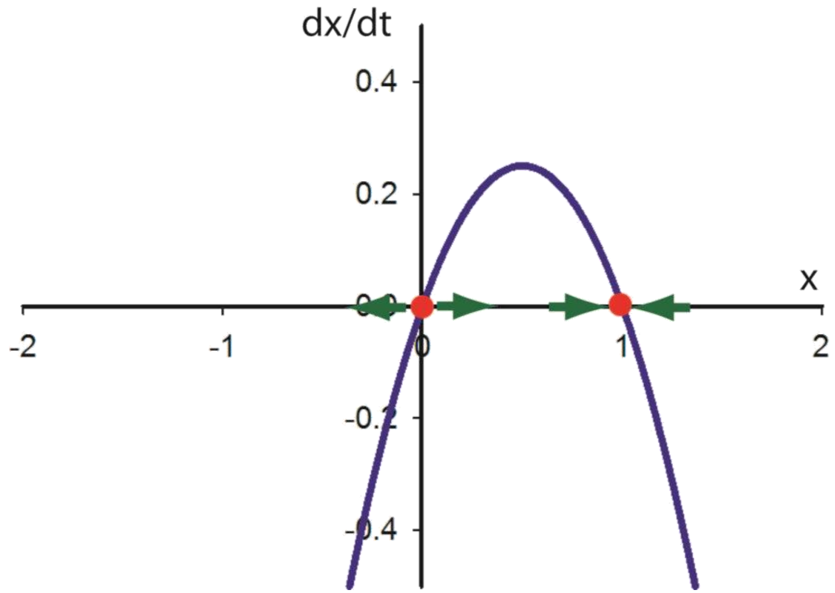


Stability of Steady States

Steady states

At steady state, dependent variable does not change with time and remains constant

For the ODE, $\frac{dx}{dt} = f(x,t)$

At steady state, $\frac{dx}{dt} = 0$

To find steady states, set

$$\frac{dx}{dt} = f(x,t) = 0$$

$$\Rightarrow f(x,t) = 0$$

Solve this relation algebraically to find steady state values of *x*

Question of stability

For the ODE, $\frac{dx}{dt} = f(x,t)$ x is at steady state, when $\frac{dx}{dt} = 0$

Say one steady state value of x is x_{ss}

Question on stability of a steady state: If x is perturbed from its steady state value x_{ss} , with time, will it return to x_{ss} or move away from x_{ss}?

