MODULE 4

LAPLACE TRANSFORMATION AND APPLICATIONS

A transform is a change in the mathematical description of a physical variable to facilitate computation.

Time domain to freq. domain

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt} \longrightarrow (1)$$

* Important functions:-

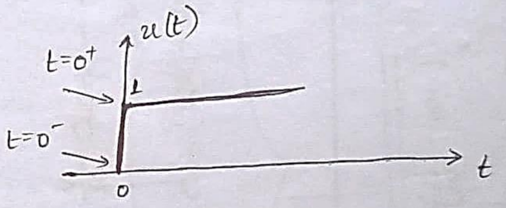
i) Unit step function, $u(t)$

ii) Delta function, $\delta(t)$

iii) Ramp function, $r(t)$.

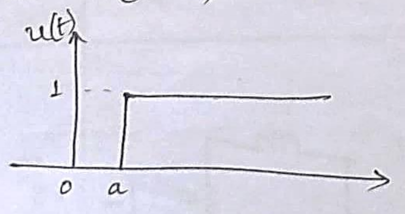
i) Unit step function, $u(t)$:-

$$u(t) = \begin{cases} 0, & t \le 0 \\ 1, & t \ge 0 \end{cases}$$

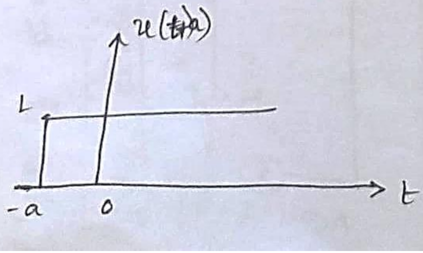


$$u(t) = \begin{cases} 0, & t \le 0^+ \\ 1, & t \ge 0^- \end{cases}$$

Unit step function that occurs at $t=a$ is expressed as $u(t-a)$



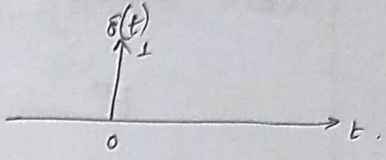
$$u(t-a) = \begin{cases} 0, & t-a<0 \text{ or } t<a \\ 1, & t-a>0 \text{ or } t>a \end{cases}$$



$$u(t+a) = \begin{cases} 0 & t+a<0 \text{ or } t<-a \\ 1 & t+a>0 \text{ or } t>-a \end{cases}$$

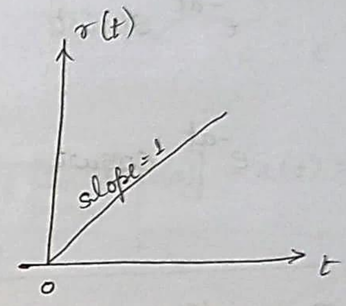
(ii) The derivative of the unit step function is the unit impulse function '$\delta(t)$'.

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t<0 \\ \text{undefined} & t=0 \\ 0 & t>0 \end{cases}$$

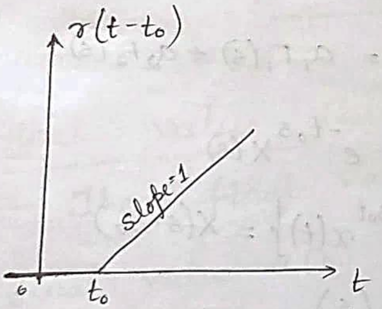


(iii) Integrating of unit step function is unit ramp function '$r(t)$'.

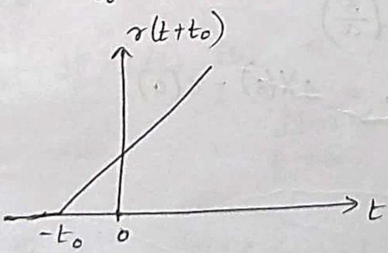
$$r(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = t\,u(t)$$



$$r(t) = \begin{cases} 0 & t \le 0 \\ t & t \ge 0 \end{cases}$$



$$r(t-t_0) = \begin{cases} 0, & t \le t_0 \\ t-t_0, & t \ge t_0 \end{cases}$$



$$r(t+t_0) = \begin{cases} 0 & t \le -t_0 \\ t+t_0 & t \ge -t_0 \end{cases}$$

* Laplace formulae :-

$f(t)$ $(t \ge 0)$	$F(s)$
$\delta(t)$	1
$u(t)$	$1/s$
t	$1/s^2$
e^{-at}	$1/s+a$

$\sin\omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\frac{s}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin\omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

* Properties of laplace transform:-

1. Linearity : $\mathcal{L}\{a_1f_1(t)+a_2f_2(t)\} = a_1F_1(s)+a_2F_2(s)$.

2. Time shifting : $\mathcal{L}\{x(t-t_0)u(t-t_0)\} = e^{-t_0 s}X(s)$.

3. Frequency domain shifting : $\mathcal{L}\{e^{s_0 t}x(t)\} = X(s-s_0)$

4. Time scaling : $\mathcal{L}\{x(at)\} = \frac{1}{a}X\left(\frac{s}{a}\right)$

5. Time differentiation : $\mathcal{L}\{dx(t)/dt\} = sX(s)-x(0)$.

6. Integration in time-domain:

If $y(t) = \int_0^t x(\tau)d\tau$.

then, $\mathcal{L}\{y(t)\} = Y(s) = \frac{X(s)}{s}$

7. Convolution : $\mathcal{L}\{x(t)*h(t)\} = X(s).H(s)$.

* Initial value theorem :-

Initial value theorem allow us to find the initial value $x(0)$ directly from its L.T. $x(s)$.

$$x(0) = \lim_{s \to \infty} s X(s).$$

Proof:- W.K.T,

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \underbrace{\int_0^{\infty} \frac{dx}{dt} e^{-st} dt}_{=0 \;\because\; e^{-\infty}=0}$$

If $s \to \infty$, then

$$\lim_{s \to \infty}\left[sX(s) - x(0)\right] = 0 \;;\; \lim_{s \to \infty}\left[sX(s)\right] - x(0) = 0$$

$$\boxed{x(0) = \lim_{s \to \infty}[s\,X(s)].}$$

* Final value theorem :-

The final value theorem allow us to find the final value $x(\infty)$ directly from its LT $x(s)$.

If $x(t)$ is a causal signal,

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} s X(s)$$

Proof :-

$$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0) = \int_0^{\infty} \frac{dx}{dt} e^{-st} dt$$

If $s \to 0$

$$\lim_{s \to 0}\left[sX(s) - x(0)\right] = \lim_{s \to 0} \int_0^{\infty} \frac{dx}{dt} e^{-st} dt$$

$$= \int_0^{\infty} \frac{dx(t)}{dt}\left[\lim_{s \to 0} e^{-st}\right] dt$$

$$= \int_0^\infty \frac{dx(t)}{dt}\, dt$$

$$= x(t)\Big|_0^\infty$$

$$= x(\infty) - x(0)$$

$$\lim_{s\to 0}\left[sX(s) - x(0)\right] = \lim_{s\to 0} sX(s) - x(0)$$

$$x(\infty) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

$$\boxed{x(\infty) = \lim_{s\to 0} sX(s)}$$

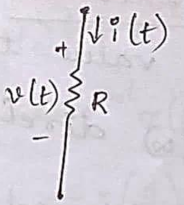
* Circuit element models & initial conditions:-

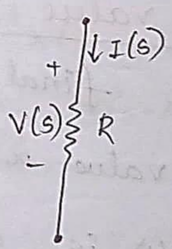
1. **Resistor :-**

$$v(t) = i(t)\, R.$$

Take LT.

