Calculus of Finite Differences

Akshaya Chakravarthy, Jack Hsieh, Emma Li, Alexandra Yang, Lauren Yoo

Counselor: Andrew Weinfeld

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1 Introduction

We compute and characterize differences of sequences. We also examine fitting polynomials to consecutive data points and general expressions of sums of k-th powers. We conclude with analogs between discrete and continuous calculus.

2 Preliminary Definitions

2.1 Differences

Definition 1. *We define* $\mathbb{N}_0 := \{n \in \mathbb{Z} \mid n \geq 0\}$.

Definition 2. *Let* $f : \mathbb{N}_0 \to \mathbb{R}$ *be an arbitrary function. We define the difference function as*

$$\Delta(f)(x) := f(x+1) - f(x).$$

We illustrate the difference operator in action with a few examples of evaluating the difference function for various polynomials:

1. $\mathbf{f(x) = 7 - 3x}$: We have

$$\begin{aligned}\Delta(f)(x) &= f(x+1) - f(x) \\ &= 7 - 3(x+1) - (7 - 3x) \\ &= \boxed{-3}.\end{aligned}$$

2. $\mathbf{f(x) = x^2}$: We have

$$\begin{aligned}\Delta(f)(x) &= f(x+1) - f(x) \\ &= (x+1)^2 - x^2 \\ &= \boxed{2x+1}.\end{aligned}$$

2.2 Iterated Differences

The operator Δ can be iterated, i.e. $\Delta^{n+1}(f) := \Delta(\Delta^n(f))$ for $n > 0$. The following are a few examples of iterations of the same functions we saw earlier:

1. $\mathbf{f(x) = 7 - 3x}$:

$$\begin{aligned}\Delta^1(f)(x) &= \boxed{-3}, \\ \Delta^2(f)(x) &= -3 - (-3) = \boxed{0}, \\ \Delta^n(f)(x) &= \boxed{0} \text{ for all } n \geq 2.\end{aligned}$$

2. $\mathbf{f}(\mathbf{x}) = \mathbf{x}^\mathbf{2}$:

$$\begin{aligned}
\Delta^1(f)(x) &= \boxed{2x+1},\\
\Delta^2(f)(x) &= 2(x+1)+1-(2x+1) = \boxed{2},\\
\Delta^3(f)(x) &= 2-2 = \boxed{0},\\
\Delta^n(f)(x) &= \boxed{0} \text{ for all } n \geq 3.
\end{aligned}$$

Observe that every time the Δ operator is iterated, the degree decreases by 1, and the coefficient of the leading term is the product of the previous leading coefficient and the previous degree (similar to a derivative). Once the output reaches 0, any further iterations will result in 0.

2.3 Antidifferences

We may also work with the Δ operator in reverse, i.e. when given some polynomial $\Delta(f)(x)$, we can solve for the polynomial $f(x)$ whose difference is $\Delta(f)(x)$. We refer to $f(x)$ as the *antidifference* of $\Delta(f)(x)$.

Definition 3. *For a function f, we define an antidifference of f as a function F such that $f = \Delta(F)$. We say $F = \Delta^{-1} f$.*

Here are a few examples of computing the antidifference:

1. $\mathbf{\Delta}(\mathbf{f})(\mathbf{x}) = \mathbf{3}$: Since the degree of $\Delta(f)(x)$ is 0, the degree of $f(x)$ must be 1. Let $f(x) = a_1x + a_0$. Then,

$$\begin{aligned}
\Delta(f)(x) &= f(x+1) - f(x)\\
&= a_1(x+1) + a_0 - (a_1x + a_0)\\
&= a_1\\
&= 3.
\end{aligned}$$

There's no restriction on a_0, so $\boxed{f(x) = 3x + c}$ for some constant c.

2. $\mathbf{\Delta}(\mathbf{f})(\mathbf{x}) = \mathbf{x}$: The degree of $\Delta(f)(x)$ is 1, so the degree of $f(x)$ must be 2. Let $f(x) = a_2x^2 + a_1x + a_0$. Then,

$$\begin{aligned}
\Delta(f)(x) &= a_2(x+1)^2 + a_1(x+1) + a_0 - (a_2x^2 + a_1x + a_0)\\
&= 2a_2x + a_2 + a_1 = x.
\end{aligned}$$

Comparing coefficients, we have $2a_2 = 1, a_2 + a_1 = 0 \implies a_2 = \frac{1}{2}, a_1 = -\frac{1}{2}$, so $\boxed{f(x) = \dfrac{1}{2}x^2 - \dfrac{1}{2}x + c}$ for some constant c.

Note that applying Δ^{-1} to some polynomial will increase the degree of that polynomial by 1 and divide the leading coefficient by the new degree, very similar to integrals. It is also only determined up to a constant (as proven below), just like an indefinite integral. The antidifference operator can be iterated just like the difference operator.

Proposition 1. *For a function $f(x)$, an antidifference of $f(x)$ exists.*

Proof. Let $F(x) = \sum_{k=0}^{x-1} f(k)$. Note that $\Delta(F(x)) = f(x)$. Thus, F is an antidifference of f. □

Proposition 2. *The antidifference is well-defined down to constant terms.*

Proof. Let $f(x), g(x)$ be functions. Suppose $\Delta f(x) = \Delta g(x)$ for all $x \in \mathbb{N}_0$. Then

$$\begin{aligned}
f(x+1) - f(x) &= g(x+1) - g(x)\\
f(x+1) - g(x+1) &= f(x) - g(x).
\end{aligned}$$

By induction, for all $x \in \mathbb{N}_0$, $f(x) - g(x) = f(0) - g(0)$. Let $c = f(0) - g(0)$. Then $f(x) = g(x) + c$ for all $x \in \mathbb{N}$. So, $f(x)$ and $g(x)$ differ only by a constant term and thus, the antidifference is well-defined down to constant terms. □

3 Important Results

3.1 Fundamental Properties of Differences

The following is an important property of the difference operator.

