ECE747:Nanophotonics
University of Wisconsin - Madison
Fall 2020

Problem Set 3

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The MATLAB scripts for all relevant questions are appended at the end of the document.

Problem 1:
Brewster's angle is a special condition which occurs when the dipoles of the second medium (index $n2$) are configured such that they're parallel to the (expected) reflected ray. In this situation, the will not emit any reflected ray, and instead emit a refracted ray exactly perpendicular to the expected reflected ray.
Ref: http://physics.bu.edu/ duffy/semester2/c27_brewster.html

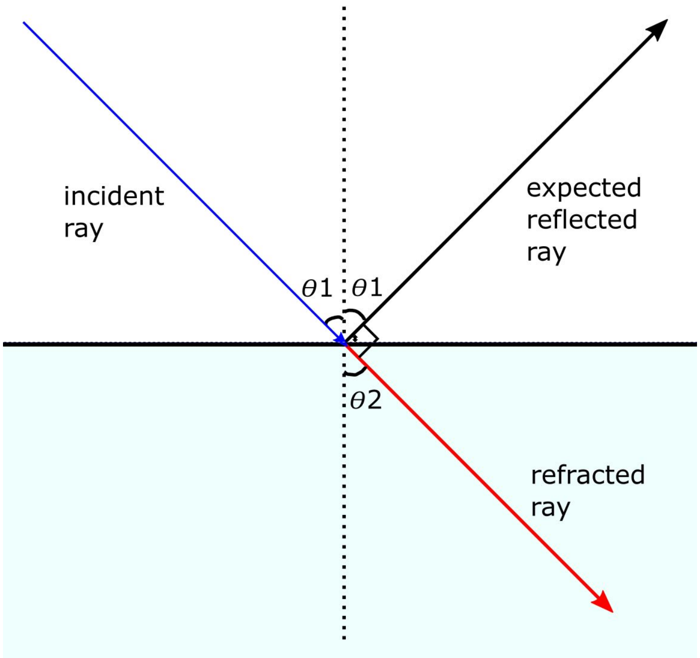

Figure 1: Diagram illustrating the condition for Brewster's angle.

Using the geometrical description in Figure 1, we can solve for the Brewster's angle.

$$\theta_1 + 90^o + \theta_2 = 180^o$$
$$\theta_1 + \theta_2 = 90^o$$

Using this expression in Snell's Law:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$
$$n_1 \sin(\theta_1) = n_2 \sin(90 - \theta_1)$$
$$n_1 \sin(\theta_1) = n_2 \cos(\theta_1)$$
$$\tan(\theta_1) = \frac{n_2}{n_1}$$
$$In\ this\ condition, \theta_1 = \theta_B$$
$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

This is the expression for the Brewster's angle ($= \theta_B$)

Another method of obtaining the expression is by using the Fresnel's equations with Snell's Law:
For P-polarisation:

$$r_p = \frac{n_1 \cos\theta_2 - n_2 \cos\theta_1}{n_1 \cos\theta_2 + n_2 \cos\theta_1}$$

For the condition when there's no reflected beam, $r_p = 0$. Hence, we get:

$$n_1 \cos\theta_2 = n_2 \cos\theta_1$$
$$n_1^2 \cos^2\theta_2 = n_2^2 \cos^2\theta_1$$
$$n_1^2(1 - \sin^2\theta_2) = n_2^2(1 - \sin^2\theta_1)$$
$$Snell's\ Law: \ \sin\theta_2 = \frac{n_1 \sin\theta_1}{n_2}$$
$$n_1^2\left(1 - \frac{n_1^2 \sin^2\theta_1}{n_2^2}\right) = n_2^2(1 - \sin^2\theta_1)$$
$$n_1^2 n_2^2 - n_1^4 \sin^2\theta_1 = n_2^4 - n_2^4 \sin^2\theta_1$$
$$\sin^2\theta_1 = \frac{n_2^2(n_2^2 - n_1^2)}{n_2^4 - n_1^4}$$
$$\sin^2\theta_1 = \frac{n_2^2}{n_2^2 + n_1^2} \ \ \&$$
$$1 - \cos^2\theta_1 = \frac{n_2^2}{n_2^2 + n_1^2}$$
$$\cos^2\theta_1 = \frac{n_1^2}{n_2^2 + n_1^2}$$

At Brewster's angle, $\theta_1 = \theta_B$. Using the expressions for sin and cos, we derive:

$$\frac{\sin^2\theta_B}{\cos^2\theta_B} = \frac{n_2^2}{n_1^2}$$
$$\tan^2\theta_B = \frac{n_2^2}{n_1^2}$$
$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right)$$

Problem 2:

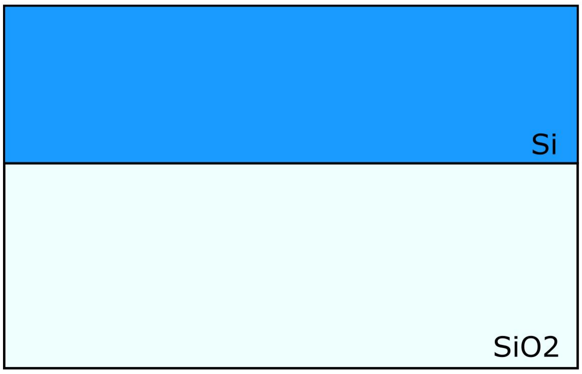

Figure 2: We start with a simple SOI wafer.

