ECE747:Nanophotonics
University of Wisconsin - Madison
Fall 2020

Problem Set 2

Sanket Deshpande

The MATLAB scripts for all relevant questions is appended at the end of the document.

Problem 1:

a. The SI units for:

  i. $\mathbf{J}$ are $A\ m^{-2}$
  ii. $\mathbf{E}$ are $V\ m^{-1}$
  iii. power are AV (derived from Joule's Law of heating)
  iv. volume are m^3

$$units\{\mathbf{J}.\mathbf{E}\} = A\ m^{-2}\ .\ V\ m^{-1}$$
$$units\{\mathbf{J}.\mathbf{E}\} = A\ V\ m^{-3}$$
$$units\{\mathbf{J}.\mathbf{E}\} = \frac{units\{power\}}{units\{volume\}}$$

Hence, we've verified that $\mathbf{J}.\mathbf{E}$ has units of power per unit volume.

b. The SI units for:

  i. $\mathbf{E}$ are $V\ m^{-1}$
  ii. $\mathbf{D}$ are $C\ m^{-2}$
  iii. $\mathbf{B}$ are $N\ m^{-1}\ A^{-1}$
  iv. $\mathbf{H}$ are $A\ m^{-1}$
  v. energy are J
  vi. volume are m^3
  vii. force are N

$$units\{U\} = units\{\mathbf{E}.\mathbf{D}\}\ +\ units\{\mathbf{B}.\mathbf{H}\}$$
$$units\{U\} = V\ m^{-1}\ .\ C\ m^{-2}\ +\ N\ m^{-1}\ A^{-1}\ .\ A\ m^{-1}$$

Since, Voltage (or potential difference between two points) is the amount of energy required to move a unit charge between two points, we have $V = J\ C^{-1}$.

$$units\{U\} = J\ m^{-3}\ +\ N\ m^{-2}$$

Since energy is the product of the force and displacement, $J = N\ m$

$$units\{U\} = J\ m^{-3}\ +\ J\ m^{-1}\ m^{-2}$$
$$units\{U\} = J\ m^{-3}\ +\ J\ m^{-3}$$
$$units\{U\} = J\ m^{-3}$$

Hence, we've verified that U has units of energy per unit volume.

c. To solve this problem, I'm assuming that the chosen volume is in free-space (vacuum). Ampere's law in this medium can be stated as:

$$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \epsilon_o \frac{\partial \mathbf{E}}{\partial t}$$

In free-space, $\mathbf{B} = \mu_o \mathbf{H}$. Therefore,

$$\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_o \frac{\partial \mathbf{E}}{\partial t}$$
$$-\epsilon_o \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J} - \nabla \times \mathbf{H}$$

Taking a dot product with $\mathbf{E}$ on both sides,

$$-\epsilon_o \mathbf{E}.\frac{\partial \mathbf{E}}{\partial t} = \mathbf{E}.\mathbf{J} - \mathbf{E}.(\nabla \times \mathbf{H})$$

Using the given "trick", $\mathbf{X}.\frac{\partial \mathbf{X}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}X^2$, and an identity, $\nabla.(\mathbf{A} \times \mathbf{B}) = \mathbf{B}.(\nabla \times \mathbf{A}) - \mathbf{A}(\nabla \times \mathbf{B})$ we have,

$$-\frac{\epsilon_o}{2}\frac{\partial}{\partial t}E^2 = \mathbf{J}.\mathbf{E} - \nabla.(\mathbf{H} \times \mathbf{E}) - \mathbf{H}.(\nabla \times \mathbf{E})$$

Using Faraday's Law, we have,

$$-\frac{\epsilon_o}{2}\frac{\partial}{\partial t}E^2 = \mathbf{J}.\mathbf{E} + \nabla.(\mathbf{E} \times \mathbf{H}) + \frac{\mathbf{B}}{\mu_o}.\frac{\partial \mathbf{B}}{\partial t}$$
$$-\frac{\epsilon_o}{2}\frac{\partial}{\partial t}E^2 - \frac{1}{2\mu_o}\frac{\partial}{\partial t}B^2 = \mathbf{J}.\mathbf{E} + \nabla.(\mathbf{E} \times \mathbf{H})$$

The energy stored per unit volume in the electric field is given as $\frac{\epsilon_o}{2}E^2$, and in the magnetic field is given as $\frac{1}{2\mu_o}B^2$. The Poynting vector is denoted as $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. Hence we have derived the Poynting's theorem as:

$$-\frac{\partial}{\partial t}U = \mathbf{J}.\mathbf{E} + \nabla.\mathbf{S}$$

where U is the rate of change of the total energy stored in the electro-magnetic field, per unit volume.

d. From the given definitions of the electric and magnetic fields, we can write,

$$\tilde{E} = E_o e^{-\iota\omega t} \quad \&$$
$$\tilde{H} = H_o e^{-\iota\omega t}$$

where E_o & H_o can be complex. Therefore,

$$\tilde{E}\tilde{H} = E_o H_o e^{-\iota 2\omega t} \quad \&$$
$$Re[\tilde{E}\tilde{H}] = E_o H_o \cos 2\omega t$$