Proposition 3. *The difference operator is a linear transformation. Equivalently, the difference operator is additive and multiplicative for constants. Equivalently, for all functions $f(x), g(x)$ and any constant c,*

$$\Delta(f(x) + g(x)) = \Delta f(x) + \Delta g(x)$$

and

$$\Delta(cf(x)) = c\Delta f(x).$$

Proof. We have

$$\begin{aligned}\Delta(f(x) + g(x)) &= (f(x+1) + g(x+1)) - (f(x) + g(x)) \\ &= (f(x+1) - f(x)) + (g(x+1) - g(x)) \\ &= \Delta(f)(x) + \Delta(g)(x).\end{aligned}$$

and

$$\begin{aligned}\Delta(cf(x)) &= cf(x+1) - cf(x) \\ &= c(f(x+1) - f(x)) \\ &= c\Delta(f)(x).\end{aligned}$$

□

Proposition 4. *Let f be a polynomial of degree $n > 0$. Then Δf is a polynomial of degree $n - 1$.*

Proof. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Then, by definition,

$$\begin{aligned}\Delta f &= f(x+1) - f(x) \\ &= (a_n(x+1)^n + a_{n-1}(x+1)^{n-1} + \ldots a_1(x+1) + a_0) - (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) \\ &= a_n((x+1)^n - x^n) + a_{n-1}((x+1)^{n-1} - x^{n-1}) + \cdots + a_2((x+1)^2 - x^2) + a_1((x+1) - x).\end{aligned}$$

Consider the leading term, $a_n((x+1)^n - x^n)$. Note that $(x+1)^n$ has a leading term of x^n, and thus, $(x+1)^n - x^n$ has degree $n-1$. So, the leading term of Δf is a non-zero constant times x^{n-1}, and the degree of Δf is $n - 1$, as desired. □

3.2 Falling Factorials

We now examine an important way of expressing and analyzing polynomials.

3.2.1 Essential Properties

Definition 4. *Let k be a nonnegative integer, we define the k-th falling factorial of x as*

$$x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+2)(x-k+1).$$

Equivalently, $x^{\underline{k}} = \frac{x!}{(x-k)!}$.

Remark 1. *For convenience, we let $x^{\underline{0}} = 1$.*

Proposition 5. *For a positive integer k, $\Delta(x^{\underline{k}}) = k(x^{\underline{k-1}})$.*

Proof.

$$\begin{aligned}\Delta(x^{\underline{k}}) &= (x+1)^{\underline{k}} - x^{\underline{k}} \\ &= \frac{(x+1)!}{(x+1-k)!} - \frac{x!}{(x-k)!} \\ &= \frac{(x+1)x!}{(x+1-k)!} - \frac{x!(x+1-k)}{(x+1-k)!} \\ &= ((x+1)-(x+1-k))\left(\frac{x!}{(x+1-k)!}\right) \\ &= k\frac{x!}{(x-(k-1))!} \\ &= kx^{\underline{k-1}}.\end{aligned}$$

□

Corollary 5.1. *For a nonnegative integer k,*

$$\Delta^{-1}(x^{\underline{k}}) = \frac{1}{k+1}x^{\underline{k+1}} + c,$$

where c is an arbitrary constant.

Proof. By Proposition 5, note that

$$\begin{aligned}\Delta\left(\frac{1}{k+1}x^{\underline{k+1}}\right) &= \frac{1}{k+1}\Delta\left(x^{\underline{k+1t}}\right) \\ &= \frac{1}{k+1}\left((k+1)x^{\underline{k}}\right) \\ &= x^{\underline{k}},\end{aligned}$$

so by Proposition 2, the antidifference is $\frac{1}{k+1}x^{\underline{k+1}} + c$ where c is an arbitrary constant. □

3.2.2 Conversions between Powers and Falling Factorials

Proposition 6. *There exist unique integers $s_{m,k}$ for $m, k \in \mathbb{N}_0$ such that*

$$x^{\underline{k}} = \sum_{m=0}^{k} s_{m,k}x^m.$$

Equivalently, there exists a unique linear mapping from polynomials of degree at most k to $\operatorname{span}\left(\{x^{\underline{k}}\}_{k\geq 0}\right)$.

Proof. Note that by definition, $x^{\underline{k}}$ is the product $x(x-1)(x-2)\ldots(x-k+1)$ and therefore can be expanded and expressed as a unique polynomial with integer coefficients. □

Corollary 6.1. *For nonnegative integers m and k,*

1. $s_{k,k} = 1$.
2. $s_{m,k} = 0$ *for* $m > k$.
3. $s_{m,0} = 0$ *for* $m > 0$.
4. $s_{0,k} = 0$ *for* $k > 0$.
5. $s_{m,k} = s_{m-1,k-1} - (k-1)s_{m,k-1}$ *for* $m, k > 0$.

Proof. We make several observations:

1. Notice that $s_{k,k}$ is the coefficient of x^k in $x^{\underline{k}} = x(x-1)(x-2)\cdots(x-k+1)$. Since $x^{\underline{k}}$ has degree k and the leading coefficient of $x-i$ (for all $i \in \{0,1,2,\cdots,k-1\}$) is 1, $s_{k,k}$is the leading coefficient of $x^{\underline{k}}$, which equals 1.

2. Recall from [1] that the degree of $x^{\underline{k}}$ is k, and thus, the coefficient of all terms in $x^{\underline{k}}$ with degree greater than k is 0. So, $s_{m,k} = 0$ for all $m > k$.

