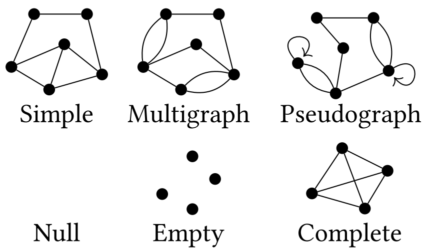
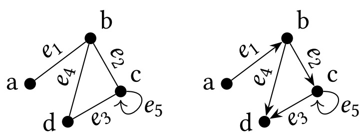
5 Graph Theory Cheatsheet

Glossary

* **Graph** is an ordered pair $G = \langle V, E \rangle$, where $V = \{v_1, \dots, v_n\}$ is a set of vertices, and $E = \{e_1, \dots, e_m\}$ is a set of edges.
  * Given a graph G, the notation $V(G)$ denotes the vertices of G.
  * Given a graph G, the notation $E(G)$ denotes the edges of G.
  * In fact, $V(\cdot)$ and $E(\cdot)$ functions allow to access "vertices" and "edges" of any object possessing them (*e.g.*, paths).
* **Order** of a graph G is the number of vertices in it: $|V(G)|$.
* **Size** of a graph G is the number of edges in it: $|E(G)|$.
* Two graphs are **equal** if their vertex sets and edge sets are equal: $G_1 = G_2$ iff $V_1 = V_2$ and $E_1 = E_2$.
* Two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ are called **isomorphic**, denoted $G_1 \simeq G_2$, if there exists an *edge-preserving* bijection $f : V_1 \to V_2$, *i.e.* any two vertices $u, v \in V_1$ are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. This means that the graphs are structurally identical *up to vertex renaming*.
* Simple **undirected** graphs have $E \subseteq V^{(2)}$, *i.e.* each edge $e_i \in E$ between vertices u and v is denoted by $\{u, v\} \in V^{(2)}$. Such *undirected edges* are also called *links* or *lines*.
  * $A^{(k)} = \{\{x_1, \dots, x_k\} \mid x_1 \neq \dots \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\}$ is the set of k-sized subsets of A.
* Simple **directed** graphs have $E \subseteq V^2$, *i.e.* each edge $e_i \in E$ from vertex u to v is denoted by an ordered pair $\langle u, v \rangle \in V^2$. Such *directed edges* are also called *arcs* or *arrows*.
  * $A^k = A \times \dots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$ is the set of k-tuples (Cartesian k-power of A).



* **Multi-edges** are edges that have the same end nodes.
* **Loop** is an edge that connects a vertex to itself.
* **Simple graph** is a graph without multi-edges and loops.
* **Multigraph** is a graph with multi-edges.
* **Pseudograph** is a multigraph with loops.
* **Null graph** is a "graph" without vertices.
* **Trivial (singleton) graph** is a graph consisting of a single vertex.
* **Empty (edgeless) graph** is a graph without edges.
* **Complete graph** K_n is a simple graph in which every pair of distinct vertices is connected by an edge.
* **Weighted graph** $G = (V, E, w)$ is a graph in which each edge has an associated numerical value (the *weight*) represented by the **weight function** $w : E \to \text{Num}$.
* **Subgraph** of a graph $G = \langle V, E \rangle$ is another graph $G' = \langle V', E' \rangle$ such that $V' \subseteq V$, $E' \subseteq E$. Designated as $G' \subseteq G$.
* **Spanning (partial) subgraph** is a subgraph that includes all vertices of a graph.
* **Induces subgraph** of a graph $G = \langle V, E \rangle$ is another graph G' formed from a subset S of the vertices of the graph and *all* the edges (from the original graph) connecting pairs of vertices in that subset. Formally, $G' = G[S] = \langle V', E' \rangle$, where $S \subseteq V$, $V' = V \cap S$, $E' = \{e \in E \mid \exists v \in S : e \mathrel{I} v\}$.



