Spatiotemporal tiling of the Kuramoto-Sivashinsky flow

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Abstract. a new spatiotemporal formulation to provide a new perspective. [1] [2]

The main goal of this reformulation is to provide a qualitative and quantitative description of the infinite space-time behavior of the Kuramoto-Sivashinsky flow.

Advances in experimental imaging, computational methods, and dynamical systems theory reveal that the unstable recurrent flows observed in moderate Reynolds number turbulent flows result from close passes to unstable invariant solutions of Navier-Stokes equations. In past decade hundreds of such solutions been computed for a variety of flow geometries, always confined to small computational domains (minimal cells). While the setting is classical, such classical field-theory advances offer new semi-classical approaches to quantum field theory and many-body problems.

The Gutkin and Osipov on many-particle quantum chaos (in particular, the spatiotemporal cat lattice models) suggests a path to determining such solutions on spatially infinite domains. Flows of interest (pipe, channel flows) often come equipped with D continuous spatial symmetries. If the theory is recast as a $(D+1)$-dimensional space-time theory, the space-time translationally recurrent invariant solutions are $(D+1)$-tori (and not the 1-dimensional periodic orbits of the traditional periodic orbit theory). Spatiotemporal cat lattice models suggest that symbolic dynamics should likewise be $(D+1)$-dimensional (rather than a single temporal string of symbols), and that the corresponding zeta functions should be sums over tori, rather than 1-dimensional periodic orbits.

Key words. relative periodic orbits, chaos, turbulence, continuous symmetry, Kuramoto-Sivashinsky equation

AMS subject classifications. 35B05, 35B10, 37L05, 37L20, 76F20, 65H10, 90C53

1. Introduction.

Recent experimental and theoretical advances [?] support a dynamical vision of turbulence: For any finite spatial resolution, a turbulent flow follows approximately for a finite time a pattern belonging to a finite alphabet of admissible patterns. The long term dynamics is a walk through the space of these unstable patterns. The question is how to characterize and classify such patterns? [3] Chaotic nonlinear systems constitute one of the few classical physics problems yet to be solved. The behaviors exhibited are so peculiar that it has permeated into popular culture via the butterfly effect. This behavior poses a serious challenge which has effects everything from weather prediction to air travel. In the recent past computational successes were made by studying turbulent flows on minimal cells. The motivation for these small domains was that they were chosen to be large enough to support turbulence but also small enough to remain computationally tractable. The main computational successes using these domains was the accurate calculation of unstable periodic solutions (because the system is hyperbolic, this is equivalent to simply saying periodic solutions). These solutions are also known as “exact coherent structures” (ECS) [**?**, **?**]. This name arose from the existence of identifiable structures which persist over the course of time

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[1] Matt 2019-03-15: An example of me commenting; magenta text marks my edit.

[2] Predrag 2019-05-13: In [?] equation numbers are on the right; here they are on the left. Check a recent issue of SIADS, fix this or not, and move this question, answered, to Sect. **??**.

[3] Matt 2020-01-20: [Background]

evolution. These solutions are important because it is their unstable and stable manifolds that dictate the dynamics [?]. Not only have conventional methods not worked on large domains, we argue that they never could have worked. The motivation behind minimal cells was to develop an intuition for turbulence which would be used to obtain results on progressively large domains.

In light of all of these difficulties we believe that new bold ideas are required to resume forward progress. We retreat from the conventional wisdom to start anew with a truly spatiotemporal theory, one that treats infinite space-time as the shadowing of a finite number of fundamental patterns which we denote as "tiles".

[4] The primary claim that we make is that in hindsight, describing turbulence via an exponentially unstable dynamical equation never could have worked. Conventional methods treat spatial dimensions as finite and fixed and time as inherently infinite. Our spatiotemporal formulation of chaos treats all continuous dimensions with translational invariance democratically as $(1 + D)$ different 'times'. The proposal is inspired by the Gutkin and Osipov [?] modelling of chain of N coupled particle by temporal evolution of a lattice of N coupled cat maps.

The alternative that we propose to describe infinite space-time chaos via the shadowing of fundamental patterns which we refer to as "tiles". These tiles are the minimal "building blocks" of turbulence; they are realized as doubly-periodic orbits which are global solutions with compact support. Finding the tiles of turbulence is fundamentally easier than finding doubly-periodic orbits on larger domains due to the exponential growth in complexity of solutions. In other words there are fewer important solutions on smaller domains. This in turn implies that there can only be a small number of fundamental tiles. This is what makes the problem tractable: if we can collect the complete set of tiles then we have the ability to construct every doubly-periodic orbit according to our theory.

The lack of exponentially unstable dynamics has powerful and immediate effects. Because there is no time integration, the problem of finding doubly-periodic orbits is now a variational one. The benefit of this is that there is no need to start an initial guess on the attractor; the optimization process handles this entirely. This allows us to find arbitrarily sized doubly-periodic orbitsbut in fact there is no need to. Our hypothesis is that we need only to find the building blocks which shadow larger doubly-periodic orbits and infinite space-time.

The spatiotemporal formulation allows a much easier categorization of what is "fundamental" by virtue of the frequency that patterns admit in the collection of doubly-periodic orbits. By identifying the most frequent patterns, we shall clip these patterns out of the doubly-periodic orbits they shadow and use them as initial conditions to search for our tiles.

The notion of "building blocks of turbulence" is one of the reasons for studying fluid flows in the first place. There is evidence that certain physical processes are fundamental, but they have yet to be used in a constructive manner. The spatiotemporal description is able to actually put these ideas in practice. The spatiotemporal completely avoids this by constructing larger doubly-periodic orbits from the combination of smaller doubly-periodic orbits. The reason why the search for the fundamental tiles is classified as "easy" is because in the small domain size limit there just aren't that many doubly-periodic orbits; the dynamics

[4]Matt 2020-01-20: [Revolution] WHY?

is relatively simple.

The first key difference is that the governing equation dictates the spatiotemporal domain size in an unsupervised fashion. The results here are not The only reason why L was treated as fixed is due to the inherent instability it includes when treated as a varying quantity. This small detail, allowing the domain size L to vary, is not as trivial as it seems. This difficulty is especially evident in the Kuramoto-Sivashinsky equation, whose spatial derivative terms are of higher order than the first order time derivative, but also there is a spatial derivative present in the nonlinear component.

