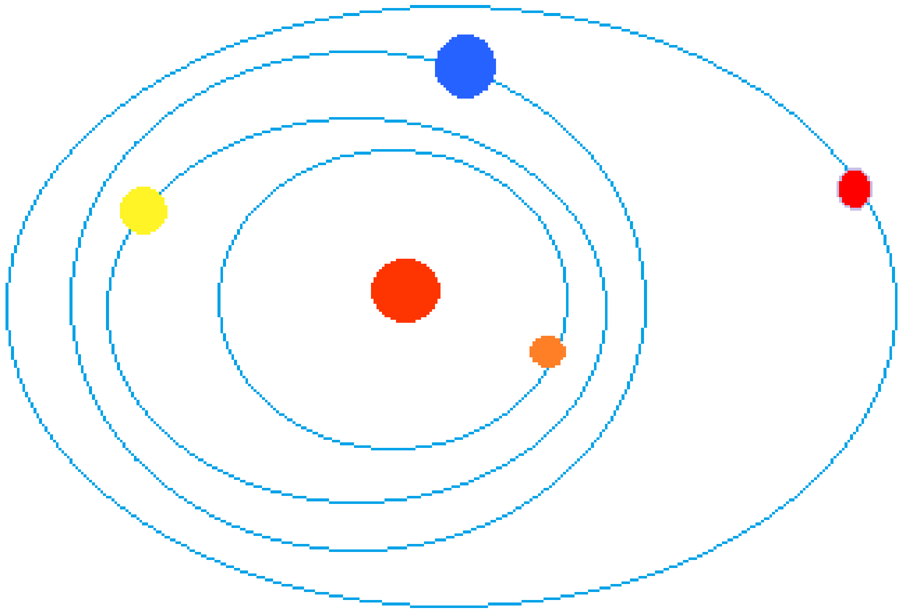
Simulation of the Solar System



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Setting up the problem

For two bodies

$$\vec{F} = -\frac{G.m_1.m_2}{|\vec{r_{12}}^2|}\hat{r_{12}} = -\frac{G.m_1.m_2}{|\vec{r_{12}}^3|}\vec{r_{12}}$$

Setting up the problem

For n bodies

$$\vec{F} = \sum_{i \neq j} -\frac{G.m_i.m_j}{|\vec{r_{ij}}^{\,3}|}\vec{r_{ij}}$$

Which translates to (for the i^{th} body):

$$\frac{dy_i^2}{dt^2} = \sum_{i \neq j} -\frac{G.m_j}{|\vec{r_{ij}}^{\,3}|}y_i \qquad \frac{dx_i^2}{dt^2} = \sum_{i \neq j} -\frac{G.m_j}{|\vec{r_{ij}}^{\,3}|}x_i$$

$$\frac{dz_i^2}{dt^2} = \sum_{i \neq j} -\frac{G.m_j}{|\vec{r_{ij}}^{\,3}|}z_i$$

Solving the problem

- Create an object “planet” and define its attributes like mass, position and velocity
- Define initial values of position and velocity to solve the initial value problem using the previous differential equations
- Calculate initial value of acceleration due to gravitational interaction
- Find future values of position of each planet using Second order Runge-Kutta Method for a given time step
- To find the value of orbital period of planet we check when the scalar product of the initial and current position vectors flip sign
- We also check when the distance from the sun is smallest and largest in a complete period
- Throughout the process we keep track off the total propagated error

2^{nd} Order Runge-Kutta Method

We aim to solve the differential equation

$$\frac{d^2y}{dt^2} = f(x)$$

We Taylor expand the function y around a point x_i for a step of h

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i'' + O(h^3)$$

Where

$$y_{i+1} = y(t_i + h) \qquad y_i = y(t_i)$$

$O(h^3) =$ Higher order terms

We truncate the series after the term of order h^2

2nd Order Runge-Kutta Method

Again we have $$y_i'' = \frac{df}{dx}.\frac{dx}{dt} = f_i'.f_i$$

So, $$y_{i+1} = y_i + h.f_i + \frac{h^2}{2} f_i'.f_i$$

Similarly we can also Taylor expand

Or $$h.f(x_i + \frac{h}{2} f) = h.f_i + \frac{h^2}{2} f_i.f_i' + O(h^3)$$

So after replacing the value in the previous equation we have the iterative equation which can be solved using initial values