3. Recall that $x^{\underline{0}} = 1$, and thus, the coefficient of x^m in $x^{\underline{0}}$ equals 0 for all $m > 0$. So, $s_{m,0} = 0$ for all $m > 0$.

4. Since $x^{\underline{k}} = x(x-1)\cdots(x-k+1)$, notice that the least degree of all non-zero terms in $x^{\underline{k}}$ is 1. Thus, the constant term is 0, and $s_{0,k} = 0$ for all $k > 0$.

5. Recall that $s_{m,k}$ is the coefficient of x^m in $x^{\underline{k}}$. Thus, $s_{m,k}$ is the coefficient of x^m in $x(x-1)\cdots(x-k+1)$. Consider a term in $x^{\underline{k}}$ with degree m. Notice that it must have coefficient $(-1)^{k-m} i_1 i_2 \cdots i_{k-m}$ where $i_1, i_2, \ldots, i_{k-m} \in \{1,2,\ldots,k-1\}$ and $i_1 \neq i_2 \neq \cdots \neq i_{k-m}$ (which can be seen by considering the linear factors of $x^{\underline{m}}$.) Thus, when adding all such terms, we obtain

$$s_{m,k} = \sum_{1 \le i_1 < i_2 < \cdots < i_{k-m} < k} (-1)^{k-m} i_1 i_2 \cdots i_{k-m} \ \textbf{[1]}.$$

Notice that we can write this sum as

$$s_{m,k} = \sum_{1 \le i_1 < i_2 < \cdots < i_{k-m} < k-1} (-1)^{k-m} i_1 i_2 \cdots i_{k-m} + \sum_{1 \le i_1 < i_2 < \cdots < i_{k-m-1} < k-1} (-1)^{k-m} i_1 i_2 \cdots i_{k-m-1}(k-1)$$

(by considering whether $i_m = k-1$). From [1], notice that

$$\sum_{1 \le i_1 < i_2 < \cdots < i_{k-m} < k-1} (-1)^{k-m} i_1 i_2 \cdots i_{k-m} = s_{m-1,k-1}$$

and similarly,

$$\sum_{1 \le i_1 < i_2 < \cdots < i_{k-m-1} < k-1} (-1)^{k-m} i_1 i_2 \cdots i_{k-m-1}(k-1) = -(k-1)s_{m,k-1}.$$

Therefore, $s_{m,k} = s_{m-1,k-1} - (k-1)s_{m,k-1}$, and the claim is proven.

□

Remark 2. *The relations described by Corollary 6.1 provide a relatively efficient method for calculating values of s. A table of values of $s_{m,k}$ can be found in the Appendix.*

Lemma 1. *Let n be a nonnegative integer and S be the $(n+1)\times(n+1)$ matrix with entries $s_{m,k}$ for all $0 \le m, k \le n$. Then S is invertible.*

Proof. By Corollary 6.1, S is an upper triangular matrix with nonzero diagonal, so the determinant of S is nonzero. Thus, S is invertible.

□

Proposition 7. *For nonnegative integers k and m, there exists unique rationals $t_{m,k}$ for $m, k \in \mathbb{N}_0$ such that*

$$x^k = \sum_{m=0}^{k} t_{m,k} x^{\underline{m}}$$

Equivalently, the set $\{x^{\underline{m}}\}_{0 \le m \le k}$ is a basis of the vector space of polynomials of degree at most k.

Proof. Note that S represents a unique linear mapping from the vector space spanned by x^k for $0 \le k \le n$ to the space spanned by $x^{\underline{k}}$ for $0 \le k \le n$. By Lemma 1, this linear mapping has a (unique) inverse expressible as matrix T with entries $t_{m,k}$. By definition, $x^k = \sum_{m=0}^{k} t_{m,k} x^{\underline{m}}$ as desired.
Note that since $S \in \mathbb{Q}^{(n+1)\times(n+1)}$ and $\mathbb{Q}^{(n+1)\times(n+1)}$ is closed under matrix inversion, $T \in \mathbb{Q}^{(n+1)\times(n+1)}$, so $t_{m,k} \in \mathbb{Q}$. □

Remark 3. *We find it convenient to define $t_{m,k} = 0$ for $m > k$ for the following proposition.*

Proposition 8. *For $m, k \in \mathbb{N}_0$, $t_{m,k}$ counts the number of ways to partition k distinct elements into m nonempty indistinct parts.*

Proof. For $m, k \in \mathbb{N}_0$, let $a_{m,k}$ count the number of ways to partition k distinct elements into m nonempty indistinct parts. Note that k elements cannot fill more than k parts, so $a_{m,k} = 0$ for $m > k$.
For $x, k \in \mathbb{N}_0$, consider the set W of mappings from $\mathbb{Z}_k$ to $\mathbb{Z}_x$. As each of the k elements of $\mathbb{Z}_k$ can map to any of the x elements of $\mathbb{Z}_x$, $|W| = x^k$.
We may also calculate the size of W by partitioning W into $W_m = \{f \in W : |\{f(u) : u \in \mathbb{Z}_x\}| = m\}$ for $0 \le k \le m$. Note that every $f \in W_m$ can be constructed by pairing a partition of $\mathbb{Z}_k$ into m nonempty indistinct parts with an ordered sequence of m distinct elements of $\mathbb{Z}_k$ by ordering the parts by least element not contained by any previous parts and then assigning all the elements in each part a common function value using the m-tuple in $\mathbb{Z}_k$. This construction is reversible, so W_m is bijective to the Cartesian product of the set of partitions of k distinct elements into m nonempty indistinct parts and the set of ordered sequence of m distinct elements of $\mathbb{Z}_k$. The former component of the Cartesian product is counted by $a_{m,k}$ while the latter is counted by $x^{\underline{m}}$. Thus,

$$x^k = |W| = \left|\cup_{m=0}^{k} W_m\right| = \sum_{i=0}^{k} |W_m| = \sum_{m=0}^{k} a_{m,k} x^{\underline{m}}.$$

Since by Proposition 7, $t_{m,k}$ is unique, so $t_{m,k} = a_{m,k}$ for $m \le k$. For $m > k$, by definition, $t_{m,k} = 0 = a_{m,k}$. Thus, $t_{m,k} = a_{m,k}$ as desired. □

Corollary 8.1. *For nonnegative integers m and k,*

1. $t_{k,k} = 1$.
2. $t_{m,0} = 0$ *for* $m > 0$.
3. $t_{0,k} = 0$ *for* $k > 0$.
4. $t_{m,k} = t_{m-1,k-1} + m t_{m,k-1}$ *for* $m, k > 0$.

