# Outline of Differential Equations *

* Definition and Classification of Differential Equations

* Some Examples of Differential Equations

* First-order Differential Equations

* Higher-order linear Differential equations

* Homogenous Linear Diff. Equations with Constant Coefficients

* Non-homogenous Diff equations with Constant Coefficient

* Using Laplace transforms to Solve Diff Equation

* Series solutions of Second order linear equations

---

## Books

* Elementary Differential Equations and Boundary Value Problems
  William E. Boyce, Richard C. DiPrima

* ~~Sch~~ Schaum's outline of Differential Equation
  Richard Bronson, Gabriel B. Costa

* Differential Equations
  Shepley L. Ross

# Differential Equations



An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

## Some examples of differential equations

- $\frac{dy}{dx} = 4x+1$ y: dependent variable (unknown function)
  x: independent variable
  - It consists of a derivative of a function of order 1
  - one dependent variable(y) with respect to one independent variable (x)

- $\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0$ y: depend variable
  x: independent variable
  - This equation consists of a derivative of y of order 2.

- $\frac{d^4x}{dt^4} + 5\frac{d^2y}{dt^2} = 0$
  - y and x dependent variables
  - t: independent variable
  - $4^{th}$ order derivative is involved

- $\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0$ y: dependent variable
  t, x: independent variables
  - we have one dependent variable and 2 independent variable
  - $2^{nd}$ order partial derivative is involved.

Differential equations occur in connection with numerous problems that are encountered in the various branches of science and engineering. For example:

- The problem of determining the motion of a projectile, rocket, satellite, or planet.
- The problem of determining the charge or current in an electric circuit.
- The problem of the conduction of heat in a rod (çubuk) or in a slab. (levha)
- The problem of determining the vibrations of a wire or a membrane
- The study of the rate of ~~reactions~~ decomposition of a radioactive substance or the rate of growth of population
- The study of ~~the~~ reaction of chemicals.
- The problem of the determination of curves that have certain geometrical properties.

Differential equations (DE) are classified by type, order and linearity.

## Classification by Type;

- Ordinary
- Partial

## Ordinary Differential Equation ( ODE )

If a differential equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, it is said to be an ordinary differential equation.

- $\dfrac{d^2y}{dx^2} + \dfrac{5dy}{dx} + 6y = 0$
- $\dfrac{dy}{dx} - 2y = e^x$
- $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + 4y$
- $L\dfrac{d^2Q(t)}{dt^2} + R\dfrac{dQ(t)}{dt} + \dfrac{1}{C}Q(t) = E(t)$ impressed voltage

( $Q(t)$ is the charge on a capacitor in a circuit with capacitance $C$, resistance $R$, and inductance $L$

## Partial Differential Equation (PDE)

If a differential equation contains partial derivatives of one or more dependent variables of two or more independent variables it is said to be a partial differential equation, or PDE for short.

- $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$
- $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2} - 2\dfrac{\partial u}{\partial t}$
- $\alpha^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial u(x,t)}{\partial t}$ (Heat conduction equation)
- $a^2 \dfrac{\partial^2 u(x,t)}{\partial x^2} = \dfrac{\partial^2 u(x,t)}{\partial t^2}$ (wave equation)

---

## Notation for ODE

**Leibniz Notation:** $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \dots, \dfrac{d^ny}{dx^n}$

$y$ → Dependent variable

$x$ → independent variable

**Prime Notation:** $y', y'', y''', y^{(4)}, \dots, y^{(n)}$

- $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 7$
- $y'' + 6y' = 7$ (We cannot see independent variable)

**Newton's Dot notation:** usually used in science and engineering specially when denoting derivatives with respect to time. $\ddot{x} = 3$

$$\frac{dx}{dt} + \frac{dy}{dt} = 3x+2y$$ (Leibniz notation)

$$\dot{x} + \dot{y} = 3x+2y$$ (Newton's Dot notation)

$$\frac{d^2s}{dt} = 12 \rightsquigarrow \ddot{s} = 12$$

## Notation for PDE

Partial derivatives are often denoted using subscript notation indicating the independent variables.

Subscript notation

Leibniz notation $\rightsquigarrow$ $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \rightsquigarrow u_{xx} + u_{yy} = 0$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t}$$

$$\downarrow$$

$$u_{xx} = u_{tt} - 2u_t$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightsquigarrow u_y = -v_x$$

---

## Classification by order

The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

(2nd order ODE)

$$\frac{d^2y}{dx^2} + 9\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$

n th order ODE $$\frac{d^n y}{dx^n} = f(x, y, y', \dots, y^{(n-1)})$$

$$y''' + 2e^t y'' + yy' = t^4$$ (3rd order ODE)

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}$$ (2nd order PDE)

## Classification by Linearity

The ordinary differential equation

$F(x, y, y', \dots, y^{(n)}) = 0$ is said to be **linear** if $F$ is a linear function of the variables $y, y', \dots, y^{(n)}$;

a similar definition applies to partial differential equations. Thus, the general linear ordinary differential equation of order $n$ is:

$$a_0(x)\, y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = g(x) \quad (*)$$

~~Is~~ a equation that is not of the form $(*)$ is a **nonlinear** equation.

---

$$a_1(x) \frac{dy}{dx} + a_0(x)\, y = g(x) \qquad n = 1$$

→ first order linear equation

**General form:**

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x)\, y = g(x)$$

$$y''' + 2e^t y'' + yy' = t^4 \quad \text{(3rd order ODE)}$$

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(2nd order PDE)}$$

## Classification by Linearity

The ordinary differential equation

$F(x, y, y', \dots, y^{(n)}) = 0$ is said to be **linear** if $F$ is a linear function of the variables $y, y', \dots, y^{(n)}$;

a similar definition applies to partial differential equations. Thus, the general linear ordinary differential equation of order $n$ is:

$$a_0(x) y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y = g(x) \quad (*)$$

an equation that is not of the form $(*)$ is a **nonlinear** equation.

---

$$a_1(x) \frac{dy}{dx} + a_0(x) y = g(x) \qquad n=1$$

→ first order liner equation

**General form:**

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

if $n=1$

$$a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

if $n=2$

$$a_2(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

Characteristics of linear diff equation

* The dependent variable $y$ and all its derivatives $y', y'', \cdots y^{(n)}$ are of the first degree, that is, the power of each term involving $y$ the dependent variable is equal to 1.

of $y, y', \cdots, y^{(n)}$

* The coefficients $a_0, a_1, a_2, a_n$ depend at most on the independent variable $x$

$$(y-x)dx+5x\,dy=0$$

$$y-x+5x\frac{dy}{dx}=0$$

$$5xy'+y=x \qquad \rightsquigarrow \quad a_1(x)\frac{dy}{dx}+a_0(x)y=g(x)$$

$y''-5y'+y=0$ ↝ Linear 2nd order ODE

$$\frac{d^3y}{dx^3}+x\frac{dy}{dx}-5y=e^x$$ Linear 3rd order ODE.

A non-linear DE contains non-linear functions of the dependent variable or its derivatives, such as $\sin y$ (Trigonometric functions) →

$e^y$ (Exponential functions) • $\ln y$ (Logarithmic functions)

$(1-y)y' + 5y = e^x$ (Non liner $1^{st}$ order ODE)

↳ coefficient depends on $y$ (Dependent variable)

$y'' + \cos y \cancel{=} 0$ ($2^{nd}$ order ODE) Non-liner

↳ Nonliner function of $y$ (dependent variable)

$y'' + \cos x = 0$ ($2^{nd}$ order ODE) Linear

↳ Nonliner function in terms of $x$ (independent variable)

$$\frac{d^4y}{dx^4} + y^2 = 0$$ (Nonlinear $4^{th}$ order ODE)

↳ Power of dependent variable not equal to 1

---

Exercise 1: Determine the order, unknown function, and the independent variable in each of the following differential equations.