$$V(s) = I(s)\, R.$$





(or) $Z(s) = \dfrac{V(s)}{I(s)}$ $\quad Z \xrightarrow{\text{only}}$ freq. domain

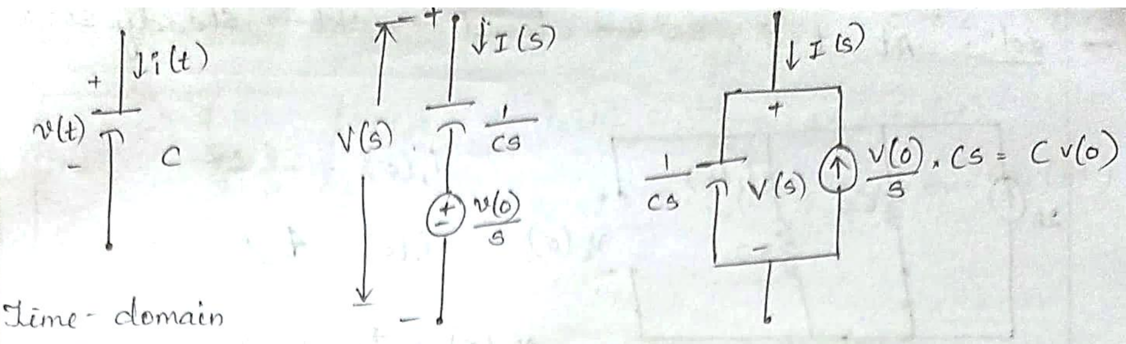
2. **Capacitor :-**

$$v(t) = \frac{1}{C}\int_0^t i(\tau)\, d\tau + v(0)$$

L.T.

$$V(s) = \frac{1}{Cs}\cdot I(s) + \frac{v(0)}{s}$$

To find impedance of 'C' set $v(0) = 0$, $\therefore Z(s) = \dfrac{V(s)}{I(s)} = \dfrac{1}{Cs}$



Time-domain

Freq. domain

3. Inductor:-

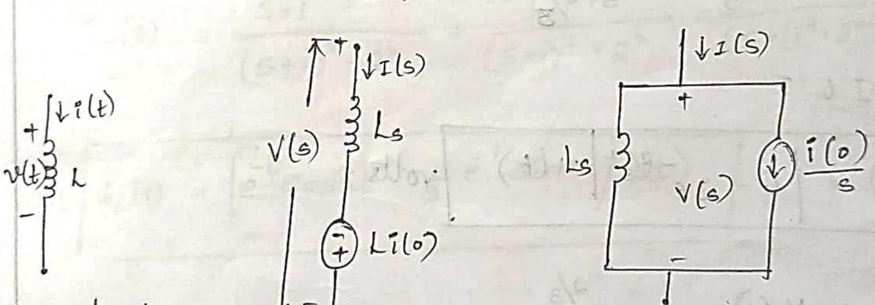
$$v(t) = L\frac{di(t)}{dt}$$

Take L.T

$$V(s) = Ls\,I(s) - Li(0)$$

To find impedance of an inductor, set $i(0) = 0$.

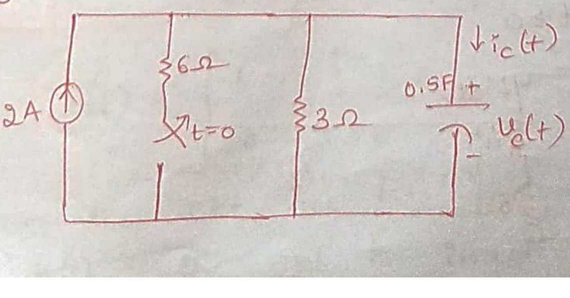
$$\therefore\ Z(s) = \frac{V(s)}{I(s)} = Ls.$$



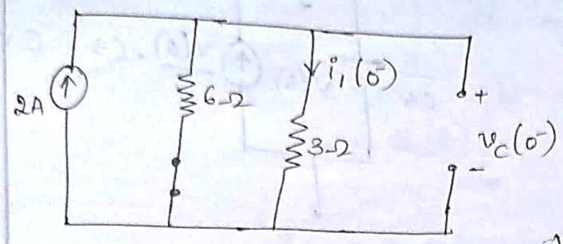
Time-domain

Freq. domain.

1. Determine the voltage $v_c(t)$ & the current $i_c(t)$ for $t \geq 0$.



→ Solⁿ: At $t = 0^-$; sw → closed → ckt → steady state

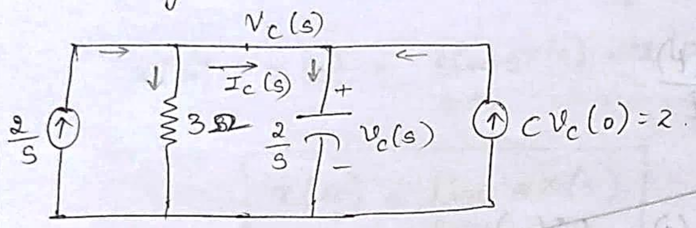


$i_1(0^-) = \frac{6 \times 2}{9}$

$i_1(0^-) = 4/3 \, A$

$v_c(0^-) = \frac{4}{3} \times 3 = 4\,V = v_c(0^+)$

Frequency domain representation is,



$i_c = C \frac{dv_c}{dt}$

$0.5 s \times V_c(s)$

$$\frac{V_c(s)}{3} + \boxed{\frac{s}{2} V_c(s)} = 2 + \frac{2}{s}$$

$$V_c(s) = \frac{6}{s} - \frac{2}{s + 2/3}$$

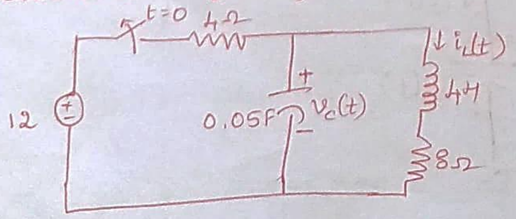
I.L.T

$$\boxed{v_c(t) = \left[6 - 2e^{-2/3\, t}\right] u(t) \text{ volts.}}$$

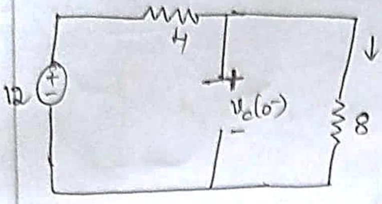
Also, $I_c(s) = \frac{V_c(s)}{2} - 2 = \frac{2/3}{s + 2/3}$

$$\boxed{i_c(t) = \frac{2}{3} e^{-2/3\, t} u(t) A.}$$

2. Determine $i_L(t)$ for $t \geq 0$



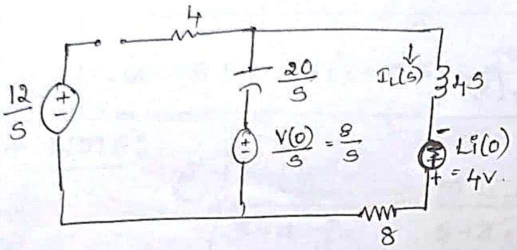
→ Soln :- At $t = 0^-$: sw → closed → steady state, $C \to o.c$; $L \to s.c$



$$i_L(0^-) = \frac{12}{12} = 1A = i_L(0^+)$$

$$v_c(0^-) = 8 \times i_L(0^-) = 8 \times 1 = 8V = v_c(0^+).$$

At $t = 0^+$:



Apply KVL to loop 2.

$$\frac{8}{s} - \frac{20}{s} I_L(s) - 4s I_L(s) + 4 - 8 I_L(s) = 0$$

$$\frac{8}{s} + 4 = \left[\frac{20}{s} + 4s + 8\right] I_L(s)$$

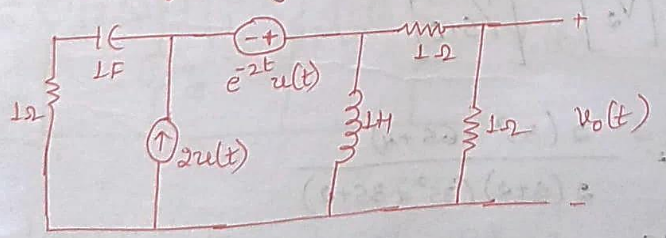
$$I_L(s) = \frac{(8+4s)/s}{\frac{20 + 4s^2 + 8s}{s}}$$

$$I_L(s) = \frac{(s+2)4}{4(s^2+2s+5)} = \frac{s+2}{s^2+2s+5} = \frac{(s+1)+1}{(s+1)^2+2^2}$$

$$I_L(s) = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2} = \frac{s+1}{(s+1)^2+2^2} + \frac{1}{2} \cdot \frac{2}{(s+1)^2+2^2}$$

$$\therefore \boxed{i_L(t) = \left[e^{-t}\cos 2t + \frac{1}{2} e^{-t} \sin 2t\right] u(t)}$$

3. Find $v_o(t)$ of the ckt



⇒ Soln :

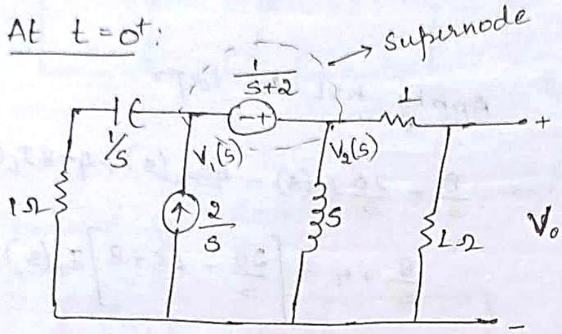
$$u(t) = \begin{cases} 1, & t \geq 0^+ \\ 0, & t \leq 0^- \end{cases}$$

Since the ckt has two independent sources energized by '$u(t)$', the ckt is inactive for $t \leq 0^-$ ∴ initial conditions are zero

$\therefore\ i_L(0^-) = i_L(0^+) = 0$

$v_c(0^-) = v_c(0^+) = 0$

At $t = 0^+$:



KCL at supernode,

$$\frac{V_1(s)}{1+\frac{1}{s}} + \frac{V_2(s)}{s} + \frac{V_2(s)}{2} = \frac{2}{s}$$

$$V_1(s)\left\{\frac{s}{s+1}\right\} + V_2(s)\left\{\frac{2+s}{2s}\right\} = \frac{2}{s} \longrightarrow (1)$$

$\therefore$ The constraint eqn. is, $V_2 - V_1 = \frac{1}{s+2} \longrightarrow (2)$

Solve (1) & (2)

$$\begin{bmatrix} \frac{s}{s+1} & \frac{2+s}{2s} \\ -1 & 1 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 2/s \\ 1/s+2 \end{bmatrix}$$

$$\therefore\ V_0(s) = \frac{1}{2}V_2(s) = \frac{2(3s^2+6s+4)}{2(s+2)(3s^2+3s+2)}$$

$$V_0(s) = \frac{\left[s^2+2s+4/3\right]}{[s+2][s+0.5-j0.646][s+0.5+j0.646]}$$

$$V_o(s) = \frac{k_1}{s+2} + \frac{k_2}{s+0.5-j0.646} + \frac{k_2^*}{s+0.5+j0.646}$$

$$\therefore \; k_1 = 0.5 \;\; \& \;\; k_2 = 0.316 \angle -37.76$$

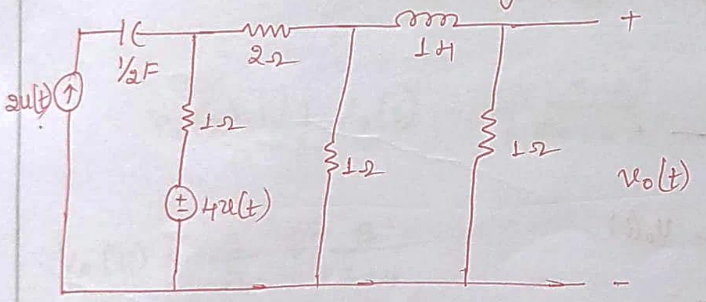
$$\therefore V_o(s) = \frac{0.5}{s+2} + \frac{0.316\angle -37.76}{s+0.5-j0.646} + \frac{0.316\angle 37.76}{s+0.5+j0.646}$$

$$\therefore \boxed{v_o(t) = 0.5e^{-2t}u(t) + 0.632e^{-0.5t}\cos\left[0.646t - 37.76\right]u(t)}$$

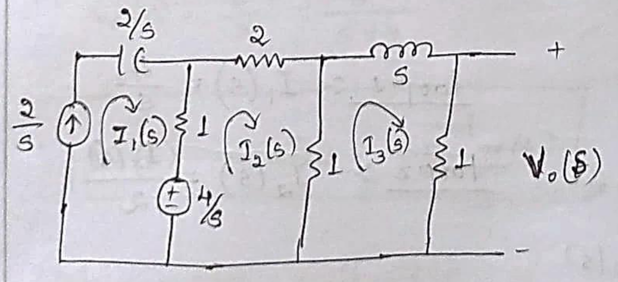
* NOTE:

$$\mathcal{L}^{-1}\left\{\frac{m\angle\theta}{s+a-j\omega} + \frac{m\angle -\theta}{s+a+j\omega}\right\} = 2me^{-at}\cos(\omega t+\theta)u(t)$$

4. Find $v_o(t)$, $t>0$ using mesh eq^{ns}.



⇒ Sol^n:- $u(t) = \begin{cases} 1 & t \geq 0^+ \\ 0 & t \leq 0^- \end{cases}$



Loop1:

$$I_1 = \frac{2}{s} \text{ A}$$

Loop2:

$$\frac{4}{s} - 1\left[I_2 - I_1\right] - 2I_2 - 1\left[I_2 - I_3\right] = 0$$

$$4I_2(s) - I_3(s) = \frac{6}{3} \longrightarrow (1)$$

Loop3: $-1\left[I_3 - I_2\right] - sI_3 - 1 \cdot I_3 = 0$

$$+I_2 - \left[s+2\right]I_3 = 0 \longrightarrow (2)$$

Solve (1) & (2), $\quad I_3 = \dfrac{1.5}{s\left[s+7/4\right]}$

$$\therefore V_0(s) = 1 \cdot I_3(s) = 1 \cdot \frac{1.5}{s\left[s+\frac{7}{4}\right]}$$