$$y_{i+1} = y_i + h.f(x_i + \frac{h}{2} f(x_i))$$

2^{nd} Order Runge-Kutta Method

But in our problem we have a second order differential equation

$$\frac{d^2y}{dt^2} = f(x)$$

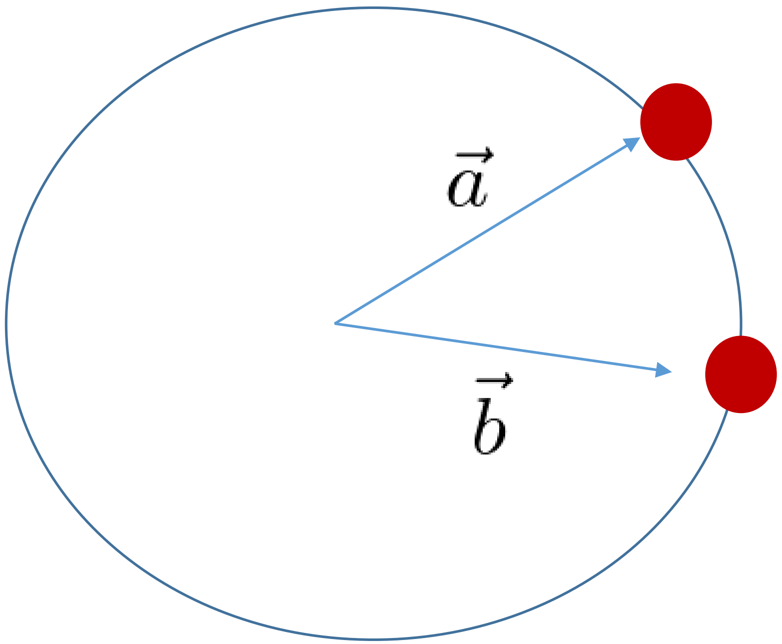
But this can be converted into a pair of differential equations by making the substitutions $\frac{dy}{dt} = z \quad \frac{dz}{dt} = f(x)$