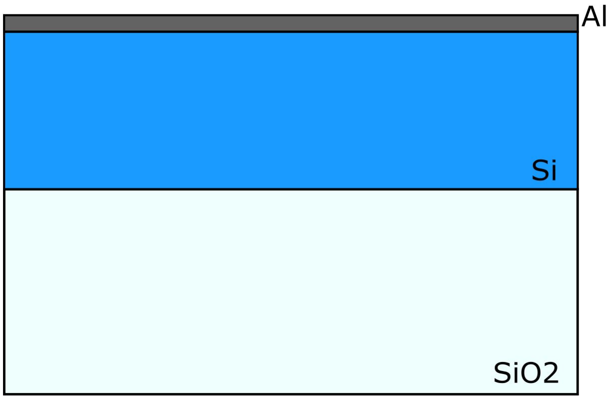

Figure 3: Metal Deposition (we use Al here)

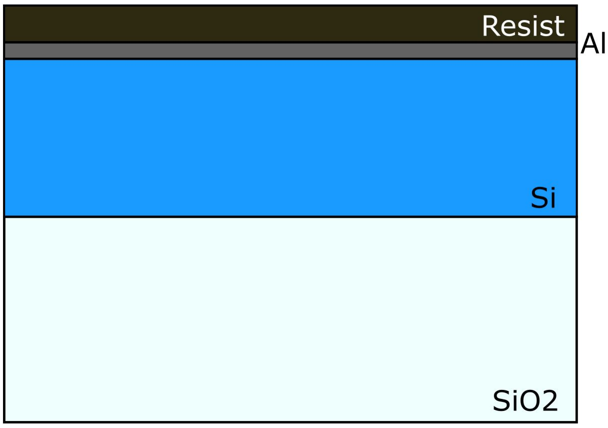

Figure 4: Photoresist Application

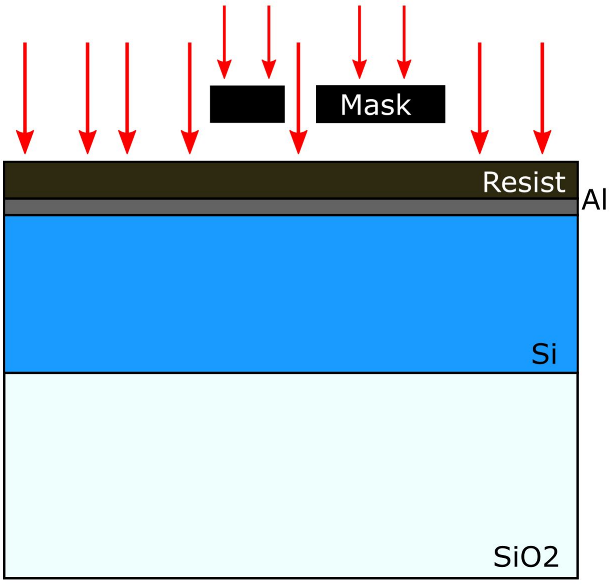

Figure 5: Exposure through Mask

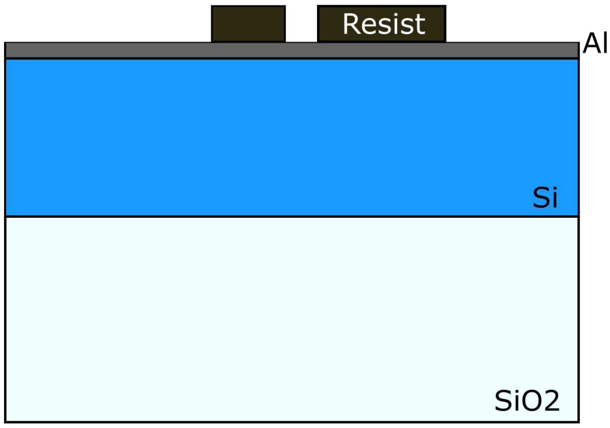

Figure 6: Development of the Resist using a wet chemical process

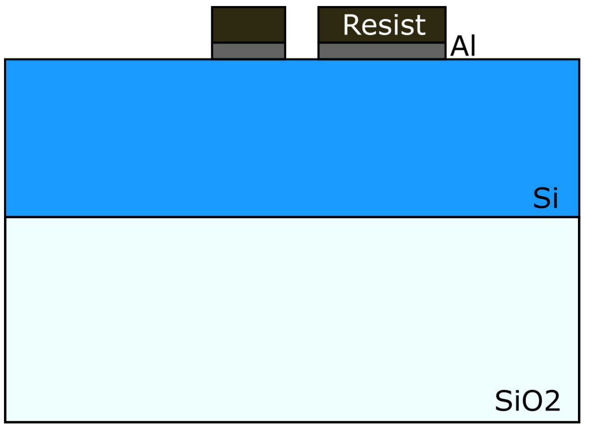

Figure 7: Metal Etch

Figure 8: Oxygen Plasma: Remove the unexposed resist

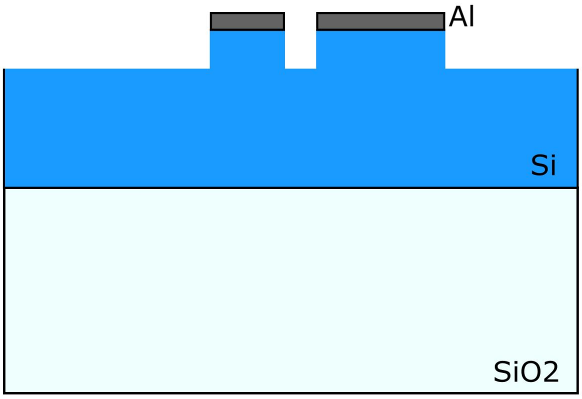

Figure 9: Reactive Ion Etching: Burns away unexposed Si

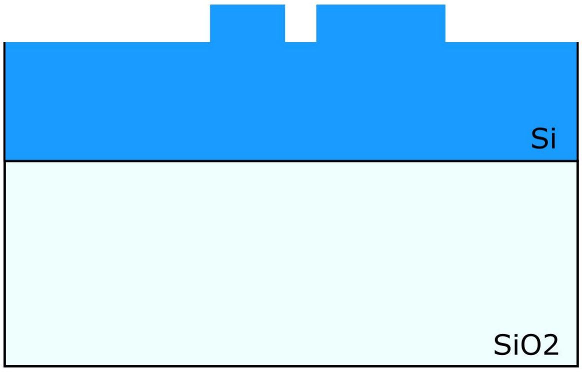

Figure 10: Metal Etch to remove away the remaining Aluminum

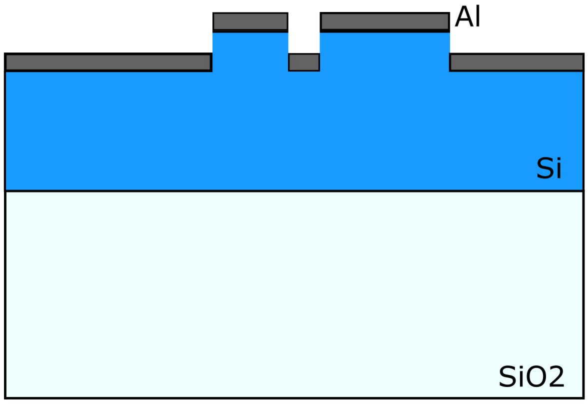

Figure 11: Metal Deposition. We again deposit Al on the Si