(a) $y''' - 5xy' = e^x + 1$

(Third order, because the highest-order derivative is the third. The unknown function is $y$; the independent variable is $x$)

(b) $t\ddot{y} + t^2\dot{y} - (\sin t)\sqrt{y} = t^2 - t + 1$

(Second order, because the highest-order derivative is the second. The unknown function is $y$; the independent variable is $t$)

(c) $s^2\frac{d^2t}{ds^2} + st\frac{dt}{ds} = s$

(~~Third~~ Second order, because the highest-order derivative is the second. The unknown function is $t$; the independent variable is $s$)

(d) $5\left(\frac{d^4 b}{dp^4}\right)^5 + 7\left(\frac{db}{dp}\right)^{10} + b^7 - b^5 = p$

9

( Fourth order, because the highest order derivative is the fourth. Raising derivatives to various powers does not alter the number of derivatives involved. The unknown function is $b$; the independent variable is $p$)

---

**Exercise 2:** State the order of the given ordinary differential equation and determine whether the equation is Linear or nonlinear.

$$(1-x)\,y'' - 4x\,y' + 5y = \cos x \qquad \text{where} \quad y = f(x)$$

Independent variable : $x$

Dependent variable : $y$

Solution

Prime notation is used.

Leibniz Notation

$$(1-x)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 5y = \cos x$$

Order: The order of a differential equation is the order of the highest derivative in the equation.

Order → $2^{nd}$ order

Linearity ↝ Determining if an ODE is linear

i-) The dependent variable $y$ and its derivatives are linear in form meaning they're raised to the power of 1.

ii) The products of the dependent variable $y$ and its derivatives are solely in terms of the independent variable $x$.

~~Linearity ↝ Linear~~

iii) Transcendental functions contain only the independent variable $x$.

( Linear $2^{nd}$ order )

Exercise 3:

State the order of the given ordinary differential equation and determine whether the equation is linear or nonlinear.

$$(1-y^2)\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = e^t$$

10

Solution:

Leibniz Notation: independent variable: t

Dependent variable: y

Dot Notation:

$$(1-y^2)\ddot{y} + t\dot{y} + y = e^t$$

Order → 2nd

Linearity: Products $(1-y^2)$ ⇝ Non-Linear

Product of second derivative contains an expression in terms of the dependent variable y.

---

Exercise 4. State the order of the given ordinary differential equation and determine whether the equation is linear non-linear

$$y^{(4)} + 2y' + e^y = x \quad \text{where } y=f(x)$$

The fourth derivative of x

prime notation.

Order ($4^{th}$)

Leibniz Notation

$$\frac{d^4y}{dx^4} + 2\frac{dy}{dx} + e^y = x$$

Linearity → 1) Check dependent variable y and its derivatives are linear in form

* inspecting the ODE we see that's the case

2) Check the products of the dependent variable y and its derivatives are solely expressed in terms of the independent variable x

* once again inspecting ODE we see that this is the case

The final step is to check if any transcendental functions are written solely in terms of the independent variable $x$.

- Notice that this ODE contains the transcendental function $e$ raised to the power $y$,

This function is in terms of the dependent variable $y$, this makes the ODE non-linear. If this function was written in terms of the independent variable $x$, then we would have a linear ODE

4th order Non-linear ODE

Exercise: State the order of the given ODE and determine whether the equation is linear or non-linear

$y = f(\theta)$

$$(\sin\theta)\, y''' - (\cos\theta)\, y' = 2 \quad \text{where} \quad y = f(\theta)$$

* Independent variable $\theta$

Dependent variable: $y$

Order: 3rd

Leibniz notation: $\sin\theta \dfrac{d^3y}{d\theta^3} - (\cos\theta)\dfrac{dy}{d\theta}$

Linearity.

1-) We first need to make sure that the dependent variable and its derivatives are linear in form meaning they're raised to the power of 1.

2-) Products of the dependent variable and its derivatives are solely expressed in terms of the independent variable

3-) Make sure that transcendental functions are also solely written in terms of the independent variable

ODE

Exercise: State the order of the given ODE and determine whether the equation is linear or non linear

$$3x^2y'' + 2\ln(x)y' + e^xy = 3\cos x$$ where $y = f(x)$

12

Prime notation

Independent variable: $x$

Dependent variable: $y$

Leibniz notation: $3x^2 \frac{d^2y}{dx^2} + 2\ln x \frac{dy}{dx} + e^xy = 3\cos x$ — $2^n$ order

Linearity?

1-) $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$ are raised to the power of 1 ✓

2-) Products of $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$ are solely expressed in terms of $x$. ✓
$(3x^2)$ $2\ln x$ $e^x$

3-) Transcendental functions are also written in terms of $x$

$\ln x, e^x, \cos x$ ✓

→ "Linear"

All three requirements are met, so we can classify this ODE as a linear ODE

---

Exercise: $\frac{d^2y}{dt^2} + \sin(t+y) = \sin t$

* Dependent variable $y$
* Independent variable $t$

Order → $2^{nd}$ order

Linearity

* Due to the terms $\sin(t+y)$ this equation is Non-linear

---

Exercise $\frac{d^2R}{dt^2} = -k/R^2$ where $k$ is a constant

* Dependent variable: $R$
* Independent variable $t$

order → $2^{nd}$ order

Due to $-k/R^2$ term this equation is non-linear

* Exercise

$$m\frac{dv}{dt} = mg - kv^2$$ where $m, g,$ and $k$ are constants

* Independent variable: $t$
* Dependent variable: $v$

First order: $\frac{dv}{dt}$

Linearity

Due to term $kv^2$ ~ non-linear

---

Exercise

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$ where $L, R,$ and $C$ are constants.

* Independent variable: $t$
* Dependent variable $Q$

Second order

Linear ODE

Linearity.

1-) $\frac{d^2Q}{dt^2}, \frac{dQ}{dt}, Q$ are raised to the power 1

2-) Products ~ $L, R, 1/C$ ~ constants (They do not depend on $Q$)

3-) No transcendental function of $Q$.

Linear

---

Partial Differential Equations (PDE)

Exercise State the order of the given partial differential equations and determine whether the equations are linear or non-linear.

* $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ ← * Leibniz notation

c is constant

Subscript notation: $u_{tt} = c^2 u_{xx}$

Independent variables: $t, x$

Dependent variables: $u$

Check for the order:

$$\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2} \rightsquigarrow 2^{nd} \text{ order}$$

Linearity!

1-) Let's make sure that all partial derivatives are raised to power 1

$\frac{\partial^2 u}{\partial t^2}$, $\frac{\partial^2 u}{\partial x^2}$ ✓ Inspecting PDE we see that this is the case.

2-) Let's check that products of partial derivatives are solely expressed in terms of the independent variables.

Partial derivative of $\frac{\partial^2 u}{\partial x^2}$ is being multiplied by a constant so technically it's not written in terms of the dependent variable u. ✓

3-) Lastly, we need to make sure that, any transcendental functions are only written in terms of the independent variables.