Proof. We heavily employ Proposition 8.

1. For $k = 0$, there is trivially one way to partition no elements into no parts. For $k > 0$, there is exactly one way to partition k distinct elements into k nonempty indistinct parts: namely, with every element in its own part. Thus, $t_{k,k} = 1$.

2. There are no ways to partition zero elements into a positive number of nonempty parts, so $t_{m,0} = 0$ for $m > 0$.

3. There are no ways to partition a positive number of elements into zero parts, so $t_{k,0} = 0$ for $k > 0$.

4. Note that for $m, k > 0$, every partition of $\mathbb{Z}_k$ into m nonempty indistinct parts can be constructed through exactly one of two options:

   (a) Taking a partition of $\mathbb{Z}_{k-1}$ into $m - 1$ parts and adding a part containing only the element k. This is counted by $t_{m-1,k-1}$.

   (b) Taking a partition of $\mathbb{Z}_{k-1}$ into m parts and inserting k into one of the m parts. This is counted by $m t_{m,k-1}$.

This procedure is reversible, so $t_{m,k} = t_{m-1,k-1} + mt_{m,k-1}$ for $m, k > 0$.

$\square$

Remark 4. *The relations described by Corollary 8.1 provide a relatively efficient method for calculating values of t. A table of values of $t_{m,k}$ can be found in the Appendix.*

3.2.3 Differences of Polynomials via Falling Factorials

Remark 5. *Proposition 7 provides us an alternate method of computing differences and antidifferences of polynomials or Taylor series expressed in normal exponents. We may convert the expression to falling factorials using Proposition 7, take the difference or antidifference using Proposition 5 or Corollary 5.1 respectively, and then return to normal exponents with Proposition 6.*

Proposition 9. *Let k be a nonnegative integer. Then*

$$\Delta\left(x^k\right) = \sum_{\ell=0}^{k-1}(\ell+1)t_{\ell+1,k}x^{\underline{\ell}} = \sum_{m=0}^{k-1} x^m \left(\sum_{\ell=m}^{k-1}(\ell+1)t_{\ell+1,k}s_{m,\ell}\right).$$

Proof. To compute $\Delta(x^k)$, we convert to falling factorials and take the difference:

$$\begin{aligned}
\Delta(x^k) &= \Delta\left(\sum_{\ell=0}^{k} t_{\ell,k}x^{\underline{\ell}}\right) \\
&= \sum_{\ell=0}^{k} t_{\ell,k}\Delta\left(x^{\underline{\ell}}\right) \\
&= \Delta(x^{\underline{0}}) + \sum_{\ell=1}^{k} t_{\ell,k}\left(\Delta\left(x^{\underline{\ell}}\right)\right) \\
&= \Delta(1) + \sum_{\ell=1}^{k} t_{\ell,k}\left(\ell x^{\underline{\ell-1}}\right) \\
&= 0 + \sum_{\ell=0}^{k-1} t_{\ell+1,k}\left((\ell+1)x^{\underline{\ell}}\right) \\
&= \sum_{\ell=0}^{k-1}(\ell+1)t_{\ell+1,k}x^{\underline{\ell}}.
\end{aligned}$$

We then may return to normal powers:

$$\begin{aligned}
\Delta(x^k) &= \sum_{\ell=0}^{k-1}(\ell+1)t_{\ell+1,k}\left(\sum_{m=0}^{\ell} s_{m,\ell}x^m\right) \\
&= \sum_{\ell=0}^{k-1}\sum_{m=0}^{\ell}(\ell+1)t_{\ell+1,k}s_{m,\ell}x^m \\
&= \sum_{m=0}^{k-1}\sum_{\ell=m}^{k-1}(\ell+1)t_{\ell+1,k}s_{m,\ell}x^m \\
&= \sum_{m=0}^{k-1} x^m\left(\sum_{\ell=m}^{k-1}(\ell+1)t_{\ell+1,k}s_{m,\ell}\right).
\end{aligned}$$

By Corollary 8.1, $t_{\ell+1,k+1} = t_{\ell,k} + (\ell+1)t_{\ell,k}$, so

$$\begin{aligned}\Delta(x^k) &= \sum_{m=0}^{k-1} x^m \left(\sum_{\ell=m}^{k-1} (t_{\ell+1,k+1} - t_{\ell,k}) s_{m,\ell}\right)\\ &= \sum_{m=0}^{k-1} x^m \left(\sum_{\ell=m}^{k-1} t_{\ell+1,k+1} s_{m,\ell} - \sum_{\ell=m}^{k-1} t_{\ell,k} s_{m,\ell}\right).\end{aligned}$$

Note that since for $\ell < m$, $s_{m,\ell} = 0$ and $t_{k,k} = 1$,

$$\sum_{\ell=m}^{k-1} t_{\ell,k} s_{m,\ell} = -s_{m,k} + \sum_{\ell=0}^{k} t_{\ell,k} s_{m,\ell}.$$