Stability also defines time evolution of x around a steady state

Stability of steady states

Spread of infection model

$$\frac{dx}{dt} = r.x.(1-x)$$

Consider $r = 1$

At steady state, $\frac{dx}{dt} = 0$

$$\therefore \quad r.x.(1-x) = 0$$

$$\therefore \text{ Either } x = 0 \text{ or } (1-x) = 0$$

$$\text{When } (1-x) = 0,$$

$$x = 1$$

So x has two steady states, $x_{ss} = 0$ and 1

Stability of steady states

Analysis of direction field of

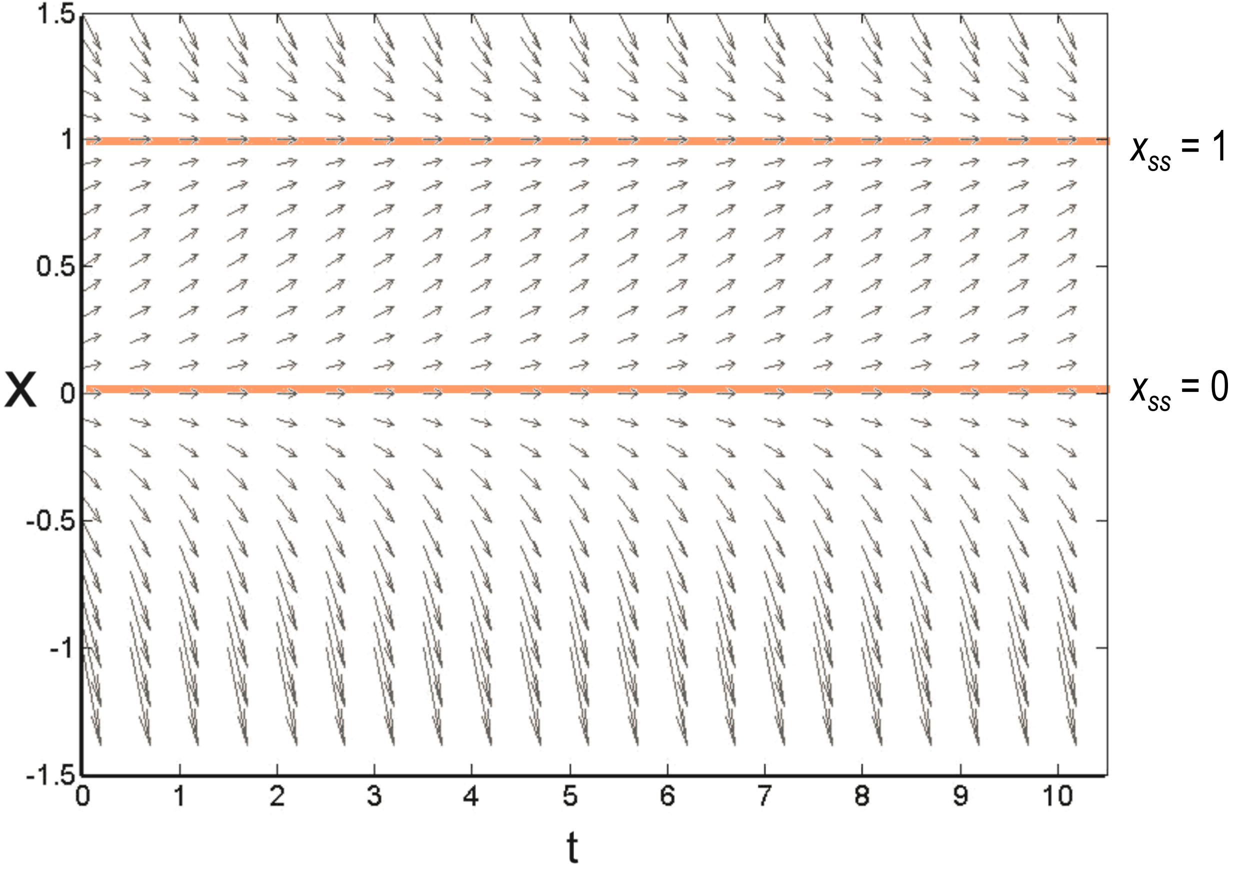
$$\frac{dx}{dt} = r.x.(1-x)$$

If x is perturbed from $x = 1$, with time, it returns back to $x = 1$.

x =1 is stable steady state

If x is perturbed from $x = 0$, with time, it moves away from $x = 0$.