* Notice that, this PDE contains no transcendental functions, so we can skip this step

* All three requirements were checked out so we can classify the linearity of this PDE as Linear. $2^{nd}$ order liner PDE

Allright let's try the next example.

* $y \frac{\partial^2 u}{\partial x \partial y} + u = 0$ ← Leibniz notation

$y u_{yx} + u = 0$ ← Subscript notation

Independent variable: x, y

Dependent variable: u

Order. We have a second partial derivative

$\frac{\partial^2 u}{\partial x \partial y}$ => $2^{nd}$ order.

Linearity.

1-) Notice that the partial derivative and the dependent variable u are raised to the power of 1.

2-) The product of the partial derivative is written in terms of the independent variable y

$y \frac{\partial^2 u}{\partial x \partial y}$

3-) Since we don't have any transcendental functions we can skip the third step

* Notice that all three criteria for a linear PDE checks out. In the end this equation can be classified as a 2nd order PDE

---

* $\frac{\partial^2 \Omega}{\partial u^2} \cdot \frac{\partial^2 \Omega}{\partial v^2} = \frac{\partial^2 \Omega}{\partial u \partial v}$ ← Leibniz notation

$\Omega_{uu} \Omega_{vv} = \Omega_{uv}$

order.

2nd order

1-) Partial derivatives are raised to the power of 1.

Check Products.

2-) Notice we have a product of partial derivatives on the left side of the equation. This automatically makes this PDE nonlinear.

Ex.4



$u_{xx} + u_{yy} + u_{zzz} = 0$ ← subscript notation

We can also write this PDE using Leibnitz notation as follows

Leibniz notation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^3 u}{\partial z^3}$

Order: $\frac{\partial^3 u}{\partial z^3} \to 3^{rd}$ order PDE

Linearity: Let's classify the linearity of the PDE.

1-) For this PDE, the partial derivatives are all raised to the power of 1

2-) In addition, partial derivatives contain no products in terms of the dependent variable

3-) This PDE contains no transcendental functions in terms of the dependent variable

* All three criteria are met so this PDE is linear

---

Example: $u_{xx} + u_{yy} + u u_y + u = 0$ ← Subscript Notation

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u \frac{\partial u}{\partial x} + \cancel{u} \frac{\partial u}{\partial y} + u = 0$ ← Leibniz notation

Independent variable: x, y

Dependent variable: u

Order $2^{nd}$ order (we have 4 partial derivatives the highest one is the $2^{nd}$ order



Linearity,

1-) Each of the partial derivatives and the dependent variable u are all raised to the power of 1.

2-) Notice that the products of these two partial derivatives are in terms of dependent variable u. This automatically makes this PDE nonlinear. In the end, this equation can be classified as a $2^{nd}$ order nonlinear PDE.

---

Let's move along to the next example

$$* \quad \frac{\partial^3 u}{\partial x^2 \partial t} = 1 + \partial u/\partial y \quad \rightarrow \text{Leibniz notation}$$

This PDE is written using Leibniz notation. Keep in mind that we can also rewrite this PDE using subscript notation as follows.

Here Independent variables: x, y, t
Dependent variable: u

$$u_{txx} = 1 + u_y$$

Let's classify the order of the PDE.

Order 1) This equation contains two partial derivatives namely, a third partial derivative and a first partial derivative

* Higher of these two is the ~~par~~ third partial derivative so this PDE is $3^{rd}$ order. Next let's check if this PDE is linear.

Linearity:

1-) Inspecting the partial derivatives, we see that they are all raised to the power of 1.

2-) We can also see that there are no products in terms of the dependent variable next to either partial derivative

3-) Lastly, there are no transcendal functions in terms of the dependent variable u.

So, this PDE is linear, in the end this equation can be classified as a $3^{rd}$ order linear PDE.

---

Alright, let's try the next example.

* $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$

Here we have a PDE written using subscript notation
We can also rewrite this PDE using Leibniz notation as follows

Independent variables: x, y
Dependent variable: u

Leibniz notation:

$$\frac{\partial^2 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial y^2 \partial x^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

Let's first classify the order of the PDE, a
order → we have 3 partial derivatives. All of them are $4^{th}$ order partial derivatives so this PDE is a $4^{th}$ order PDE.

Linearity:
1-) Notice that all partial derivatives are raised the power of 1.
2-) In addition, the products of the partial derivatives are not in terms of the dependent variable u.
3-) Lastly, there are no transcendental functions in terms of the dependent variable.

→ So, $4^{th}$ order linear PDE

× So, Let's work on the next example

$$x^2 \frac{\partial f}{\partial x} + z^2 \frac{\partial^2 f}{\partial y^2} = e^{zxy}$$

Here we a PDE written in Leibniz notation. We can also rewrite this PDE using subscript notation as follows

$$x^2 f_x + z^2 f_{yy} = e^{zxy}$$

Here the dependent variable is $f$, and independent variables are $x, y, z$

Let's first classify the order of the PDE.

Order: This equation contains two partial derivatives $\left(\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial y^2}\right)$. The higher one is the second partial derivative, so the order of this PDE is second order.

Now Let's check to see if this PDE is Linear.

Linearity:

1) Inspecting the partial derivatives we see that they are raised to the power of 1.

2) Also, the products of these partial derivatives are solely expressed in terms of independent variables

3) Lastly the exponential function is all also solely expressed in terms of the independent variables, which make this PDE linear.

In the end, this PDE can be classified as a $2^{nd}$ order linear PDE

## Solutions of Differential Equations

A solution of a differential equation in the unknown function y and the independent variable x on the interval I, is a function y(x) that satisfies the differential equation identically for all x in I.

* A solution of a differential equation is an expression for the dependent variable in terms of the independent one(s) which satisfies the relation. $\rightsquigarrow y' = 2x \rightsquigarrow y = x^2 + c \rightsquigarrow$ solution(1) $y = x^2 + 5 \rightsquigarrow$ solution(2)

A general solution of a differential equation includes all possible solutions (solution 1) and typically includes arbitrary constants (in the case of ODE) or arbitrary functions (in the case of a PDE). A solution without arbitrary constants/functions is called a particular solution. (solution 2) Often we find a particular solution to a differential equation by giving extra conditions in the form of initial or boundary conditions.

- The solution y(x) and/or its derivatives are required to have specific values at a single point, for example, $y(0) = 2$ and $y'(0) = 4$. Such problems are traditionally called initial value problems because the system is assumed to start evolving from the fixed initial point (in this case, at 0)

- The solution y(x) is required to have specific values at a pair of points, for example, $y(0) = 3$ and $y(1) = 5$. These problems are known as boundary value problems because the points 0 and 1 are regarded as boundary points (or edges) of the domain of interest in the application

## Example:



* The problem $y'' + 2y' = e^x$; $y(\pi) = 1$, $y'(\pi) = 2$ is an initial value problem, because two specific values ($y(\pi)$ and $y'(\pi)$) are both given at $x = \pi$.

* The problem; $y'' + 2y' = e^x$; $y(0) = 1$, $y(1) = 1$ is a boundary value problem, because two specific values are given at the different values $x = 0$ and $x = 1$

---

* $y' = 3x^2$ → ~~Leibniz~~ (Prime notation)
  ↓
  $\frac{dy}{dx} = 3x^2$ (Leibniz notation)

The general solution of $y' = 3x^2$ will come out to be ~~dy~~ $y = x^3 + c$ ~~$y = x^3 + c$~~ ~~indicates~~ where c denotes the arbitrary constant.

