Causal Estimation (continued)

Robin J. Evans and Vanessa Didelez

University of Oxford and BIPS, University of Bremen

APTS Week 4, Oxford
September 2024

Outline

1. Efficient Adjustment
   - Doubly-Robust Approaches
2. Machine Learning Methods
   - Post Double Selection Inference
   - Double Machine Learning

Adjustment Sets

We know that, in a directed acyclic graph model, we can estimate the causal effect of (say) A on Y by **adjusting** for a set of variables that forms a **back-door** (adjustment) set.

That is, if we want the causal effect of A on Y, and there is a back-door adjustment set $\boldsymbol{C}$, evaluate

$$\mathbb{E}Y(a) := \mathbb{E}_{\boldsymbol{C}}\mathbb{E}[Y \mid A = a, \boldsymbol{C}].$$

Adjustment Sets

We know that, in a directed acyclic graph model, we can estimate the causal effect of (say) A on Y by **adjusting** for a set of variables that forms a **back-door** (adjustment) set.

That is, if we want the causal effect of A on Y, and there is a back-door adjustment set $\boldsymbol{C}$, evaluate

$$\mathbb{E}Y(a) := \mathbb{E}_{\boldsymbol{C}}\mathbb{E}[Y \mid A = a, \boldsymbol{C}].$$

What other sets might be valid (i.e. lead to a consistent estimate)?

Which is the most **efficient** set to use?

Here we take 'efficient' to mean having the smallest variance over all possible valid adjustment sets.

Variances

For illustrative purposes, we consider the linear case, but the results extend to general models (Rotnitzky and Smucler, 2020).

Lemma

Suppose we have a multivariate model with covariance matrix Σ*. In a linear regression of* Y *on* $\boldsymbol{C}$*, the covariance matrix of the regression coefficient vector is*

$$(n\Sigma_{\boldsymbol{CC}})^{-1}\sigma_{yy\cdot\boldsymbol{C}}.$$

Variances

For illustrative purposes, we consider the linear case, but the results extend to general models (Rotnitzky and Smucler, 2020).

Lemma

Suppose we have a multivariate model with covariance matrix Σ. In a linear regression of Y on $\boldsymbol{C}$, the covariance matrix of the regression coefficient vector is

$$(n\Sigma_{\boldsymbol{CC}})^{-1}\sigma_{yy\cdot\boldsymbol{C}}.$$

This leads to a corollary that if we regress Y on A and the collection of variables $\boldsymbol{C}$, the variance of that coefficient is

$$n^{-1}\frac{\sigma_{yy\cdot a\boldsymbol{C}}}{\sigma_{aa\cdot\boldsymbol{C}}}.$$

Independences

Theorem

Suppose that $\boldsymbol{C}, \boldsymbol{D} \subseteq \boldsymbol{V} \setminus \{A, Y\}$*, and let* $\boldsymbol{C}' = \boldsymbol{C} \setminus \boldsymbol{D}$ *and* $\boldsymbol{D}' = \boldsymbol{D} \setminus \boldsymbol{C}$*. Then if* $Y \perp_d \boldsymbol{D}' \mid \boldsymbol{C}, A$ *and* $A \perp_d \boldsymbol{C}' \mid \boldsymbol{D}$*, we have*

$$\frac{\sigma_{yy \cdot a\boldsymbol{C}}}{\sigma_{aa \cdot \boldsymbol{C}}} \leq \frac{\sigma_{yy \cdot a\boldsymbol{D}}}{\sigma_{aa \cdot \boldsymbol{D}}}.$$

In other words, $\boldsymbol{C}$ is a more efficient set to use than $\boldsymbol{D}$!

Proof.

□

Independences

Theorem

Suppose that $\boldsymbol{C}, \boldsymbol{D} \subseteq \boldsymbol{V} \setminus \{A, Y\}$, *and let* $\boldsymbol{C}' = \boldsymbol{C} \setminus \boldsymbol{D}$ *and* $\boldsymbol{D}' = \boldsymbol{D} \setminus \boldsymbol{C}$. *Then if* $Y \perp_d \boldsymbol{D}' \mid \boldsymbol{C}, A$ *and* $A \perp_d \boldsymbol{C}' \mid \boldsymbol{D}$, *we have*

$$\frac{\sigma_{yy \cdot a\boldsymbol{C}}}{\sigma_{aa \cdot \boldsymbol{C}}} \leq \frac{\sigma_{yy \cdot a\boldsymbol{D}}}{\sigma_{aa \cdot \boldsymbol{D}}}.$$

In other words, $\boldsymbol{C}$ is a more efficient set to use than $\boldsymbol{D}$!

Proof.

Note that $\boldsymbol{C}' \cup \boldsymbol{D} = \boldsymbol{C} \cup \boldsymbol{D}'$. By the first d-separation, we have

$$\sigma_{yy \cdot a\boldsymbol{C}} = \sigma_{yy \cdot a\boldsymbol{C}\boldsymbol{D}'} = \sigma_{yy \cdot a\boldsymbol{C}'\boldsymbol{D}}.$$

Removing entries from the conditioning set will only increase the residual variance, so $\sigma_{yy \cdot a\boldsymbol{C}} \leq \sigma_{yy \cdot a\boldsymbol{D}}$.

□

Independences

Theorem

Suppose that $\boldsymbol{C}, \boldsymbol{D} \subseteq \boldsymbol{V} \setminus \{A, Y\}$*, and let* $\boldsymbol{C}' = \boldsymbol{C} \setminus \boldsymbol{D}$ *and* $\boldsymbol{D}' = \boldsymbol{D} \setminus \boldsymbol{C}$*. Then if* $Y \perp_d \boldsymbol{D}' \mid \boldsymbol{C}, A$ *and* $A \perp_d \boldsymbol{C}' \mid \boldsymbol{D}$*, we have*

$$\frac{\sigma_{yy \cdot a\boldsymbol{C}}}{\sigma_{aa \cdot \boldsymbol{C}}} \leq \frac{\sigma_{yy \cdot a\boldsymbol{D}}}{\sigma_{aa \cdot \boldsymbol{D}}}.$$

In other words, $\boldsymbol{C}$ is a more efficient set to use than $\boldsymbol{D}$!

Proof.

Note that $\boldsymbol{C}' \cup \boldsymbol{D} = \boldsymbol{C} \cup \boldsymbol{D}'$. By the first d-separation, we have

$$\sigma_{yy \cdot a\boldsymbol{C}} = \sigma_{yy \cdot a\boldsymbol{C}\boldsymbol{D}'} = \sigma_{yy \cdot a\boldsymbol{C}'\boldsymbol{D}}.$$

Removing entries from the conditioning set will only increase the residual variance, so $\sigma_{yy \cdot a\boldsymbol{C}} \leq \sigma_{yy \cdot a\boldsymbol{D}}$.