So now we have two iterative equations

$$y_{i+1} = y_i + h(z_i + \frac{h}{2}f(x_i))$$

$$z_{i+1} = z_i + h.f(x_i + \frac{h}{2}z_i)$$

Measuring the orbital period

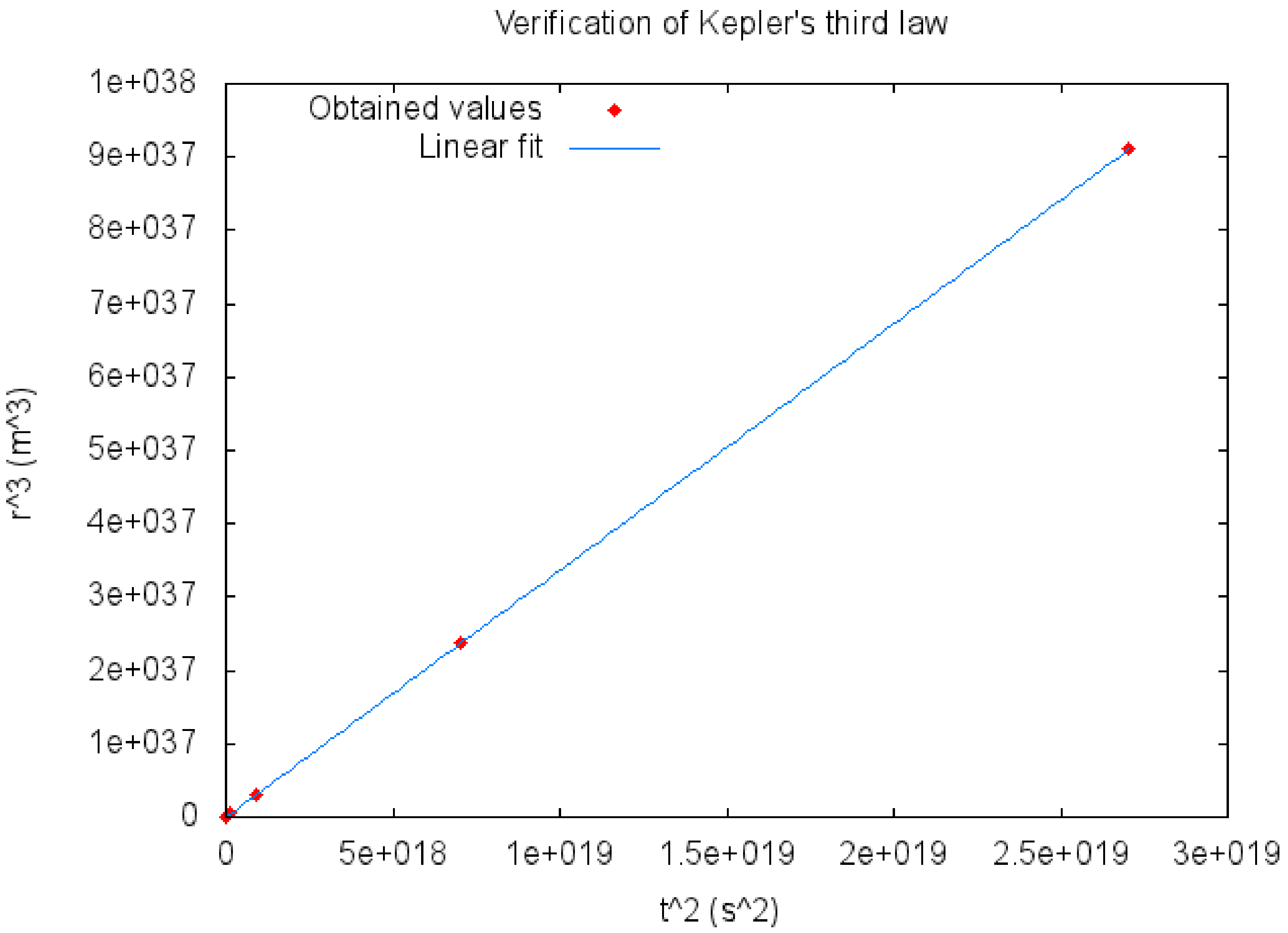


- When the initial($\vec{a}$) and final($\vec{b}$) vectors are perpendicular their scalar product is zero
- It gradually increases (or decreases) and flips its sign when they are perpendicular
- We consider the time interval between alternate sign flips as a full orbital period

Comparison between obtained and literature values

Planets	Computed Value				Observed Value		
	Orbital period (earth day)	Apehelion Distance (Km)	Perihelion Distance (Km)	Error (km)	Orbital period (earth day)	Apehelion Distance (Km)	Perihelion Distance (Km)
Mercury	88.05	69817023.9	46030315.22	283658.9078	87.969	6.98E+07	4.60E+07
Venus	224.9	108999817	107513313.3	323821.3941	224.701	1.09E+08	1.07E+08
Earth	365.55	152173282.3	147102830.9	282257.1427	365.256	1.52E+08	1.47E+10
Mars	687.05	249444828.9	206420572.8	248686.7516	686.98	2.49E+08	2.07E+08
Jupiter	4341.45	818684253.2	739457429.7	123001.145	4332.589	8.17E+08	7.41E+08
Saturn	10854.25	1522092612	1350198180	88927.51881	10759.22	1.51E+09	1.35E+09
Uranus	30733	3004315176	2741952384	1170581.866	30685.4	3.00E+09	2.74E+09
Neptune	60205	4550283810	4446572488	990535.1568	60189	4.55E+03	4.44E+09
Pluto	90426	7356377758	4444270376	1388174.857	90560	7.38E+03	4.44E+03
Moon	27.25	406479.1024	359601.4889	1435.771875	27.3217	4.06E+05	3.63E+05

Verifying Kepler's Third Law



Slope of graph= $(3.36423 \pm 0.00007) \times 10^{18}\ \mathrm{m^3 s^{-2}}$ Theoretical Value= $\frac{GM_{\mathrm{Sun}}}{4\pi^2} = 3.36048 \times 10^{18}\ \mathrm{m^3 s^{-2}}$

Estimation of errors

There can be three major sources of error in the numerical computation:

- Round-off error Error due to discreteness of computer memory

- Truncation error Error due to approximation of infinite series

- Incomplete modelling Error introduced due to non accountancy of gravitational forces due to asteroid belts, smaller satellites, etc. and non gravitational forces like gaseous drag, photon pressure, etc. Moreover the model considered is non relativistic .

Estimation of round off error

0.15624 in 32-bit representation would be

32 bit representation	0	1	1	1	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Sign Bit | Exponent (8 bits) | Mantissa (23 bits)

There are infinite real numbers between any two real numbers but we have a finite and discrete memory to represent them.
So numbers are rounded of to accommodate them into the finite space.

So cumulative error= $\sqrt{N}\epsilon$ *

Where, N = Number of arithmetic steps done
ϵ= The smallest float when added to 1 produces a float different from 1

**See Reference 2*

Estimation of truncation error

We truncated the Taylors series after first three terms. The terms left out of order h^3 and higher give rise to truncation error

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i'' + O(h^3)$$

The standard truncation error in each step can be estimated by the doubling method.

$$Error = \frac{y(2h) - y(h+h)}{2^p - 1} \quad *$$

Where,

$y(2h)$ = new position calculated with a step size of $2h$

$y(h+h)$ = new position calculated by twice using a step h

**For proof see Reference 3*

Acknowledgements & References

1. HORIZONS System created by Jet Propulsion Laboratory, NASA for Ephemerides data
2. The Python Software Foundation and David Scherer (for Visual Python)
3. Local error estimation by doubling (1985) by L.F. Shampine, *Computing*
4. Numerical Recipes in C++: The Art of Scientific Computing (Second Edition) by W.H. Press et al.