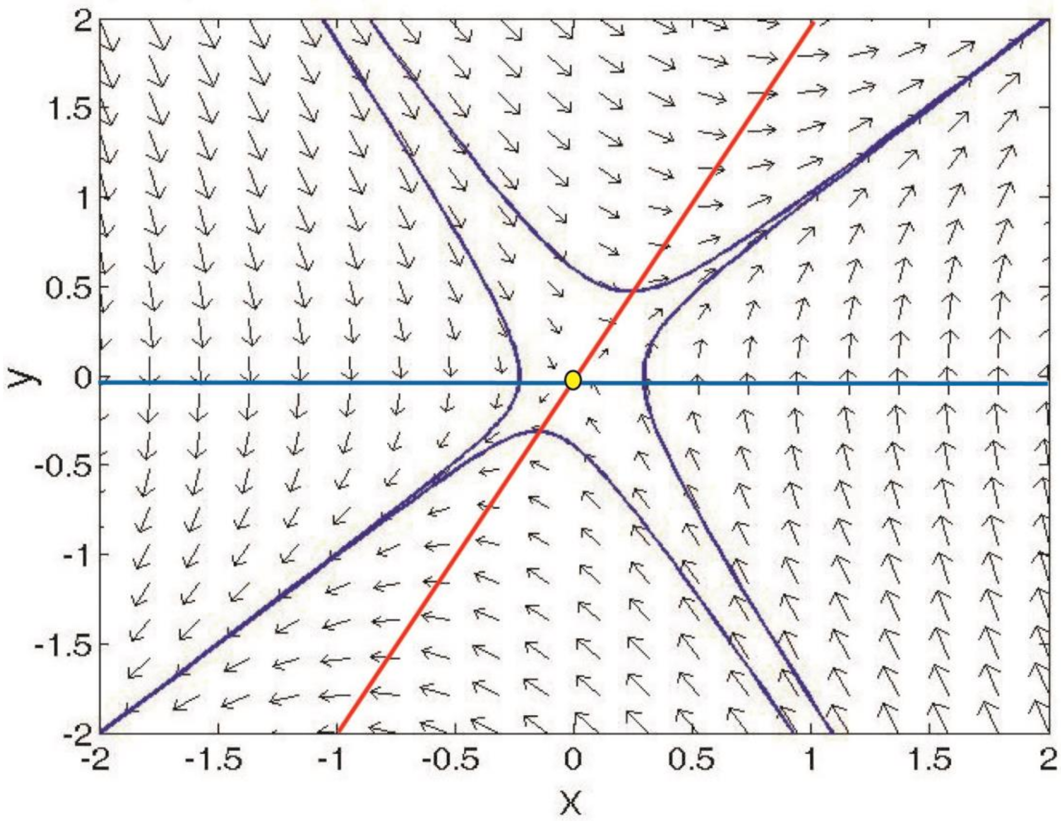
# Phase Portrait

Saddle point

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = 2.x - y$$

From one direction trajectories move towards the steady state and move away from it in other direction