Similarly, using the other independence $\sigma_{aa \cdot \boldsymbol{C}} \geq \sigma_{aa \cdot \boldsymbol{D}}$; hence the result. □

Efficient Adjustment Set

Definition

Let $\mathrm{cn}_{\mathcal{G}}(A \to Y)$ be all nodes on any causal path from A to Y, excluding A.

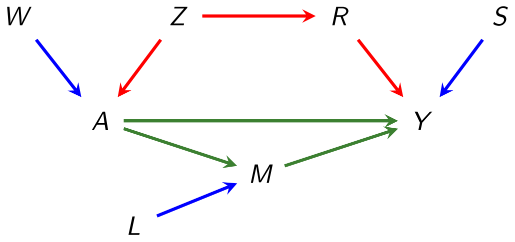
The following result was first proven by Henckel et al. (2022) (preprint 2019), and later extended to the general semi-parametric case by Rotnitzky and Smucler (2020).

Theorem (Efficient adjustment)

Let $$O_{\mathcal{G}}(A \to Y) := \mathrm{pa}_{\mathcal{G}}(\mathrm{cn}_{\mathcal{G}}(A \to Y)) \setminus (\mathrm{cn}_{\mathcal{G}}(A \to Y) \cup \{A\}).$$

Then $O_{\mathcal{G}}(A \to Y)$ is the optimal adjustment set for estimating the causal effect of A on Y, in the sense that the variance of the A-coefficient is minimized.

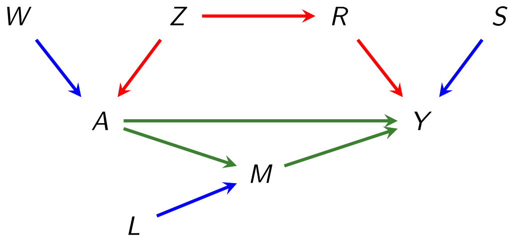
Adjustment Sets



In this graph we:

- must leave causal paths open, so do **not** adjust for M (or A or Y);

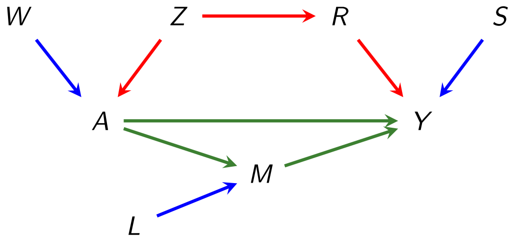
Adjustment Sets



In this graph we:

- must leave causal paths open, so do **not** adjust for M (or A or Y);
- need to block back-door path, so **must** adjust for Z, R or both;

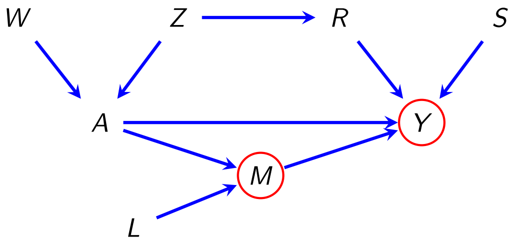
Adjustment Sets



In this graph we:

- must leave causal paths open, so do **not** adjust for M (or A or Y);
- need to block back-door path, so **must** adjust for Z, R or both;
- can *decide whether* to adjust for any of W, L, S.

Efficient Adjustment

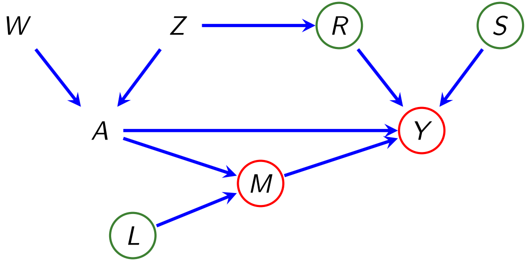


We have

$$O_{\mathcal{G}}(A \to Y) := \mathrm{pa}_{\mathcal{G}}(\mathrm{cn}_{\mathcal{G}}(A \to Y)) \setminus (\mathrm{cn}_{\mathcal{G}}(A \to Y) \cup \{A\}),$$

where $\mathrm{cn}_{\mathcal{G}}(A \to Y) = \{M, Y\}$, and therefore the first set is $\{A, S, M, L, R\}$.

Efficient Adjustment



We have

$$O_{\mathcal{G}}(A \to Y) := \mathrm{pa}_{\mathcal{G}}(\mathrm{cn}_{\mathcal{G}}(A \to Y)) \setminus (\mathrm{cn}_{\mathcal{G}}(A \to Y) \cup \{A\}),$$

where $\mathrm{cn}_{\mathcal{G}}(A \to Y) = \{M, Y\}$, and therefore the first set is $\{A, S, M, L, R\}$.

Hence $O_{\mathcal{G}}(A \to Y) = \{L, R, S\}$.

Forbidden Projection

Another way to characterize $O_{\mathcal{G}}(A \to Y)$ is via the **forbidden projection**.

The **forbidden set** is $\mathrm{forb}_{\mathcal{G}}(A \to Y) = \mathrm{de}_{\mathcal{G}}(\mathrm{cn}_{\mathcal{G}}(A \to Y)) \cup \{A\}$.

This is the set of nodes that never appear in **any** valid adjustment set.

Forbidden Projection

Another way to characterize $O_{\mathcal{G}}(A \to Y)$ is via the **forbidden projection**.

The **forbidden set** is $\mathrm{forb}_{\mathcal{G}}(A \to Y) = \mathrm{de}_{\mathcal{G}}(\mathrm{cn}_{\mathcal{G}}(A \to Y)) \cup \{A\}$.

This is the set of nodes that never appear in **any** valid adjustment set.

Forbidden Projection

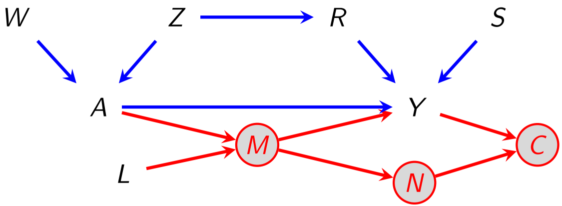
This consists of:

(i) projecting out the forbidden nodes (except A and Y) to obtain $\mathcal{G}'$;

(ii) setting $O_{\mathcal{G}}(A \to Y) = \mathrm{pa}_{\mathcal{G}'}(Y) \setminus \{A\}$.

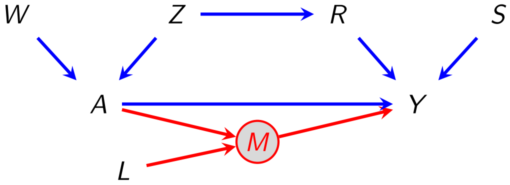
To perform **latent projection**, any vertices to be dropped that have no children are simply removed. (Why?)