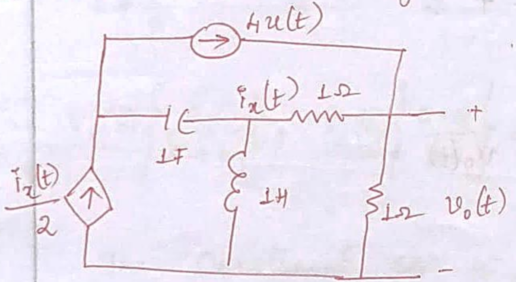
$$V_0(s) = \frac{K_1}{s} + \frac{K_2}{s+7/4}$$

$$\therefore K_1 = 6/7 \quad \& \quad K_2 = -6/7$$

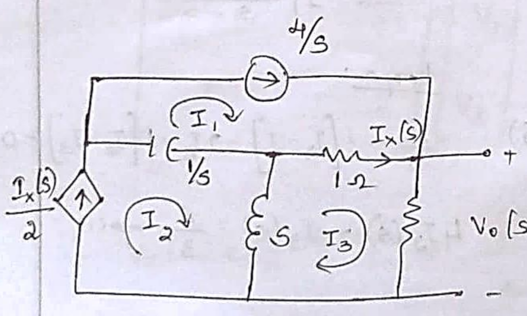
$$\therefore V_0(s) = \frac{6/7}{s} - \frac{6/7}{s+7/4}$$

$$\boxed{v_0(t) = \frac{6}{7}\left[1 - e^{-7/4 t}\right] u(t)}$$

5. Use mesh analysis & find $v_0(t)$, $t>0$



→ Soln: At $t \le 0^-$: ckt is not energized ∵ of independent c.s. $[4u(t)]$ ∴ Initial conditions are zero.



loop 1 : $I_1(s) = \frac{4}{s}$

loop 2 : $I_2(s) = \frac{I_x(s)}{2}$

But $I_x(s) = I_3 - I_1 = I_3 - \frac{4}{s}$

$$\therefore I_2(s) = \frac{1}{2}\left[I_3 - \frac{4}{s}\right]$$

Loop3 :-

$$-s\left[I_3 - I_2\right] - 1\left[I_3 - I_1\right] - I_3 = 0$$

$$s\left[I_3 - I_2\right] + 1\left[I_3 - I_1\right] + I_3 = 0$$

$$-I_1 - sI_2 + \left[s+2\right]I_3 = 0$$

$$-\frac{4}{s} - s\left[\frac{1}{2}\left\{I_3 - \frac{4}{s}\right\}\right] + \left[s+2\right]I_3 = 0$$

$$-\frac{4}{s} - \frac{s}{2}I_3 + 2 + (s+2)I_3 = 0$$

$$\left[-\frac{4}{s} + 2\right] + \left[s+2 - s/2\right]I_3 = 0$$

$$I_3 = \frac{-\left[\frac{2s-4}{s}\right]}{\frac{2s+4-s}{2}} = \frac{-4\left[s-2\right]}{\left[4+s\right]s}$$

$$\therefore V_o(s) = 1 \cdot I_3(s) = \frac{-4(s-2)}{s(s+4)}$$

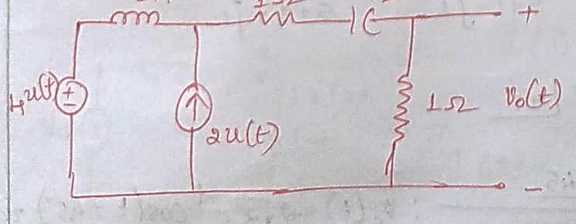
$$V_o(s) = \frac{A}{s} + \frac{B}{s+4}$$

$$\therefore A = 2,\ B = -6$$

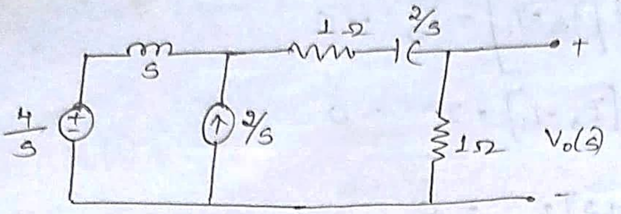
$$\therefore V_o(s) = \frac{2}{s} - \frac{6}{s+4}$$

$$\therefore \boxed{v_o(t) = 2u(t) - 6e^{-4t}u(t)}$$

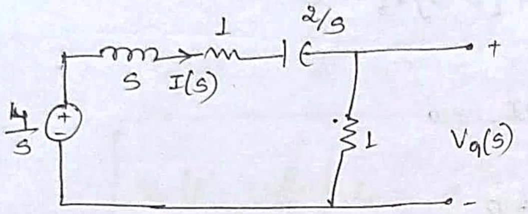
6. Using the principle of superposition, find $v_o(t)$ for $t>0$



Sol^n :- At $t \le 0^-$; No initial condition.



Case(1) :- Let '$\frac{4}{s}$' be present



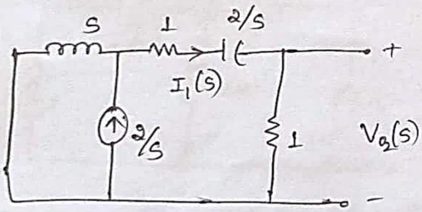
$$\therefore I(s) = \frac{\frac{4}{s}}{s+1+\frac{2}{s}+1}$$

$$= \frac{4/\cancel{s}}{\frac{s^2+2s+2}{\cancel{s}}}$$

$$I(s) = \frac{4}{s^2+2s+2}$$

$$\therefore V_{01}(s) = 1 I(s) = \frac{4}{s^2+2s+2}$$

Case(2) :- Let '$2/s$' be present



$$I_1(s) = \frac{s \times 2/s}{s+1+2/s+1}$$

$$= \frac{2s}{s^2+2s+2}$$

$$\therefore V_{02}(s) = 1 \times I_1(s) = \frac{2s}{s^2+2s+2}$$

$$\therefore V_0(s) = V_{01}(s) + V_{02}(s)$$

$$= \frac{4}{s^2+2s+2} + \frac{2s}{s^2+2s+2}$$

$$\hookrightarrow = \frac{4+2s}{s^2+2s+2} = \frac{A}{s+1-j1} + \frac{A^*}{s+1+j1}$$

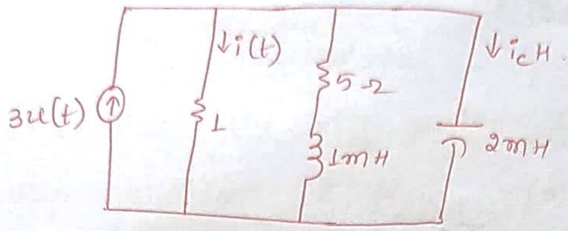
$$\therefore A = \sqrt{2}\angle{-45^\circ} \; ; \; A^* = \sqrt{2}\angle{45^\circ}$$

$$V_0(s) = \frac{\sqrt{2}\angle{-45^\circ}}{s+1-j1} + \frac{\sqrt{2}\angle{45^\circ}}{s+1+j1} \; ; \; v_0(t) = 2\sqrt{2}\, e^{-t} \cos(t+45^\circ)\, u(t)$$

7\. a) Convert the ckt. to an appropriate 's' domain representation

b) Find thevenin equivalent seen by 1Ω resistor

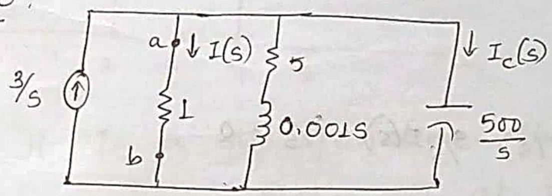
c) Analyze the simplified ckt to find an expression for $i(t)$



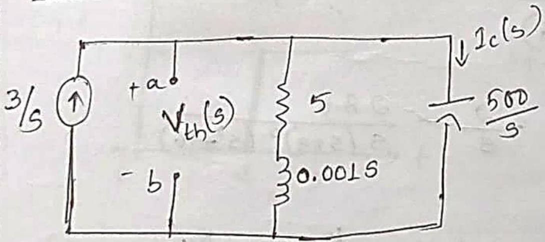
Soln :-

a) At $t \le 0^-$; ckt. is not active. ∴ the initial conditions are zero.