A solution without arbitrary constants is termed a particular solution. In the figure below, we see the plots of $y = x^3 + c$ where $c = -2, -1, 0, 1, 2$. Each of them is a particular solution of $y' = 3x^2$. These curves are called <u>one-parameter family of curves</u>.

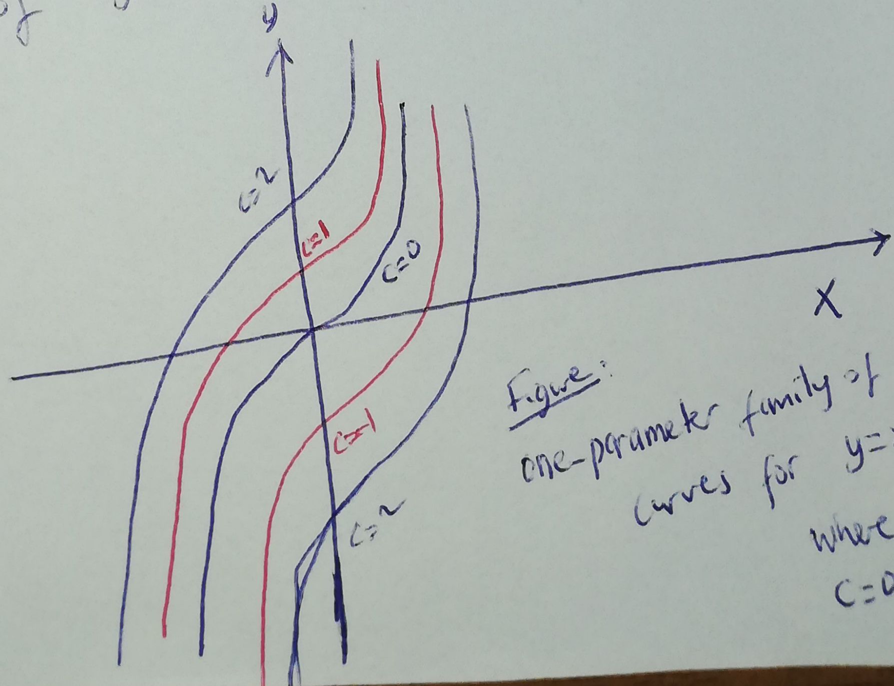



Figure: one-parameter family of curves for $y = x^3 + c$ where $c = 0, \mp 1, \mp 2$

**Example** Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$, where $c_1$ and $c_2$ are arbitrary constants, a solution of $y'' + 4y = 0$

**Solution** Differentiating $y$, we find

$$y' = 2c_1 \cos 2x - 2c_2 \sin 2x$$

$$y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$$

$$y'' + 4y = (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(c_1 \sin 2x + c_2 \cos 2x)$$

$$= (-4c_1 + 4c_1)\sin 2x + (-4c_2 + 4c_2)\cos 2x$$

$$= 0$$

Thus, $y = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of $x$ and is a solution on the interval $(-\infty, \infty)$

$y = c_1 \sin 2x + c_2 \cos 2x$ is the general solution of the $y'' + 4y = 0$. A few particular solutions are,

* $y = 5 \sin 2x - 3 \cos 2x$ (choose $c_1 = 5$ and $c_2 = -3$)
* $y = \sin 2x$ (choose $c_1 = 1$ and $c_2 = 0$)
* $y = 0$ (choose $c_1 = c_2 = 0$)

* Our aim in this course is to solve differential equations. (Finding the unknown function)

For example let's write $1^{st}$ order linear ordinary differential equation:

$$\frac{dy}{dx} = 4x+1$$ Let's solve this equation.

$$dy = (4x+1)\,dx$$ (Integrate both sides!)

$$\int dy = \int (4x+1)\,dx$$

$$y = \frac{4x^{1+1}}{1+1} + x + c$$

$$y = \frac{4x^2}{2} + x + c \rightarrow y = 2x^2 + x + c$$ ⇝ integration constant

* Be aware that you must be familiar with following calculus concepts in order not to get into trouble

Calculus concepts that you are considered to know

- Definition of a derivative
- Rules of differentiation
- Partial derivatives
- Definition of the Indefinite Integral
- Rules of Integration

# A brief review of Calculus

## Definition of the derivative

The derivative of a function $f$ is another function $f'$ defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

at all points $x$ for which the limit exists. If $f'(x)$ exists, we say that $f$ is differentiable at $x$.

## Notations for derivatives

If $y = f(x)$ we can use the dependent variable $y$ to represent the function, and we can denote the derivative of the function with respect to $x$ in any of the following ways:

$$y' = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x) = D f(x) = D_x y$$

These are used very often

These are used very rarely.

## Differentiation Rules

- If $g(x) = c$ (constant) $g'(x) = 0$
- If $f(x) = x^r$, then $f'(x) = r x^{r-1}$ (General power rule)

- $(f+g)'(x) = f'(x) + g'(x)$
- $(f-g)'(x) = f'(x) - g'(x)$
- $(c\,f(x))' = c\,f'(x)$
- $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ (Product Rule)
- $(f_1 \cdot f_2 \cdot f_3 \cdots f_n)' = f_1' f_2 \cdots f_n + f_1 f_2' \cdots f_n + f_1 f_2 f_3' \cdots f_n + \dots + f_1 f_2 f_3 \cdots f_n'$
- $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{(f(x))^2}$ (Reciprocal rule)
- $\left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x)\,g'(x)}{(g(x))^2}$ (The Quotient Rule)
- $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$ (The Chain Rule)
- $(f \circ g)'(x) = f'(g(x))\,g'(x)$

$f \circ g \rightarrow$ ("f composed with g")

Composite function

- $\frac{d}{dx}\sin x = \cos x$ • $\frac{d}{dx}\cos x = -\sin x$
- $\frac{d}{dx}\tan x = \sec^2 x$ • $\frac{d}{dx}\cot x = -\csc^2 x$
- $\frac{d}{dx}\sec x = \sec x \cdot \tan x$ • $\frac{d}{dx}\csc x = -\csc x \cdot \cot x$
- $\frac{d}{dx}e^x = e^x$ • $\frac{d}{dx}\ln x = \frac{1}{x}$ • $\frac{d}{dx}a^x = a^x \ln a$
- $\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$

- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$
- $\sec x = \frac{1}{\cos x}$
- $\csc x = \frac{1}{\sin x}$

$\tan x = \frac{\sin x}{\cos x}$ $\quad \frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$

$= \frac{\cos x \cdot \cos x - (-\sin x)\sin x}{\cos^2 x}$

$\sec^2 x = \frac{1}{\cos^2 x} = \frac{\overbrace{\cos^2 x + \sin^2 x}^{1}}{\cos^2 x}$

# Definition of the Indefinite Integral

**Definition:** A function $F(x)$ is called an antiderivative of $f(x)$ on an interval $I$, if $F'(x) = f(x)$ for all $x$ in $I$.

$$f(x) \underset{\text{derivative}}{\overset{\text{anti derivative}}{\rightleftarrows}} F(x)$$

**Definition:** The collection of all anti-derivatives of $f$ is called the indefinite integral of $f$ with respect to $x$, and is denoted by $\int f(x)\,dx$.

- The symbol $\int$ is an integral sign. The function $f$ is the integrand of the integral, and $x$ is the variable of integration.