Forbidden Projection



To project out:

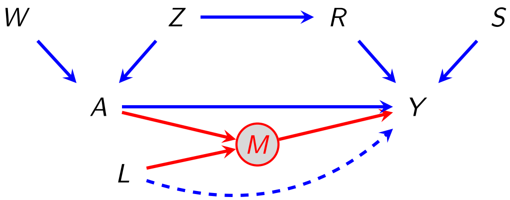
Forbidden Projection



To project out:

- remove vertices with no observed descendants;

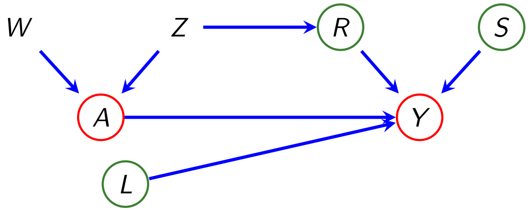
Forbidden Projection



To project out:

- remove vertices with no observed descendants;
- replace any mediators with edges directly from parents to their children.

Forbidden Projection



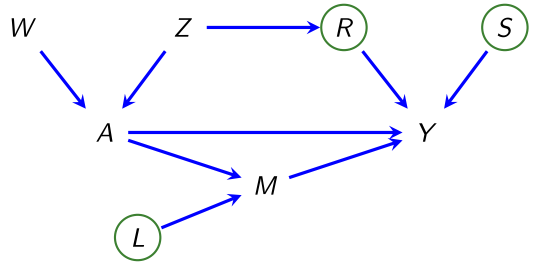
To project out:

- remove vertices with no observed descendants;
- replace any mediators with edges directly from parents to their children.
- then drop the mediators.

Now notice that $O_{\mathcal{G}}(A \to Y) = \mathrm{pa}_{\mathcal{G}'}(Y) \setminus \{A\}$.

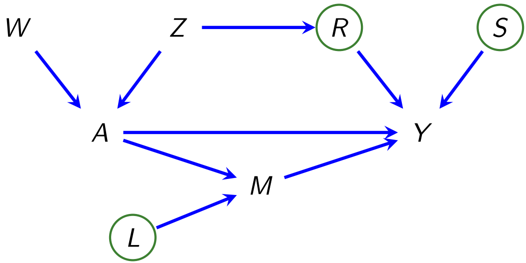
Intuition behind Efficient Adjustment

Notice that we adjust for some variables (L and S), even though these are **not** actually confounders.



Intuition behind Efficient Adjustment

Notice that we adjust for some variables (L and S), even though these are **not** actually confounders.



Notice also that we **do not** control for instruments. (i.e. variables affecting only treatment).

In theory conditioning on an instrument will **increase** the variance in the estimate, because it **reduces** variance in A.

In practice, conditioning on an instrument will also induce **bias**.

Intuition behind Efficient Adjustment

Think of effect estimation as a regression.

```
X <- rnorm(100, sd=1)
Y <- X + rnorm(100, sd=1)
summary(lm(Y ~ X))$coef[,1:2]
```

```
##             Estimate Std. Error
## (Intercept)    -0.10      0.098
## X               0.95      0.107
```

```
X <- rnorm(100, sd=0.1)
Y <- X + rnorm(100, sd=1)
summary(lm(Y ~ X))$coef[,1:2]
```

```
##             Estimate Std. Error
## (Intercept)    -0.03       0.11
## X               0.51       1.10
```

Reducing the variation in X **increases** the standard error.

Intuition behind Efficient Adjustment

Think of effect estimation as a regression.

```
X <- rnorm(100, sd=1)
Y <- X + rnorm(100, sd=1)
summary(lm(Y ~ X))$coef[,1:2]
```

```
##             Estimate Std. Error
## (Intercept)    -0.10      0.098
## X               0.95      0.107
```

```
X <- rnorm(100, sd=1)
Y <- X + rnorm(100, sd=0.1)
summary(lm(Y ~ X))$coef[,1:2]
```

```
##             Estimate Std. Error
## (Intercept)   -0.003      0.011
## X              0.995      0.011
```

However, reducing the variation in Y **decreases** the standard error.

More about Efficient Adjustment

The key quantity is:

$$\frac{\text{variance in } Y}{\text{variance in } X}.$$

We want the top to be small and the bottom to be large for good precision.

More about Efficient Adjustment

The key quantity is:

$$\frac{\text{variance in } Y}{\text{variance in } X}.$$

We want the top to be small and the bottom to be large for good precision.

The results in the earlier slides were first proved in the multivariate Gaussian case by Henckel et al. (2022) (preprint 2019).

The forbidden projection idea comes from Witte et al. (2020).
Gives a nice duality:

- **worst** adjustment set is parents of treatment;
- **best** adjustment set is parents of outcome (after projection!)

More about Efficient Adjustment

It was extended to the general semi-parametric case by Rotnitzky and Smucler (2020).

In fact, they show that any non-parametric estimation method (i.e. without making use of parameteric assumptions in the conditional distributions) can be performed most efficiently using the optimal adjustment set.
This includes **propensity score** methods, **doubly robust** approaches and **double machine learning** methods.

More about Efficient Adjustment

It was extended to the general semi-parametric case by Rotnitzky and Smucler (2020).

In fact, they show that any non-parametric estimation method (i.e. without making use of parameteric assumptions in the conditional distributions) can be performed most efficiently using the optimal adjustment set.
This includes **propensity score** methods, **doubly robust** approaches and **double machine learning** methods.

It has also been generalized to models that have hidden variables, though the results are not always so nice (see Smucler et al., 2020).

References

Henckel et al. – Graphical criteria for efficient average causal treatment effect estimation via adjustment in causal linear models. *JRSS-B*, 2022.

Rotnitzky and Smucler – Efficient adjustment sets for population average causal treatment effect estimation in graphical models. *JMLR*, 2020.

Smucler et al. – Efficient adjustment sets in causal graphical models with hidden variables. *Biometrika*, 2022.

Witte et al. – On Efficient Adjustment in Causal Graphs, *JMLR*, 2020.

When is the naïve estimate correct?

We know that, if $\boldsymbol{X}$ are sufficient to control for confounding, then

$$P(Y(a)) = \sum_{\boldsymbol{x}} P(\boldsymbol{x}) \cdot P(Y \mid \boldsymbol{x}, A = a).$$

When does

$$P(Y(a)) = P(Y \mid A = a) \qquad ?$$

When is the naïve estimate correct?

We know that, if $\boldsymbol{X}$ are sufficient to control for confounding, then

$$P(Y(a)) = \sum_{\boldsymbol{x}} P(\boldsymbol{x}) \cdot P(Y \mid \boldsymbol{x}, A = a).$$

When does

$$P(Y(a)) = P(Y \mid A = a) \qquad ?$$

Suppose that $A \perp\!\!\!\perp \boldsymbol{X}$. Then:

$$\begin{aligned}
P(Y \mid A = a) &= \sum_{\boldsymbol{x}} P(\boldsymbol{X}, Y \mid A = a) \\
&= \sum_{\boldsymbol{x}} P(\boldsymbol{X} \mid A = a) \cdot P(Y \mid \boldsymbol{X}, A = a) \\
&= \sum_{\boldsymbol{x}} P(\boldsymbol{X}) \cdot P(Y \mid \boldsymbol{X}, A = a) \\
&= P(Y(a)).
\end{aligned}$$

When is the naïve estimate correct?