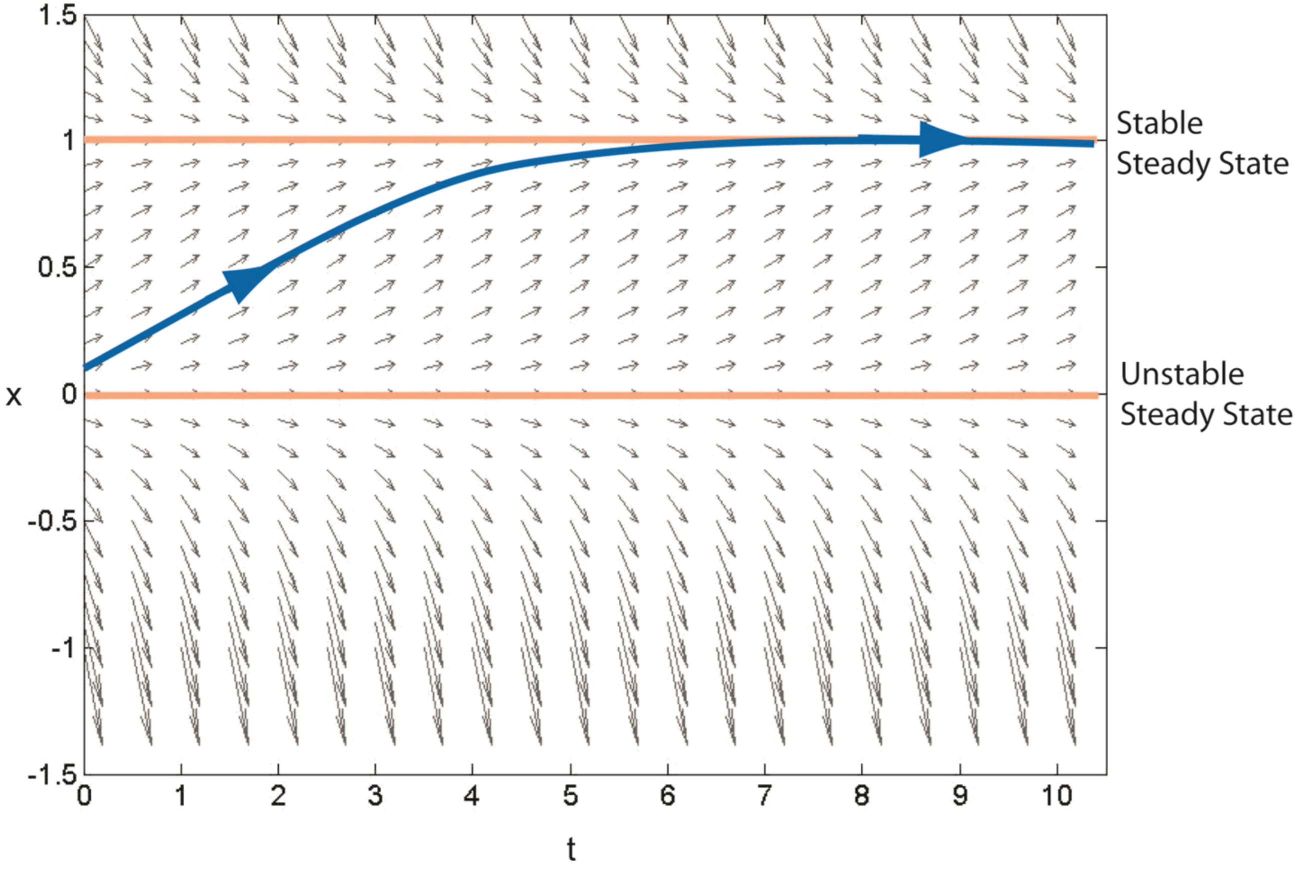
x = 0 is unstable steady state



Stability of steady states

Analysis of direction field of

$$\frac{dx}{dt} = r.x.(1-x)$$



Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$

With r = 1

Slighty higher value

Steady state value

Slighty lower value

x	dx/dt	sign	arrow

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$

With r = 1

x	dx/dt	sign	arrow
0	0		

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$ *With r = 1*

x	dx/dt	sign	arrow
0.1	0.09	(+)ve	↑
0	0		→

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$ *With r = 1*

x	dx/dt	sign	arrow
0.1	0.09	(+)ve	↑
0	0		→
-0.1	-0.11	(-)ve	↓

Both the arrows, for higher and lower values of *x*, are moving away from steady state.

So this steady state is unstable.

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$

With r = 1

x	dx/dt	sign	arrow
1	0		→

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$ *With r = 1*

x	dx/dt	sign	arrow
1.1	-0.11	(-)ve	$\downarrow$
1	0		$\rightarrow$

Checking stability numerically

$$\frac{dx}{dt} = r.x.(1-x)$$

With r = 1

x	dx/dt	sign	arrow
1.1	-0.11	(-)ve	$\downarrow$
1	0		$\rightarrow$
0.9	0.09	(+)ve	$\uparrow$

Both the arrows, for higher and lower values of *x*, pointing towards steady state.