Specifically, we propose to study the evolution of Kuramoto-Sivashinsky on the 2-dimensional infinite spatiotemporaldomain and develop a 2-dimensional symbolic dynamics for it: the columns coding admissible time itineraries, and rows coding the admissible spatial profiles. Our spatiotemporal method is the clear winner in both a computational and theoretical sense. By converting to a tile based shadowing description we have essentially removed the confounding notion of an infinite number of infinitely complex doubly-periodic orbits from the discussion. Now we must put these ideas into practice. The testing grounds for these ideas will be the spatiotemporal Kuramoto-Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + u u_x = 0 \quad \text{where} \quad x \in [0, \tilde{L}], t \in [0, T] \tag{1.1}$$

where $u = u(x,t)$ represents a spatiotemporal velocity field. This equation has been used to model many different processes such as the laminar flame front velocity of Bunsen burners. While (**??**) is much simpler than the spatiotemporal Navier-Stokes equation, we would argue that the main benefit is the simplicity of visualizing its two-dimensional space-time. This visualization makes the arguments more understandable and compelling in addition to making the tiles easier to identify. The translational invariance and periodicity of (**??**) make spatiotemporal Fourier modes a natural choice. The inherently infinitely dimensional equations are approximated by a Galerkin truncation of these spatiotemporal Fourier modes. The Kuramoto-Sivashinsky equation (**??**) in terms of the Fourier coefficients $\hat{\mathbf{u}}$ is a system of differential algebraic equations $\hat{\mathbf{u}}$

$$F(\hat{\mathbf{u}}, \tilde{L}, T) \equiv (\omega - k^2 + k^4)\, \hat{\mathbf{u}} + \frac{k}{2}\mathcal{F}(\mathcal{F}^{-1}(\hat{\mathbf{u}})^2)\,. \tag{1.2}$$

The nonlinear term is computed in a *pseudospectral* fashion: a method which computes the nonlinear term as a product in physical space as opposed to a convolution in spectral space. The definitions of each term is as follows; $\mathcal{F}$ and $\mathcal{F}^{-1}$ represent the forward and backwards spatiotemporal Fourier transform operators. Likewise, ω and k contain the appropriate temporal and spatial frequencies to produce the corresponding derivatives. Any and all indices are withheld to avoid unnecessary confusion at this stage. The spatiotemporal system of differential algebraic equations (**??**) is of the form $F(\hat{\mathbf{u}}, \tilde{L}, T) = 0$. This type of optimization problem is ubiquitous in engineering and optimization literature. Therefore solving (**??**) is a matter of adapting known numerical methods to its idiosyncracies. Once we have the ability to solve (**??**) we need to first create a collection of doubly-periodic orbits. The only requirement that the collection must satisfy is that it must capture all fundamental patterns by adequately sampling the set of doubly-periodic orbits. In other words an exhaustive search is not our aim;

not only that, but also the collection need not sample all spatiotemporal domain sizes. We hypothesize that there is some upper bound on the spatiotemporal size of fundamental tiles due to spatiotemporal correlation lengths. Once the collection is deemed sufficient we proceed to visual inspection. In this manner we determine the most frequent patterns and single them out as tile candidates. This is done by literally clipping them out of the doubly-periodic orbits that they shadow. Each clipping is then treated as an initial guess for a fundamental tile which is itself a doubly-periodic orbit. Therefore, these represent initial conditions for the optimization method. It is not a guarantee that every clipping converges to a doubly-periodic orbit; therefore the number of attempts to find a tile should continue until it does in fact converge. The number of convergence attempts is typically proportional to how confident we are that the pattern being scrutinized is in fact a tile. Once a collection of tiles is collected, we can construct new and reproduce known doubly-periodic orbits. This is completed with a method we refer to as "gluing". It is as straightforward as one might infer: tiles are combined in a spatiotemporal array to form initial conditions used to find larger doubly-periodic orbits. Methods of gluing temporal sequences of doubly-periodic orbits exist but never has the ability to glue doubly-periodic orbits spatiotemporally existed before. With the implementation of the gluing method can begin to probe the 2-dimensional spatiotemporal symbolic dynamics previously mentioned. A fully determined symbolic dynamics is sufficient to describe infinite space-time completely. We already have the two edges of this symbol plane - the $\tilde{L} = 22$ minimal cell [**?**, **?**] is sufficiently small that we can think of it as a low-dimensional ("few-body" in Gutkin and Klaus Richter [**?**, **?**, **?**, **?**] condensed matter parlance) dynamical system, the left-most column in the Gutkin and Osipov [**?**] $2D$ symbolic dynamics spatiotemporal table (not a 1-dimensional symbol sequence block), a column whose temporal symbolic dynamics we will know, sooner or later. Michelson [**?**] has described the bottom row. The remainder of the theory will be developed from the bottom up, starting with small spatiotemporal blocks.

The plans for our spatiotemporal formulation have been laid bare. The main concept is that the infinities of turbulence can be described by spatiotemporal symbolic dynamics whose letters are fundamental spatiotemporal patterns. Consequentially, we have created numerical methods which not only perform better than conventional methods but also present incredible newfound capabilities. These newfound capabilities include but are not limited to finding small doubly-periodic orbits which shadow larger doubly-periodic orbits but also constructing larger doubly-periodic orbits from smaller ones. These new and robust methods alone present a way forward for turbulence research, hence their is merit in a spatiotemporal formulation even though the theory has not been fully fleshed out. To test our spatiotemporal ideas we require three separate numerical methods: the first should be able to find doubly-periodic orbits of arbitrary domain size. The second needs to be able to clip or extract tiles from these doubly-periodic orbits. Lastly, we need a method of gluing these tiles together. All three of these techniques require the ability to solve the optimization problem $F(\hat{\mathbf{u}}, \tilde{L}, T) = 0$ on an arbitrarily sized doubly periodic domain.

[5] As previously discussed, this work does not use approximate recurrences or time integration to generate initial conditions. Instead we simply initialize a lattice of Fourier modes by first deciding on the dimensions of the lattice and then assigning random values to the modes.

[5] Matt 2020-02-18: How?

Specifically, random values in this case are drawn from the standard normal distribution and then normalized such that the physical field $u(x,t)$ has the assigned maximum value. Manipulations of the Fourier spectrum can also be made but we have no specific recommendation for how to do so as it can be rather nonintuitive. The first method substitutes an equivalent optimization problem instead of directly solving $F = 0$. The optimization problem is formed by the construction of a scalar cost function

$$\mathcal{I}(\hat{\mathbf{u}}, \tilde{L}, T) = \frac{1}{2}||F(\hat{\mathbf{u}}, \tilde{L}, T)||_2^2 \,. \tag{1.3}$$

taking a derivative with respect to a fictitious time τ

$$\begin{aligned} \frac{\partial \mathcal{I}}{\partial \tau} &= \nabla \left(\frac{1}{2}||F(\hat{\mathbf{u}}, \tilde{L}, T)||_2^2 \right) \partial_\tau[\hat{\mathbf{u}}, \tilde{L}, T] \\ &= \left(\left[\frac{\partial F}{\partial \hat{\mathbf{u}}}, \frac{\partial F}{\partial \tilde{L}}, \frac{\partial F}{\partial T} \right]^\top F(\hat{\mathbf{u}}, \tilde{L}, T) \right) \cdot \partial_\tau[\hat{\mathbf{u}}, \tilde{L}, T] \\ &\equiv \left(J^\top F \right) \cdot \partial_\tau[\hat{\mathbf{u}}, \tilde{L}, T] \,. \end{aligned} \tag{1.4}$$

This equation (**??**) by itself does not provide us with a descent direction because $\partial_\tau[\hat{\mathbf{u}}, \tilde{L}, T]$ remains unspecified. The simplest choice is the negative gradient of the cost function; this choice corresponds to the gradient descent algorithm.

$$\partial_\tau[\hat{\mathbf{u}}, \tilde{L}, T] = -\left(J^\top F \right) \,, \tag{1.5}$$

such that

$$\frac{\partial \mathcal{I}}{\partial \tau} = -\left|\left| \left(J^\top F \right) \right|\right|_2^2 \leq 0 \,. \tag{1.6}$$

In order to "descend" we use Euler's method to integrate in the descent direction. Note that this integration is with respect to fictitious time and represents making successive variational corrections; it is not dynamically unstable time integration. We elect to use a combination of step limit and absolute tolerance to determine when the descent terminates. If the cost function doesn't cross the threshold by the step limit then the descent is terminated. The descent algorithm can be viewed as a method of converging approximate solutions close enough to a final doubly-periodic orbit such that the least-squares algorithm can converge them, akin to [**?**].