We know that, if $\boldsymbol{X}$ are sufficient to control for confounding, then

$$P(Y(a)) = \sum_{\boldsymbol{x}} P(\boldsymbol{x}) \cdot P(Y \mid \boldsymbol{x}, A = a).$$

When does

$$P(Y(a)) = P(Y \mid A = a) \qquad ?$$

In summary, if **either** $A \perp\!\!\!\perp \boldsymbol{X}$ **or** $Y \perp\!\!\!\perp \boldsymbol{X} \mid A$ then $Y(a)$ and $Y \mid A = a$ have the same distributions.

This is perhaps unsurprising, given that in either of those cases, $\boldsymbol{X}$ is not really a confounder at all!

Doubly Robust Approaches

Note we've seen that if we specify

- the **outcome model** (i.e. $Y \mid A, \boldsymbol{X}$) correctly, we can obtain a consistent estimate of the ACE by averaging over the empirical $\boldsymbol{X}$ values;
- the **propensity score model** (i.e. $A \mid \boldsymbol{X}$) correctly, we can use the Horvitz-Thompson estimator which is also consistent.

Is there an estimator that uses both of these models, but only requires one of them to be correct?

Doubly Robust Approaches

Note we've seen that if we specify

- the **outcome model** (i.e. $Y \mid A, \boldsymbol{X}$) correctly, we can obtain a consistent estimate of the ACE by averaging over the empirical $\boldsymbol{X}$ values;
- the **propensity score model** (i.e. $A \mid \boldsymbol{X}$) correctly, we can use the Horvitz-Thompson estimator which is also consistent.

Is there an estimator that uses both of these models, but only requires one of them to be correct?

Yes!

We can use the following approach: suppose we believe that

$$\mathbb{E}[Y \mid a, \boldsymbol{x}] = Q_a(\boldsymbol{x}; \beta, \gamma) \qquad \text{and} \qquad \pi(\boldsymbol{x}) = \pi(\boldsymbol{x}; \eta)$$

for **parametric** models Q_0, Q_1, and π.

These are sometimes called **working models**.

Doubly Robust Methods

Notice that the following function has expectation $Y(1)$ if **either** Q_1 or π is specified correctly:

$$\begin{aligned}\mu_1^{dr}(O) &= Q_1(\boldsymbol{X}) + \frac{A}{\pi(\boldsymbol{X})}\{Y - Q_1(\boldsymbol{X})\} \\ &= \frac{AY}{\pi(\boldsymbol{X})} + \left\{1 - \frac{A}{\pi(\boldsymbol{X})}\right\} Q_1(\boldsymbol{X}).\end{aligned}$$

Doubly Robust Methods

Notice that the following function has expectation $Y(1)$ if **either** Q_1 or π is specified correctly:

$$\mu_1^{dr}(O) = Q_1(\boldsymbol{X}) + \frac{A}{\pi(\boldsymbol{X})}\{Y - Q_1(\boldsymbol{X})\}$$
$$= \frac{AY}{\pi(\boldsymbol{X})} + \left\{1 - \frac{A}{\pi(\boldsymbol{X})}\right\} Q_1(\boldsymbol{X}).$$

So fit 'nuisance' models Q and π to the data (e.g. by maximum likelihood). This gives parameter estimates $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\eta}$.

Then consider the following estimator of $\mathbb{E}Y(1)$:

$$\hat{\mu}_1^{dr} = \frac{1}{n}\sum_{i=1}^{n}\left\{\frac{A_i\{Y_i - Q_{A_i}(\boldsymbol{X}_i;\hat{\beta},\hat{\gamma})\}}{\pi(\boldsymbol{X}_i;\hat{\eta})} + Q_1(\boldsymbol{X}_i;\hat{\beta},\hat{\gamma})\right\}.$$

If **either** model is correctly specified, then by the above we can see that the estimate will be consistent.

This property is called **double robustness**.

Doubly Robust Methods

We can do something similar for $\hat{\mu}_0^{dr}$, and then

$$\hat{\beta}^{dr} := \hat{\mu}_1^{dr} - \hat{\mu}_0^{dr}. \qquad (*)$$

We call this the **augmented** inverse probability weighted estimator (AIPW).

Doubly Robust Methods

We can do something similar for $\hat{\mu}_0^{dr}$, and then

$$\hat{\beta}^{dr} := \hat{\mu}_1^{dr} - \hat{\mu}_0^{dr}. \tag{$*$}$$

We call this the **augmented** inverse probability weighted estimator (AIPW).

In addition, each $\hat{\mu}_a^{dr}$ is **semi-parametric efficient** if both parametric models are correct, so it achieves the same rate (asymptotically) as maximum likelihood estimation.

If Q_a is wrong then MLEs will be difficult to interpret.

In practice, even under moderate misspecifications of both models, the doubly robust estimator mostly performs well in practice.

Demonstrations

The R package `causl`[1] allows one to simulate data from a parametrically specified causal model.

Suppose we want to have:

$$Z \sim \text{Exponential}(\lambda)$$
$$A \mid Z = z \sim \text{Bernoulli}\left(\text{logit}(\alpha_0 + \alpha_1 z)\right)$$
$$Y \mid do(A = a) \sim N(\beta a,\ \sigma^2)$$

with $\lambda = 2$, $\alpha_0 = 0$, $\alpha_1 = 1$ and $\beta = 1/2$.

```
library(causl)
forms <- list(Z ~ 1,
              A ~ Z,
              Y ~ A,
              ~ 1)  ## for the copula
```

[1] https://github.com/rje42/causl

Demonstrations

The R package causl[1] allows one to simulate data from a parametrically specified causal model.

Suppose we want to have:

$$Z \sim \text{Exponential}(\lambda)$$
$$A \mid Z = z \sim \text{Bernoulli}\left(\text{logit}(\alpha_0 + \alpha_1 z)\right)$$
$$Y \mid do(A = a) \sim N(\beta a,\ \sigma^2)$$

with $\lambda = 2$, $\alpha_0 = 0$, $\alpha_1 = 1$ and $\beta = 1/2$.

```
library(causl)
forms <- list(Z ~ 1,
              A ~ Z,
              Y ~ A,
              ~ 1)  ## for the copula
```

```
pars <- list(Z = list(beta = -log(2), phi=1),  ## we use log-link
             A = list(beta = c(0,1)),
             Y = list(beta = c(0,0.5), phi = 1),
             cop = list(beta = 1))
fam <- list(3,5,1,1)  # distributions: 1=normal, 3=Gamma, 5=binomial
```

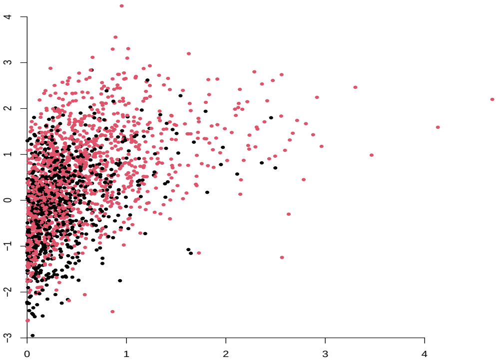
[1] https://github.com/rje42/causl

Demonstrations

We can then use the `causalSamp` function to simulate our data:

```
set.seed(123)
dat <- rfrugalParam(1e4, formulas=forms, pars=pars, family=fam)

## Inversion method selected:  using pair-copula parameterization
```



The plot shows the first 2000 data points.

