Notes For Bachelor of Science (TU)

BSC NOTES PDF COLLECTION

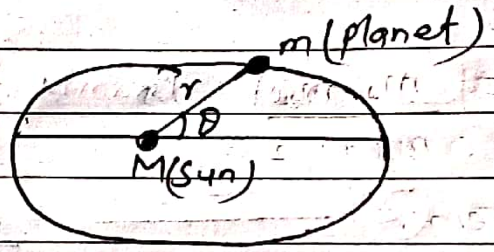
Kepler's law of Planetary motion

Kepler's gave the three laws of Planetary motion which can be explained as follows:

① Kepler's first law

→ each planet revolves arround the sun in an elliptical orbit being sun at on of the focus the elliptical orbit. It gives the nature of the path of the planet arround sun.

Proof:



Consider a planet of mass m ~~being~~ having position vector r with respect to sun of mass M at the focus of the orbit. the force between sun and planet is central force. So, in central force field, The angular moment of the partical remains constant.

ie.

$$J = mvr$$

$$J = m(\omega r)r = \text{constant}$$

$$J = m\omega r^2$$

$$\omega = \frac{J}{mr^2}$$

$$\omega = \frac{h}{r^2} \qquad \left(\text{where, } h = \frac{J}{m} = \text{constant}\right)$$

$$\therefore \frac{d\theta}{dt} = \frac{h}{r^2} \quad \text{—— (I)}$$

Where r and θ be the polar coordinate of planet with respect to sun.

Let us defined the variable u such that.

$$u = \frac{1}{r}$$

$$r = \frac{1}{u}$$

$$\frac{dr}{dt} = \frac{d}{du}(u^{-1}) \frac{du}{d\theta} \cdot \frac{d\theta}{dt}$$

$$\frac{dr}{dt} = \frac{-1}{u^2} \frac{du}{d\theta}\left(\frac{h}{r^2}\right) \quad [\because \text{from (i)}]$$

$$\frac{dr}{dt} = -h \frac{du}{d\theta} \quad \left\{\because u = \frac{1}{r} \Rightarrow ur = 1 \Rightarrow u^2 r^2 = 1\right\}$$

Again differentiating w.r.t 't' we get,

$$\frac{d^2r}{dt^2} = -h \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) \frac{d\theta}{dt}$$

$$\frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot \frac{h}{r^2}$$

or, $$\frac{d^2r}{dt^2} = -h^2 u^2 \frac{d^2u}{d\theta^2} \quad \text{(2)} \quad \left[\because u^2 = \frac{1}{r^2}\right]$$

Now, the force acting on a planet due to Sun is

F = reduced mass x radial acceleration

$$F = -\mu \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 \right] \text{———} (3)$$

Where,

$$\mu = \frac{mM}{m+M}$$

where,

m = mass of the Planet

M = mass of Sun

M is more massive than m (M > m)

$$\therefore \mu = m \text{———} (4)$$

Also,

The gravitational force between sun and Planet is,

$$F = \frac{GMm}{r^2} \text{———} (5)$$

from eqn (3), (4) and (5) we get,

$$\frac{GMm}{r^2} = -m\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

$$\frac{GM}{r^2} = -\frac{d^2r}{dt^2} + r\left(\frac{d\theta}{dt}\right)^2$$

$$GMu^2 = h^2u^2\frac{d^2u}{d\theta^2} + \frac{1}{u}\left(\frac{h}{r^2}\right)^2 \quad \left[\frac{1}{r^2} = u^2\right]$$

$$GMu^2 = h^2u^2\frac{d^2u}{d\theta^2} + \frac{1}{u}h^2u^4 \quad \left[\left(\frac{1}{r^2}\right)^2 = (u^2)^2 = u^4\right]$$

$$GMu^2 = h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3$$

$$\frac{GM}{h^2} = \frac{d^2u}{d\theta^2} + u \quad \text{--- (6)}$$

eqⁿ (6) is the differential eqⁿ of the orbit of the planet arround the sun.

The general solution of differential equation of the orbit of the planet of sun is

$$u = \frac{GM}{h^2} + A\cos(\theta - \theta_0) \quad \text{--- (7)}$$

Where, A and θ_0 are two constant.

By rotating the planet about an axis $\theta = 0$

$$u = \frac{GM}{h^2} + A\cos\theta$$

$$\frac{1}{r} = \frac{GM}{h^2} + A\cos\theta$$

$$\frac{1}{r} = \frac{GM}{h^2}\left(1 + \frac{Ah^2}{GM}\cos\theta\right)$$

$$\frac{h^2/GM}{r} = 1 + \frac{Ah^2}{GM}\cos\theta \quad \text{--- (8)}$$

Compairing eqⁿ (8) with general eqⁿ of conic

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{--- (9)}$$

$$l = \frac{h^2}{GM} \text{ and } e = \frac{Ah^2}{GM} \quad \text{--- (10)}$$

By sustituting the value of A, h, G and M in eqⁿ (10) the value of eccentricity is found to be less than 1. Therefore the path of the planet arround sun is elliptical nature.