Adjacency matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \end{array}$$

Incidence matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

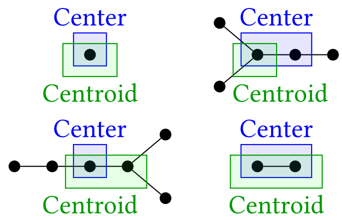
* **Adjacency** is the relation between two vertices connected with an edge.
* **Adjacency matrix** is a square matrix $A_{V \times V}$ of an adjacency relation.
  * For simple graphs, adjacency matrix is binary, *i.e.* $A_{ij} \in \{0, 1\}$.
  * For directed graphs, $A_{ij} \in \{0, 1, -1\}$.
  * For multigraphs, adjacency matrix contains edge multiplicities, *i.e.* $A_{ij} \in \mathbb{N}_0$.
* **Incidence** is a relation between an edge and its endpoints.
* **Incidence matrix** is a Boolean matrix $B_{V \times E}$ of an incidence relation.
* **Degree** $\deg(v)$ the number of edges incident to v (loops are counted twice).
  * $\delta(G) = \min_{v \in V} \deg(v)$ is the **minimum degree**.
  * $\Delta(G) = \max_{v \in V} \deg(v)$ is the **maximum degree**.
  * Handshaking lemma. $\sum_{v \in V} \deg(v) = 2|E|$.
* A graph is called r-**regular** if all its vertices have the same degree: $\forall v \in V : \deg(v) = r$.
* **Complement graph** of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent iff they are non-adjacent in G.
* **Intersection graph** of a family of sets $F = \{S_i\}$ is a graph $G = \Omega(F) = \langle V, E \rangle$ such that each vertex $v_i \in V$ denotes the set S_i, *i.e.* $V = F$, and the two vertices v_i and v_j are adjacent whenever the corresponding sets S_i and S_j have a non-empty intersection, *i.e.* $E = \{\langle v_i, v_j \rangle \mid i \neq j, S_i \cap S_j \neq \varnothing\}$.

- **Line graph** of a graph $G = \langle V, E \rangle$ is another graph $L(G) = \Omega(E)$ that represents the adjacencies between edges of G. Each vertex of $L(G)$ represents an edge of G, and two vertices of $L(G)$ are adjacent iff the corresponding edges share a common endpoint in G (*i.e.* edges are "adjacent"/"incident").
- **Walk** is an alternating sequence of vertices and edges: $l = v_1 e_1 v_2 \dots e_{n-1} v_n$.
  - **Trail** is a walk with distinct edges.
  - **Path** is a walk with distinct vertices (and therefore distinct edges).
  - A walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
  - **Circuit** is a closed trail.
  - **Cycle** is a closed path.

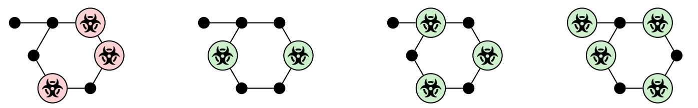
Term	V[1]	E[2]	"Closed" term
Walk	+	+	Closed walk
Trail	+	–	Circuit
Path	–	–	Cycle
	–	+	(*impossible*)

[1] Can vertices be repeated?
[2] Can edges be repeated?

- **Length** of a path (walk, trail) $l = u \rightsquigarrow v$ is the number of edges in it: $|l| = |E(l)|$.
- **Girth** is the length of the shortest cycle in the graph.
- **Distance** $\operatorname{dist}(u, v)$ between two vertices is the length of the shortest path $u \rightsquigarrow v$.
  - $\varepsilon(v) = \max_{u \in V} \operatorname{dist}(v, u)$ is the **eccentricity** of the vertex v.
  - $\operatorname{rad}(G) = \min_{v \in V} \varepsilon(v)$ is the **radius** of the graph G.
  - $\operatorname{diam}(G) = \max_{v \in V} \varepsilon(v)$ is the **diameter** of the graph G.
  - $\operatorname{center}(G) = \{v \mid \varepsilon(v) = \operatorname{rad}(G)\}$ is the **center** of the graph G.