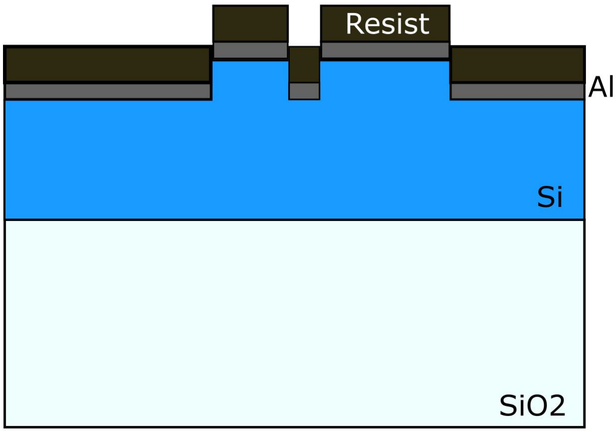

Figure 12: Photoresist Application for the second time

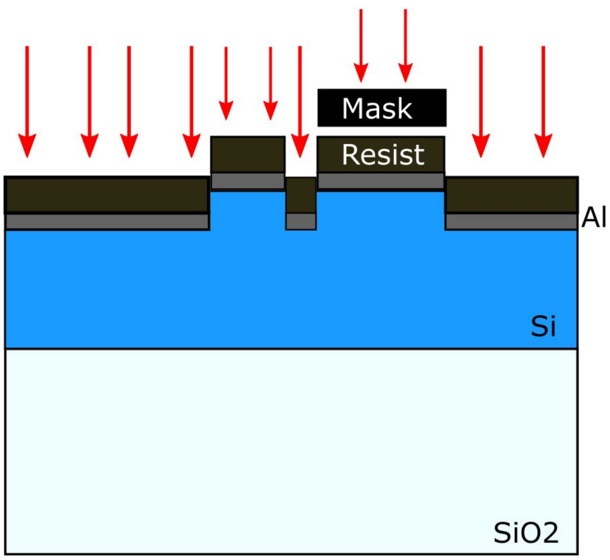

Figure 13: Exposure through Mask

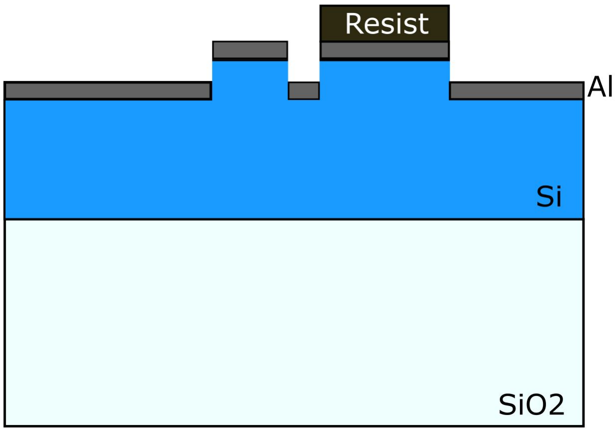

Figure 14: Development of the Resist using a wet chemical process

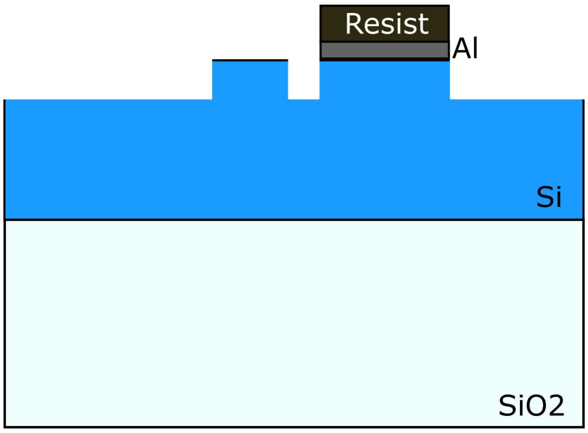

Figure 15: Metal Etch

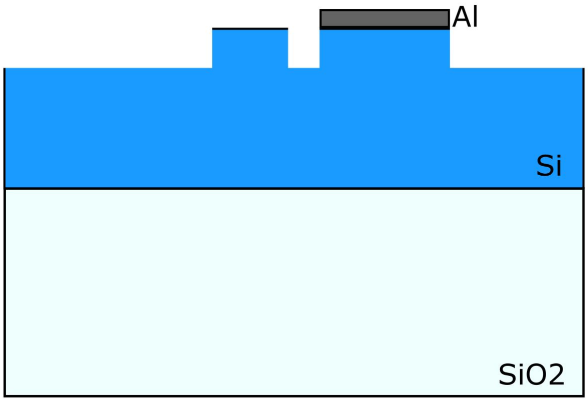

Figure 16: Oxygen Plasma: Remove the unexposed resist

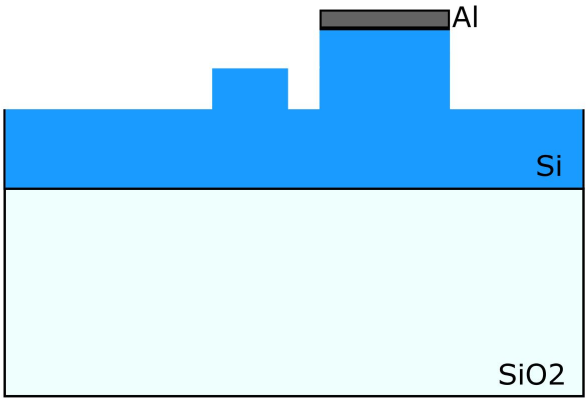

Figure 17: Reactive Ion Etching: Burns away unexposed Si

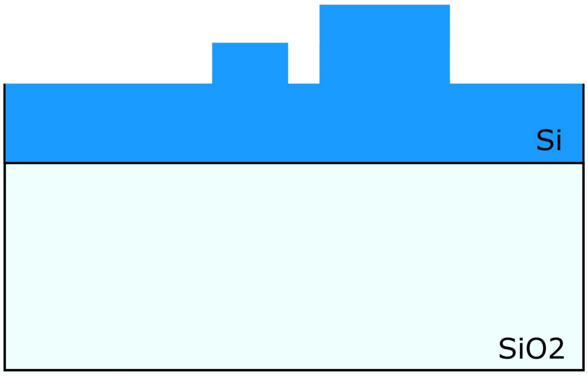

Figure 18: Metal Etch to remove away the remaining Aluminum

Problem 3:

a. The effective index of the given waveguide is $\eta_{eff} = 1.5766$. I observed two degenerated modes exhibitng the same effective index.