At $t \ge 0^+$;



b) Thevenin's equivalent :-



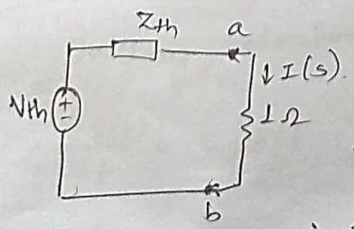
$$Z_{th}(s) = (5 + 0.001s) \| \frac{500}{s}$$

$$\Rightarrow = \frac{2500 + 0.5s}{0.001s^2 + 5s + 500}\ \Omega$$

$$\therefore V_{th}(s) = Z_{th}(s) \cdot 3/s$$

$$\Rightarrow = \frac{7.5 \times 10^6 + 1500s}{s(s^2 + 5000s + 5\times10^5)}$$

∴ Thevenin's equivalent ckt. is,



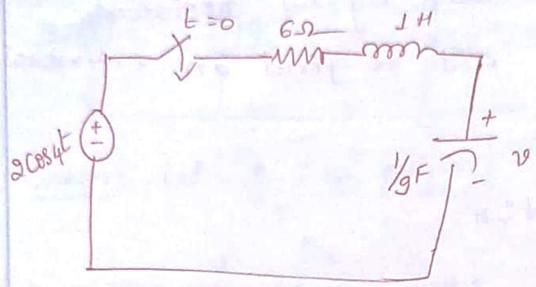
$$I(s) = \frac{V_{th}}{Z_{th} + 1} = \frac{7.5\times10^6 + 1500s}{s(s+4886)(s+614)}$$

$$\therefore I(s) = \frac{2.5}{s} + \frac{0.008}{s+4886} - \frac{2.508}{s+614}$$

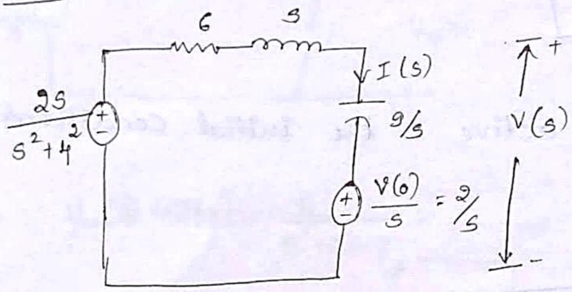
$$\therefore i(t) = \left[2.5 + 0.008e^{-4886t} - 2.508e^{-614t}\right] u(t)$$

8. Find the complete response for

Take $v(0) = 2V$



→ soln:



Apply KVL,

$$\frac{2s}{s^2+4^2} - 6I(s) - sI(s) - \frac{9}{s}I(s) - \frac{2}{s} = 0$$

$$\therefore I(s) = \frac{-32}{(s^2+6s+9)(s^2+16)}$$

Hence, $V(s) = \frac{2}{s} + \frac{9}{s}I(s) = \frac{2}{s} - \frac{288}{s(s+3)^2(s^2+16)}$

$$\therefore V(s) = \frac{2}{s} + \frac{K_1}{s} + \frac{K_2}{s+3} + \frac{K_3}{(s+3)^2} + \frac{K_4}{s-j4} + \frac{K_4^*}{s+j4}$$

$$\therefore K_1 = \left.\frac{-288}{(s+3)^2(s^2+16)}\right|_{s=0} = -2$$

$K_2 = -2.2$

$K_3 = 3.84$

$K_4 = 0.36 \angle -106.2$

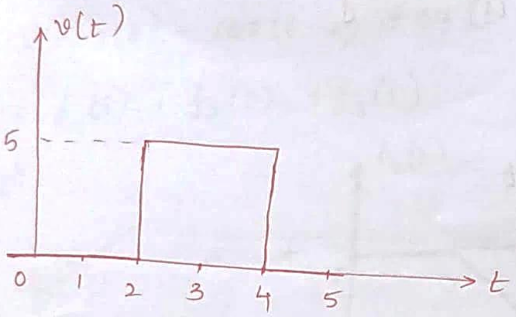
$K_5 = 0.36 \angle 106.2$

$$\therefore V(s) = \frac{2}{s} - \frac{2}{s} + \frac{2.2}{s+3} + \frac{3.84}{(s+3)^2} + \frac{0.36\angle -106.2}{s-j4} + \frac{0.36\angle 106.2}{s+j4}$$

$$\therefore v(t) = \left[2.2e^{-3t} + 3.84te^{-3t} + 0.72\cos(4t - 106.2^\circ)\right]u(t)$$

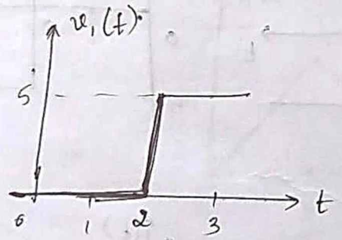
* Waveform Synthesis :-

1. Express the vtg. pulse in fig interms of unit step function & find V(s). Also find $\mathcal{L}\left\{\frac{dv(t)}{dt}\right\}$.

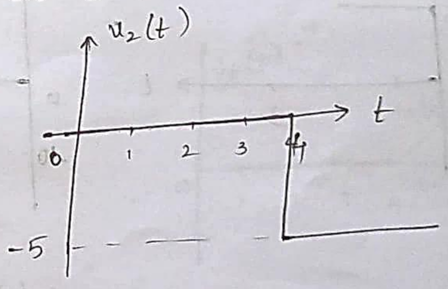


→ Soln: It is a step function, that switches on at $t = 2$ secs & switches off at $t = 4$ secs.

i.e.



\+



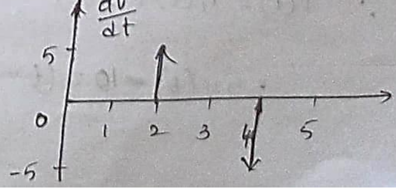
$\therefore v(t) = v_1(t) + v_2(t)$

$\hookrightarrow = 5u(t-2) - 5u(t-4)$.

$$\therefore V(s) = \frac{5}{s}e^{-2s} - \frac{5}{s}e^{-4s} = \frac{5}{s}\left[e^{-2s} - e^{-4s}\right].$$

Taking derivative of $v(t)$, yields.

$$\frac{dv(t)}{dt} = 5\left[\delta(t-2) - \delta(t-4)\right]$$

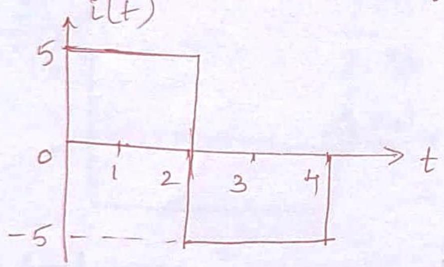


$$\mathcal{L}\{\delta(t-a)\} = e^{-as}\,\mathcal{L}\{\delta(t)\} = e^{-as}$$

$$\mathcal{L}\left\{\frac{du(t)}{dt}\right\} = \mathcal{L}\left\{5\left[\delta(t-2) - \delta(t-4)\right]\right\}$$