Furthermore, since the matrices S and T are inverses, $\sum_{\ell=m}^{k-1} t_{\ell,k} s_{m,\ell} = 0$ if $m \neq k$. Since $m \leq k-1$, $m \neq k$, so

$$\Delta(x^k) = \sum_{m=0}^{k-1} x^m \left(-s_{m,k} + \sum_{\ell=m}^{k-1} t_{\ell+1,k+1} s_{m,\ell}\right).$$

$\square$

Proposition 10. *Let k be a nonnegative integer. Then*

$$\Delta^{-1}\left(x^k\right) = c'' + \sum_{\ell=1}^{k+1} \frac{t_{\ell-1,k}}{\ell} x^{\underline{\ell}} = c + \sum_{m=0}^{k+1} x^m \left(\sum_{\ell=m}^{k+1} \frac{t_{\ell-1,k} s_{m,\ell}}{\ell}\right)$$

where c is an arbitrary constant.

Proof. To compute $\Delta^{-1}(x^k)$, we convert to falling factorials and take the difference:

$$\begin{aligned}\Delta^{-1}(x^k) &= \Delta^{-1}\left(\sum_{\ell=0}^{k} t_{\ell,k} x^{\underline{\ell}}\right)\\ &= \sum_{\ell=0}^{k} t_{\ell,k} \Delta^{-1}\left(x^{\underline{\ell}}\right)\\ &= c'' + \sum_{\ell=0}^{k} t_{\ell,k}\left(\frac{1}{\ell+1} x^{\underline{\ell+1}}\right)\\ &= c'' + \sum_{\ell=1}^{k+1} \frac{t_{\ell-1,k}}{\ell} x^{\underline{\ell}}.\\ &= c' + \sum_{\ell=0}^{k+1} \frac{t_{\ell-1,k}}{\ell} x^{\underline{\ell}}.\end{aligned}$$

We then may return to normal powers:

$$\begin{aligned}
\Delta^{-1}(x^k) &= c' + \sum_{\ell=0}^{k+1} \frac{t_{\ell-1,k}}{\ell}\left(\sum_{m=0}^{\ell} s_{m,\ell} x^m\right) \\
&= c' + \sum_{\ell=0}^{k+1}\sum_{m=0}^{\ell} \frac{t_{\ell-1,k}s_{m,\ell}}{\ell} x^m \\
&= c' + \sum_{m=0}^{k+1}\sum_{\ell=m}^{k+1} \frac{t_{\ell-1,k}s_{m,\ell}}{\ell} x^m \\
&= c' + \sum_{m=0}^{k+1} x^m \left(\sum_{\ell=m}^{k+1} \frac{t_{\ell-1,k}s_{m,\ell}}{\ell}\right) \\
&= c + \sum_{m=0}^{k+1} x^m \left(\sum_{\ell=m}^{k+1} \frac{t_{\ell-1,k}s_{m,\ell}}{\ell}\right).
\end{aligned}$$

□

3.2.4 Polynomial Curve Fitting

Lemma 2. *For $m, k, n \in \mathbb{N}_0$ such that $n \geq m$,*

$$\sum_{\ell=0}^{n} (-1)^{m-\ell}\binom{m}{\ell}\ell^{\underline{k}} = \begin{cases} m! & m = k \\ 0 & m \neq k \end{cases}.$$

Proof. Note that $\binom{m}{\ell} = 0$ for $\ell > m$ and $\ell^{\underline{k}} = 0$ for $\ell < k$. Thus,

$$\begin{aligned}
\sum_{\ell=0}^{n} (-1)^{m-\ell}\binom{m}{\ell}\ell^{\underline{k}} &= \sum_{\ell=k}^{m} (-1)^{m-\ell}\binom{m}{\ell}\ell^{\underline{k}} \\
&= \sum_{\ell=0}^{m-k} (-1)^{m-k-\ell}\binom{m}{\ell+k}(\ell+k)^{\underline{k}} \\
&= \sum_{\ell=0}^{m-k} (-1)^{m-k-\ell}\frac{m!}{(\ell+k)!(m-\ell-k)!}\cdot\frac{(\ell+k)!}{\ell!} \\
&= \sum_{\ell=0}^{m-k} (-1)^{m-k-\ell}\frac{m^{\underline{k}}(m-k)!}{(m-\ell-k)!\ell!} \\
&= m^{\underline{k}}\sum_{\ell=0}^{m-k} (-1)^{m-k-\ell}\binom{m-k}{\ell}.
\end{aligned}$$

For $k = m$, the sum evaluates to $(-1)^{-0}\binom{0}{0} = 1$, yielding value $m^{\underline{k}} = m^{\underline{m}} = m!$ as desired. For $k \neq m$, by the binomial theorem,

$$\sum_{\ell=0}^{n} (-1)^{m-k-\ell}\binom{m}{\ell}\ell^{\underline{k}} = (1 + -1)^{m-k} = 0.$$

as desired. □

Proposition 11. *For a nonnegative integer n, let A be the $(n+1)\times(n+1)$ matrices with entries $(A)_{m,k} = m^{\underline{k}}$ for $0 \leq m, k, \leq n$. Let B be the $(n+1)\times(n+1)$ matrix with entries $(B)_{m,k} = \frac{(-1)^{m-k}}{k!(m-k)!}$ for $0 \leq k \leq m \leq n$ and 0 otherwise. Then A and B are inverses.*

Proof. Note that

$$
\begin{aligned}
(BA)_{m,k} &= \sum_{\ell=0}^{m} \frac{(-1)^{m-\ell}}{\ell!(m-\ell)!}\ell^{\underline{k}} \\
&= \sum_{\ell=0}^{m}(-1)^{m-\ell}\frac{1}{m!}\cdot\frac{m!}{\ell!(m-\ell)!}\ell^{\underline{k}} \\
&= \frac{1}{m!}\sum_{\ell=0}^{m}(-1)^{m-\ell}\binom{m}{\ell}\ell^{\underline{k}}.
\end{aligned}
$$

