ECE 747 – Nanophotonics
Problem Set 1
Due: September 28, 2020, 11:59pm; submitted via Canvas

Notes:

(a) You are encouraged to discuss the homework with other students in the class, and you can work together. However, all of your code must be your own, all of the figures must be your own, and all of the writeup must be your own.
(b) Your homework writeup must be in pdf format, with your code in the text (at the end, or throughout the writeup).
  - Your neat and properly commented working code should also be uploaded as well (e.g., working .m files if written in Matlab)
  - **Exception**: for one problem, you are allowed to embed a photo of your handwritten solution. This exception is in place because occasionally you may have a lot of messy algebra, and I want to save you the time needed to type it all up.

(c) All plots should be clearly labeled (axis labels, units, etc.), and should be computer generated. Figures should be drawn using some sort of figure-making software (PowerPoint, Illustrator, etc.), though neat touch-screen-drawn sketches may be acceptable.
(d) If you used data from a reference (something from a publication, internet database, etc.), the reference should be given in your writeup.
(e) Your writeup should be typed up, including all necessary equations using a proper math font, and all figures appropriately labeled and captioned. I recommend using either MS Word (with its built-in equation editor) or Latex to prepare your writeups, though you are free to use whatever you prefer.

Problem 1:

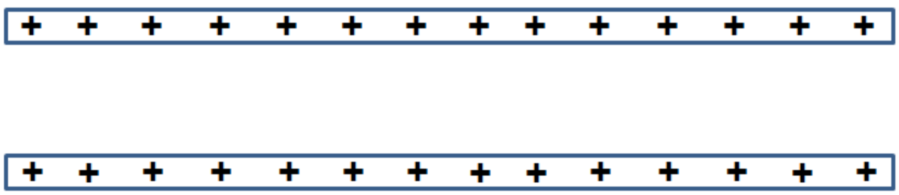
Assume a plane wave $\mathbf{E} = \mathbf{E_0} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, with the notation that **bold** = vector

(a) (2 pts) Show that $\nabla \times \mathbf{E} = i\mathbf{k} \times \mathbf{E}$
(b) (2 pts) Find $\nabla \cdot \mathbf{E}$ in terms of $\mathbf{k}$ and $\mathbf{E}$, similar to the form above

Problem 2:

We briefly mentioned in lecture that Maxwell's equations are compatible with the theory of special relativity. This is quite remarkable, because they are based on empirical observations and were assembled by 1865, whereas Einstein's special relativity did not emerge until 1905. In this problem, you will derive a key result of special relativity—length contraction—from just Maxwell's equations. In some ways, this problem will be retracing Einstein's steps from 1905.

Assume two infinitely long, infinitely thin, perfectly conducting parallel wires, each one of them has a linear charge density *b* that has units of charge/length. Assume that the charges are not moving in the lab reference frame. The distance between the wires is *d*.



a) (2 pt) Calculate the electric force per unit length between the wires

b) (2 pts) Assume you are now moving along the wires with speed *v*. Now you perceive a current flowing along both wires. Calculate the magnetic force per unit length between the wires.
Hint relevant to part (c): assume, for the sake of argument, that the linear charge density on the wire is now b', instead of b.

c) (4 pts) Using the above results, derive the factor by which the length contracts, for a particular speed v. This is called the Lorentz factor, $\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$. The Lorentz factor is used to account for both length contraction and time dilation in the theory of special relativity.

Problem 3:

(a) (1 pt) Given light with a frequency $f = 300$ THz, calculate ω, λ, k, and E_{ph}, both in air and in silica glass (make a table with 8 entries).

(b) (1 pt) Plot the dispersion relation (ω vs. k) for light of frequency f from 300 THz to 600 THz in air and in silica glass.

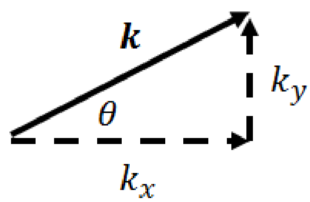
(c) (2 pts) Assume that this same light can be described by a plane wave given by $\widetilde{\boldsymbol{E}}_{\mathbf{1}} = \tilde{A}e^{i(kz-\omega t)}\widehat{\boldsymbol{x}}$, where $\tilde{A} = 1 + 1i$. Write down the expression for actual, physical (i.e. real) electric field $E_1(z,t)\widehat{\boldsymbol{x}}$ using no complex numbers at all. Then, plot $E_1(z,t)$ at $t = 0$, for z from -2000 nm to $+2000$ nm. You can assume that $f = 300$ THz, and the wave is in air.