Doubly Robust Methods

Let us suppose that $\mathbb{E}Y$ is linear in A and $\boldsymbol{X}$ separately, so

$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta_A a + \beta_{\boldsymbol{X}} \boldsymbol{x}.$$

```
# get propensity score
ps <- fitted(glm(A ~ Z, data=dat, family="binomial"))
dat <- dplyr::mutate(dat, ps = ps)  # add est. propensity score

# outcome model
modY <- lm(Y ~ A + Z, data=dat)

dat0 <- dat1 <- dat    ## set 0 and 1 in mock datasets
dat0$A <- 0; dat1$A <- 1

## compute mu_x for x = {0,1}
mu1 <- mean(dat$A*(dat$Y - predict(modY))/dat$ps
            + predict(modY, dat1))

mu0 <- mean((1-dat$A)*(dat$Y - predict(modY))/(1-dat$ps)
            + predict(modY, dat0))

mu1 - mu0


## [1] 0.48
```

References

Bang, H. and Robins, J.M. Doubly robust estimation in missing data and causal inference models. *Biometrics*, 61(4), pp.962-973, 2005.

Moore and van der Laan. Covariate adjustment in randomized trials with binary outcomes, *Statistics in Medicine*, 2008.

Robins, J.M. and Rotnitzky, A. Semiparametric efficiency in multivariate regression models with missing data. *Journal of the American Statistical Association*, 90(429), 122-129, 1995.

Rosenbaum, P.R. and Rubin, D.B. The central role of the propensity score in observational studies for causal effects. *Biometrika*, 70(1), 41-55, 1983.

Scharfstein, D.O., Rotnitzky, A. and Robins, J.M. Adjusting for nonignorable drop-out using semiparametric nonresponse models. *Journal of the American Statistical Association*, 94(448), pp.1096-1120. 1999.

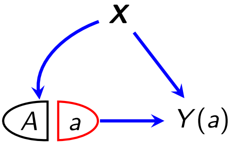
Outline

1. Efficient Adjustment
   - Doubly-Robust Approaches

2. Machine Learning Methods
   - Post Double Selection Inference
   - Double Machine Learning

Post 'Double Selection' Inference

Suppose we have the following set up, where $\boldsymbol{X}$, is high-dimensional (say $|\boldsymbol{X}| = p$).



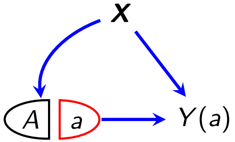
It is clear that we can **identify** the causal effect of A on Y, since assuming independent observations and the model implied by the SWIG:

$$\mathbb{E}Y(a) \;=\; \sum_{\boldsymbol{x}} P(\boldsymbol{x}) \cdot \mathbb{E}[Y \mid a, \boldsymbol{x}] \;=\; \mathbb{E}\left[\frac{Y \mathbb{1}_{\{A=a\}}}{P(A = a \mid \boldsymbol{X})}\right];$$

however, statistically we may still have difficulties.

Post 'Double Selection' Inference

Suppose we have the following set up, where $\boldsymbol{X}$, is high-dimensional (say $|\boldsymbol{X}| = p$).



It is clear that we can **identify** the causal effect of A on Y, since assuming independent observations and the model implied by the SWIG:

$$\mathbb{E}Y(a) \;=\; \sum_{\boldsymbol{x}} P(\boldsymbol{x}) \cdot \mathbb{E}[Y \mid a, \boldsymbol{x}] \;=\; \mathbb{E}\left[\frac{Y \mathbb{1}_{\{A=a\}}}{P(A = a \mid \boldsymbol{X})}\right];$$

however, statistically we may still have difficulties.

- We do not know what form the expressions for $\mathbb{E}[Y \mid a, \boldsymbol{x}]$, $P(\boldsymbol{x})$, or $P(a \mid \boldsymbol{x})$ should take.
- Even if we knew the families, actually estimating the parameters may be infeasible with a finite dataset of reasonable size.

Frisch-Waugh-Lovell Theorem

Suppose we have n i.i.d. observations $(\boldsymbol{X}_i, A_i, Y_i)$ such that

$$A_i = \alpha^T \boldsymbol{X}_i + \delta_i \qquad\qquad Y_i = \beta A_i + \gamma^T \boldsymbol{X}_i + \varepsilon_i,$$

where $\boldsymbol{X}_i$ has fewer than $n-1$ entries.

Consider two different ways of obtaining an estimate of β:

1. regress Y on $\boldsymbol{X}$ and A using OLS, and look at $\hat{\beta}$;

Frisch-Waugh-Lovell Theorem

Suppose we have n i.i.d. observations $(\boldsymbol{X}_i, A_i, Y_i)$ such that

$$A_i = \alpha^T \boldsymbol{X}_i + \delta_i \qquad\qquad Y_i = \beta A_i + \gamma^T \boldsymbol{X}_i + \varepsilon_i,$$

where $\boldsymbol{X}_i$ has fewer than $n - 1$ entries.

Consider two different ways of obtaining an estimate of β:

1. regress Y on $\boldsymbol{X}$ and A using OLS, and look at $\hat{\beta}$;
2. regress Y on $\boldsymbol{X}$ to obtain residual r_Y; and then A on $\boldsymbol{X}$ to obtain r_A; then regress r_Y on r_A, and take the linear coefficient $\tilde{\beta}$.

Frisch-Waugh-Lovell Theorem

Suppose we have n i.i.d. observations $(\boldsymbol{X}_i, A_i, Y_i)$ such that

$$A_i = \alpha^T \boldsymbol{X}_i + \delta_i \qquad\qquad Y_i = \beta A_i + \gamma^T \boldsymbol{X}_i + \varepsilon_i,$$

where $\boldsymbol{X}_i$ has fewer than $n - 1$ entries.

Consider two different ways of obtaining an estimate of β:

1. regress Y on $\boldsymbol{X}$ and A using OLS, and look at $\hat{\beta}$;
2. regress Y on $\boldsymbol{X}$ to obtain residual r_Y; and then A on $\boldsymbol{X}$ to obtain r_A; then regress r_Y on r_A, and take the linear coefficient $\tilde{\beta}$.

Theorem (Frisch and Waugh (1933), Lovell (1963))

The estimates for β from methods 1 and 2 are the same.

Intuition

Why does this result hold?

Proof.