By Lemma 2, this evaluates to $\frac{1}{m!}(m!) = 1$ if $m = k$ and 0 otherwise. Thus, $BA = I_{n+1}$, so A and B are inverses. ☐

Proposition 12. *Let $f(x) = \sum_{m=0}^{n} c_m x^{\underline{m}}$. Then for $0 \le m \le n$,*

$$c_m = \sum_{k=0}^{m}\left(\frac{(-1)^{k-i}}{k!(m-k)!}f(k)\right).$$

Proof. Let row vectors $\mathbf{c}$ and $\mathbf{f}$ be defined as $\mathbf{c} = [c_0 \;\; c_1 \;\; c_2 \ldots c_n]$ and $\mathbf{f} = [f(0) \;\; f(1) \;\; f(2) \ldots f(n)]$. Let A and B be the matrices defined in Proposition 11. Note that for $0 \le k \le n$,

$$f(k) = \sum_{m=0}^{n} c_m k^{\underline{m}}.$$

Thus, $\mathbf{c}A^{\intercal} = \mathbf{f}$, so $\mathbf{c} = \mathbf{f}B^{\intercal}$, and so

$$c_m = \sum_{k=0}^{m}\frac{(-1)^{k-i}}{k!(m-k)!}f(k)$$

as desired. ☐

Remark 6. *Proposition 12 implies that, given a polynomial of degree n that passes through points (k, y_k) for $0 \le k \le n$, the polynomial can be made to pass through any $(n+1, y_{n+1})$ merely by adding a multiple of $x^{\underline{n+1}}$.*

Corollary 12.1. *For $n \in \mathbb{N}_0$, for $(a_i)_{0\le i\le n} \in \mathbb{R}$, there exists a polynomial of degree n such that $f(i) = a_i$ for $i \in \mathbb{Z}$, $0 \le i \le n$.*

3.3 Sum of Powers

Let $f_k(n) := \sum_{i=0}^{n-1} i^k$. Note that $\Delta(f_k(n)) = \sum_{i=0}^{n} i^k - \sum_{i=0}^{n-1} i^k = n^k$.

Theorem 1. *Let $f_k(n) := \sum_{i=0}^{n-1} i^k$. Then $f_k(n)$ is a polynomial in n.*

We examine three approaches to proving the theorem and characterizing $f_k(n)$.

3.3.1 Binomial Identities

Lemma 3. $\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$.

Proof. Consider a set of $n+1$ elements. Suppose we wish to choose a set of $k+1$ elements from this set. Notice that there are $\binom{n+1}{k+1}$ ways to do this. Let us now count this number in a different way. Let A be the set of $n+1$ elements, $A = \{a_1, a_2, \ldots, a_{n+1}\}$ and S be the set of $k+1$ elements, $S = \{s_1, s_2, \ldots, s_{k+1}\}$. Consider the following:

1. $a_1 \in A$. Notice that if $a_1 \in S$, then there are $\binom{n}{k}$ ways to choose the remaining k elements from n.

2. $a_1 \notin A$. Then, consider a_2. If $a_2 \in S$, then there are $\binom{n-1}{k}$ ways to choose the remaining k elements from $n-1$. If $a_2 \notin S$, then we must choose $k+1$ elements from $n-1$ (as both a_1, a_2 $\notin S$.) Now, suppose $a_2 \notin S$, and consider a_3. If $a_3 \in S$, then we must choose k elements from $n-2$, and there are $\binom{n-2}{k}$ such ways. If $a_3 \notin S$, then we must choose $k+1$ elements from $n-2$.

When continuing this process of considering whether $a_i \in S$ for $i \in \{1, 2, \ldots, k\}$, we obtain that there are a total of $\sum_{i=k}^{n} \binom{i}{k}$ such ways, and thus, the lemma is proven. $\square$

Theorem 2. $\sum_{i=1}^{k} s_{i,k} f_i(n+1) = \frac{(n+1)^{\underline{k+1}}}{k+1}$

Proof. From Lemma 3, we obtain $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$. Thus,

$$\begin{aligned}
\sum_{i=k}^{n} \frac{i(i-1)\cdots(i-k+1)}{k!} &= \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)!} \\
\sum_{i=k}^{n} i(i-1)\cdots(i-k+1) &= \frac{(n+1)n(n-1)\cdots(n-k+1)}{(k+1)} \\
&= \frac{(n+1)^{\underline{k+1}}}{(k+1)}.
\end{aligned}$$

When expanding the terms on the LHS and noticing that $i(i-1)\cdots(i-k+1) = 0$ when $i = 1, 2, \cdots (k-1)$, we obtain

$$\begin{aligned}
\sum_{i=1}^{n} (s_{k,k} i^k + s_{k-1,k} i^{k-1} + \cdots s_{1,k} i) &= \frac{(n+1)^{\underline{k+1}}}{(k+1)} \\
\implies s_{k,k} f_k(n+1) + s_{k-1,k} f_{k-1}(n+1) + \cdots s_{1,k} f_1(n+1) &= \frac{(n+1)^{\underline{k+1}}}{(k+1)},
\end{aligned}$$

and thus, $\sum_{i=1}^{k} s_{i,k} f_i(n+1) = \frac{(n+1)^{\underline{k+1}}}{k+1}$ and the theorem is proven. $\square$

Lemma 4. $\sum_{i=1}^{k} f_i(n+1)(s_{i,k} - \frac{k(n-k+1)}{k+1} s_{i,k-1}) = 0$, *given* $f_1(n+1) = \frac{n(n+1)}{2}$.