The second method is application of a least-squares solver to the root finding problem $F = 0$. The Newton system is derived here for context.

$$F(\hat{\mathbf{u}} + \delta\hat{\mathbf{u}}, \tilde{L} + \delta\tilde{L}, T + \delta T) \approx F(\hat{\mathbf{u}}, \tilde{L}, T) + J \cdot [\delta\hat{\mathbf{u}}, \delta\tilde{L}, \delta T] + \dots \,. \tag{1.7}$$

substitution of zero for the LHS (the root) yields

$$J \cdot [\delta\hat{\mathbf{u}}, \delta\tilde{L}, \delta T] = -F(\hat{\mathbf{u}}, \tilde{L}, T) \,. \tag{1.8}$$

where

$$J \equiv \left[\frac{\partial F}{\partial \hat{\mathbf{u}}}, \frac{\partial F}{\partial \tilde{L}}, \frac{\partial F}{\partial T}\right] . \tag{1.9}$$

Technically this equation is solved iteratively, each time producing its own least-squares solution which guides the field to doubly-periodic orbit. The equations are augmented to include variations in $\tilde{L}, T$ and as such the linear system is actually rectangular. We chose to solve the equations in a least-squares manner as we are not focused on finding a unique solution; any member of a doubly-periodic orbits group orbit will do. The price of this indefiniteness is that we might collect doubly-periodic orbits which belong to the same group orbit. To improve the convergence rate of the algorithm we also include backtracking: the length of the Newton step is reduced until either a minimum length is reached (failure) or the cost function decreases. As a caveat, our specific least-squares implementation is memory limited. That is, we can only apply it to some maximum dimension as it requires the explicit construction of a large, dense matrix. Currently it suits our purposes such that we do not include any other numerical methods in this discussion. The primary numerical methods that we apply have been described. Now we can move onto describing exactly how we used these method to further our spatiotemporal theory.

As previously mentioned, we must first find a collection of doubly-periodic orbits which we believe adequately samples the space of doubly-periodic orbits, up to some maximum size. We automated the search over a range of periods and domain sizes. Periods were chosen from the range $T \in [20, 180]$. Meanwhile, the spatial range was $\tilde{L} \in [22, 88]$. The discretization size depended on the spatiotemporal domain size; more modes are needed to resolve larger solutions. The number of lattice points in each dimension were typically chosen to be powers of two in order because of their interaction with discrete Fourier transforms. A strict rule for lattice size was never developed so we offer is the approximate guidelines

$$M = 2^{\mathrm{int}(\log_2(\tilde{L})+1)} \tag{1.10}$$

for space and

$$N = 2^{\mathrm{int}(\log_2(T))} . \tag{1.11}$$

for time. The tolerance of the cost function for the gradient descent was typically set at 10^{-4} and the step limit was set as a function of the size of the lattice. For the least-squares with backtracking the tolerance for termination was originally 10^{-14} and the step limit was 500. The large step limit was because of the allowance of back-tracking, which reduces the step length. The final tolerance can likely be relaxed as there is minimal change in solutions over many orders of magnitude of the cost function; an indication that a different norm should be used. As a reminder, our claim is that the tiles are doubly-periodic orbits which shadow larger doubly-periodic orbits. Therefore we should be able to converge subdomains which have been numerically clipped out of larger doubly-periodic orbits. After visual inspection, we believed the number of fundamental tiles to be small. Therefore, a precise and unsupervised algorithm for clipping was not developed. Instead the only criteria we abided by is that the clipping must include the tile being sought after; of course, clippings that were closer to being doubly periodic were sought after. For the original doubly-periodic orbit with dimensions $x \in [0, \tilde{L}_0]$ and $t \in [0, T_0]$ defined on a lattice, the clipping can be described as follows. Find the approximate domain on which the shadowing occurs and then literally extract the subregion of the parent lattice, setting the new spatiotemporal dimensions according to the smaller lattice. In other words, the same grid spacing was maintained throughout this procedure. This process in combination with our numerical methods was sufficient for finding tiles. It is one thing to claim that certain spatiotemporal doubly-periodic orbits are the building blocks of turbulence for the Kuramoto-Sivashinsky equation. It is another thing entirely to put our money where our mouth is by actually using them in this manner. We would like to remind the audience that the ability to construct and find solutions in this manner has not been witnessed in the literature. With this in mind our choices should be treated as preliminary ones; it is entirely possible and likely that many improvements could be made. Much like the clipping process used to find tiles combining solutions in space-time, the overarching idea of gluing is straightforward and intuitive. Specifically, the tiles represent the Brillouin zone, fundamental domain, unit cell of a lattice, etc. of each fundamental doubly-periodic orbit. The general case is that we have a general $s_n \times s_m$ sized mosaic of tiles. The admissibility of the gluing is determined by the (currently unknown) symbolic dynamics. Gluing is only well defined if the lattices being combined have the same number of grid points along the gluing boundary. This creates a problem, however, as different tiles will have different spatiotemporal dimensions $\tilde{L}, T$ because they are fundamentally different solutions. This actually helps provide a precise meaning to the term "gluing". Gluing is a method of creating initial conditions which approximates a non-uniform rectangular lattice (combination of tiles) as uniform. This of course introduces local error which depends on the grid size; therefore there should not be an extreme discrepancy between the doubly-periodic orbits or tiles being glued. With this in mind, we simply rediscretize and concatenate the new lattices. The dimensions of the new lattice are determined by the sum or average of the original dimensions. For example, if gluing two tiles together in time, the new period would be $T = T_1 + T_2$ but the new spatial period is $\tilde{L} = \frac{\tilde{L}_1 + \tilde{L}_2}{2}$. In this case the number of spatial grid *points* and temporal grid *spacing* should be the same. There are many more complicated alternatives, limited only by the imagination.

2. Results.

3. Transition and summary so far. So far we have motivated a spatiotemporal theory of turbulence which replaces unstable dynamics with spatiotemporal patterns. We formulated these ideas using the Kuramoto-Sivashinsky equation and then described a few methods with which to solve the corresponding equations. The following section describes the results of our numerical investigation.