Mode list

mode #	effective index	wavelength (µm)	loss (dB/cm)	group index	TE polarization fraction (Ex)	waveguide TE/TM fraction (%)
1	1.576676	1.55	0.0000	N/A	92	58.73 / 83.3
2	1.576676	1.55	0.0000	N/A	10	58.73 / 83.3

Figure 19: Degenerate modes exhibiting the same effective index

b. The E-Field intensities in the two degenerate modes can be shown as:

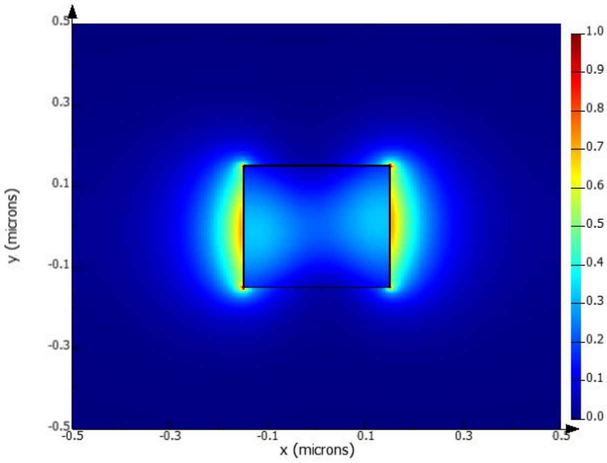

Figure 20: Degenerate Mode 1

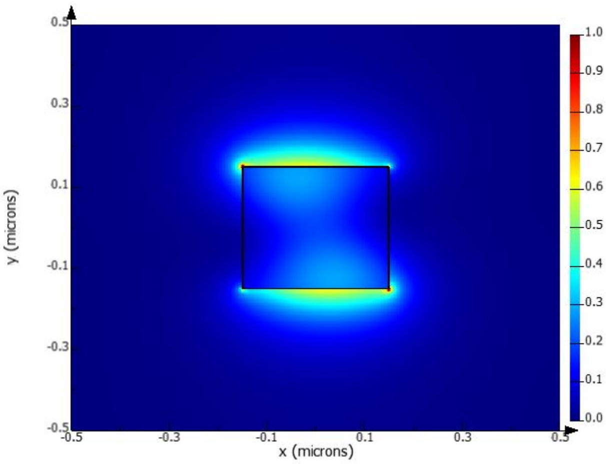

Figure 21: Degenerate Mode 2

c. The dispersion relation plotted using 10 data points.

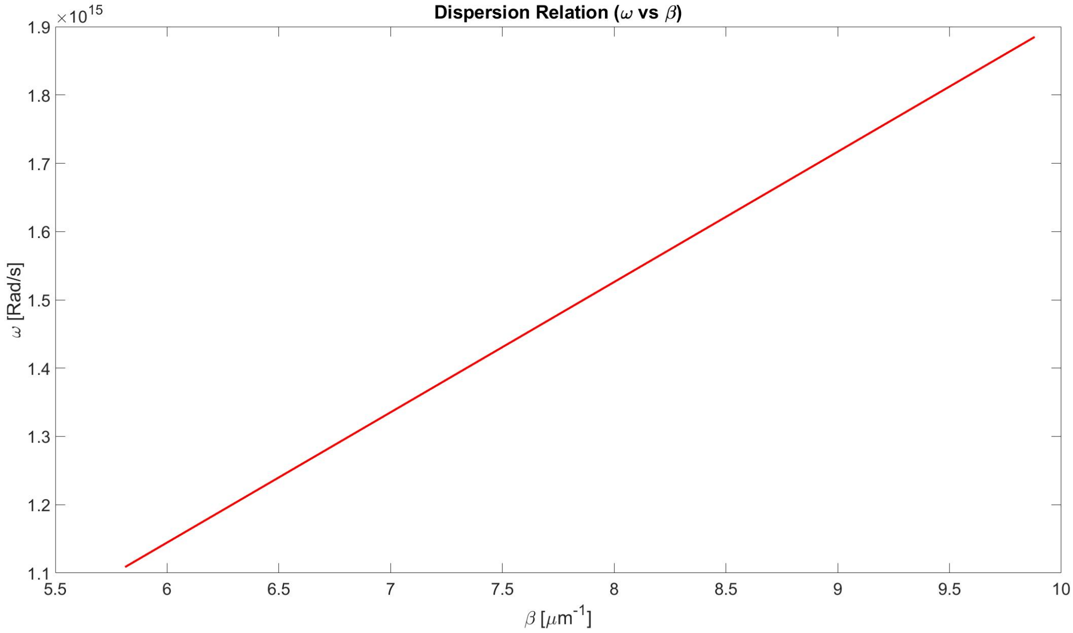

Figure 22: Dispersion relation of the given waveguide

d. Group velocity vs angular frequency relation in the given waveguide.