Proof. Notice that

$$\frac{(n+1)^{\underline{k+1}}}{(k+1)} = \frac{k(n-k+1)}{k+1} \frac{(n+1)^{\underline{k}}}{k},$$

and thus, from Theorem 2, since

$$\sum_{i=1}^{k} s_{i,k} f_i(n+1) = \frac{(n+1)^{\underline{k+1}}}{k+1}$$

and

$$\sum_{i=1}^{k-1} s_{i,k-1} f_i(n+1) = \frac{(n+1)^{\underline{k}}}{k},$$

we obtain

$$\sum_{i=1}^{k} s_{i,k} f_i(n+1) = \frac{k(n-k+1)}{k+1} \frac{(n+1)^{\underline{k}}}{k}$$

and

$$\frac{k(n-k+1)}{k+1} \sum_{i=1}^{k-1} s_{i,k-1} f_i(n+1) = \frac{k(n-k+1)}{k+1} \frac{(n+1)^{\underline{k}}}{k}.$$

When subtracting the two previous equations, we get the desired equation. Notice that when we subtracted the two previous equations, we obtained the difference between consecutive recurrences, but when doing so, we disregarded the base case of the recursion (or when $n = 1$). When computing $f_1(n+1)$ from Theorem 2, we obtain $f_1(n+1) = \frac{n(n+1)}{2}$, and thus the lemma is proven.

$\square$

Theorem 3. *Let* $a_{m,k} = s_{m,k} - \frac{k(n-k+1)}{k+1} s_{m,k-1}$*. Then*

$$f_k(n+1) = \frac{n(n+1)}{2}\left(-a_{1,k} + \sum_{1<i_1<k} a_{1,i_1}a_{i_1,k} - \sum_{1<i_1<i_2<k} a_{1,i_1}a_{i_1,i_2}a_{i_2,k} + \cdots + (-1)^{k+1}a_{1,2}a_{2,3}\cdots a_{k-1,k}\right).$$

for all $k > 1$ *and* $f_1(n+1) = \frac{n(n+1)}{2}$.

Proof. Consider $f_k(n+1)$ and suppose $a_{m,k} = s_{m,k} - \frac{k(n-k+1)}{k+1} s_{m,k-1}$. From Lemma 4, notice that since $a_{n,n} = s_{n,n} = 1$ for all $n \in \mathbb{N}$ (as shown in Corollary 6.1),

$$f_k(n+1) + f_{k-1}(n+1)a_{k-1,k} + \cdots + f_1(n+1)a_{1,k} = 0 \text{ [1]}.$$

Similarly, we obtain

$$f_{k-1}(n+1) + f_{k-2}(n+1)a_{k-2,k-1} \cdots + f_1(n+1)a_{1,k-1} = 0 \text{ [2]}$$

and so on, until $f_1(n+1) = \frac{n(n+1)}{2}$ (as proved in Lemma 4). To obtain a closed form for $f_k(n+1)$, notice that we can solve this system of equations. By multiplying [2] by $a_{k-1,k}$ and subtracting from [1], we can solve for $f_k(n)$ in terms of $f_{k-2}(n), f_{k-3}(n), \ldots f_1(n)$. When continuing this process, we obtain the above theorem. ☐

Proposition 13. $f_k(n+1)$ *is a polynomial in* n.

Proof. Notice that $a_{m,k}$ is a polynomial in n for all $m, k \in \mathbb{N}$ (as $s_{m,k}, k$ are constants and $a_{m,k} = s_{m,k} - \frac{k(n-k+1)}{k+1} s_{m,k-1}$). Thus, from Theorem 3, since $f_k(n+1)$ is the product of polynomials in n, $f_k(n+1)$ is also a polynomial in n and the proposition is proven. ☐

3.3.2 Differences of $f_k(n)$

Notice that

$$f_k(n) = \sum_{i=0}^{n-1} i^k,$$

where i^k can be written as $i \cdot i^{k-1}$. This means that $f_k(n+1)$ can be visualized in the following way:

$$\begin{aligned}
&1^{k-1}+\\
&2^{k-1} + 2^{k+1}+\\
&3^{k-1} + 3^{k-1} + 3^{k-1}+\\
&\vdots\\
&n^{k-1} + n^{k-1} + n^{k-1} + \ldots + n^{k-1}.
\end{aligned}$$

Note that the n elements in the first column sum to $1^{k-1} + 2^{k-1} + \ldots + n^{k-1} = f_{k-1}(n+1)$. Similarly, there are $n-1$ elements in the second column that sum to $2^{k-1} + 3^{k-1} + \ldots + n^{k-1} = f_{k-1}(n+1) - 1^{k-1}$. Continuing, we can observe that the sum of each column in order follows the following pattern:

$$\begin{aligned}
&f_{k-1}(n+1)+\\
&f_{k-1}(n+1) - 1^{k-1}+\\
&f_{k-1}(n+1) - 1^{k-1} - 2^{k-1}+\\
&\vdots\\
&f_{k-1}(n+1) - 1^{k-1} - 2^{k-1} - \ldots - (n-1)^{k-1},
\end{aligned}$$

which can be expressed as

$$f_k(n+1) = \sum_{i=1}^{n} (f_{k-1}(n+1) - f_{k-1}(i)) = nf_{k-1}(n+1) - \sum_{i=1}^{n} f_{k-1}(i).$$

This provides a recurrence formula for $f_k(x)$ based on k.

Proposition 14. *$f_k(n)$ is a polynomial expression.*

Proof. Note that $f_1(n) = \frac{n(n+1)}{2}$ is a polynomial. If $f_k(n)$ is a polynomial, $f_{k+1}(n)$ is the repeated addition and subtraction of $f_k(n)$ and is therefore a polynomial as well. Then, by induction, $f_k(n)$ is a polynomial in n for all k.