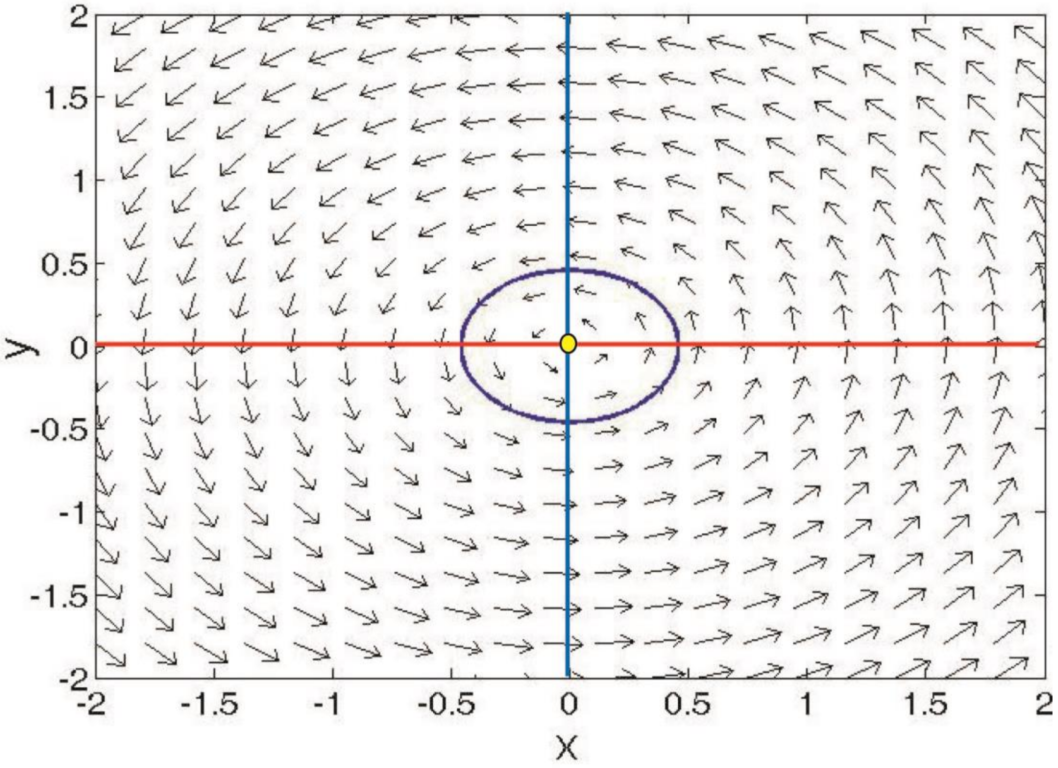
## Phase Portrait

Center type

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

Trajectories move around the steady state in closed paths.