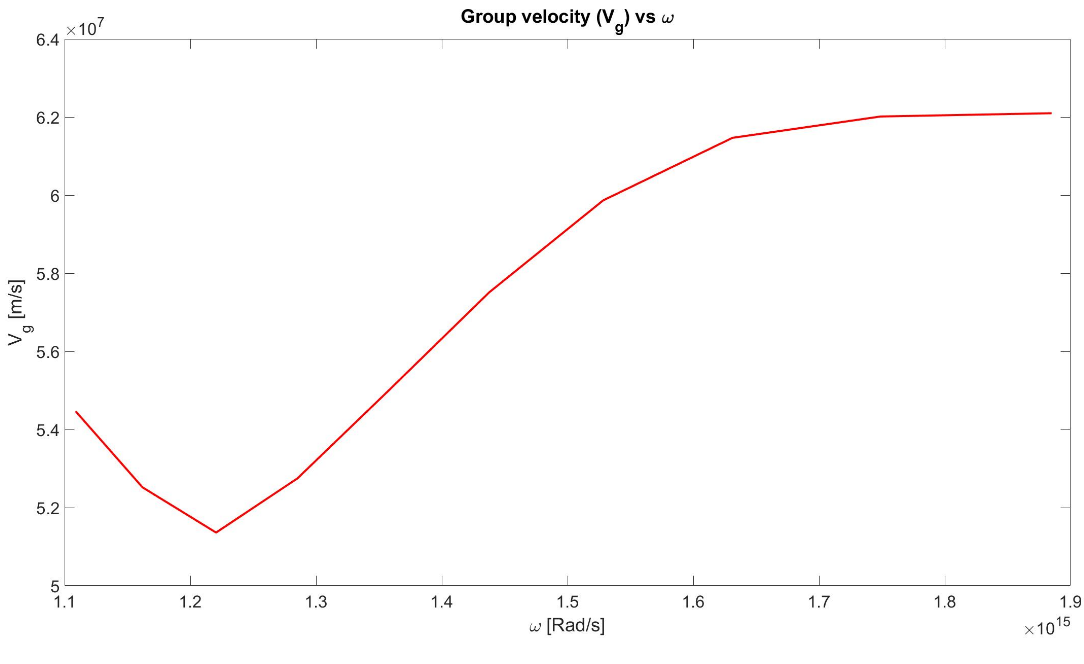

Figure 23: Group velocity vs angular frequency

e. The E-field intensity pattern for the two fundamental modes can be visualised as:

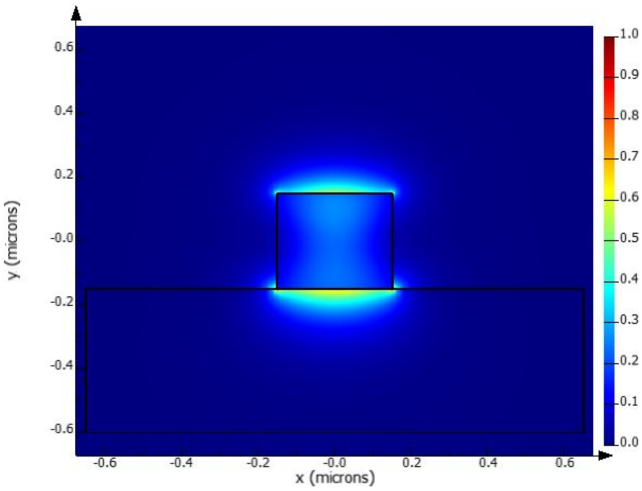

Figure 24: E-Field intensity for GaAs on SiO2; Fundamental mode 1.

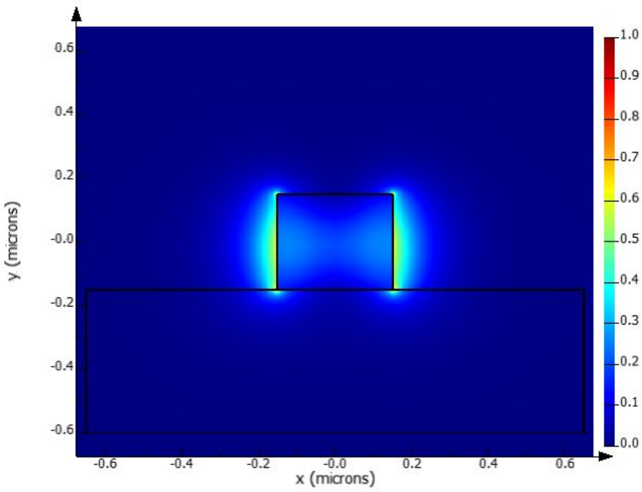

Figure 25: E-Field intensity for GaAs on SiO2; Fundamental mode 2.

The effective index in this case increases to $n_{eff} = 1.7816$ and $n_{eff} = 1.6754$. The higher refractive index can be attributed to the addition of SiO2 at the base of the waveguide. Previously, most of the higher intensity was moving through air (n=1), but is now moving through SiO2 (n=3.37), and therefore the effective index increases.