□

3.3.3 Falling Factorials Approach

Since $\Delta(f_k)(n) = n^k$ and $f_k(0) = 0$, by Proposition 10,

$$f_k(n) = \sum_{\ell=1}^{k+1} \frac{t_{\ell-1,k}}{\ell} \cdot n^{\underline{\ell}} = c + \sum_{m=0}^{k+1} n^m \left(\sum_{\ell=m}^{k+1} \frac{t_{\ell-1,k} s_{m,\ell}}{\ell} \right)$$

. Clearly, $f_k(n)$ is a polynomial in n.

3.3.4 Corollaries

Corollary 1.1. *The antidifference of a polynomial is a polynomial.*

Proof. Since n^k forms a basis of the vector space of polynomials of degree and, by Theorem 1, $\Delta^{-1}(n^k) = f_k(n)$ is a polynomial of n, the antidifference of any polynomial of n is therefore a linear combination of polynomials and thus is a polynomial. □

3.4 Analogies to Derivatives

Let g and h be functions. Then, for $f(x) = g(x)h(x)$,

$$f'(x) = g'(x)h(x) + h'(x)g(x).$$

Similarly, for $f(x) = \frac{g(x)}{h(x)}$,

$$f'(x) = \frac{g'(x)h(x) - h'(x)g(x)}{h(x)^2}.$$

Proposition 15 (Product Rule)**.** *Let g and h be polynomial functions. Then, for $f(x) = g(x)h(x)$, the analogous product formula for Δ is*

$$\Delta(f)(x) = \Delta(g)(x)\Delta(h)(x) + \Delta(g)(x)h(x) + \Delta(h)(x)g(x).$$

Proof. Since $f(x) = g(x)h(x)$, we have $f(x+1) = g(x+1)h(x+1)$, and so

$$\begin{aligned}
\Delta(f)(x) &= g(x+1)h(x+1) - g(x)h(x) \\
&= (g(x+1) - g(x))(h(x+1) - h(x)) + (g(x+1) - g(x))h(x) + (h(x+1) - h(x))g(x) \\
&= \Delta(g)(x)\Delta(h)(x) + \Delta(g)(x)h(x) + \Delta(h)(x)g(x).
\end{aligned}$$

□

Proposition 16 (Quotient Rule)**.** *For $f(x) = \frac{g(x)}{h(x)}$, the analogous quotient formula is*

$$\frac{\Delta(g)(x)h(x) - \Delta(h)(x)g(x)}{\Delta(h)(x)h(x) + (h(x))^2}.$$

Proof. $f(x) = \frac{g(x)}{h(x)}$, so $f(x+1) = \frac{g(x+1)}{h(x+1)}$. Hence,

$$\begin{aligned}\Delta(f)(x) &= \frac{g(x+1)}{h(x+1)} - \frac{g(x)}{h(x)} \\ &= \frac{g(x+1)h(x) - h(x+1)g(x)}{h(x+1)h(x)} \\ &= \frac{(g(x+1)-g(x))h(x) - (h(x+1)-h(x))g(x)}{(h(x+1)-h(x))h(x) + h(x)^2} \\ &= \frac{\Delta(g)(x)h(x) - \Delta(h)(x)g(x)}{\Delta(h)(x)h(x) + h(x)^2}.\end{aligned}$$

□

Remark 7. *We notice similarities in structure between derivative formulas and those of the difference operator; particularly, the $g'(x)h(x) \pm h'(x)g(x)$ structure in the derivative product and quotient rules, respectively, are translated across to the difference formulas.*

4 Further Questions

4.1 Concerning Differences and Antidifferences

1. How can we solve difference equations? Can we find analogies connecting differential equations and difference equations?
2. Can we find analogies connecting integrals and antiderivatives? Integrals can be described as calculating areas of curves. What do antiderivatives describe?
3. How can we extend the concept of finite differences to multivariable functions? For instance, double and triple integrals can be used to calculate volume in 3D space; do there exist analogs for antidifferences with similar uses?
4. How and for what uses can differences and antidifferences be applied to Taylor series?

4.2 Concerning $s_{m,k}$ and $t_{m,k}$

1. We found a simple combinatorial interpretation for $t_{m,k}$. Does there exist a combinatorial interpretation of its counterpart, $s_{m,k}$?
2. Are there closed formulas for $s_{m,k}$ and $t_{m,k}$? What do their generating functions reveal about their structure and asymptotics?
3. What further (combinatorial?) properties do $s_{m,k}$ and $t_{m,k}$ exhibit?
4. What further (combinatorial?) relations exist between $s_{m,k}$ and $t_{m,k}$?

4.3 Concerning $f_k(n)$

1. Can we solve the binomial coefficient recurrence of $f_k(n)$ using finite differences?
2. Can we formulate a similar recurrence of a q-analog of $f_k(n)$ using q-binomial coefficients?

5 Appendix

5.1 Select Values for $s_{m,k}$ and $t_{m,k}$

k \ m	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	-1	1					
3	0	2	-3	1				
4	0	-6	11	-6	1			
5	0	24	-50	35	-10	1		
6	0	-120	274	-225	85	-15	1	
7	0	720	-1764	1624	-735	175	-21	1

Table 1: Table of values of $s_{m,k}$ for $0 \leq k, m \leq 7$. Values for $m > k$ are zero and excluded.

k \ m	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	1	3	1				
4	0	1	7	6	1			
5	0	1	15	25	10	1		
6	0	1	31	90	65	15	1	
7	0	1	63	301	350	140	21	1

Table 2: Table of values of $t_{k,m}$ for $0 \leq k, m \leq 7$. Values for $m > k$ are zero and excluded.

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