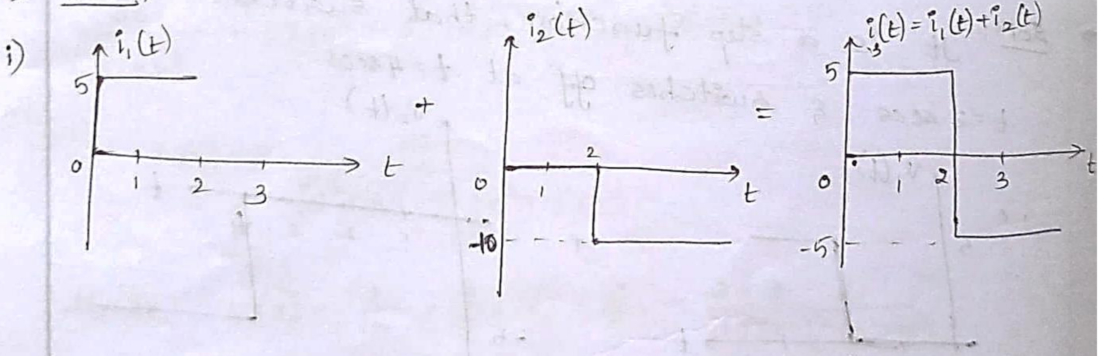
$$= 5\left[e^{-2s} - e^{-4s}\right]$$

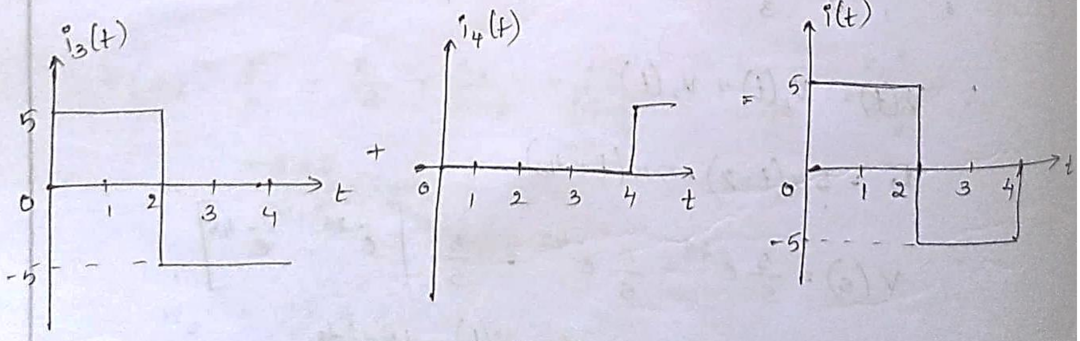
2. Express the current pulse in terms of unit st[ep]. Find (i) $\mathcal{L}\{i(t)\}$ (ii) $\mathcal{L}\left\{\int i(t)\,dt\right\}$.



→ Sol^n:-

i)





$\therefore\ i(t) = i_1(t) + i_2(t) + i_4(t)$

$= 5u(t) - 10\,u(t-2) + 5\,u(t-4)$

$$\therefore I(s) = \frac{5}{s} - \frac{10}{s}e^{-2s} + \frac{5}{s}e^{-4s}$$

$$= \frac{5}{s}\left[1 - 2e^{-2s} + e^{-4s}\right]$$

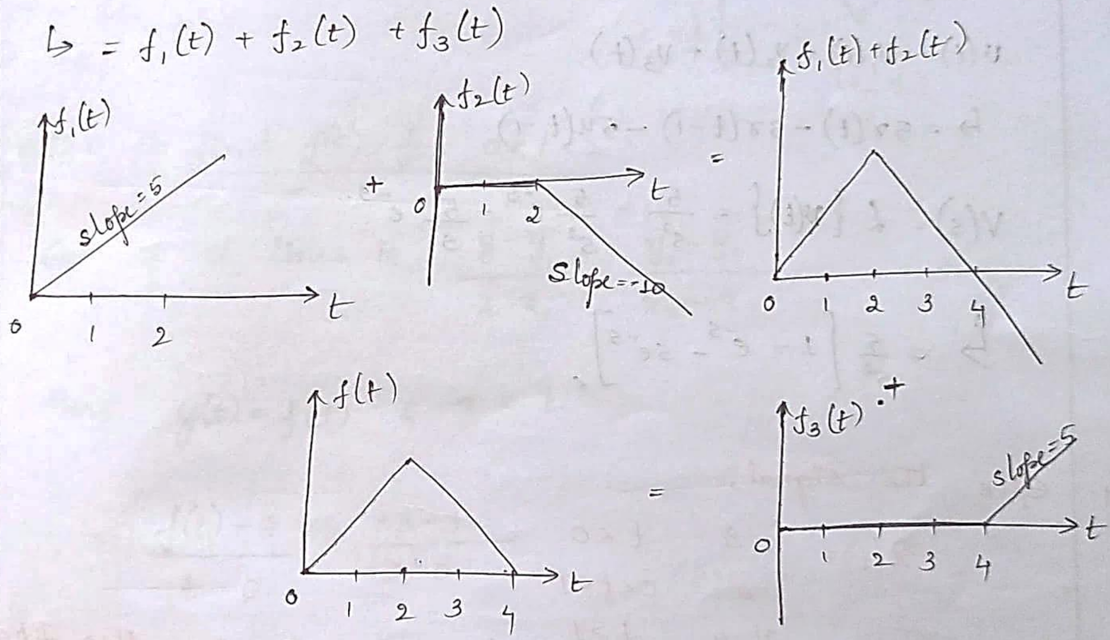
$$= \frac{5}{s}\left[1 - e^{-2s}\right]^2$$

ii) $f(t) = \int i(t)\,dt$

$$= \int \left[5u(t) - 10u(t-2) + 5u(t-4)\right]dt$$

$$= 5r(t) - 10r(t-2) + 5r(t-4)$$

$$= f_1(t) + f_2(t) + f_3(t)$$

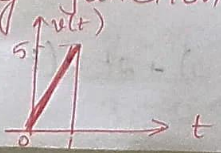


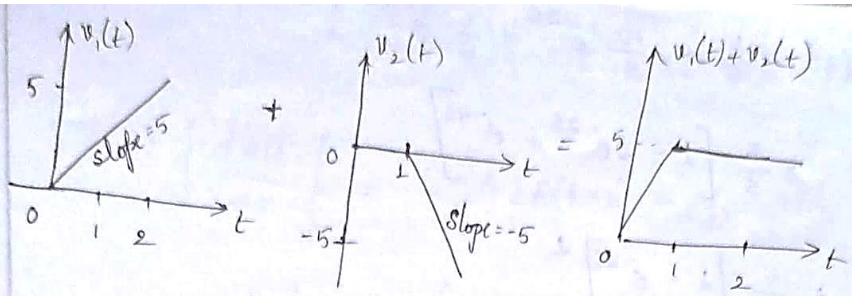
$$\therefore \mathcal{L}[f(t)] = \mathcal{L}\left[5r(t) - 10r(t-2) + 5r(t-4)\right]$$

