2012 IMO Problems/Problem 2

Problem

Let $a_2, a_3, \cdots, a_n$ be positive real numbers that satisfy $a_2 \cdot a_3 \cdots a_n = 1$. Prove that

$$(a_2+1)^2 \cdot (a_3+1)^3 \cdots (a_n+1)^n \gneq n^n$$

Solution

The inequality between arithmetic and geometric mean implies

$$(a_k+1)^k = \left(a_k + \frac{1}{k-1} + \frac{1}{k-1} + \cdots + \frac{1}{k-1}\right)^k \geq k^k \cdot a_k \cdot \frac{1}{(k-1)^{k-1}} = \frac{k^k}{(k-1)^{k-1}} \cdot a_k$$

The inequality is strict unless $a_k = \dfrac{1}{k-1}$. Multiplying analogous inequalities for $k = 2,\ 3,\ \cdots,\ n$ yields

$$(a_2+1)^2 \cdot (a_3+1)^3 \cdots (a_n+n)^n \gneq \frac{2^2}{1^1} \cdot \frac{3^3}{2^2} \cdot \frac{4^4}{3^3} \cdots \frac{n^n}{(n-1)^{n-1}} \cdot a_2 \cdot a_3 \cdots a_n = n^n$$