Problem 4:

a. We can start by assuming a mode propagating, along the z-direction, in the waveguide (1-D confinement) such that the total power across its wavefront (along y-direction) carries unit power.
Let us assume the wave is sinusoidal in nature. Hence, we can write the expression for its wavefront as:

$$U(y) \ = \ A \ \sin\left(\frac{ky}{d}\right)$$

where, d is the separation between the plates and k and A are constants which can be determined using boundary conditions.
BC 1: Symmetric about the midpoint
If $U(y=0)=0$, then $U(y=d)=0$. Therefore:

$$A\sin(k)=0$$
$$\mathbf{k}=\mathbf{m}\pi \quad m\in 0,1,2,3,4...$$

We have $U(y) \ = \ A \ \sin(\frac{m\pi y}{d})$
BC 2: Unit power over the wavefront

$$\int_0^d A^2 \ \sin^2\left(\frac{m\pi y}{d}\right) dy = 1$$
$$A \ = \ \sqrt{\frac{2}{d}}$$

Hence, the equation for the wavefront is given as :

$$U(y) \ = \ \sqrt{\frac{2}{d}} \ \sin\left(\frac{m\pi y}{d}\right)$$

We'll use the above expression in the Helmholtz Equation in 1-D confinement, given as:

$$\left(\frac{d^2}{dy^2}-\beta^2+\frac{\omega^2 n^2}{c^2}\right)U(y)=0$$
$$\frac{d^2}{dy^2}\left[\sin\left(\frac{m\pi y}{d}\right)\right]=\sin\left(\frac{m\pi y}{d}\right)\left[\beta^2-\frac{\omega^2 n^2}{c^2}\right]$$
$$-\left(\frac{m\pi}{d}\right)^2=\beta^2-\frac{\omega^2 n^2}{c^2}$$
$$\beta=\sqrt{\frac{\omega^2 n^2}{c^2}-\left(\frac{m\pi}{d}\right)^2}$$

b. For a wave propagating in the +z-direction, the field profile can be given as:

$$\mathbf{E}(y,\omega,t)=\sqrt{\frac{2}{d}} \ \sin\left(\frac{m\pi y}{d}\right) \ e^{\iota(\beta z-\omega t)}\hat{x}$$

c. For the given free-space wavelength, we can define the angular frequency as $\omega = \frac{2\pi c}{\lambda}$. Using this and the given values for 'd' and 'n', we can calculate the value of the propagation constant in the waveguide for m=1, $\beta = 1.2468 \times 10^7 + \iota\, 6332.9$.

The intensity can approximately be given as $I \approx e^{-2\ Img[\beta]\ z}$. We use this expression to calculate the ratio of intensities at z=0 and z=1cm:

$$\frac{I(@z = 1cm)}{I(@z = 0cm)} = exp(-2\ \times Img[1.2468 \times 10^7 + \iota\, 6332.9]\ \times 0.01)$$

$$\frac{I(@z = 1cm)}{I(@z = 0cm)} = exp(-2\ \times 6332.9\ \times 0.01)$$

$$\frac{I(@z = 1cm)}{I(@z = 0cm)} = 9.8518e - 56$$

Problem 5:

a. The propagation constants for the right-moving wave (β_R) and left-moving wave (β_L) are equal and have opposite sign. Let's assume the right-moving wave is positive, and left-moving wave has a negative sign. The wavenumber of grating therefore must be able to convert β_R to β_L.

$$
\begin{aligned}
k_g + \beta_R &= \beta_L \\
k_g + \frac{2\pi n_{eff}}{\lambda_o} &= -\frac{2\pi n_{eff}}{\lambda_o} \\
k_g &= -2 \times \frac{2\pi n_{eff}}{\lambda_o} \\
\frac{2\pi}{\Lambda} &= -2 \times \frac{2\pi n_{eff}}{\lambda_o}
\end{aligned}
$$

Hence, the grating wavelength (taking the abs value) can be expressed as

$$\Lambda = \frac{\lambda_o}{2n_{eff}}$$

b. For the given waveguide, I observed the following modes, using Lumerical FDTD.

Mode list

mode #	effective index	wavelength (μm)	loss (dB/cm)	group index	TE polarization fraction (Ey)	waveguide TE/TM fraction (%)
1	2.755966	1.55	0.0000	N/A	0	100 / 77.26
2	1.715762	1.55	0.0000	N/A	100	66.35 / 100
3	1.308407	1.55	0.0000	N/A	0	100 / 80.09
4	1.268683	1.55	0.0000	N/A	100	81.13 / 100
5	1.042050	1.55	0.0000	N/A	100	89.91 / 100
6	0.8494239	1.55	0.0000	N/A	0	100 / 31.12
7	0.7946841	1.55	0.0000	N/A	100	52.83 / 100

Figure 26: I observed 7 lossless modes for this waveguide.

The fundamental mode for $\lambda_o = 1.55\mu m$ occurs at an effective index of $\mathbf{n_{eff}} = \mathbf{2.755966}$, and can be visualised as:

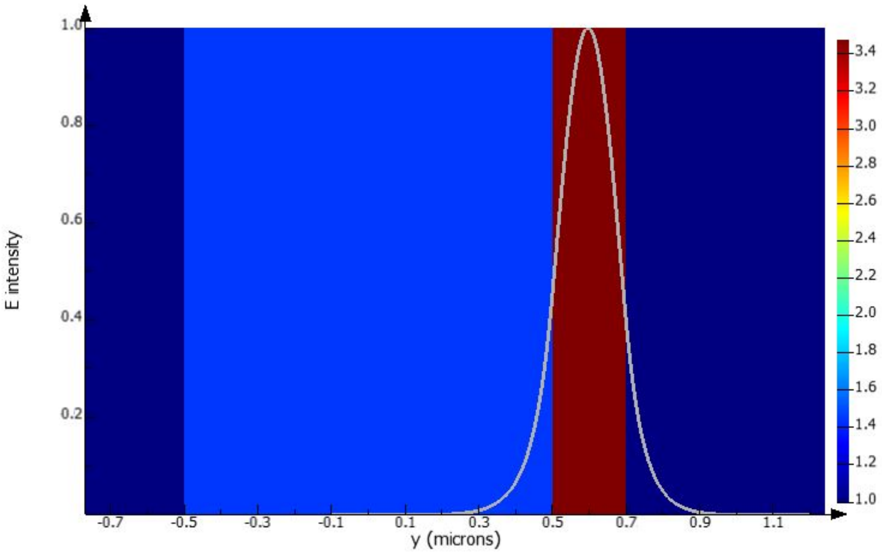

Figure 27: Visualization of the fundamental mode for the given waveguide.