4. Results. Before our numerical investigation began in earnest we first tested its efficacy using known periodic orbits [?]. The first test was to find fixed points using coarse spatiotemporal discretizations of these periodic orbits. Other spatiotemporal methods such as the Newton descent method developed in [?] gave an indication as to the typical spatiotemporal discretization size required to resolve periodic orbits with $L = 22$ but not in the context of (??). Regardless of the discretization size, these known solutions would never be solutions to (??), due to the intrinsic error introduced by numerical integration. [?] summarizes this nicely by saying "solving a discretized system of ODEs is different than solving the underlying PDE". These solutions converged to periodic orbits as expected; albeit with slightly different spatiotemporal domain sizes. One could argue that the domain size changing disqualifies our statement that we found the "same" solution. This debate arises organically in later sections so it is postponed until then. With this we confirmed that our numerical methods worked in the sense that we could now solve (**??**). It is worthwhile to also demonstrate how robust and powerful these methods are. The use of "powerful" here denotes computational improvements over other methods. For example, in [**?**] the number of points used to discretize periodic orbits was stated to be $M = 32$ points in space (for a constant $L = 22$) and in time either $N = 512$ or $N = 1024$ depending on the period. In our computational tests and later calculations on the same domain size the spatial discretization $M = 32$ is maintained by the temporal discretization is reduced to either $N = 32$ or $N - 64$. This is an improvement by a factor of 16 and should not be overlooked, as a common criticisms of spatiotemporal methods is that they require too much computational memory to be viable. Likewise there seems to be an incredible robustness to noise that we believe arises from the inherent topology of the spatiotemporal formulation. In fact, the noise that was withstood for a small spatiotemporal solution was larger in magnitude than that of the original field itself. In this specific case, the additive noise was an *aperiodic* lattice of values drawn from the standard normal distribution, multiplied by the L_∞ norm of the periodic orbit. The sum of these two fields converged to the same solution, up to symmetry operations (translations) and domain size changes. This indeed exceeded our expectations; the sum of periodic orbit and noise is nearly indistinguishable from the noise alone. Note that the perturbation did not include changes to domain size, the "memory" of the domain size might be what ultimately lead to this resiliency. It was these preliminary results that enabled us to abandon recurrence methods entirely. This allowed for a broad search for periodic orbits that spanned all symmetry types and a wide variety of domain sizes, starting with initial conditions lacking any knowledge of the Kuramoto-Sivashinsky equation. Perhaps a more practical description would be that with our codes, the layman could find periodic orbits of a nonlinear chaotic PDE without any knowledge of the Kuramoto-Sivashinsky equation or nonlinear dynamics. The process of finding periodic orbits, described in Sect. **??** was automated to sample the space of domain sizes spanned by $L \times T \in [22, 66] \times [20, 200]$. Again we remind the reader that our hypothesis regarding fundamental patterns does not require an exceptionally large domain size; it must only be large enough so that all of these fundamental patterns are exhibited. With these spatiotemporal tiles and their accompanying states, the numerical methods described in Sect. **??** were applied. With these methods and some trial and error with the parameters, we were able to form a collection of hundreds of periodic orbits within a short period of time. To give a very rough estimate, the total time required to find a periodic orbit ranges from a handful of seconds to tens of minutes, highly dependent on dimensionality. Strictly speaking, we implicitly did not allow for longer times as we limited the dimensionality by virtue of our range of domain sizes.

The first reassuring detail of this collection of periodic orbits is that all periodic orbits look more or less the same (outliers discussed later on). More precisely, if one were to zoom in on the local features and patterns in a single spatiotemporal region neither the global spatiotemporal symmetry nor the spatiotemporal domain size would be ascertainable. The fundamental pattern hypothesis is a much stricter requirement than this, however. It says that periodic orbits must be quantitatively similar on spatiotemporal windows of predetermined (by the equations) size. To demonstrate this, the figure Figure **??** displays periodic orbits of the four different spatiotemporal symmetry classes we consider in our investigation. They are essentially indistinguishable when plotted in their fundamental domains as shown here.

As a result of the least-squares approach, it is also possible to find orbits that look very similar and are defined on similarly sized domains. The reasons for these results are two-fold: least-squares lacking unique solutions and also continuous families of orbits.

Another means of developing our pattern intuition is to look at the *atypical* patterns of the Kuramoto-Sivashinsky equation. The discussion that follows concerns the patterns and periodic orbits which we consider to be outliers. This classification consists of three main categories: periodic orbits which have are highly symmetric, contain uncommon patterns, or which are shadowed by an equilibrium for a substantial portion of time (reminiscient of homoclinic cycles). For our purposes, "outliers" is a blanket term for very isolated (unstable) solutions which are hardly ever realized. The reason we do not just denote these solutions as isolated is because there are (presumably) isolated periodic orbits which are shadowed by fundamental patterns. In other words, we want to stress that these solutions really are oddballs that you would never seen in simulation. Our searches find these outliers because stability has no affect on the optimization problem. The claim that these are isolated solutions is supported by the fact is supported by the fact that we find them more frequently in the antisymmetric subspace of solutions; which is by definition isolated. It is numerically possible to find antisymmetric solutions even if the search is not constrained to the antisymmetric subspace.

Normally this would not be possible due to large instabilities, but again we reiterate that stability does not affect our spatiotemporal optimization method. Our belief is that it is possible to be tricked into thinking that isolated patterns exist "everywhere" but in fact they only exist in the antisymmetric subspace. What does this mean? It depends on the symmetry constraints but here are two examples. For no imposed symmetry, demonstrates an "antisymmetric" periodic orbit with broken reflection symmetry. Likewise, if the imposed symmetry is shift-reflection, then by its nature it can realize even multiples of prime periods of antisymmetric periodic orbits; i.e. two, four, etc. repeats of an antisymmetric solution. These aren't just assumptions either; these solutions can actually be converged to the antisymmetric subspace (if the reflection axis is restored via translation, at least).

We have displayed patterns that we consider to be both typical and atypical. This developed our as to what patterns are fundamental periodic orbit candidates. To do so we can both find patterns which occur frequently over our entire collection of solutions as well as patterns that occur frequently in individual periodic orbits. Our task goes beyond pattern identification in the next subsection, as we must find candidates which will converge to the desired pattern.

The solution demonstrates a behavior that is not particularly common on smaller domain sizes.

Now that we have a collection of solutions and some intuition we pursue the fundamental periodic orbits. To further motivate some of our guesses for patterns, we direct the reader to where a very large time integrated trajectory (aperiodic in time) is displayed. In addition, we attempt to display the incredible frequency that a single pattern shadows this trajectory.

Note that because these are shadowing events, the patterns clipped out from the trajectory look similar but not identical; in both size and shape.