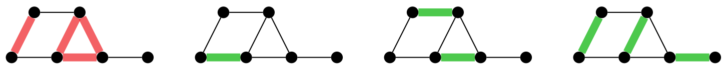


- **Clique** $Q \subseteq V$ is a set of vertices inducing a complete subgraph.
- **Stable set** $S \subseteq V$ is a set of independent (pairwise non-adjacent) vertices.



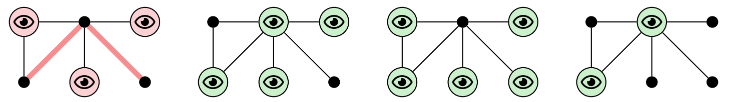
not stable set	**stable set**	**stable set**	**stable set**
	not maximal	maximal	maximal
	not maximum	not maximum	maximum

- **Matching** $M \subseteq E$ is a set of independent (pairwise non-adjacent) edges.



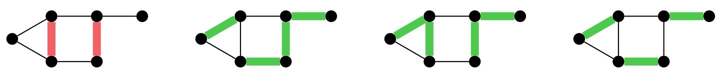
not matching	**matching**	**matching**	**matching**
	not maximal	maximal	maximal
	not maximum	not maximum	maximum

- **Perfect matching** is a matching that covers all vertices in the graph.
  - A perfect matching (if it exists) is always a minimum edge cover (*but not vice-versa!*).
- **Vertex cover** $R \subseteq V$ is a set of vertices "covering" all edges.



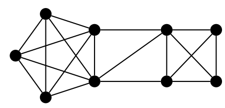
not vertex cover	**vertex cover**	**vertex cover**	**vertex cover**
	not minimal	minimal	minimal
	not minimum	not minimum	minimum

- **Edge cover** $F \subseteq E$ is a set of edges "covering" all vertices.



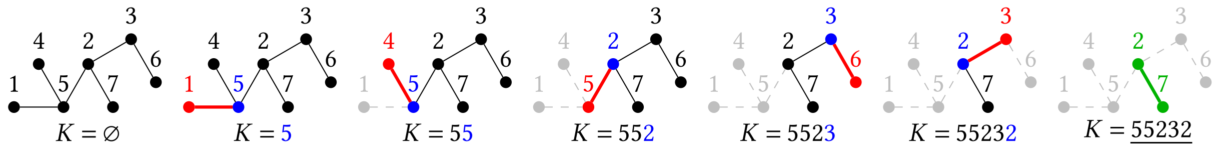
not edge cover	**edge cover**	**edge cover**	**edge cover**
	not minimal	minimal	minimal
	not minimum	not minimum	minimum