## Some Integration Rules

- $\int dx = x + c$, $c$ is the integration constant.
- $\int x^2\,dx = \frac{x^3}{3} + c$
- $\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \quad (r \neq -1)$ if $r=1$ • $\int \frac{1}{x}\,dx = \ln|x| + c$
- $\int \cos ax\,dx = \frac{1}{a}\sin ax + c$
- $\int \sin ax\,dx = -\frac{1}{a}\cos ax + c$
- $\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + c$
- $\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + c$
- $\int \cot x\,dx = \ln|\sin x| + c$
- $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
- $\int \tan x\,dx = \ln|\sec x| + c = -\ln|\cos x| + c$

## Integration by parts

$$\int u\,dv = uv - \int v\,du$$

LAPTÜ
- L → Logarithmic
- A → Algebraic (Polynomials)
- P → Polynomials
- T → Trigonometric functions
- Ü → exponential
- Inverse Trigonometric (arc functions)

ILATE
- I → Inverse
- L → Logarithmic
- A → Algebraic (Polynomials)
- T → Trigonometric
- E → Exponential

## Partial derivatives

$$f(x,y) = x^4 y^3 + x^2 y$$

• Partial derivative with respect to x;

$$\frac{\partial f}{\partial x}\left(x^4 y^3 + x^2 y\right) = 4x^3 y^3 + 2xy$$

We take y as a constant. Then we obtain the derivative of $x^4$ and $x^2$

"$\partial$ del symbol"

• Partial derivative with respect to y;

$$\frac{\partial f}{\partial x}\left(x^4 y^3 + x^2 y\right) = x^4 3y^2 + \cancel{2}x^2$$

we take x as a constant. Then we obtain the derivative of $y^3$ and y.

## Solutions of First Order Differential Equations

* Separable Equations
* First order linear differential Equations (using integrating factor)
* Exact Differential Equations
* Substitutions and Transformation
* Bernoulli Equations.

# Separable Differential Equations

A simple class of first-order differential equations that can be solved using integration is the class of separable equations. These are equations

$$\frac{dy}{dx} = f(x,y) \quad (1),$$

that can be rewritten to isolate the variables x and y (together with their differentials dx and dy) on opposite sides of the equation, as in

$$h(y)\, dy = g(x)\, dx$$

- So the original right-hand side $f(x,y)$ (1) must have the factored form

$$f(x,y) = g(x) \cdot \frac{1}{h(y)}$$

More formally, we write $p(y) = \frac{1}{h(y)}$ and present the following definition.

**Definition**

If the righthand side of the equation

$$\frac{dy}{dx} = f(x,y)$$

can be expressed as a function $g(x)$ that →

depends only on y, then differential equation is called separable.

* In other words, a first order equation is separable if it can written in the form

$$\frac{dy}{dx} = g(x)\, p(y)$$

* For example, the equation

$$\frac{dy}{dx} = \frac{2x + xy}{y^2+1}$$

(factorization) "It is separable"

$$\frac{2x+xy}{y^2+1} = \frac{x\,(2+y)}{y^2+1} = x\,\frac{2+y}{y^2+1} = g(x)\, p(y)$$

* However the equation

$$\frac{dy}{dx} = 1 + xy$$ is not separable.

* Informally speaking, one solves separable equations by performing the separation and then integrating each side.

## Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)\, p(y) \quad (2)$$

multiply by $dx$ and by $h(y) = \frac{1}{p(y)}$ to obtain

$$h(y)\, dy = g(x)\, dx$$

Then integrate both sides

$$\int h(y)\,dy = \int g(x)\,dx$$

$$H(y) = G(x) + C$$

where we have merged the two constants of integration into a single symbol C

The last equation gives an implicit solution to the differential equation.

Example Solve the differential equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}$$

Solution

$$y^2\,dy = (x-5)\,dx \quad \text{(Integrate both side)}$$

$$\int y^2\,dy = \int (x-5)\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y = \left(\frac{3x^2}{2} - 15x + 3C\right)^{1/3}$$

*Replacing 3C by the single symbol K, we then have;

$$y = \left(\frac{3x^2}{2} - 15x + K\right)^{1/3}$$

**Example**

Solve the equation

- $x \dfrac{dy}{dx} = \dfrac{1}{y^3}$

$$\int y^3\, dy = \int \frac{dx}{x} \rightarrow$$

$$\frac{y^4}{4} = \ln|x| + c \rightarrow y^4 = 4 \ln x + c$$

$$y^4 = \ln x^4 + c$$

$$y = \pm \sqrt[4]{\ln x^4 + c}$$

Explicit Solution

---

**Example** $\dfrac{dy}{dx} = \dfrac{x^2+2}{y}$

$$\int y\, dy = \int (x^2+2)\, dx$$

$$\frac{y^2}{2} = \frac{x^3}{3} + 2x + c$$

$$y^2 = \frac{2}{3} x^3 + 4x + 2c$$

$$2c = K$$

$$y^2 = \frac{2}{3} x^3 + 4x + K$$

$$y = \sqrt{\frac{2}{3} x^3 + 4x + K}$$

$$y = -\sqrt{\frac{2}{3} x^3 + 4x + K}$$

Explicit Solution

Example Solve the equation

• $y' = \frac{x+1}{y^4+1}$

$y' = \frac{dy}{dx}$

Prime notation

Leibniz notation

$$\frac{dy}{dx} = \frac{x+1}{y^4+1}$$

$$(y^4+1)\,dy = (x+1)\,dx$$

$$\frac{y^5}{5} + y = \frac{x^2}{2} + x + c$$

$$y^5 + 5y = \frac{5}{2}x^2 + 5x + 5c$$

$$\frac{x^2}{2} + x - \frac{y^5}{5} - y = c \rightarrow \text{Implicit solution.}$$

Since it is impossible algebraically to solve this equation explicitly for $y$, the solution must be left in its present implicit form

Question / Solve $\frac{dx}{dt} = \frac{9t}{x e^{x+3t}}$ (Separable equation)



$$\frac{dx}{dt} = \frac{9t}{x e^{x} e^{3t}} \Rightarrow dx\, x e^{x} = \frac{9t\, dt}{e^{3t}}$$

$$\int x e^{x} dx = \int \frac{9t\, dt}{e^{3t}}$$

$\int u\,dv = uv - \int v\,dx$

$u = x \quad du = dx$

$dv = e^{x} dx \quad v = e^{x}$

$u = 9t \quad du = 9\,dt$

$dv = e^{-3t} dt \rightarrow v = \frac{e^{-3t}}{-3}$

$$x \cdot e^{x} - \int e^{x} dx = 9t\left(\frac{e^{-3t}}{-3}\right) - \int \frac{e^{-3t}}{-3}\, 9\,dt$$

$$x \cdot e^{x} - (e^{x}) = -3t\, e^{-3t} - \left(\frac{-3\, e^{-3t}}{-3}\right) + c$$

$$e^{x}(x-1) = -3t\, e^{-3t} - e^{-3t} + c$$

$$e^{x}(x-1) = (-3t-1)\, e^{-3t} + c$$

$$e^{x}(x-1) + (3t+1)\, e^{-3t} = c \quad \leftarrow \text{Implicit Solution}$$

---

## Some Examples of Differential Equations

**Example**: (Free falling object)

Suppose that an object is falling in the atmosphere near sea level. Formulate a differential that describes the motion.