The first guess pattern was the spatiotemporal wiggle which appears in The motivation behind this guess was that it very distinctly repeats twice with respect to time. This was the inaugural fundamental periodic orbit search and because we were fairly certain it was fundamental, we proceeded to apply the "clipping" procedure from Sect. 1 in an iterative fashion. The results are as displayed in The final result was an antisymmetric fundamental periodic orbit whose velocity field on the fundamental domain consisted of a wavelength "wiggling" in time. Comparison of this pattern back to our reference solution collection, we notice that while converged in the antisymmetric subspapce, the pattern more often (if not always) occurs with its reflection partner. The general spatiotemporal trajectory does not exist in the antisymmetric subspace; we believe that this is a manifestation of that. Even though the fundamental periodic orbit exists in the antisymmetric subspace, it might just be that this is a special case of a broader family of solutions which do not have reflection symmetry. This result, not only provided us with our first fundamental periodic orbit but also provided us with an obvious guess of the second fundamental periodic orbit. If we look back at the iterative clipping procedure we see that the spatiotemporal wiggle is spatially adjacent to a single wavelength. Therefore, it would make sense that a single wavelength equilibrium solution would be our second fundamental periodic orbit. With these two fundamental periodic orbits we could not explain the spatiotemporally chaotic behavior of arbitrary solutions; in fact, we could not explain anywhere near a majority of space-time. This indicated that perhaps the most important fundamental periodic orbit was yet missing. To come up with our third guess, we appealed to a space-time trajectory defined on a large domain and produced via time integration. Doing so had immediate consequences. Arguably the most common pattern in this trajectory is what we now refer to as a spatiotemporal defect. The spatiotemporal defect is the occurence where two spatially adjacent wavelengths merge into one. This is an important mechanism because it somehow encapsulates how the number of wavelengths defined at any given instant in time fluctuates about the most unstable wavelength number. An addition property of the spatiotemporal defect that creates some difficulty in finding the corresponding fundamental periodic orbit is that it also incorporates a phase shift; that is, it is a relative periodic fundamental periodic orbit. The defect is the most important of all fundamental periodic orbits, its frequency and the physical properties dominate the large spatiotemporal trajectory. This can be seen in where image editing software was used to clip out believed shadowings by the spatiotemporal defect. where both the number of wavelengths as well as spatial drift velocity are constantly fluctuating. Out of our three fundamental periodic orbits, only the spatiotemporal defect can be used to explain these two observations. The relative periodic nature of the spatiotemporal defect made finding the corresponding fundamental periodic orbit was more difficult than both the spatiotemporal streak and spatiotemporal wiggle. By "more difficult", we mean that multiple attempts as well as initial conditions were required. This process continued, as we still believed that more unique fundamental periodic orbits existed. Indeed, we did find what we believed at first to be unique fundamental periodic orbits but upon numerical continuation, we are able to show that these were redundant. This was a critically important revelation; the fundamental periodic orbits and tiles are realized on infinitely many spatiotemporal domains, existing in continuous families.

By itself this is not shocking nor new; it is equivalent to the existence of solution branches that could be and have been explored in bifurcation analyses. In the context of our spatiotemporal theory and fundamental periodic orbits, however, we believe it is the most essential piece of information that we have discovered to this point. In the context of our anticipated spatiotemporal symbolic dynamics, there is no discrete alphabet. Instead our tiles are rubber such that in our to represent a periodic orbit symbolically we need both the configuration of tiles as well as their periods. It is even more complicated than this, as we infer that each symbol in the configuration can exist as a subset of the continuous family; determined by its spatiotemporal neighbors.

It might be accurate to say that all periodic orbits exist in continuous families *because* fundamental periodic orbits exist in continuous families. This betrays our intuition; in hyperbolic systems, periodic orbits are isolated by virtue of their unstable manifolds. One manner of reconciliation is to say that periodic orbits are dynamically isolated but also beholden to continuous deformations of the local geometry. The periods exhibited by each continuous family seems to exist on finite intervals, which we believe are punctuated by bifurcations. Exploration of every fundamental periodic orbit's continuous family actually reduced our number of unique tiles to three. To reiterate; there may only be three fundamental periodic orbits needed to describe the infinite space-time solution of the Kuramoto-Sivashinsky equation. This results exceeded our expectations and we believe shows the merit of spatiotemporal formulations. White these results are very informative they leave us with more open questions than we Sect. **??**.

It is absolutely essential to understand the notion of continuous families of fundamental periodic orbits and so we summarize it here, before moving on to the gluing of spatiotemporal tiles. We postulated and then found a collection of fundamental periodic orbits, whose shadowing describes the entirety of space-time. Further analysis showed that our tiles exist in continuous families. Consequently, any spatiotemporal symbolic dynamics will necessarily have a continuous alphabet as well as grammar. This is completely foreign to us and hence we do not see an obvious path ahead for future investigations. In the absence of a systematic way to proceed, we investigate whether gluing fundamental periodic orbits together to find periodic orbits is indeed possible. This was the natural step to take, because eventually it will be required for any symbolic dynamical investigations.

Our claim is that only three spatiotemporal tiles are required to describe every solution to the Kuramoto-Sivashinsky equation. In order to develop this theory further, we need to discover what is known as the grammar of our spatiotemporal symbolic dynamics. The grammar is the rule-book dictating which symbolic combinations are *admissible*, or realized as periodic orbits. To probe the grammar, we needed the ability to numerically combine our spatiotemporal tiles and converge them to periodic orbits.

To test our implementation we again appeal to known solutions. The process is summarized in where tile approximations were combined to create a crude initial condition representative of the target solution. Because we had a specific target in mind, we initialized the tile periods with numbers close to the target's periods (specifically the integer component of each original period). This was a custom tailored initial condition and the amount of supervision can not be overstated, however, it was a test of the question: "Can approximate solutions be glued to find another, known solution?". It turns out the answer is yes. As can be seen by the known solution is essentially reproduced by the periodic orbit found. We say "essentially" because of course there are small differences in its periods and field due to not finding a periodic orbit with exactly the same periods. Encouraged by these results, the next test was to take known periodic orbits and glue them temporally, a familiar notion to periodic orbit theory. In other words, find periodic orbits shadowed by two smaller periodic orbits. Because at no point are periods being fixed (unless they are being used as a continuation parameter), every periodic orbit has unique periods. Even though combining periodic orbits in time is familiar, they have *which have differing spatial periods*; therefore, it is entirely different than what others have done in the past. We demonstrate this method in

In line with our identical treatment of space and time, we can also glue periodic orbits spatially. The figures show exactly this, the constituent periodic orbits and the converged result after spatial gluing.

In two periodic orbits with shift-reflection symmetry were chosen to be glued in space. As described in Sect. 1 the initial condition was created in a symmetry preserving manner such that this specific initial condition had shift-reflection symmetry imposed on it. The specific manner of doing so is

Much like clipping this gluing technique can also be applied iteratively, to form a sequence of progressively larger periodic orbits.

The only limit is really the imagination (and when they get very large, the numerical methods). Gluing in space, time, alternating between the two, etc. leads to the culmination of the method, gluing fundamental periodic orbits in a spatiotemporal fashion.