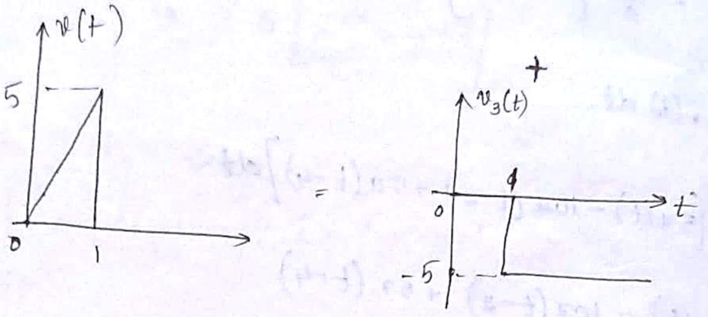
$$= \frac{5}{s^2} - \frac{10}{s^2}e^{-2s} + \frac{5}{s^2}e^{-4s}$$

$$= \frac{5}{s^2}\left[1 - 2e^{-2s} + e^{-4s}\right]$$

3. Express the sawtooth function in terms of singularity function. Then find $\mathcal{L}\{v(t)\}$.







$v(t) = v_1(t) + v_2(t) + v_3(t)$

$= 5r(t) - 5r(t-1) - 5u(t-1)$

$V(s) = \mathcal{L}\{v(t)\} = \frac{5}{s^2} - \frac{5}{s^2}e^{-s} - \frac{5}{s}e^{-s}$

$= \frac{5}{s^2}\left[1 - e^{-s} - se^{-s}\right]$

4. Give the signal,

$$x(t) = \begin{cases} 3 & t < 0 \\ -2 & 0 < t < 1 \\ 2t-4 & t > 1 \end{cases}$$

Express $x(t)$ in terms of singularity functions. Also find $\mathcal{L}\{x(t)\}$.

⇒ Soln:

i) At $t < 0$, $x(t) = 3u(-t)$

ii) $0 < t < 1$, $x(t) = -2\left[u(t) - u(t-1)\right]$

iii) $t > 0$, $x(t) = (2t-4)\,u(t-1)$

$\therefore x(t) = 3u(-t) - 2\left[u(t) - u(t-1)\right] + (2t-4)\,u(t-1)$

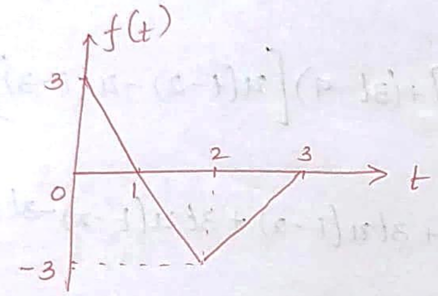
$= 3\left[1 - u(t)\right] - 2u(t) + 2u(t-1) + 2t\,u(t-1) - 4u(t-1)$

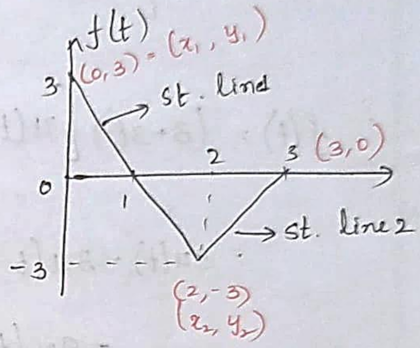
$$x'(t) = 3 - 5u(t) - 2u(t-1) + 2(t-1+1)\,u(t-1)$$

$$= 3 - 5u(t) - 2u(t-1) + 2(t-1)u(t-1) + 2u(t-1)$$

$$\hookrightarrow = 3 - 5u(t) + 2r(t-1)$$

5. Express `f(t)' interms of singularity functions & then find F(s).





→ soln:- To find f(t) for 0<t<2 :

Eqn. of st. line1 is, $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$

Here, $y(t) = f(t)$ & $x = t$

$$\frac{f(t)-3}{t-0} = \frac{-3-3}{2-0}$$

$$\frac{f(t)-3}{t} = \frac{-6}{2}$$

$$2\left[f(t)-3\right] = -6t$$

$$f(t) - 3 = -3t$$

$$(or)\quad f(t) = -3t+3$$

For 2<t<3 :

St. line 2, $\dfrac{y_1-y_2}{x_1-x_2} = \dfrac{y_3-y_2}{x_3-x_2}$; $\dfrac{f(t)+3}{t-2} = \dfrac{0+3}{3-2}$

$f(t) + 3 = 3t - 6$

$f(t) = 3t - 9$

$$\therefore f(t) = \begin{cases} 3 - 3t & 0 < t < 2 \\ 3t - 9 & 2 < t < 3 \\ 0 & \text{otherwise} \end{cases}$$

$\therefore f(t) = (3-3t)\left[u(t) - u(t-2)\right] + (3t-9)\left[u(t-2) - u(t-3)\right]$

$= 3u(t) - 3u(t-2) - 3tu(t) + 3tu(t-2) + 3tu(t-2) - 3tu(t-3)$
$- 9u(t-2) + 9u(t-3)$

$= 3u(t) - 3tu(t) - 12u(t-2) + 6tu(t-2) - 3tu(t-3) +$
$9u(t-3)$

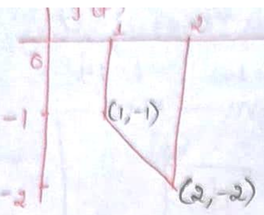
$= 3u(t) - 3tu(t) - 12u(t-2) + 9u(t-3) + 6(t+2-2)u(t-2)$
$-3(t+3-3)u(t-3)$

$= 3u(t) - 3tu(t) \cancel{-12u(t-2)} + \cancel{9u(t-3)} + 6(t-2)u(t-2) + 12u(t-2)$
$- 3(t-3)u(t-3) - \cancel{9u(t-3)}$

$= 3u(t) - 3tu(t) + 6(t-2)u(t-2) - 3(t-3)u(t-3)$

$\therefore F(s) = \dfrac{3}{s} - \dfrac{3}{s^2} + \dfrac{6}{s^2}e^{-2s} - \dfrac{3}{s^2}e^{-3s}$

6. Express the function $f(t)$ using singularity functions & find $F(s)$



→ Soln:

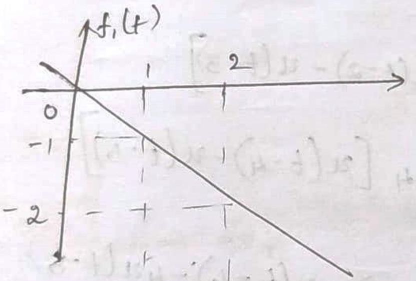
$$\frac{f_1(t)+1}{t-1} = \frac{-2+1}{2-1}$$

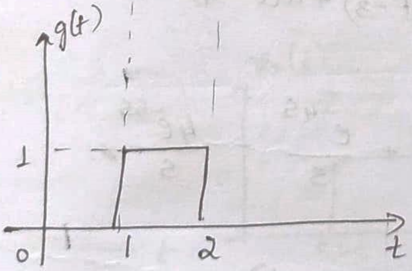
$$\frac{f_1(t)+1}{t-1} = \frac{-1}{1}$$

$$f_1(t)+1 = 1-t$$

$$f_1(t) = -t \quad [\text{'t' lying b/w '1' \& '2'}]$$

$$\therefore \; f(t) = f_1(t)\, g(t)$$





$$\therefore \; f(t) = -t\left[u(t-1) - u(t-2)\right]$$

$$f(t) = -t\,u(t-1) + t\,u(t-2)$$

$$f(t) = -(t-1+1)\,u(t-1) + (t-2+2)\,u(t-2)$$

$$= -(t-1)\,u(t-1) - u(t-1) + (t-2)\,u(t-2) + 2u(t-2)$$

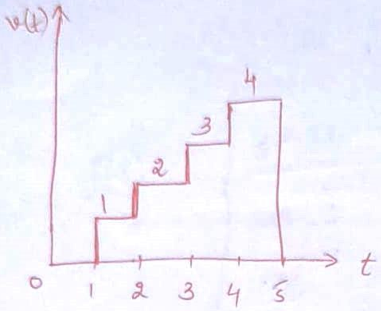
$$\boxed{= -r(t-1) - u(t-1) + r(t-2) + 2u(t-2)}$$

$$\therefore \; F(s) = \frac{-1}{s^2}e^{-s} - \frac{1}{s}\cdot e^{-s} + \frac{1}{s^2}e^{-2s} + \frac{2\cdot e^{-2s}}{s}$$