* The motion takes place during a certain time interval, so let us use "t" to denote time.
Also let us use "v" to represent the velocity of the falling object. The velocity will change with time, so we think of v as a function of t.

t → independent variable
unit: seconds (s)

v → dependent variable
unit: meters/second (m/s)

"$v(t)$"

* We will assume that v is positive in the downward direction, that is, when the object is falling.
The physical law that governs the motion of the object is Newton's second law expressing by the equation;

$$F = ma \quad (1)$$

where m is the mass of the object, a is its acceleration and F is the net force exerted on the object.
To keep our units consistent, we will measure m in kilograms, a in meters/second$^2$ and F in Newtons. Of course, a is related to v by a $a = \frac{dv}{dt}$, so we can rewrite Eq(1) in the form;

$$F = m\frac{dv}{dt} \quad (2)$$

Next, consider the forces that act object as it falls

Gravity exerts a force equal to the weight of the object, or $mg$, where "$g$" is the acceleration due to gravity. "$g$" is approximately equal to $9.8\,m/s^2$ near the Earth's surface. There is also a force due to air resistance, or drag, that is more diffucult to model. For the sake of convenience, Let's assume that the drag force has the magnitude $\gamma v$ where $\gamma$ is a constant called the drag coefficient. The physical units for $\gamma$ are mass/time or $kg/s$ for this problem. In writing an expression for the net force, we need to remember that gravity always acts in the downward (positive) direction, whereas drag acts in the upward (negative) direction as shown in the following figure.

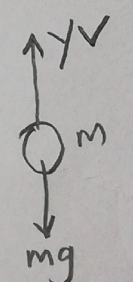



Figure: Free-body diagram of the forces on a falling object.

$$F_{net} = ma = mg - \gamma v \quad (3)$$

$$m \cdot \frac{dv}{dt} = mg - \gamma v \quad (4) \quad \left(a = \frac{dv}{dt}\right)$$

- Eq (4) is a mathematical model of an object falling in the atmosphere near sea level.

Note that the model contains the three constants "m", "g", and "γ". The constants m and γ depend on very much on the particular object that is falling, and they are usually different for different objects. It's common to refer them as parameters, since they take on a range of values during the course of an experiment. On the other hand "g" is a physical constant, whose value is the same for all objects.

* To solve Eq(4), we need to find a function $v = v(t)$ that satisfies the equation.

* Let us suppose that $m = 10\,kg$ and $\gamma = 2\,kg/s$.

Then Eq(4) can be rewritten as;

$$10 \frac{dv}{dt} = 10 \cdot 9.8 - 2v$$ (Divide both sides by 10)

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \quad (5)$$

$$\frac{dv}{dt} = \frac{49 - v}{5} \Rightarrow \frac{dv}{49 - v} = \frac{dt}{5}$$ (Separable first order linear equation)

Let's investigate the behaviour of solutions of Eq (5) without solving the differential equation.

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \quad (5)$$

* If $v = 40$, then $\frac{dv}{dt} = 9.8 - \frac{40}{5} = 1.8$

This means that the slope of a solution $v = v(t)$ has the value 1.8 at any point where $V = 40$. We can display this information graphically in the $t$-$v$ plane by drawing short line segments on the line with slope 1.8 at several points on the line $v = 40$. Similarly, if $v = 50$, then $\frac{dv}{dt} = -0.2$, so we draw line segments with slope $-0.2$ at several points on the line $v = 50$. We obtain the figure below by ~~proceeding~~ proceeding in the same way with other values of $v$. This figure is an example of what is called a direction field or sometimes a slope field

For $\frac{dv}{dt} = 0 \Rightarrow 0 = 9.8 - \frac{v}{5} \Rightarrow v = 49 \rightarrow$ This is critical value.

$v(t) = 49$ is called equilibrium solution.

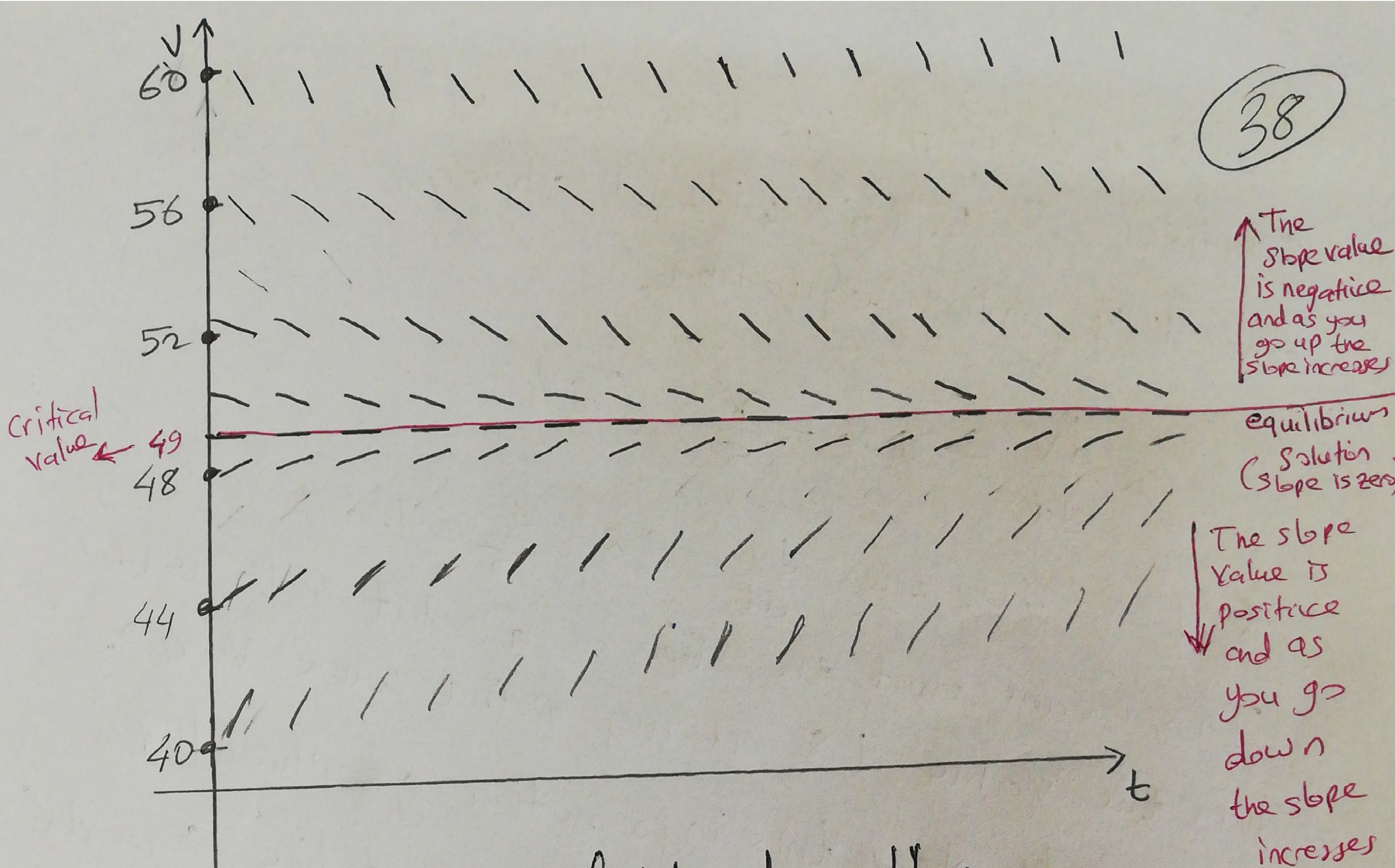



Figure Direction field and equilibrium solution for $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

---

* If v is less than a critical value ($v=49$), then all the line segments have positive slopes, and the speed of the falling object increases as it falls. On the other hand, if v is greater than the critical value ($v=49$), then the line segments have negative slopes and the falling object slows down as it falls.