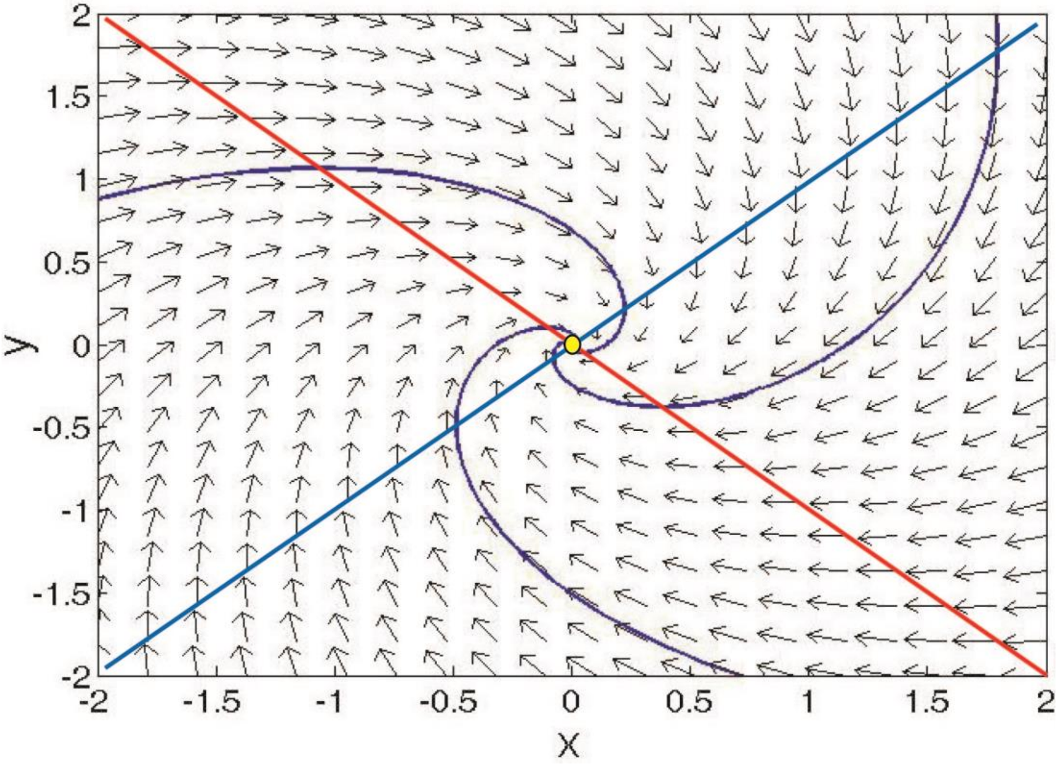
## Phase Portrait

Stable spiral or spiral sink

$$\frac{dx}{dt} = -x + y$$

$$\frac{dy}{dt} = -x - y$$

Spiral trajectories collapse at the steady state



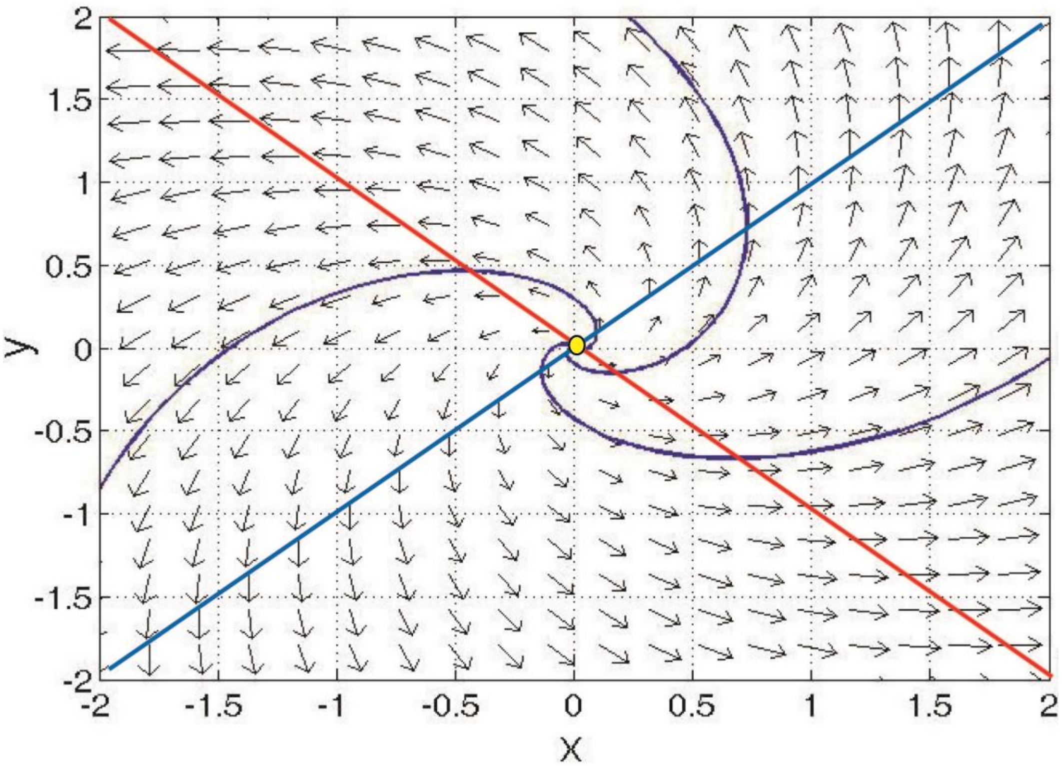
## Phase Portrait

Unstable spiral or spiral source

$$\frac{dx}{dt} = x - y$$

$$\frac{dy}{dt} = x + y$$

Spiral trajectories originate from steady state and diverge

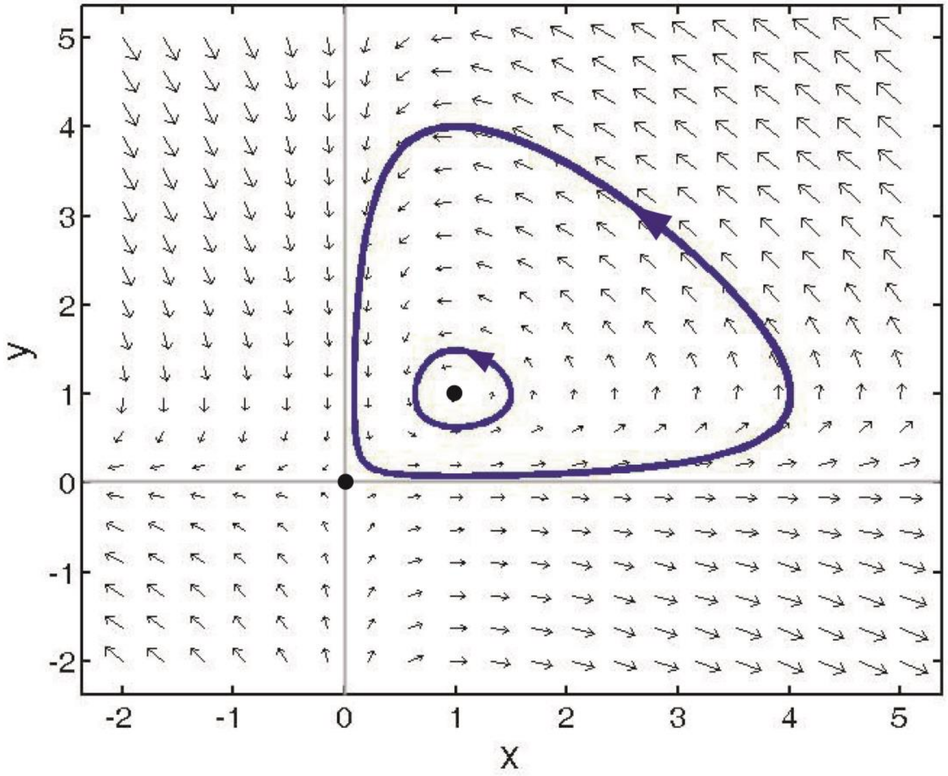


## Phase portrait for nonlinear system

- More than one steady states

- Combination of trajectories around

each steady state → Complete phase portrait



## Key points:

1. For a system of ODEs, dynamics of the system is visualized in phase portrait.
2. A phase portrait is a geometric representation of the trajectories in the phase plane
3. It tells how the system evolves around the steady state
4. To create the phase portrait: take equidistant grid points on phase plane and place arrows on each grid point with slope equal to derivate of one dependent variable with respect to another.
5. There are several types of phase portraits for linear systems: stable/sink node, unstable/source node, saddle, center, stable/sink spiral, unstable/source spiral
6. Phase portrait of a nonlinear system is combination of trajectories around each steady states
7. **Phase portraits, for both linear and nonlinear systems can be predicted/analyzed using algebraic method.**