Note that $r_A = A - \hat{\alpha}^T \boldsymbol{X}$, so $r_A \perp\!\!\!\perp \boldsymbol{X}$.
Then

$$\begin{aligned}
\mathbb{E}[Y \mid \boldsymbol{X}, A] &= \beta A + \gamma^T \boldsymbol{X} \\
&= \beta(r_A + \alpha^T \boldsymbol{X}) + \gamma^T \boldsymbol{X} \\
&= \beta r_A + (\alpha + \gamma)^T \boldsymbol{X}.
\end{aligned}$$

Then, since $\boldsymbol{X} \perp\!\!\!\perp r_A$, we must have that regressing Y on $\boldsymbol{X}$ gives an estimate of $\alpha + \gamma$.

Intuition

Why does this result hold?

Proof.

Note that $r_A = A - \hat{\alpha}^T \boldsymbol{X}$, so $r_A \perp\!\!\!\perp \boldsymbol{X}$.
Then

$$\begin{aligned}\mathbb{E}[Y \mid \boldsymbol{X}, A] &= \beta A + \gamma^T \boldsymbol{X} \\ &= \beta(r_A + \alpha^T \boldsymbol{X}) + \gamma^T \boldsymbol{X} \\ &= \beta r_A + (\alpha + \gamma)^T \boldsymbol{X}.\end{aligned}$$

Then, since $\boldsymbol{X} \perp\!\!\!\perp r_A$, we must have that regressing Y on $\boldsymbol{X}$ gives an estimate of $\alpha + \gamma$.
Hence

$$\mathbb{E} r_Y = \beta \mathbb{E} r_A,$$

giving the result. □

Sparsity

Suppose that we have

$$\mathbb{E}[A \mid \boldsymbol{X} = \boldsymbol{x}] = \alpha^T \boldsymbol{x}$$
$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta a + \gamma^T \boldsymbol{x}.$$

Sparsity

Suppose that we have

$$\mathbb{E}[A \mid \boldsymbol{X} = \boldsymbol{x}] = \alpha^T \boldsymbol{x}$$
$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta a + \gamma^T \boldsymbol{x}.$$

Assume also that $\log p = o(n^{1/3})$ and there exist subsets $\boldsymbol{B}$ and $\boldsymbol{D}$ of size at most $s_n \ll n$ such that:

$$\mathbb{E}[A \mid \boldsymbol{x}] = \alpha_{\boldsymbol{B}}^T \boldsymbol{x} + r_n$$
$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta a + \gamma_{\boldsymbol{D}}^T \boldsymbol{x} + t_n,$$

where the approximation error is stochastically smaller than the estimation error: i.e.

$$\mathbb{E}\|r_n\|_2 \lesssim \sqrt{\frac{s_n}{n}} \qquad \text{and} \qquad \mathbb{E}\|t_n\|_2 \lesssim \sqrt{\frac{s_n}{n}}.$$

Sparsity

Suppose that we have

$$\mathbb{E}[A \mid \boldsymbol{X} = \boldsymbol{x}] = \alpha^T \boldsymbol{x}$$
$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta a + \gamma^T \boldsymbol{x}.$$

Assume also that $\log p = o(n^{1/3})$ and there exist subsets $\boldsymbol{B}$ and $\boldsymbol{D}$ of size at most $s_n \ll n$ such that:

$$\mathbb{E}[A \mid \boldsymbol{x}] = \alpha_{\boldsymbol{B}}^T \boldsymbol{x} + r_n$$
$$\mathbb{E}[Y \mid A = a, \boldsymbol{X} = \boldsymbol{x}] = \beta a + \gamma_{\boldsymbol{D}}^T \boldsymbol{x} + t_n,$$

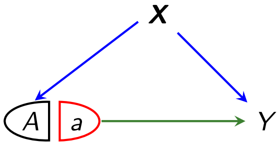
where the approximation error is stochastically smaller than the estimation error: i.e.

$$\mathbb{E}\|r_n\|_2 \lesssim \sqrt{\frac{s_n}{n}} \qquad \text{and} \qquad \mathbb{E}\|t_n\|_2 \lesssim \sqrt{\frac{s_n}{n}}.$$

In other words, a much smaller subset of covariates is sufficient to **approximately** make A and Y unconfounded.

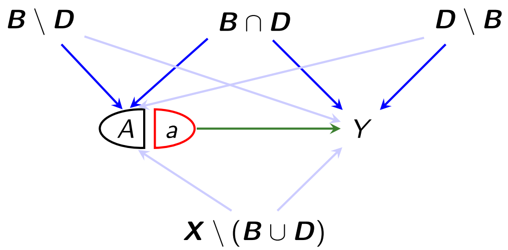
Post ‘Double Selection’ Inference

Graphical representation:



Post 'Double Selection' Inference

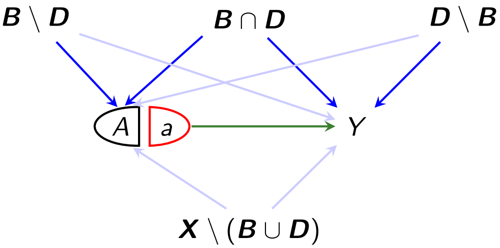
Graphical representation:



The idea is that if we account for variables in **both $\boldsymbol{B}$ and $\boldsymbol{D}$**, then we will be guaranteed to have good control of the bias in estimating β.

Post 'Double Selection' Inference

Graphical representation:



The idea is that if we account for variables in **both $\boldsymbol{B}$ and $\boldsymbol{D}$**, then we will be guaranteed to have good control of the bias in estimating β.

In principle we can use any consistent selection method to choose $\boldsymbol{B}$ and $\boldsymbol{D}$. In practice, Belloni et al. recommend a version of the lasso.

Post 'Double Selection' Inference

Here we perform a simulated example. Suppose that

$$A_i = \alpha \sum_{i=1}^{7} X_i + \delta_i$$

$$Y_i = \beta A_i + \gamma \sum_{i=4}^{10} X_i + \varepsilon_i$$

where $\delta_i, \varepsilon_i \overset{\text{i.i.d.}}{\sim} N(0,1)$ (independently), and we are given 1000 covariates in $\boldsymbol{X}$, where each $X_{ij} \sim N(0,1)$ independently.

Post 'Double Selection' Inference

Here we perform a simulated example. Suppose that

$$A_i = \alpha \sum_{i=1}^{7} X_i + \delta_i$$

$$Y_i = \beta A_i + \gamma \sum_{i=4}^{10} X_i + \varepsilon_i$$

where $\delta_i, \varepsilon_i \overset{\text{i.i.d.}}{\sim} N(0,1)$ (independently), and we are given 1000 covariates in $\boldsymbol{X}$, where each $X_{ij} \sim N(0,1)$ independently.

Set $\beta = \gamma = 2$ and $\alpha = 1$, and pick $n = 100$.