c. The given configuration was set up as follows:

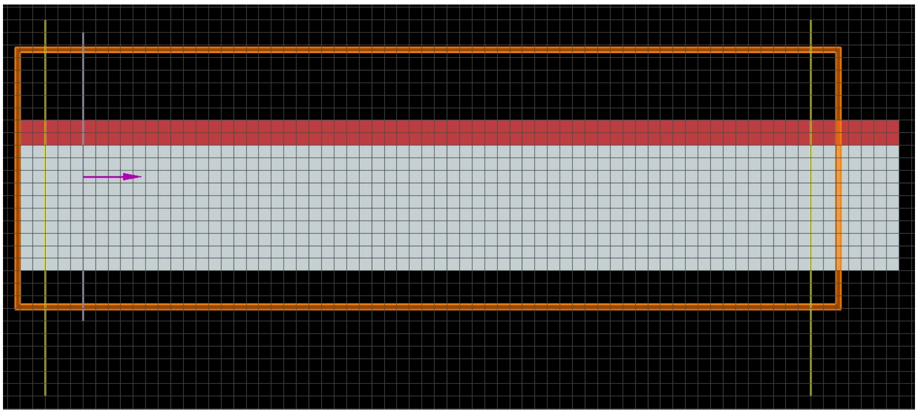

Figure 28: XY view of the configuration.

The reflection is seen using the monitor on the left hand side of the souce and the transmission is seen using the monitor on the right hand side. The following plots were obtained for the free-space wavelength range $\lambda_o = 1.3\mu m$ to $\lambda_o = 1.7\mu m$

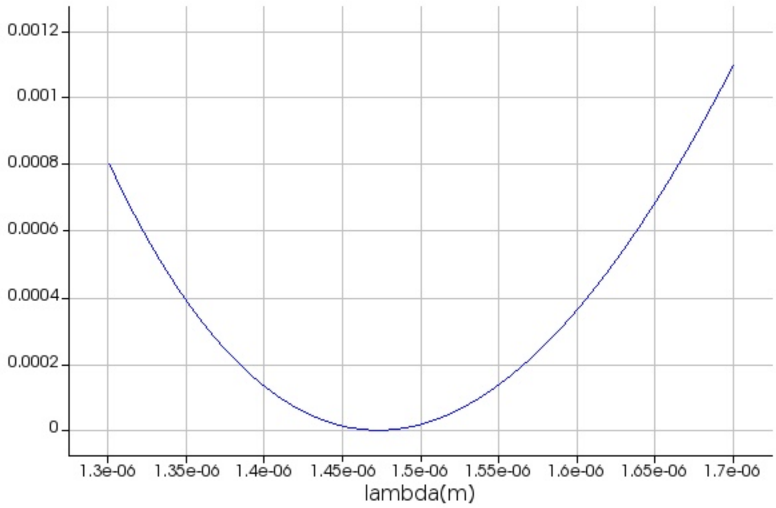

Figure 29: Reflection (abs value) as a function of wavelength

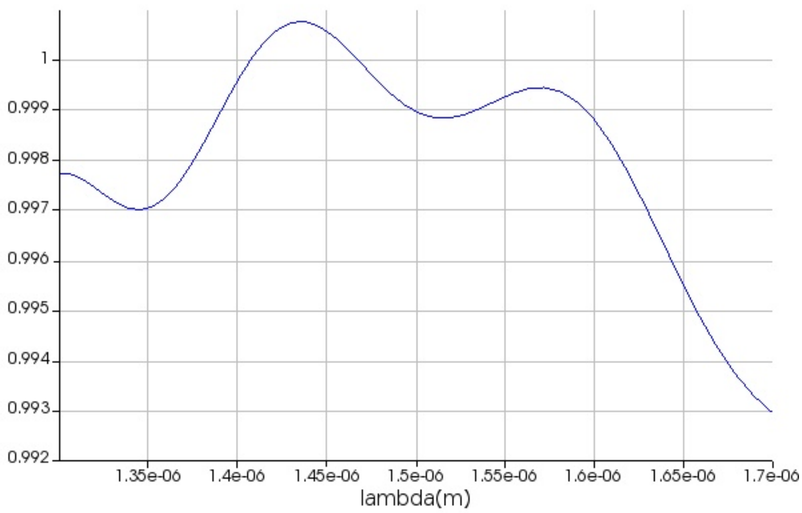

Figure 30: Transmission as a function of wavelength

d. The grating wavelength is calculated using the formula derived in part (a) for the effective index determined in part (b), and has the value $\lambda_g = 0.2812\,\mu m$. The waveguide with this grating is illustrated in the following figure:

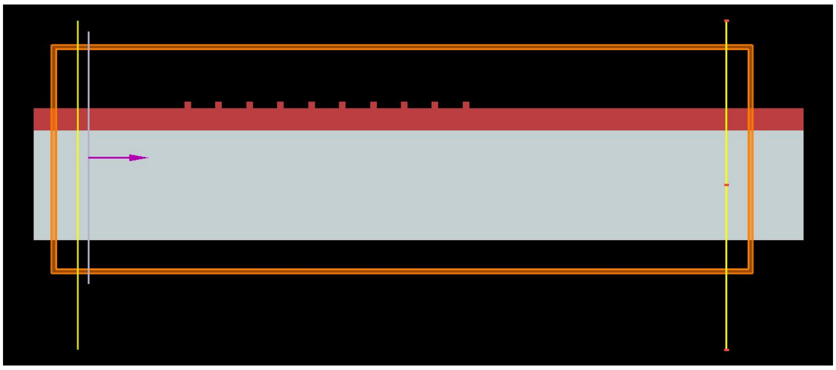

Figure 31: XY view of the waveguide the grating

The reflection and transmission plots vs the free-space wavelength are found to be as:

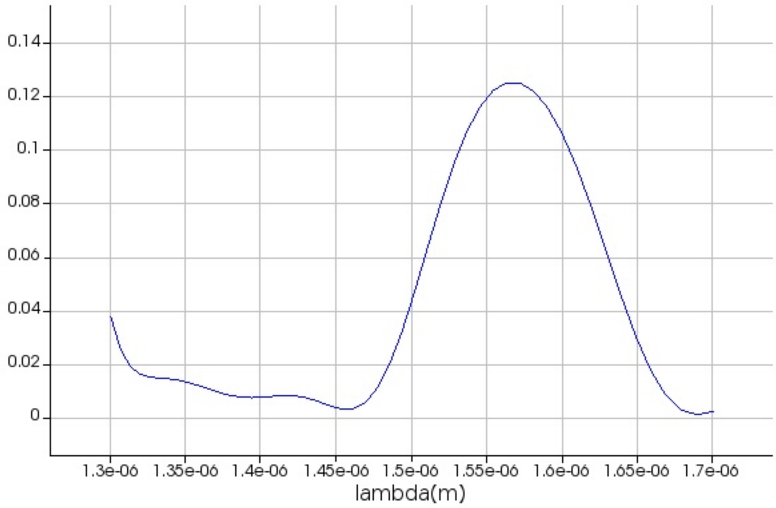

Figure 32: Reflection (abs value) as a function of wavelength

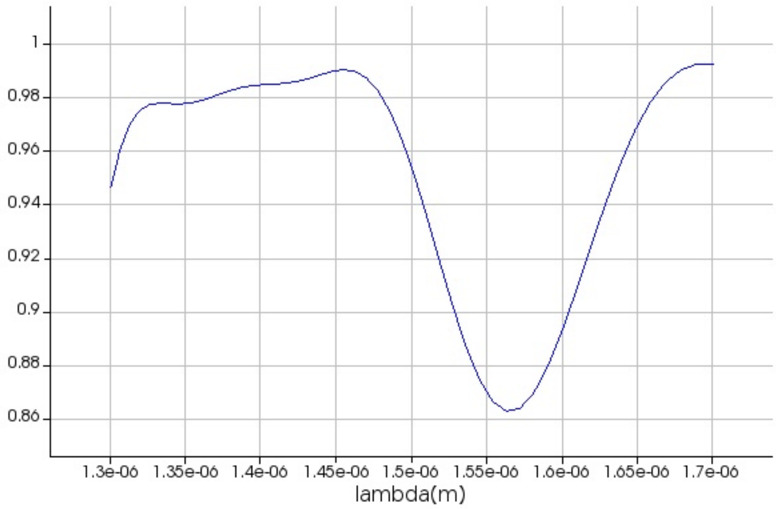

Figure 33: Transmission as a function of wavelength

We can observe that the reflection increases (and transmission decreases) near the wavelength of the fundamental mode. This reinforces our calculation in part (a), for using the grating to couple the right-moving wave to the left-moving wave in the same medium.

MATLAB Scripts

Problem 3:

```
%MATLAB code for ECE747 Q3 (c,d)

%defining all the parameters
c=3e+08;%speed of light
wave_range=linspace(1e-06,1.7e-06,10); %free-space wavelength range
n_eff_range=zeros(1,10); %empty array of effective indices corresponding to each free-
%Run Lumerical mode solver for each of the wavelength in wave_range, to
%obtain the corresponding n_eff
n_eff_range(1)=2.837678; %corresponding to wave_range(1)=1e-06
n_eff_range(2)=2.682639; %corresponding to wave_range(2)=1.078e-06
n_eff_range(3)=2.526582; %corresponding to wave_range(3)=1.156e-06
n_eff_range(4)=2.365418; %corresponding to wave_range(4)=1.233e-06
n_eff_range(5)=2.192126; %corresponding to wave_range(5)=1.311e-06
n_eff_range(6)=2.005841; %corresponding to wave_range(6)=1.389e-06
n_eff_range(7)=1.804829; %corresponding to wave_range(7)=1.467e-06
n_eff_range(8)=1.593506; %corresponding to wave_range(8)=1.544e-06
n_eff_range(9)=1.376746; %corresponding to wave_range(9)=1.622e-06
n_eff_range(10)=1.178673; %corresponding to wave_range(10)=1.7e-06

w_range=2*pi*c./wave_range; %corresponding angular frequency range
beta_range=2*pi.*n_eff_range./wave_range; %corresponding wave number range

%making the plot of the dispersion relation
%plot(beta_range*10^(-6),w_range,'Color','r','Linewidth',3)
%title('Dispersion Relation (\omega vs \beta)')
%xlabel('\beta [\mum^{-1}]')
%ylabel('\omega [Rad/s]')
%ax = gca;
%ax.FontSize = 25;

v_g=real(gradient(w_range)./gradient(beta_range)); %group velocity

%making the plot of the group velocity vs angular frequency
plot(w_range, v_g,'Color','r','Linewidth',3)
title('Group velocity (V_g) vs \omega')
xlabel('\omega [Rad/s]')
ylabel('V_g [m/s]')
ax = gca;
ax.FontSize = 25;
```

Problem 4:

```
%MATLAB code for ECE747 Q4

%defining all the parameters
c=3e+08;%speed of light
d=2e-06; %separation between the waveguide plates in meters
wl=1e-06; %free space wavelength
z=0.01; %propagation length
n=2+0.001i; %complex refractive index of the medium in the waveguide

omega=2*pi*c/wl; %angular frequency at given wavelength

beta = sqrt(omega^2*n^2/c^2-pi^2/d^2) %propagation constant inside the waveguide
I_ratio = exp(-2*imag(beta)*z) %ratio of intensity at z vs at source
```