So this steady state is stable.

Analyzing stability graphically

Plot x vs dx/dt curve.

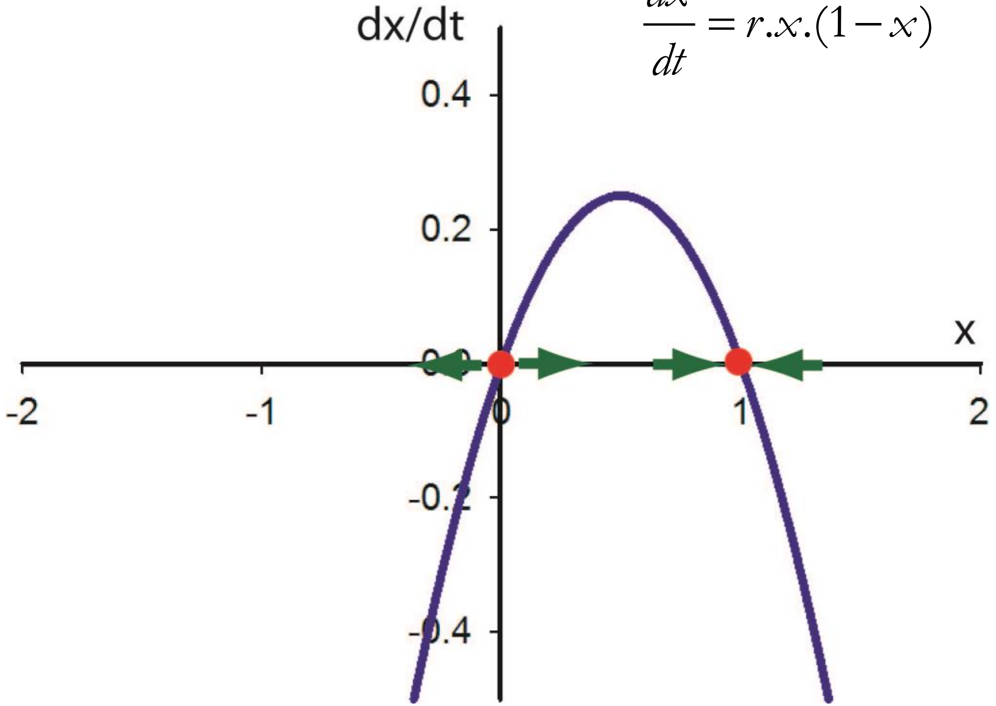
Steady states: Points of intersection of the curve with axes for x

Calculate the sign of dx/dt near each steady states. Place the arrows on the graph.

Stable steady state: both arrows points to steady state

Unstable steady state: both arrows points away from steady states.