$\frac{dv}{dt} = 9.8 - v/5$ (5)

Suppose this object is dropped from a height of 300 m.
Let's find its velocity at any time t,

The word "dropped" in the statement of the problem suggests that the initial velocity is zero, so we will use the initial condition $v(0)=0$.

$$\frac{dv}{dt} = \frac{49 - v}{5} \Rightarrow$$

$$\int \frac{dv/dt}{(v-49)} = \int -1/5$$ By integrating both sides we obtain;

$$\ln|v-49| = -t/5 + C$$

$$v - 49 = e^{-t/5 + C} = e^{-t/5} \cdot e^{C}$$ ← small another constant

$$\boxed{v = 49 + ce^{-t/5}}$$ (6)

↳ This is the general solution of the Eq (5).

To find c, we can use initial condition $v(t)=0$

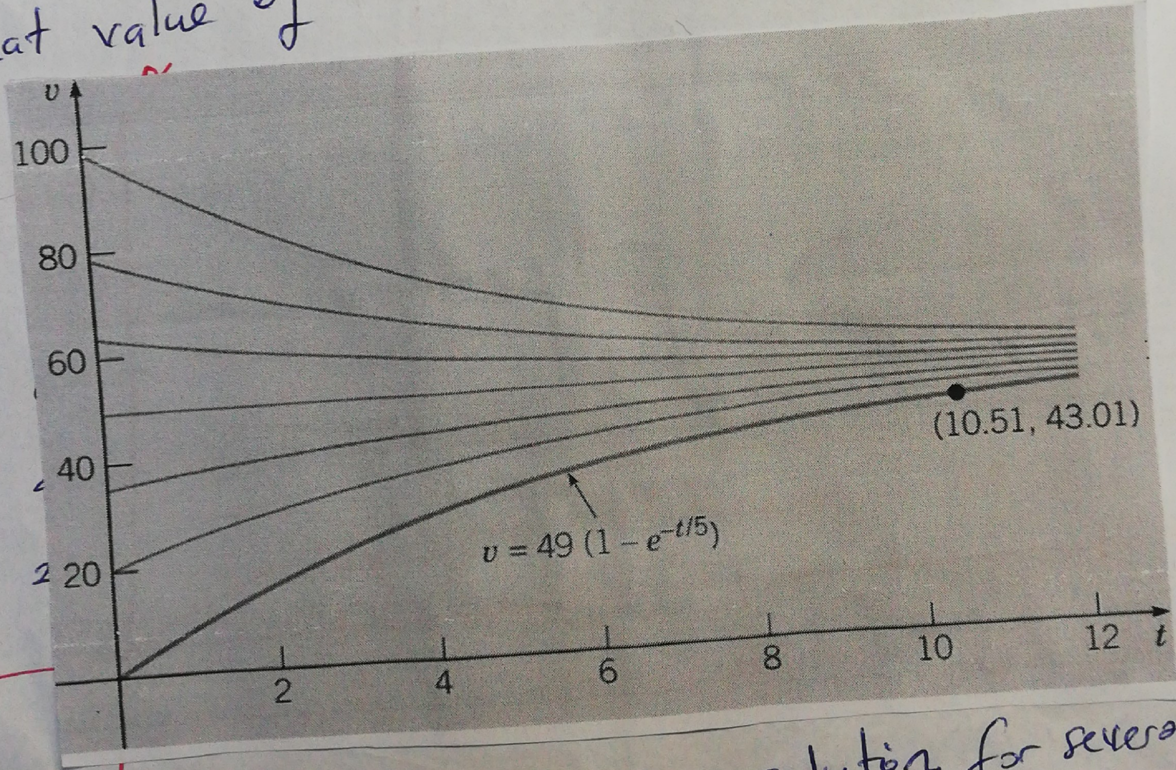
$v(0)=0$ $\quad 0 = 49 + ce^{-0}$ ↳ $c=-49$

$$\boxed{v = 49(1-e^{-t/5})}$$ (7) if $v(0)=0$

H.W
This equation gives the velocity of the falling object at any positive time (before it hits the ground)

Graphs of the general solution (6) for several values of c are shown in figure 3. The geometrical representation of the general solution (6) is an infinite family of curves called <u>integral curves</u>.

Each integral curve is associated with a particular value of c and is the graph of the solution corresponding to that value of c.




Figure(3) Graphs of the solution for several values of c.

It's evident that all solutions tend to approach the equilibrium solution $v = 49$.

## Linear Equations (First order linear differential equations)

A linear first-order equation is an equation that can be expressed in the form

$$\boxed{a_1(x)\frac{dy}{dx} + a_0(x)\,y = b(x)} \quad (1)$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable $x$, not on $y$.

* If $a_0(x) = 0$, then we have

* $a_1(x)\dfrac{dy}{dx} = b(x) \rightarrow \int dy = \int \dfrac{b(x)}{a_1(x)}\,dx$

* $y = \int \dfrac{b(x)}{a_1(x)}\,dx + c \quad (2)$

Suppose that $a_0(x) = a_1'(x)$

$$a_1(x)\frac{dy}{dx} + a_0(x)\cdot y = a_1(x)\,y' + a_0(x)\,y = a_1(x)\,y' + a_1'(x)\,y$$

$$a_1(x)\,y' + a_1'(x)\,y = \frac{d}{dx}\left[a_1(x)\cdot y\right]$$

Therefore Equation (1) becomes

$$\frac{d}{dx}\left[a_1(x)\,y\right] = b(x) \quad (3)$$

$$a_1(x)\,y = \int b(x)\,dx + C$$

$$y(x) = \frac{1}{a_1(x)}\left[\int b(x)\,dx + C\right]$$

The form (3) can be achieved through multiplication of the original equation (1) by a well_chosen function $\mu(x)$.

Such a function $\mu(x)$ is called an integrating factor for equation (1). The easiest way to see this is first divide the original equation (1) by $q_1(x)$ and put it into standard form

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad (4)$$

where $P(x) = \frac{q_0(x)}{q_1(x)}$ and $Q(x) = \frac{b(x)}{q_1(x)}$.

Next we wish to determine $\mu(x)$ so that the left-hand side of the multiplied equation

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \mu(x)\,Q(x) \quad (5)$$

is just derivative of the product $\mu(x)\,y$:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \frac{d}{dx}[\mu(x)\cdot y] = \mu(x)\frac{dy}{dx} + \mu'(x)\cdot y.$$

Clearly, this requires that $\mu$ satisfy

$$\mu'(x) = \mu(x)\cdot P(x) \quad (6)$$

To find such a function, we recognize that Eq(6) is a seperable differential equation, which we can write as

$$\frac{d\mu}{dx} = \mu(x)\,P(x)$$

$$\int \left(\frac{1}{\mu}\right) d\mu = \int P(x)\,dx$$

$$\ln\mu = \int P(x)\,dx \qquad \mu = e^{\int P(x)\,dx} \quad \left(\text{Take integrating constant} = 0\right)$$

With this choice for $\mu(x)$ equation (5) becomes

$$\frac{d}{dx}[\mu(x) \cdot y] = \mu(x) \cdot Q(x)$$

which has the solution

$$y(x) = \frac{1}{\mu(x)} \left[ \int \mu(x) Q(x)\, dx + C \right] \quad (8)$$

Here $C$ is an arbitrary constant, so (8) gives a one parameter family of solutions to Eq (4). This form is known as the general solution to Eq (4).

