AMC10 2019 A15

A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$. Then, a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. What is $p+q$?

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Using the recursive formula, we find $a_3 = \frac{3}{11}$, $a_4 = \frac{3}{15}$, and so on. It appears that $a_n = \frac{3}{4n-1}$ for all n. Setting $n = 2019$, we find $a_{2019} = \frac{3}{8075}$, so the answer is E (8078).

By our assumption, $a_{m-1} = \frac{3}{4m-5}$ and $a_m = \frac{3}{4m-1}$.

$$a_{m+1} = \frac{a_{m-1} \cdot a_m}{2a_{m-1} - a_m} = \frac{\frac{3}{4m-5} \cdot \frac{3}{4m-1}}{2 \cdot \frac{3}{4m-5} - \frac{3}{4m-1}} = \frac{\frac{9}{(4m-5)(4m-1)}}{\frac{6(4m-1)-3(4m-5)}{(4m-5)(4m-1)}}$$

$$= \frac{9}{6(4m-1) - 3(4m-5)} = \frac{3}{4(m+1)-1}$$

so our induction is complete.