This spatiotemporal gluing is the foundation for a spatiotemporal symbolic dynamics. There are many unknowns in regards to best practices; we have only implemented the most basic of details. At this point in time the only capability is to take a symbolic representation and then create the corresponding configuration of fundamental periodic orbits. Each of the fundamental periodic orbits used in this creation process are static members of each of their continuous families, specifically the spatiotemporal defect, spatiotemporal wiggle, and spatiotemporal streak displayed in

5. Summary.

5.1. Summary. We have put for a spatiotemporal formulation of turbulence which replaces exponentially unstable dynamics for a collection of fundamental spatiotemporal patterns. We have demonstrated that arbitrary spatiotemporal solutions can be described by combinations of these special doubly-periodic orbits, which we denote as "tiles". Additionally, preliminary investigations support the claim that only three tiles are required for these spatiotemporal constructions. Our results were made possible by the implementation of numerical methods which are robust due to topological considerations. This lead to newfound capabilities and techniques that so far have not been witnessed elsewhere. Specifically, by allowing the entire spatiotemporal domain to vary during the optimization process we are able to extract small doubly-periodic orbits from larger doubly-periodic orbits(clipping) as well as build large doubly-periodic orbits by combining smaller doubly-periodic orbits(gluing). The latter of these two techniques is very powerful as it allows us to find doubly-periodic orbits of progressively larger spatiotemporal extent. Not only that, it acts as a staging ground for the determination of a 2-dimensional spatiotemporal symbolic dynamics.

5.2. Open Challenges and Future work. Our techniques, if they can be applied to the Navier-Stokes equations, would allow for the construction of larger spatiotemporal solutions using the known exact coherent structures defined on minimal cells. While the spatiotemporal "tiles" of the Navier-Stokes equations are not currently known, there is much more intuition as to what physical processes are fundamental; it is our hope that this knowledge can be leveraged to find the four dimensional space-time tiles. With these new ideas come an amalgam of fundamental questions, most of which have yet to be unanswered.

5.2.1. Theory of tile families, rubberized, tiles identified globally, not localized. The most important open question is how to realize a 2-dimensional spatiotemporal symbolic dynamics in the face of continuous tile families. If we are lucky, admissibility is not dependent on the size of the tiles but merely their spatiotemporal configurations (and possibly symmetry considerations). It is of course always possible that the admissibility depends on the family in a (fractal?) manner. Another unfortunate detail is that the grammar could be obfuscated by the potency numerical methods; in other words, a symbolic combination may be admissible but the specific numerical methods employed might not be able to find it. Almost assuredly better numerical methods exist as the ones currently employed are towards the simple end of the complexity continuum. The lack of description of the symbolic dynamics also leaves much to be desired in terms of the implementation of the gluing method. There are a number of important details that need to be considered for a systematic gluing method. Three examples are: including tile-wise local Galilean velocities, continuous tile family considerations, as well as symmetry considerations. The guiding principal would be to minimize the extent of the discontinuities at the tile boundaries. We know that each tile will locally solve the Kuramoto-Sivashinsky equation on their interior such that the effect of solving the gluing optimization problem is mainly to smooth out these discontinuities.

5.2.2. Have no metric to tell if final tile is the realization of the symbolic initial condition. For every gluing combination it is essential that the result be a realization of the initial symbolic combination. False positives lead to an incorrect grammar which can be quite deadly if the corresponding 2-dimensional itinerary defines a small tile. It is possible (and currently believed to happen very often) that the gluing combinations converge to the "wrong" periodic orbits. In order to avoid an incorrect grammar we are trying to develop a means of testing the underlying topology as a means of classifying any errors. Currently the pursuit involves applying persistent homology to attempt to detect topological signatures of the fundamental periodic orbits.

5.2.3. no physical predictions yet. Without a grammar we cannot determine the smallest admissible The lack of a systematic gluing method and symbolic dynamics prevents us from predicting or calculating any physical quantities.

5.2.4. Navier-Stokes or Kolmogorov or Fitz-Hugh Nagumo. As a reminder, this was a testing ground for these ideas. The main goal is to eventually apply this to systems which have experimental setups. This requires a large amount of effort that will hopefully be reduced by collaborative efforts once the research code is released to the public. The code that we have developed is being developed such that it has a modular form. That is, the numerical methods are being developed to be mostly agnostic to the equations.

5.2.5. Subdivision of domain; can't use Fourier. Looking forward, the foreseeable numerical challenges are of course solving spatiotemporal equations in systems with more continuous dimensions. Our entire argument is that progressively larger periodic orbits are comprised of combinations of fundamental periodic orbits. This may be powerful enough to permit the use of our current spatiotemporal numerical methods. If this is not possible, then there are a number of options for how to proceed. Fourier spectral methods may not be the best choice as spatiotemporal Fourier modes are the part of a global description. Another potential and perhaps necessary component of the spatiotemporal formulation is the ability to divide and conquer by performing parallel spatiotemporal computations. That is, the convergence of large spatiotemporal domains by solving the equations locally on subdomains which communicate with each other after each parallel computation. This type of computation is forbidden by Fourier spectral methods as it assumes periodic boundary conditions. The general case would need to be either finite element or (we believe the better choice) a spectral method which can handle non-periodic boundary conditions such as Chebyshev spectral methods.

Acknowledgments. We are grateful to N.B. Budanur for the derivation of the Kuramoto-Sivashinsky spatial evolution PDEs (**??**) and many spirited exchanges, and the anonymous referee for many perspicacious observations. P.C. and M.N.G. thank the family of G. Robinson, Jr. for partial support.

6. Tiles' GuBuCv17 clippings and notes. Move good text not used in [?] to this file, for possible reuse later.

2016-11-05 Predrag A theory of turbulence that has done away with *dynamics*? We rest our case.

2019-03-19 Predrag Dropped this:

In what follows we shall state results of all calculations either in units of the 'dimensionless system size' $\tilde{L}$, or the system size $\tilde{L} = 2\pi\tilde{L}$.

Due to the hyperviscous damping u_{xxxx}, long time solutions of Kuramoto-Sivashinsky equation are smooth, a_k drop off fast with k, and truncations of (**??**) to $16 \leq N \leq 128$ terms yield accurate solutions for system sizes considered here (see Appendix **??**).

For the case investigated here, the state space representation dimension $d \sim 10^2$ is set by requiring that the exact invariant solutions that we compute are accurate to $\sim 10^{-5}$.

6.1. GuBuCv17 to do's. Internal discussions of [?] edits.

2019-03-17 Predrag to Matt My main problem in writing this up is that I see nothing in the blog that formulates the variational methods that you use, in a mathematically clear and presentable form. Perhaps there is some text from
`siminos/gudorf/thesisProposal/proposal.tex`
that you can use to start writing up variational justification for your numerical codes, section **??** *Variational methods.*

2019-03-17 Predrag to Matt Please write up *tile extraction* and *glueing* in the style of a SIADS article.

2019-03-17 Predrag to Matt Should any of Appendix **??** *Fourier transform normalization factors* be incorporated into **GuBuCv17** [?]?

2019-04-10 Matt writing To begin `variational.tex` I included two equivalent formulations of the variational problem; the first is written in a more concise manner while the second is written in a more explicit manner. The longer of the two is commented out. The more explicit description uses dummy variables (Lagrange multipliers) which replace parameters $(\tilde{L}, T)$ as independent variables.