## Method for solving First order Linear Equations

(a) Write the equation in the standart form

$$\frac{dy}{dx} + P(x) y = Q(x)$$

(b) Calculate the integrating factor $\mu(x)$ by the formula;

$$\mu(x) = \exp\left[ \int P(x)\, dx \right] \longrightarrow \boxed{\mu(x) = e^{\int P(x) dx}}$$

(c) Multiply the equation in standart form by $\mu(x)$ and recalling that the left-hand side is just $\frac{d}{dx}[\mu(x) y]$, obtain

$$\underbrace{\mu(x) \frac{dy}{dx} + P(x) \mu(x) y}_{\frac{d}{dx}[\mu(x) y]} = \mu(x) \cdot Q(x)$$

$$\frac{d}{dx}\left[\mu(x)\cdot y\right] = \mu(x)\cdot Q(x)$$

(d) Integrate the last equation and solve for $y$ by dividing $\mu(x)$ to obtain (8).

$$y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)\,Q(x)\,dx + C\right]$$

---

**Example 1**: Find the general solution of

$$y' - 3y = 6$$

**Solution**: $\frac{dy}{dx} - 3y = 6 \qquad \left(\frac{dy}{dx} + P(x)\cdot y = Q(x)\right)$

1. step: $P(x) = -3$, $Q(x) = 6$

2. step (Integrating factor) $\mu(x) = e^{\int P(x)\,dx} = e^{\int -3\,dx} = e^{-3x}$ (Integration constant is 0)

Multiply both sides of the equation by $\mu(x) = e^{-3x}$.

3. step

$$\underbrace{e^{-3x}\frac{dy}{dx} - e^{-3x}\,3y}_{\frac{d}{dx}\left(e^{-3x}\cdot y\right)} = 6e^{-3x}$$

$$\frac{d}{dx}\left(e^{-3x}\cdot y\right) = 6e^{-3x}$$

Integrating both sides of this last equation with respect to $x$,

$$\int \frac{d}{dx}\left(e^{-3x}y\right)dx = \int 6e^{-3x}\,dx$$ (45)

$$e^{-3x}y = \frac{6}{-3}e^{-3x} + C$$

$$e^{-3x}y = -2e^{-3x} + C$$

$$y = \frac{-2e^{-3x}}{e^{-3x}} + \frac{C}{e^{-3x}}$$

$$\boxed{y = -2 + Ce^{3x}}$$

**Example 2** find the general solution of

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x, \quad x>0.$$

**Solution**

**1. step**: Rewrite the equation in the standart form

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cdot\cos x \qquad \text{(Multiply by } x\text{)}$$

$$\boxed{\frac{dy}{dx} - \frac{2}{x}y = x^2\cos x.}$$

- $P(x) = -\frac{2}{x}$

**2. step**. Calculate the integrating factor $\mu(x)$.

$$\mu(x) = e^{\int P(x)\,dx}$$

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|}$$

$$\mu(x) = e^{\ln x^{-2}} = x^{-2}$$

46

$\ln x^{-2} = n$

$e^n = x^{-2}$

$e^{\ln x^{-2}} = e^n = x^{-2}$

Step 3
Multiply by $\mu(x)$.

$$\frac{dy}{dx} - \frac{2}{x}y = x^2 \cdot \cos x$$

$$\mu(x) \cdot \frac{dy}{dx} - \frac{2}{x}y = \mu(x)\, x^2 \cos x$$

$$\underbrace{x^{-2}\frac{dy}{dx} - x^{-2}\frac{2}{x}y}_{\frac{d}{dx}\left(x^{-2}y\right)} = x^{-2}x^2 \cos x$$

$$\frac{d}{dx}\left(x^{-2}y\right) = \cos x$$

Step 4: We now integrate both sides and solve for $y$;

$$x^{-2}y = \int \cos x \, dx$$

$$x^{-2}y = \sin x + c$$

$$\boxed{y = x^2 \sin x + cx^2}$$

Example 3: Solve the initial-value problem that consists of the differential equation

$$(x^2+1)\frac{dy}{dx} + 4xy = x$$

and the initial condition $y(2) = 1$.

Step_1

$$(x^2+1)\frac{dy}{dx} + 4xy = x$$

47

- Divide by $x^2+1$ to obtain the standart form.

$$\frac{dy}{dx} + \frac{4x}{(x^2+1)} \cdot y = \frac{x}{x^2+1} \qquad \left(\frac{dy}{dx} + P(x) \cdot y = Q(x)\right)$$

- $P(x) = \dfrac{4x}{x^2+1}$

Step 2 Calculate $\mu(x)$;

$$\mu(x) = e^{\int \frac{4x}{x^2+1} dx}$$

$$x^2+1 = u$$

$$\int \frac{4x}{x^2+1} dx \qquad 2x\,dx = du$$

$$\int \frac{2du}{u} = 2\ln u \rightsquigarrow 2\ln(x^2+1)$$

$$\mu(x) = e^{2\ln(x^2+1)} = e^{\ln(x^2+1)^2} = (x^2+1)^2$$

Step 3 Multiple both sites by $\mu(x)$

$$\underbrace{(x^2+1)^2 \frac{dy}{dx} + (x^2+1)^2\, 4x\, y}_{\frac{d}{dx}\left[(x^2+1)^2 \cdot y\right]} = \frac{x \cdot (x^2+1)^2}{x^2+1}$$

$$\frac{d}{dx}\left[(x^2+1)^2 \cdot y\right] = x \cdot (x^2+1) \qquad \text{Integrate both Sides}$$

Step 4

$$(x^2+1)^2 \cdot y = \int x \cdot (x^2+1)\,dx$$

$$(x^2+1)^2 \cdot y = \int (x^3+x)\,dx \rightarrow (x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + c$$

$$y(2) = 1$$

$$25 = \frac{16}{4} + \frac{4}{2} + c$$

$$25 = 6 + c \qquad c = 19$$

$$\boxed{(x^2+1)\,y = \frac{x^4}{4} + \frac{x^2}{2} + 19}$$

**Example 4.** Solve the initial value problem,

$$t\frac{dy}{dt} + 2y = 4t^2 \quad \text{where } y(1) = 2.$$

**Solution**

**Step 1.** Write the equation in the standart form

$$\frac{dy}{dt} + \frac{2}{t}y = 4t \qquad \left(\frac{dy}{dt} + P(t)\,y = Q(t)\right)$$

**Step 2.** Calculate the integrating factor $\mu(t)$

$$\mu(t) = e^{\int P(t)\,dt}$$

$$\mu(t) = e^{\int \frac{2}{t}dt} = e^{2\ln t} = e^{\ln t^2} = t^2$$

**Step 3.** Multiply the equation by $\mu(t)$

$$\underbrace{\frac{dy}{dt}t^2 + \frac{2}{t}t^2 y} = 4t\,t^2$$

**Step 4.**

$$\int \frac{d}{dt}(t^2 \cdot y) = \int 4t^3 \qquad \text{(Integrate both sides)}$$

$$t^2 \cdot y = t^4 + c \qquad y(1) = 2 \quad \text{for } t=1,\ y=2$$

$$y = t^2 + \frac{c}{t^2} \qquad 2 = 1 + c, \quad c = 1$$

$$\boxed{y = t^2 + \frac{1}{t^2}}$$