Eigen Values and Eigen Vectors :
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Characteristic Equation of a Matrix

The roots of the characteristic equation of a matrix $A \in \mathbb{R}^{n \times n}$ gives us the Eigen values of the matrix. There would be n Eigen values corresponding to n Eigen vectors for a square matrix of order n.

For an Eigen vector $v \in \mathbb{R}^{n \times 1}$ corresponding to an Eigen value of λ, we have

$$Av = \lambda v$$

$$=> (A - \lambda I)v = 0$$

Now, v being an Eigen vector is non-zero, and hence $(A - \lambda I)$ must be singular for the preceding to hold true.

For $(A - \lambda I)$ to be singular, $det(A - \lambda I) = 0$, which is the characteristics equation for matrix A. The roots of the characteristics equation gives us the Eigen values. Substituting the Eigen values in the $Av = \lambda v$ equation and then solving for v gives the Eigen vector corresponding to the Eigen value.

For example, the Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

can be computed as seen next.

The characteristics equation for the matrix A is $\det(A - \lambda I) = 0$.

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 0 \quad => \lambda^2 + 3\lambda + 2 = 0 => \lambda = -2, -1$$

The two Eigen values are -2 and -1.

Let the Eigen vector corresponding to the Eigen value of -2 be $u = [a\, b]^T$.

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = -2 \begin{bmatrix} a \\ b \end{bmatrix}$$

This gives us the following two equations:

$$0a + 1b = -2a \quad => 2a + b = 0 \qquad - \quad (1)$$

$$-2a - 3b = -2b \quad => 2a + b = 0 \qquad - \quad (2)$$

Both the equations are the same; i.e., $2a + b = 0 => \dfrac{a}{b} = \dfrac{1}{-2}$.

Let $a = k_1$ and $b = -2k_1$, where k_1 is a constant.

Therefore, the Eigen vector corresponding to the Eigen value -2 is $u = k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Using the same process, the Eigen vector v corresponding to the Eigen value of -1 is $v = k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

One thing to note is that Eigen vectors and Eigen values are always related to a specific operator (in the preceding case, matrix A is the operator) working on a vector space. Eigen values and Eigen vectors are not specific to any vector space.