Post ‘Double Selection’ Inference

```
alpha <- 1
gamma <- beta <- 2
n <- 100; p <- 1000

## simulate data
set.seed(123)
Z <- matrix(rnorm(n*p), n, p)
X <- Z %*% c(rep(alpha, 7), rep(0,p-7)) + rnorm(n)
Y <- Z %*% c(rep(0,3), rep(gamma, 7), rep(0,p-10)) + beta*X + rnorm(n)
dat <- data.frame(Y=Y, X=X, Z)
names(dat) <- c("Y","X",paste0("Z",seq_len(p)))

head(dat[,1:9])
```

```
##         Y      X      Z1     Z2     Z3     Z4      Z5      Z6      Z7
## 1  -1.932  0.876 -0.5605 -0.710  2.199 -0.715 -0.0736 -0.6019  1.0740
## 2 -11.460  0.227 -0.2302  0.257  1.312 -0.753 -1.1687 -0.9937 -0.0273
## 3   0.821  0.408  1.5587 -0.247 -0.265 -0.939 -0.6347  1.0268 -0.0333
## 4  -0.752 -1.633  0.0705 -0.348  0.543 -1.053 -0.0288  0.7511 -1.5161
## 5  -4.478 -1.284  0.1293 -0.952 -0.414 -0.437  0.6707 -1.5092  0.7904
## 6  -2.355  0.906  1.7151 -0.045 -0.476  0.331 -1.6505 -0.0951 -0.2107
```

Post 'Double Selection' Inference

We can try a naïve model, and obtain the wrong answer.

```
sum_lm <- summary(lm(Y ~ X, data=dat))
sum_lm$coef
```

```
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)    0.244      0.492   0.496 6.21e-01
## X              3.067      0.184  16.649 2.52e-30
```

```
coef <- sum_lm$coef
```

Notice that the estimate $\hat{\beta} = 3.07$ is not within 2 s.e.s (0.37) of $\beta = 2$.

Post 'Double Selection' Inference

Then we can try using the R package `hdm`, which implements double selection.

```
library(hdm) ## library for implementation
lasso_out = rlassoEffect(y=dat[,"Y",drop=FALSE],
                         d=dat[,"X",drop=FALSE],
                         x=Z, method="double selection")

sum_out <- summary(lasso_out)
sum_out


## [1] "Estimates and significance testing of the effect of target variables"
##   Estimate. Std. Error t value Pr(>|t|)
## X     2.018      0.119    16.9   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Note this solution $\tilde{\beta} = 2.02$, is (well) within two s.e.s (0.24) of $\beta = 2$.

Post ‘Double Selection’ Inference: Application

Let us try applying double selection to a wage dataset.

```
X <- model.matrix(~ -1 + female + (widowed +divorced + separated +
                  nevermarried + hsd08 + hsd911 + hsg + cg + ad + mw + so +
                  we + exp1 + exp2 + exp3)^2, data = cps2012)
X <- X[, apply(X, 2, var) != 0] # exclude all constant variables
y <- cps2012$lnw
effects_female <- rlassoEffects(x = X, y = y, index = "female")
summary(effects_female)
```

```
## [1] "Estimates and significance testing of the effect of target variables"
##        Estimate. Std. Error t value Pr(>|t|)
## female  -0.28067    0.00692   -40.5   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Post 'Double Selection' Inference: Application

Now let's try fitting the other covariates too (note some are causally subsequent to sex).

```
data(cps2012)
X <- model.matrix(~ -1 + female + female:(widowed + divorced + separated +
                  nevermarried +hsd08 + hsd911 + hsg + cg + ad + mw + so +
                  we + exp1 + exp2 + exp3) + (widowed +divorced + separated +
                  nevermarried + hsd08 + hsd911 + hsg + cg + ad + mw + so +
                  we + exp1 + exp2 + exp3)^2, data = cps2012)
X <- X[, apply(X, 2, var) != 0] # exclude all constant variables
index.gender <- grep("female", colnames(X))
y <- cps2012$lnw
```