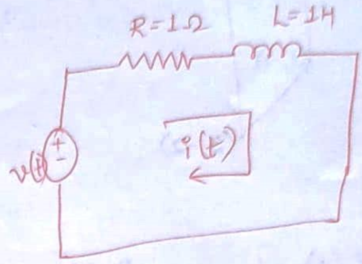
7. a) Find the L.T. of the staircase w/f [fig. a]

b) If this vtg. were applied to an RL series ckt.

with $R=1\Omega$ & $L=1H$, find the current i(t) fig b



fig(a)



→ soln:-

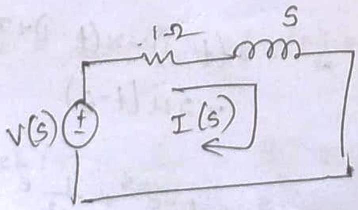
(a)
$$v(t)=\begin{cases}1 & 1<t<2\\ 2 & 2<t<3\\ 3 & 3<t<4\\ 4 & 4<t<5\\ 0 & \text{elsewhere}\end{cases}$$

$$v(t)=[u(t-1)-u(t-2)]+2[u(t-2)-u(t-3)]$$
$$+3[u(t-3)-u(t-4)]+4[u(t-4)-u(t-5)]$$

$$\hookrightarrow = u(t-1)+u(t-2)+u(t-3)+u(t-4)-4u(t-5)$$

$$\therefore V(s)=\frac{1}{s}e^{-s}+\frac{e^{-2s}}{s}+\frac{e^{-3s}}{s}+\frac{e^{-4s}}{s}-\frac{4e^{-5s}}{s}$$

(b)



$$\therefore I(s)=\frac{V(s)}{s+1}$$

$$\therefore I(s)=\frac{e^{-s}}{s(s+1)}+\frac{e^{-2s}}{s(s+1)}+\frac{e^{-3s}}{s(s+1)}+\frac{e^{-4s}}{s(s+1)}-\frac{4e^{-5s}}{s(s+1)}$$

$$=\left[\frac{1}{s}-\frac{1}{s+1}\right]e^{-s}+\left[\frac{1}{s}-\frac{1}{s+1}\right]e^{-2s}+\left[\frac{1}{s}-\frac{1}{s+1}\right]e^{-3s}+\left[\frac{1}{s}-\frac{1}{s+1}\right]e^{-4s}$$
$$-4\left[\frac{1}{s}-\frac{1}{s+1}\right]e^{-5s}$$

$$\therefore i(t) = \left[u(t) - e^{-t}u(t)\right]_{t\to t-1} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-2} +$$ → doubt

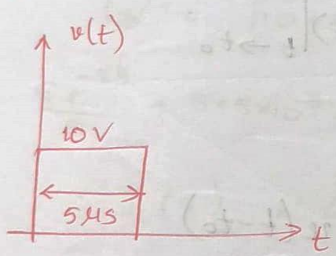
$$\left[u(t) - e^{-t}u(t)\right]_{t\to t-3} + \left[u(t) - e^{-t}u(t)\right]_{t\to t-4}$$

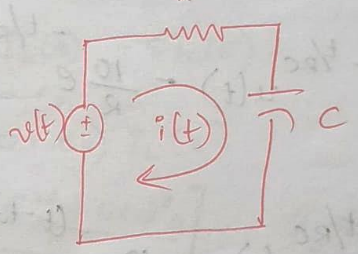
$$-4\left[u(t) - e^{-t}u(t)\right]_{t\to t-5}$$

$$= \left[1 - e^{-(t-1)}\right]u(t-1) + \left[1 - e^{-(t-2)}\right]u(t-2) + \left[1 - e^{-(t-3)}\right]u(t-3)$$

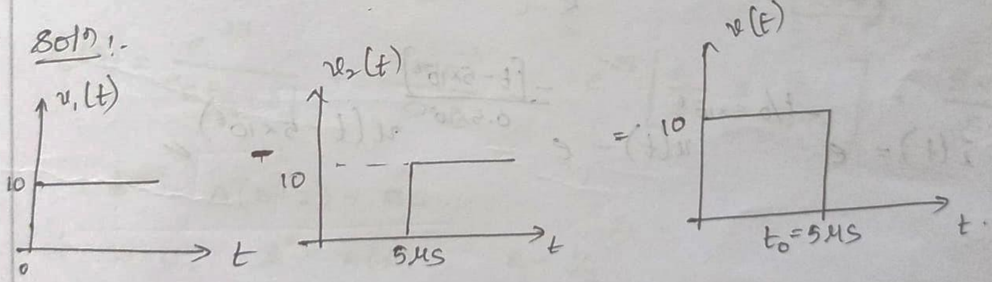
$$+ \left[1 - e^{-(t-4)}\right]u(t-4) - 4\left[1 - e^{-(t-5)}\right]u(t-5).$$

8. A vtg. pulse of 10V magnitude & 5 μsec duration is applied to RC n/w. Find i(t) if R = 10Ω & C = 0.05 μF.





→ Soln :-

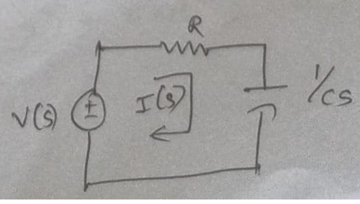


$\therefore v(t) = v_1(t) - v_2(t)$

$\Rightarrow = 10u(t) - 10u(t - t_0)$

$$\therefore V(s) = \frac{10}{s} - \frac{10}{s}e^{-t_0 s} = \frac{10}{s}\left[1 - e^{-t_0 s}\right].$$

Assuming all IC are zero.



$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}}$$

$$I(s) = \frac{10\left[1 - e^{-t_0 s}\right]}{s\left[R + \frac{1}{Cs}\right]}$$

$$I(s) = \frac{10C\cancel{s}}{\cancel{s}(RCs+1)} \cdot \left(1 - e^{-t_0 s}\right)$$

$$= \frac{10}{R}\left[\frac{1}{s + \frac{1}{RC}}\right]\left(1 - e^{-t_0 s}\right)$$

$$\hookrightarrow = \frac{10}{R}\left[\frac{1}{s+\frac{1}{RC}} - \frac{e^{-t_0 s}}{s + \frac{1}{RC}}\right]$$

ILT

$$i(t) = \frac{10}{R} e^{-t/RC} u(t) - \frac{10}{R} e^{-t/RC} u(t)\Big|_{t \to t_0}$$

$$\hookrightarrow = \frac{10}{R} e^{-t/RC} u(t) - \frac{10}{R} e^{-(t-t_0)/RC} u(t-t_0)$$

$$\therefore \; i(t) = e^{-t/0.5\times10^{-6}} u(t) - e^{-\frac{\left[t - 5\times10^{-6}\right]}{0.5\times10^{-6}}} u(t - 5\times10^{-6})$$