* **Cut vertex** (**articulation point**) is a vertex whose removal increases the number of connected components.
* **Bridge** is an edge whose removal increases the number of connected components.
* **Biconnected graph** is a connected "nonseparable" graph, which means that the removal of any vertex does not make the graph disconnected. Alternatively, this is a graph without *cut vertices*.
* **Biconnectivity** can be defined as a relation on edges $R \subseteq E^2$:
  - Two edges are called *biconnected* if there exist two *vertex-disjoint* paths between the ends of these edges.
  - Trivially, this relation is an equivalence relation.
  - Equivalence classes of this relation are called **biconnected components**, also known as **blocks**.
* **Edge biconnectivity** can be defined as a relation on vertices $R \subseteq V^2$:
  - Two vertices are called *edge-biconnected* if there exist two *edge-disjoint* paths between them.
  - Trivially, this relation is an equivalence relation.
  - Equivalence classes of this relation are called **edge-biconnected components** (or *2-edge-connected components*).
* **Vertex connectivity** $\varkappa(G)$ is the minimum number of vertices that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-vertex-connected.
* k**-vertex-connected graph** is a graph that remains connected after less than k vertices are removed, *i.e.* $\varkappa(G) \geq k$.
  - Corollary of Menger's theorem: graph $G = \langle V, E \rangle$ is k-vertex-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *vertex-independent* (*internally vertex-disjoint*) paths between u and v.
  - k-vertex-connected graphs are also called simply *k-connected*.
  - 1-connected graphs are called *connected*, 2-connected are *biconnected*, 3-connected are *triconnected*, *etc.*
  - Note the "exceptions":
    - Singleton graph K_1 has $\varkappa(K_1) = 0$, so it is **not** *1-connected*, but still considered *connected*.
    - Graph K_2 has $\varkappa(K_2) = 1$, so it is **not** *2-connected*, but considered *biconnected*, so it can be a block.
* **Edge connectivity** $\lambda(G)$ is the minimum number of edges that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-edge-connected.
* k**-edge-connected graph** is a graph that remains connected after less than k edges are removed, *i.e.* $\lambda(G) \geq k$.
  - Corollary of Menger's theorem: graph $G = \langle V, E \rangle$ is k-edge-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *edge-disjoint* paths between u and v.
  - 2-edge-connected are called *edge-biconnected*, 3-edge-connected are *edge-triconnected*, *etc.*
  - Note the "exception":
    - Singleton graph K_1 has $\lambda(K_1) = 0$, so it is **not** *2-edge-connected*, but considered *edge-biconnected*, so it can be a *2-edge-connected component*.
* Whitney's theorem. For any graph G, $\varkappa(G) \leq \lambda(G) \leq \delta(G)$.



$\varkappa(G) = 2$, $\lambda(G) = 3$, $\delta(G) = 3$, $\Delta(G) = 6$

- **Tree** is a connected undirected acyclic graph.
- **Forest** is an undirected acyclic graph, *i.e.* a disjoint union of trees.
- An **unrooted tree** (**free tree**) is a tree without any designated *root*.
- A **rooted tree** is a tree in which one vertex has been designated the *root*.
  - In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path to the root.
  - A **child** of a vertex v is a vertex of which v is the parent.
  - A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v.
  - A **leaf** is a vertex with no children. Equivalently, **leaf** is a *pendant vertex*, *i.e.* $\deg(v) = 1$.
  - An **internal vertex** is a vertex that is not a leaf.
  - A k-**ary tree** is a rooted tree in which each vertex has at most k children. *2-ary trees* are called **binary trees**.
- A **labeled tree** is a tree in which each vertex is given a unique *label*, *e.g.*, $1, 2, \dots, n$.
- CAYLEY'S FORMULA. Number of labeled trees on n vertices is n^{n-2}.
- **Prüfer code** is a unique sequence of labels $\{1, \dots, n\}$ of length $(n-2)$ associated with the labeled tree on n vertices.
  - **Encoding** (iterative algorithm for converting tree T labeled with $\{1, \dots, n\}$ into a Prüfer sequence K):
    - On each iteration, remove the leaf with *the smallest label*, and extend K with *a single neighbour* of this leaf.
    - After $(n-2)$ iterations, the tree will be left with *two adjacent* vertices — there is no need to encode them, because there is only one unique tree on 2 vertices, which requires 0 bits of information to encode.



  - **Decoding** (iterative algorithm for converting a Prüfer sequence K into a tree T):
    - Given a Prüfer code K of length $(n-2)$, construct a set of "leaves" $W = \{1, \dots, n\} \setminus K$.
    - On each iteration:
      (1) Pop the *first* element of K (denote it as k) and the *minimum* label in W (denote it as w).
      (2) Connect k and w with an edge $\langle k, w \rangle$ in the tree T.
      (3) If $k \notin K$, then extend the set of "leaves" $W \coloneqq W \cup \{k\}$.
    - After $(n-2)$ iterations, the sequence K will be empty, and the set W will contain exactly two vertices — connect them with an edge.