$$\frac{dx}{dt} = r.x.(1-x)$$



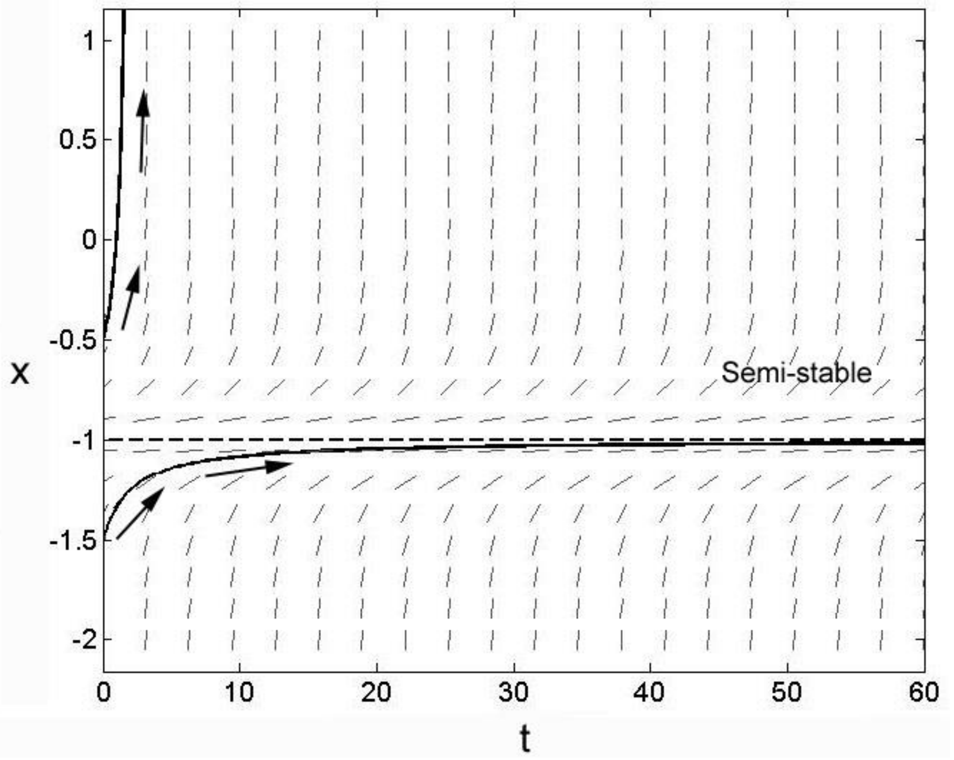
Semi-stable steady state

$$\frac{dx}{dt} = (x+1)^2$$

Steady state of *x* = -1

x	dx/dt	sign	arrow
-0.9	0.01	(+)ve	↑
-1	0		→
-1.1	0.01	(+)ve	↑

This steady state is semi-stable



Semi-stable steady state

$$\frac{dx}{dt} = (x+1)^2$$

Steady state of x = -1

x	dx/dt	sign	arrow
-0.9	0.01	(+)ve	↑
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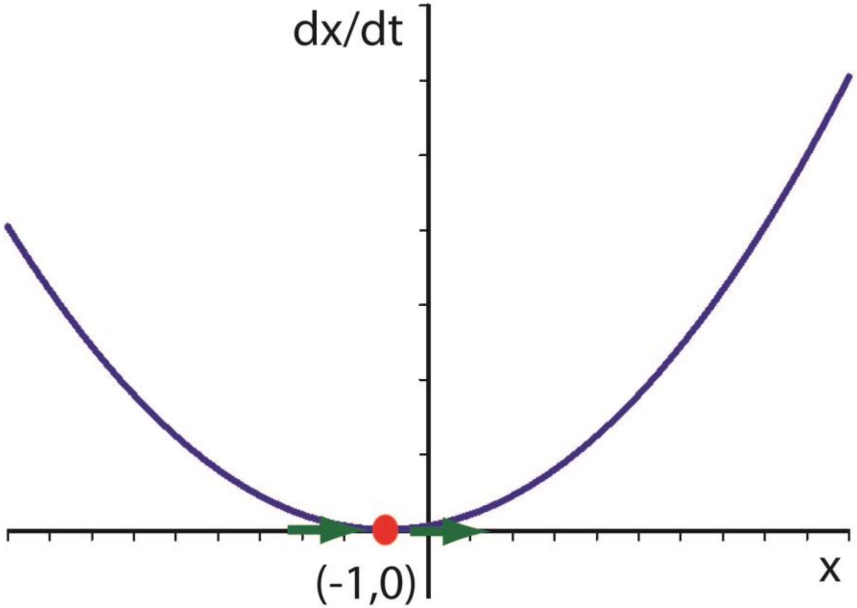
This steady state is semi-stable



Key points:

1. Steady states are of three types: Stable, unstable, semi-stable.
2. Stability decides time-evolution of a system around a steady state.
3. If perturbed from its stable steady state, with time, the system returns back to the stable steady state.
4. Stability can be analyzed using direction filed, numerical method