

2012 IMO Problems/Problem 2

Problem

Let a_2, a_3, \dots, a_n be positive real numbers that satisfy $a_2 \cdot a_3 \cdots a_n = 1$. Prove that

$$(a_2 + 1)^2 \cdot (a_3 + 1)^3 \cdots (a_n + 1)^n \geq n^n$$

Solution

The inequality between arithmetic and geometric mean implies

$$(a_k + 1)^k = \left(a_k + \frac{1}{k-1} + \frac{1}{k-1} + \cdots + \frac{1}{k-1} \right)^k \geq k^k \cdot a_k \cdot \frac{1}{(k-1)^{k-1}} = \frac{k^k}{(k-1)^{k-1}} \cdot a_k$$

The inequality is strict unless $a_k = \frac{1}{k-1}$. Multiplying analogous inequalities for $k = 2, 3, \dots, n$ yields

$$(a_2 + 1)^2 \cdot (a_3 + 1)^3 \cdots (a_n + 1)^n \geq \frac{2^2}{1^1} \cdot \frac{3^3}{2^2} \cdot \frac{4^4}{3^3} \cdots \frac{n^n}{(n-1)^{n-1}} \cdot a_2 \cdot a_3 \cdots a_n = n^n$$

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