I'm including explanations of the numerical algorithms but I don't think I should present them in their style for algorithms, because we didn't invent them just applied them in a unique way. If desired I think the easiest way of including them per SIADS style guide is to use the algorithm package they suggest: `algpseudocode` and `algorithmic` are the package names.

I feel conflicted as to whether to define the gradient matrix using a new letter or the "mathematician way". e.g. $A(x)$ or $DG(x)$. Also, I started using $\mathbf{z}$ to represent state space vectors. I'm not a fan of using z but I don't want to confuse people by using u,x, etc.

I need to get better at writing or stop being OCD over how sentences are written.

2019-04-16 Matt update In an effort to make the chapters and `GuBuCv17.tex` more modular, I've split apart some of the chapters into smaller, more manageable pieces. For example, `variational.tex` was covering too many topics to be reflected by the file name and `numerics.tex` predominately covered discrete lagrangian systems and Noether's theorem. The algorithms (matrix free adjoint descent, matrix free GMRES and Gauss-Newton) have yet to be discussed in excruciating detail. This is my fault, in hindsight I've done a poor job with recording what I do and how I do it. I'm going to get better at this.

For the time being, until it is deemed unnecessary or unintelligent, I am going to break the chapters into the files `adjointdescent.tex` and `iterativemethods.tex`. I'm going to change the discourse so that instead of requiring the current order, namely, `variational.tex-¿adjointdescent.tex-¿iterativemethods.tex` the pieces will be written as to be independent of one another.

In order to get specific, I needed to include the Kuramoto-Sivashinsky equation written in the Fourier-Fourier basis; I put this in `sFb.tex`

2019-04-17 MNG update Realized that in order to get specific with the numerical methods I need to include both an exposition on the spatiotemporal Fourier modes as well as the matrix-free computations. The latter really stresses the improvements over the finite-difference approximation of the Jacobian that requires time integration ubiquitous in plane-couette and pipe numerics. Expanding on `adjointdescent.tex` and `iterativemethods.tex`. Again, the main stratagem is to make the separate `.tex` files as independent as possible to avoid "long distance references".

2019-04-18 MNG Heavy edits to `tiles.tex` Added section on preconditioning `preconditioning.tex`

Formatting edits to `matrixfree.tex` can be ignored.

Added details in `iterativemethods.tex` regarding GMRES and SciPy wrapper for LAPACK solver GELSD

2019-04-23 MNG Converting indices to abide by the conventions: physical space indices $u(x_m, t_n)$, and spatiotemporal Fourier space indices $\hat{\mathbf{u}}_{kj}$.

2018-05-09 PC can do. Also, remember that $u(x_m, t_n)$ implies that everywhere the ordering is $(\tilde{L}, T)$, and not $(T, \tilde{L})$.

Luca Dieci asked (borderline pleaded) to abide by the mathematics convention that n is the index for discrete time. I'm avoiding ℓ and τ_t due to the unnecessary confusion with domain size $\tilde{L}$ and period T.

2018-05-09 PC Agreed. τ_t we usually control by macro $\backslash zeit$, so currently t_n.

2019-04-24 MNG Discussion of how I foresee paper(s) playing out in `blogMNG.tex` by considering subject matter, narratives, and paper length. Perhaps unsurprisingly I lean towards structuring a paper similar to my thesis.

I'm unsure how to approach spatiotemporal symmetries in a practical manner. Projection operators which produces symmetry invariant subspaces are nice and complements the selection rules for different symmetries nicely. Specifically it provides the reason for why the selection rules exist and motivates the use of symmetry constrained Fourier transforms. The only issue I have with this is that the results of the formal derivation are not really used beyong that. I think this is likely a case of "It-is-trivial-now-that-I-know-it' syndrome. Perhaps it would be sufficient to say that the selection rules constitute these subspaces without the formalism?

2019-04-29 MNG Rewrite of `KSsymm.tex` after double checking the derivations. Going to rewrite `sFb.tex`, I'm paying for the expedient manner in which is was written; in other words just use a single Fourier basis as opposed to a real basis and a complex basis, Matt.

2019-04-30 MNG Rewrites to describe the spatiotemporal Kuramoto-Sivashinsky equation only in terms of real valued Fourier coefficients for consistency. The index notation gets a little rough but the pseudospectral form of the equation is nice enough.
Tried to find the most concise description of how I handle relative periodic orbits using mean velocity frame (time dependent rotation transformation).

2019-05-02 MNG Is it necessary to recap all of the results in Sect. **??** in this paper? Other than the spatial integration calculation the results described in [**?**, **?**]. I'm unsure how to connect the spatiotemporal calculations to results pertaining to the dynamical system formulation, e.g. temporal stability and energy budget.
Moved `SpatTempSymbDyb.tex` to after `tiles.tex` such that it proceeds from finding tiles to using tiles.
The bulk of each section is complete; perhaps need to add some more detail to `glue.tex` and `tiles.tex` but mostly need to work on picking, producing, and inserting figures.
Going to list suggestions for figures at the top of each section in commented text.

2019-05-02 MNG Added tile figures: Extraction and converged results in `tiles.tex`.
Modifying scripts to produce figures of general numerical convergence (initial condition to final converged doubly-periodic orbit), produce figures demonstrating step-by-step gluing for repeated gluing, and produce figures for the "frankenstein" plots (combining tiles to produce doubly-periodic orbits). Basically just producing more figures.

2019-05-11 PC moved Ibragimov to `gudorf/thesis/thesis.tex` until we find it useful.

2019-05-13 MNG
- Added spatial gluing figures
- Added description of gluing procedure

2019-05-13 PC Figures are looking great, and in my talks people seem to "get" tile extraction and gluing, so they are very important. A few notes, before you produce the next versions:
- I think you should label all u color bars in multiples of 1, or or 0.5 if that is really needed, not different units in every plot.
- Once you have improved a given figure, keep the same name rather than renaming it (they are often shared between different articles, presentations and blogs)

2019-07-05 PC dropped from trawl.tex: " In both formulations there is no guarantee of convergence but it is clearly better to take less time regardless of convergence.
In our formulation, convergence can not be guaranteed either, but the resources committed to the initial guesses generation are negligible. "

$$
\begin{aligned}
q_k &= 2\pi \frac{k}{\tilde{L}}\,, &\qquad k &= 1, \cdots, M/2-1 \\
\omega_j &= 2\pi \frac{j}{T}\,, &\qquad j &= 0, \cdots, N/2-1 \\
x_m &= \frac{m}{M}\tilde{L}\,, &\qquad m &= 0, \cdots, M-1 \\
t_n &= \frac{n}{N}T\,, &\qquad n &= 0, \cdots, N-1\,.
\end{aligned}
\tag{6.1}
$$

2019-08-21 MNG Moved discussion of recurrence plots and multiple shooting from trawl.tex to variational.tex

It seemed more coherent to first describe the disadvantages of the IVP to motivate the variational problem. I'm going to refer to what I do as "solving a variational problem" as opposed to boundary value problem because it insinuates (at least to me) that we're solving a Dirichlet BC in $1+1$ dimensions problem.