Post ‘Double Selection’ Inference: Application

```r
effects_female <- rlassoEffects(x = X, y = y, index = index.gender)
summary(effects_female)
```

```
## [1] "Estimates and significance testing of the effect of target variables"
##                      Estimate. Std. Error t value Pr(>|t|)
## female                -0.15492    0.05016   -3.09  0.00201 **
## female:widowed         0.13610    0.09066    1.50  0.13332
## female:divorced        0.13694    0.02218    6.17  6.7e-10 ***
## female:separated       0.02330    0.05321    0.44  0.66144
## female:nevermarried    0.18685    0.01994    9.37  < 2e-16 ***
## female:hsd08           0.02781    0.12091    0.23  0.81809
## female:hsd911         -0.11934    0.05188   -2.30  0.02144 *
## female:hsg            -0.01289    0.01922   -0.67  0.50252
## female:cg              0.01014    0.01833    0.55  0.58011
## female:ad             -0.03046    0.02181   -1.40  0.16241
## female:mw             -0.00106    0.01919   -0.06  0.95581
## female:so             -0.00818    0.01936   -0.42  0.67247
## female:we             -0.00423    0.02117   -0.20  0.84176
## female:exp1            0.00494    0.00780    0.63  0.52714
## female:exp2           -0.15952    0.04530   -3.52  0.00043 ***
## female:exp3            0.03845    0.00786    4.89  1.0e-06 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

References

Belloni, A., Chernozhukov, V. and Hansen, C. (2014). Inference on treatment effects after selection among high-dimensional controls. *The Review of Economic Studies*, 81(2), 608–650.

Frisch, R. and F.V. Waugh (1933). Partial time regression as compared with individual trends. *Econometrica* 1 (October): 387–401.

Lovell, M.C. (1963). Seasonal adjustment of economic time series and multiple regression analysis. *JASA* 58 (December): 993–1010.

Double Machine Learning

Double (or **debiased**) **machine learning** is an increasingly common approach to estimating causal effects. See, e.g. Chernozhukov et al. (2018).

The basic idea is the same as the approach of Belloni et al. (2014).

We estimate separate **high-dimensional models** for the treatment and outcome.

Double Machine Learning

Double (or **debiased**) **machine learning** is an increasingly common approach to estimating causal effects. See, e.g. Chernozhukov et al. (2018).

The basic idea is the same as the approach of Belloni et al. (2014).

We estimate separate **high-dimensional models** for the treatment and outcome.

The methods make extensive use of **cross-fitting**, i.e. splitting the data into separate components and using each to predict the other.

This allows for estimation while preventing **over-fitting**.

Mathematically speaking, much more **complicated models** can be used but still give an unbiased estimator of a (low-dimensional) causal effect.

Conditions for Double ML

A crucial condition for double ML to work is **Neyman orthogonality**, which says that the derivative of the estimating equation (at the true parameters) with respect to any nuisance parameters should be zero.

Suppose our score function is $\psi(W; \theta, \eta)$, with parameters of interest θ and nuisance parameters η. Then we need:

$$\frac{\partial}{\partial \eta} \mathbb{E}\psi(W; \theta_0, \eta)\Big|_{\eta=\eta_0} = 0,$$

where (θ_0, η_0) are the true parameters.

Conditions for Double ML

A crucial condition for double ML to work is **Neyman orthogonality**, which says that the derivative of the estimating equation (at the true parameters) with respect to any nuisance parameters should be zero.

Suppose our score function is $\psi(W; \theta, \eta)$, with parameters of interest θ and nuisance parameters η. Then we need:

$$\frac{\partial}{\partial \eta} \mathbb{E}\psi(W; \theta_0, \eta)\Big|_{\eta=\eta_0} = 0,$$

where (θ_0, η_0) are the true parameters.

If we are given a score function that is **not** Neyman orthogonal, we can often change it to become so.

Conditions for Double ML

Consider the linear model example, where the usual score is

$$\tilde{\psi}_\beta(W; \beta, \gamma) = (Y - \beta A - \gamma^T \boldsymbol{X}) \cdot A$$
$$\tilde{\psi}_\gamma(W; \beta, \gamma) = (Y - \beta A - \gamma^T \boldsymbol{X}) \cdot \boldsymbol{X}.$$

Conditions for Double ML

Consider the linear model example, where the usual score is

$$\tilde{\psi}_\beta(W; \beta, \gamma) = (Y - \beta A - \gamma^T \boldsymbol{X}) \cdot A$$
$$\tilde{\psi}_\gamma(W; \beta, \gamma) = (Y - \beta A - \gamma^T \boldsymbol{X}) \cdot \boldsymbol{X}.$$

Suppose we consider a directional derivative $\delta \cdot h$ with $h \in \mathbb{R}^{|\boldsymbol{X}|}$, then we have

$$\begin{aligned}
&\frac{\partial}{\partial \gamma} \tilde{\psi}_\beta(W; \beta, \gamma_0 + \delta h)\bigg|_{\delta \to 0} \\
&= \lim_{\delta \to 0} \frac{(Y - \beta A - (\gamma_0 + \delta h)^T \boldsymbol{X}) \cdot A - (Y - \beta A - \gamma_0^T \boldsymbol{X}) \cdot A}{\delta} \\
&= -h^T \boldsymbol{X}.
\end{aligned}$$

In particular, this is **not** zero!

Conditions for Double ML

Now, we can reparametrize the nuisance parameter γ as $\eta = (\gamma, \mu)$, where we choose μ so that the new score for β is

$$\begin{aligned}\psi_\beta(W; \beta, \eta) &= \tilde{\psi}_\beta(W; \beta, \gamma) - \mu^T \tilde{\psi}_\gamma(W; \beta, \gamma) \\ &= (Y - \beta A - \gamma^T \boldsymbol{X})(A - \mu^T \boldsymbol{X}).\end{aligned}$$

If we pick $\mu = \alpha$, then note that the expectation of second factor is 0!

Hence, **small** errors in the estimation of γ and α will **not** affect the estimate of β.

Conditions for Double ML

Now, we can reparametrize the nuisance parameter γ as $\eta = (\gamma, \mu)$, where we choose μ so that the new score for β is

$$\begin{aligned}\psi_\beta(W; \beta, \eta) &= \tilde{\psi}_\beta(W; \beta, \gamma) - \mu^T \tilde{\psi}_\gamma(W; \beta, \gamma) \\ &= (Y - \beta A - \gamma^T \boldsymbol{X})(A - \mu^T \boldsymbol{X}).\end{aligned}$$

If we pick $\mu = \alpha$, then note that the expectation of second factor is 0!

Hence, **small** errors in the estimation of γ and α will **not** affect the estimate of β.

In particular:

$$\begin{aligned} & \frac{\partial}{\partial \gamma} \psi_\beta(W; \beta, \gamma, \alpha) = -\boldsymbol{X}(A - \alpha^T \boldsymbol{X}) \\ \text{and} \quad & \frac{\partial}{\partial \alpha} \psi_\beta(W; \beta, \gamma, \alpha) = -\boldsymbol{X}(Y - \beta A - \gamma^T \boldsymbol{X}), \end{aligned}$$

and these both have expectation 0.

Moral: Neyman orthogonality is very helpful for robustness to misspecification.

401(k) Example

Chernozhukov et al. (2018) analyse data on 401(k) savings plans, and whether eligibility to enroll leads to an increase in net assets.

They consider a dataset of 9,915 individuals, measuring:

age	age in years;
inc	income;
educ	years of education;
fsize	family size;
marr	indicator of being married;
twoearn	two earners in household;
db	member of defined benefit pension scheme;
pira	eligible for Individual Retirement Allowance;
hown	homeowner.

DML for 401(k) Example

```
library(DoubleML)
library(mlr3)
library(data.table)
library(dplyr)

## note that the DoubleML package uses data.table objects
dat <- fetch_401k(return_type = "data.table", instrument = TRUE)

# Initialize DoubleMLData (data-backend of DoubleML)
dml = DoubleMLData$new(dat,
                       y_col = "net_tfa",
                       d_cols = "e401",
                       x_cols = c("age", "inc", "educ", "fsize",
                        "marr", "twoearn", "db", "pira", "hown"))
mod <- DoubleMLIRM$new(dml,
              ml_m = lrn("classif.cv_glmnet", s = "lambda.min"),
              ml_g = lrn("regr.cv_glmnet",s = "lambda.min"),
              n_folds = 10, n_rep = 10)
mod$fit()   ## fit the model
```

```
c(beta=mod$coef, se=mod$se)


## beta.e401   se.e401
##      1669      3752
```

DML for 401(k) Example

We can also try using a more flexible set of covariates.

```
## add quadratic terms to age, income, education and family size
formula_flex = formula(" ~ -1 + poly(age, 2, raw=TRUE) +
  poly(inc, 2, raw=TRUE) + poly(educ, 2, raw=TRUE) +
  poly(fsize, 2, raw=TRUE) + marr + twoearn + db + pira + hown")
features_flex = data.frame(model.matrix(formula_flex, dat))
model_data = data.table("net_tfa" = dat[, net_tfa],
                        "e401" = dat[, e401], features_flex)

## initialize and fit model
dml_f <- DoubleMLData$new(model_data, y_col = "net_tfa",
                          d_cols = "e401")
mod_f <- DoubleMLIRM$new(dml_f,
              ml_m = lrn("classif.cv_glmnet", s = "lambda.min"),
              ml_g = lrn("regr.cv_glmnet",s = "lambda.min"),
              n_folds = 10, n_rep = 5)
mod_f$fit()
```

We obtain a much smaller standard error.

```
c(beta=mod_f$coef, se=mod_f$se)


## beta.e401   se.e401
##      8538      1258
```

References

V. Chernozhukov, D. Chetverikov, M. Demirer, E. Duflo, C. Hansen, W. Newey and J.M. Robins (2018). Double/debiased machine learning for treatment and structural parameters. *The Econometrics Journal*, 21(1) C1–C68.