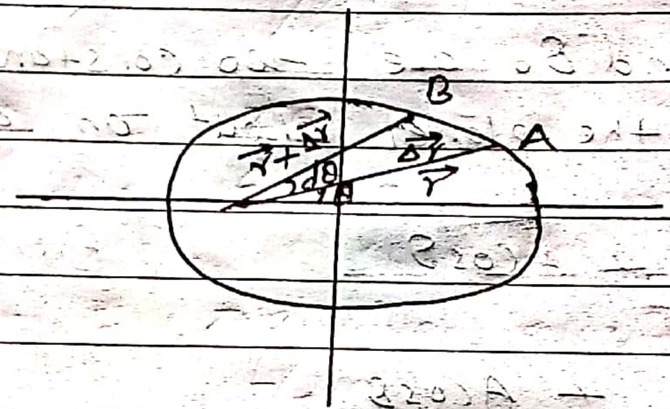
Kepler's Second Law

The Areal velocity of the planet with respect to Sun always remain constant. The rate of change of area swept by radius vector of planet with respect to sun is called Areal velocity, that is,

$$\vartheta = \frac{dA}{dt}$$

Where dA is the area swept small time dt.

Proof:



Consider a planet of a mass m is revolving around an elliptical orbit with ~~an~~ sun of mass M being at one the focus of the ellipse.

Suppose (r, θ) be the position of the planet at any instant of time t. After time $t + \Delta t$ the position of the planet will be $(r+\Delta r), (\Delta + d\theta)$.

The area swept by radius vector in small time Δt is given by,

$$\vec{\Delta A} = \frac{1}{2} \vec{r} \times \vec{\Delta r} \quad \text{———} \quad (I)$$

In limiting condition for small Δt.

$$\lim_{\Delta t} \frac{\vec{\Delta A}}{\Delta t} = \frac{1}{2} \vec{r} \times \lim_{\Delta t \to 0} \frac{\vec{\Delta r}}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2} \vec{r} \times \vec{v} \quad \text{--- (2)}$$

Since, the force between sun and planet is central force. In central field the angular momentum remains constant.

ie.

$$\vec{L} = \vec{r} \times \vec{P}$$

$$\vec{L} = \vec{r} \times m\vec{v}$$

$$\frac{\vec{L}}{m} = \vec{r} \times \vec{v} \quad \text{--- (3)}$$

From eq^n (2) and (3) we get.

$$\boxed{\frac{d\vec{A}}{dt} = \frac{1}{2}\frac{\vec{L}}{m} = \frac{\vec{L}}{2m} = \text{constant}}$$

hence the Areal velocity remains constant.

Kepler's Third Law

The square of period of revolution of planet arround sun is directly proportional to ~~to~~ the cube of the semi-major axis of the elliptical orbit of the planet.

If T be the time period of revolution of the planet arround the sun and a be the semi-major axis of the path of the elliptical orbit of the planet then.

$$T^2 \propto a^3$$

if T_1 and T_2 be the periods for the orbit with the semi-major axis a_1 and a_2 then

$$\Rightarrow \boxed{\frac{T_1^2}{T_2^2} \propto \frac{a_1^3}{a_2^3}}$$

Proof:

Let T be the period of the revolution of planet arround the sun and 'a' and 'b' be the semi-major axis and semi-minor axis respectively. then,

$$\text{Areal velocity} = \frac{\text{Total area covered in time period 'T'}}{\text{Time period}}$$

$$\frac{dA}{dt} = \frac{\pi ab}{T}$$

$$T = \frac{\pi ab}{dA/dt}$$

Since,

$$\frac{dA}{dt} = \frac{L}{2m}$$

$$T = \frac{\pi ab}{L/2m}$$

$$T = \frac{2\pi mab}{L}$$

$$T^2 = \frac{4\pi^2 m^2 a^2 b^2}{L^2} \quad \text{—— (1)}$$

we have,

Semi-latus rectum $(l) = \frac{b^2}{a}$

$$b^2 = al \quad \text{—— (2)}$$

from (1) and (2) we get,

$$T^2 = \frac{4\pi^2 m^2 a^2 (al)}{L^2}$$

$$T^2 = \frac{4\pi^2 m^2 \cancel{l}}{L^2} \times a^3 \qquad \left[L = \frac{h^2}{GM} \right]$$

$$T^2 = \left(\frac{4\pi^2 m^2}{l^2} \cdot \frac{h^2}{GM} \right) a^3$$

~~where,~~

$$\cancel{\frac{4\pi^2 m^2}{l^2} \cdot \frac{h^2}{GM}}$$

$$\boxed{T^2 \propto a^3}$$

Where, $\frac{4\pi^2 m^2}{l^2} \cdot \frac{h^2}{GM}$ is constant.