General narrative of variational.tex

- Exponential instability bad
- Variational formulation good
- How to solve variational problem (general description of optimization)
- Losses from variational formulation (notion of dynamics, stability, bifurcation analysis).
- How to recoup from these losses (adjoint sensitivity, Lagrangian, Hill's formula)

It's currently a hot mess.

2019-09-20 MNG Input references to topological defects and motifs in complex networks. Renamed the "defect tile" to the "merger tile" but also made the connection that similar patterns in crystals are referred to as "edge dislocations".

Just clean up and rewriting `tiles.tex` mainly; it's almost in shape.

2018-05-09 PC Dropped: The following definitions will be devoid of symmetry considerations such that the equations represent the general case.

For $\tilde{L} < 1$ the only equilibrium of the system is the globally attracting constant solution $u(x,t) = 0$, denoted E_0 from now on. With increasing system size $\tilde{L}$ the system undergoes a series of bifurcations. The resulting equilibria and relative equilibria are described in the classical papers of Kevrekidis, Nicolaenko and Scovel [**?**], and Greene and Kim [**?**], among others. The relevant bifurcations up to the system size investigated here are summarized in Figure **??**: at $\tilde{L} = 22/2\pi = 3.5014\cdots$, the equilibria are the constant solution E_0, the equilibrium E_1 called GLMRT by Greene and Kim [**?**, **?**], the 2- and 3-cell states E_2 and E_3, and the pairs of relative equilibria $\mathrm{TW}_{\pm 1}$, $\mathrm{TW}_{\pm 2}$. All equilibria are in the antisymmetric subspace $\mathbb{U}^+$, while E_2 is also invariant under D_2 and E_3 under D_3.

Due to the translational invariance of Kuramoto-Sivashinsky equation, they form invariant circles in the full state space. In the $\mathbb{U}^+$ subspace considered here, they correspond to $2n$ points, each shifted by $\tilde{L}/2n$. For a sufficiently small $\tilde{L}$ the number of equilibria is small and concentrated on the low wave-number end of the Fourier spectrum.

dropped this: G, the group of actions $g \in G$ on a state space (reflections, translations, *etc.*) is a spatial symmetry of a given system if $gu_t = F(g\,u)$.

An instructive example is offered by the dynamics for the $(\tilde{L}, T) = (22, T)$ system that [?] specializes to. The size of this small system is ~ 2.5 mean wavelengths ($\tilde{L}/\sqrt{2} = 2.4758\ldots$), and the competition between states with wavenumbers 2 and 3. The two zero Lyapunov exponents are due to the time and space translational symmetries of the Kuramoto-Sivashinsky equation.

For large system size, as the one shown in Figure **??**, it is hard to imagine a scenario under which attractive periodic states (as shown in [?], they do exist) would have significantly large immediate basins of attraction.

2019-10-17 MNG : Merged symmetry discussions. `KSsymmMNG1` was deleted because seems to be an old discussion predating the spatiotemporal symmetry group discussion as it still mentions equivariance. The focus should only be on invariance under symmetry operations, as invariance gives rise to the the practical application of the symemtry discussion which is constraints on the spatiotemporal Fourier coefficients. `KSsymmMNG` was deleted because it is just an older version of `KSsymm`. `KSsymmPC` uses different notation and says things better than I do so I'll have to figure out how to merge it in.

2019-10-25 PC dropped from *variational.tex*:

Linear stability analysis has been used in bifurcation analysis of describe the existence and bifurcations of solutions as well as the geometry of state spaces corresponding to different flows [?, ?, ?].

Commonly time variational integrators preserve symplectic structure

2019-09-05 MNG Dropped from *variational.tex*: multishooting optimization of cost functional because it doesn't jive with spatiotemporal methods (based on integration)

Adjoint sensitivity and Hill's formula sections when I figure them out or they seem useful:

Section on adjoint sensitivity The spatiotemporal reformulation of a dynamical problem also requires a reformulation of its linear stability analysis.

Nevertheless, we still have the notions of tangent spaces and derivatives so the natural replacement is the notion of sensitivity. In the context of finite element (finite difference) representation, instead of computing a derivative and transporting it around a periodic orbit, it instead computes the derivative of the temporal average of the quantity with respect to whichever parameter is desired [?, ?, ?]. Because there is no transport, one need not worry about the exponential instability present. Essentially sensitivity is to stability as boundary value problem is to initial value problem in this context. Because the spatiotemporal boundary problem is defined on a compact domain on which the scalar field does not diverge, dynamical observables are bounded; they do not experience numerical overflow (underflow) associated with unstable (stable) manifolds.

$$S = \int_{\mathcal{M}} \mathcal{L}(u, v, u_x, v_x, u_t, v_t, u_x x, v_x x) dx dt \tag{6.2}$$

such that the matrix of second variations, or Hessian, of this action functional is defined as

$$H = \nabla \nabla^\top S \tag{6.3}$$

such that the derivatives are taken with respect to the infinite dimensional scalar fields $u, v, \ldots$, such that the Hessian matrix is infinite dimensional prior to discretization of the scalar fields. The resultant discrete Lagrangian system and subsequent Hessian should be the Hessian of Hill's formula, I believe. If one is trying to derive Hamilton's action principle as a result of discretization (that is, finite differences) as in [**?**] then one must take care to define spatiotemporal differentiation operators in a manner consistent with an action principle. A large amount of the derivation of the discrete action principle and discrete Noether's theorem of [**?**] relates to using a finite element discretization in physical space. I am unsure how these ideas extend to a Fourier basis; I currently am assuming that as long as the differentiation operators, and hence the derivatives (jet bundle) is properly defined then everything should work out. When two total derivatives of the Lagrangian density are taken, one arrives at the following matrix representation of the Hessian. Keep in mind that we have ordered the variables in terms of the order of the corresponding derivatives $(u, v, u_t, v_t, u_x, v_x, u_x x, v_x x)$.

$$\begin{bmatrix} -v_x(t,x)/3 & u_x(t,x)/3 & 0 & -1/2 & v(t,x)/3 & -2u(t,x)/3 & 0 & 0 & \\ u_x(t,x)/3 & 0 & 1/2 & 0 & u/3 & 0 & 0 & 0 & \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ v(t,x)/3 & u(t,x)/3 & 0 & 0 & 0 & & -1 & 0 & 0 \\ -2u(t,x)/3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \end{bmatrix}. \tag{6.4}$$

This is an infinite dimensional matrix, but upon discretization each block will represent a diagonal matrix whose diagonal contains the scalar field values of the corresponding spacetime coordinates. For instance, $u_x/3 \equiv \frac{1}{3}u_x(x,t) \to \frac{1}{3}u_x(t_n, x_m)$. Because each of the blocks are diagonal, that is, $1 \equiv \mathcal{I}^{N*M}$, the determinant expansion is long but not impossible to decipher. Note the presence of the adjoint variables v, v_x. There is freedom in the choice of what these variables should be, because they are non-physical.

2020-02-28 MNG Reformatted the paper into sections which follow the outline so far. ,`tileoutline.tex`, `tileintro.tex`, `tilebody.tex`,`tilesummary.tex`, `tilefuture.tex`

Note to Predrag - send this paper to