AMC12 2010 B22

Let $ABCD$ be a cyclic quadrilateral. The side lengths of $ABCD$ are distinct integers less than 15 such that $BC \cdot CD = AB \cdot DA$. What is the largest possible value of BD?

A. $\sqrt{\frac{325}{2}}$ B. $\sqrt{185}$ C. $\sqrt{\frac{389}{2}}$ D. $\sqrt{\frac{425}{2}}$ E. $\sqrt{\frac{533}{2}}$

Let AB = a, BC = b, CD = c, and DA = d. By the Law of Cosines, $BD^2 = a^2 + d^2 - 2ad\cos\angle BAD$ and $BD^2 = b^2 + c^2 - 2bc\cos\angle BCD$. $ad = bc$ and $\angle BAD = 180 - \angle BCD$, which gives us $\cos\angle BAD = -\cos\angle BCD$. Therefore, $2ad\cos\angle BAD = -2bc\cos\angle BCD$. Adding, we get $2BD^2 = a^2 + b^2 + c^2 + d^2$. The maximum values of a, b, c, and d are 14, 6, 12, and 7, respectively. $BD = \sqrt{\frac{14^2+6^2+12^2+7^2}{2}} = \sqrt{\frac{425}{2}}$, so our answer is D.