(d) (1 pt) By plotting several real electric-field distributions at slightly different times, show that waves described by $\widetilde{\boldsymbol{E}}_{\mathbf{1}} = \tilde{A}e^{i(kz-\omega t)}\widehat{\boldsymbol{x}}$ and $\widetilde{\boldsymbol{E}}_{\mathbf{2}} = \tilde{A}e^{i(-kz-\omega t)}\widehat{\boldsymbol{x}}$ move in opposite directions

Problem 4:

This problem is designed to (i) get you to feel more comfortable with the concept of plane waves, and (ii) demonstrate that Gaussian beams can be understood as a superposition of many plane waves. The technical term for breaking up a field into a series of plane waves is the "angular spectrum representation" or "angular spectrum method".

Let's say you have a z-polarized plane wave that is propagating in free space along some direction in the $x - y$ plane, at a wavelength of $\lambda_0 = 500$ nm. The wave can be described by $E = E_z = e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, along a direction defined by θ, where θ is the angle away from the x axis:



(a) (3 pts) Plot the real field at $t = 0$ as a function of x and y, in a window covering $x = -5\ \mu$m to 5 μm, and $y = -5\ \mu$m to 5 μm. Make two plots – one for $\theta = 0°$ and one for $\theta = 10°$. You should see the wavefronts typical of a plane wave.

(b) (2 pts) Now assume that you have three waves of equal amplitude, E_1, E_2, and E_3, propagating simultaneously along the directions defined by $\theta = -10°$, $0°$, and $+10°$. Because Maxwell's equations are linear, you can simply add the electric fields up. Plot the total real field at $t = 0$. You should see that the resulting pattern is reminiscent of an array of propagating Gaussian beams.

(c) (5 pts) Now assume that, instead of three identical waves, you have $N = 30$ waves. These waves have the standard plane-wave form, but they can vary with amplitude: $A_n e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, where A_n is a real number from 0 to 1, and n is simply a counter for the waves, going from 1 to 30. Assume that these waves have a range of angles θ, ranging from $-10°$ to $+10°$, corresponding to a range in k_y, and that the 30 values of k_y are linearly distributed from the minimum to the maximum value.

Assume that A_n is Gaussian in k_y, centered about $k_y = 0$: $A_n = e^{-k_{y,n}^2/(2(\sigma)^2)}$, where σ indicates how spread out the distribution is. Let's assume that $\sigma = \frac{k_y(10°)}{2}$, meaning that the σ is half of the maximum value of k_y. The max value of k_y corresponds to 10 °.

Now, plot the total real field of all 30 plane waves together, at $t = 0$. You should observe that this corresponds to a Gaussian wave moving along the x direction.

(d) (1 pt) Plot the irradiance of the wave in part (c). This result demonstrates that a Gaussian beam can be understood as the superposition of plane waves with a range of transverse k (i.e. k_y) and a Gaussian distribution of amplitudes.

<u>Problem 5:</u> (2 pts) Starting with the Macroscopic Maxwell's equations and the constitutive relations, derive the wave equation for $\mathbf{H}$

<u>Problem 6:</u>

Though, in general, any material can be described by a combination of Lorentz (and related) oscillators, we sometimes assume that there are more oscillators at very high frequencies, but they are so far away that they can be wrapped up into a single quantity called ϵ_∞ as follows:

$$\epsilon = \epsilon_\infty + \sum_j \frac{A_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

This way, we don't need to take into account all of the Lorentz oscillator terms, only the ones that are near our frequency of interest. Note that I've also combined the A and f_j terms from lecture as $A_j = A \times f_j$.

A fictitious material I just made up is described by a Lorentz model with two oscillators, with resonant frequencies corresponding to free space wavelengths $\lambda_{0,1} = 10\ \mu m$ and $\lambda_{0,2} = 5\ \mu\text{m}$.

The other Lorentz parameters are as follows:

ϵ_∞	A_1	A_2	γ_1	γ_2
4	$2\omega_1^2$	$2\omega_2^2$	$\omega_1/3$	$\omega_2/5$

(a) (3 pts) Plot the real and imaginary part of the complex refractive index of this material vs. free-space wavelength over the range from 0.5 μm to 20 μm. Point out the on the plot the refractive index corresponding to ϵ_∞.

As we will cover later, the speed with which an optical pulse (carrying information and/or energy) propagates through a medium is called the group velocity, defined as $v_g = Re\left[\frac{\partial\omega}{\partial k}\right]$, where k is the wave-number.

(b) (2 pts) Plot the group velocity vs. the free-space wavelength of the above material.