From the given definitions of the Poynting vector, we can write,

$$S = E\ H$$
$$S = Re[E_o e^{-\iota\omega t}] * Re[H_o e^{-\iota\omega t}]$$
$$S = E_o \cos\omega t * H_o \cos\omega t$$
$$S = E_o H_o \cos^2\omega t$$

Hence, $S \neq Re[\tilde{E}\tilde{H}]$

e. Simplifying the given expression for the magnitude of the time-average of the Poynting vector:

$$|< \mathbf{S} >| = \frac{1}{2} Re[\tilde{E}\tilde{H}^*]$$
$$|< \mathbf{S} >| = \frac{1}{2} Re[E_o e^{-\iota\omega t}\ H_o^* e^{\iota\omega t}]$$
$$|< \mathbf{S} >| = \frac{1}{2} Re[E_o H_o^*]$$

To prove this, we start with the Poynting vector,

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$
$$\mathbf{S} = Re[\tilde{\mathbf{E}}] \times Re[\tilde{\mathbf{H}}]$$
$$\mathbf{S} = Re[\mathbf{E_o} e^{-\iota\omega t}] \times Re[\mathbf{H_o} e^{-\iota\omega t}]$$
$$\mathbf{S} = \frac{1}{2}(\mathbf{E_o} e^{-\iota\omega t} + \mathbf{E_o^*} e^{\iota\omega t}) \times \frac{1}{2}(\mathbf{H_o} e^{-\iota\omega t} + \mathbf{H_o^*} e^{\iota\omega t})$$
$$\mathbf{S} = \frac{1}{4}(\mathbf{E_o} \times \mathbf{H_o} e^{-\iota 2\omega t} + \mathbf{E_o} \times \mathbf{H_o^*} + \mathbf{E_o^*} \times \mathbf{H_o} + \mathbf{E_o^*} \times \mathbf{H_o^*} e^{\iota 2\omega t})$$
$$\mathbf{S} = \frac{1}{4}(\mathbf{E_o} \times \mathbf{H_o^*} + \mathbf{E_o^*} \times \mathbf{H_o} + \mathbf{E_o^*} \times \mathbf{H_o^*} e^{\iota 2\omega t} + \mathbf{E_o} \times \mathbf{H_o} e^{-\iota 2\omega t})$$
$$\mathbf{S} = \frac{1}{2}(Re[\mathbf{E_o} \times \mathbf{H_o^*}] + Re[\mathbf{E_o^*} \times \mathbf{H_o^*} e^{\iota 2\omega t}])$$

The time average of this quantity over the entire time period, $T = 2\pi/\omega$, can be denoted as:

$$< \mathbf{S} > = \frac{1}{T}\int_0^T \frac{1}{2}(Re[\mathbf{E_o} \times \mathbf{H_o^*}] + Re[\mathbf{E_o^*} \times \mathbf{H_o^*} e^{\iota 2\omega t}])\, dt$$

The second term goes to 0 for the integral over the entire time period. Therefore, we get the expression,

$$< \mathbf{S} > = \frac{1}{2}\, Re[\mathbf{E_o} \times \mathbf{H_o^*}]$$

Taking modulus on both sides, we get,

$$|< \mathbf{S} >| = \left|\frac{1}{2} Re[\mathbf{E_o} \times \mathbf{H_o^*}]\right|$$
$$|< \mathbf{S} >| = \frac{1}{2} Re[\ |\mathbf{E_o} \times \mathbf{H_o^*}|\ ]$$
$$|< \mathbf{S} >| = \frac{1}{2} Re[E_o\ H_o^*]$$

Hence proved

f. We derived that Irradiance is given as $I = |< \mathbf{S} >| = \frac{1}{2} Re[E_o\ H_o^*]$. We can use the definition of impedance as $\eta = \frac{E_o}{H_o}$, where E_o and H_o can be complex amplitudes. Taking the complex conjugate of η, we get:

$$H_o^* = \frac{E_o^*}{\eta^*}$$

Therefore, we can use this in the expression of irradiance,

$$I = \frac{1}{2}\ Re\Big[\frac{E_o\ E_o^*}{\eta^*}\Big]$$
$$I = \frac{1}{2}\ Re\Big[\frac{|E_o|^2}{\eta^*}\Big]$$
$$I = \frac{|E_o|^2}{2} Re\Big[\frac{1}{\eta^*}\Big]$$

g. For a lossless, non-magnetic material, the impedance is real and we can write it as $\eta = \sqrt{\frac{\mu_o}{\epsilon_r\ \epsilon_o}}$. Using this in the expression for Irradiance:

$$I = \frac{|E_o|^2}{2\sqrt{\frac{\mu_o}{\epsilon_r\ \epsilon_o}}}$$
$$I = \frac{|E_o|^2\ n}{2}\sqrt{\frac{\epsilon_o}{\mu_o}}$$
$$I = \frac{1}{2}\epsilon_o\ n\ c\ |E_o|^2$$

Here, $n = \sqrt{\epsilon_r}$ is the refractive index of the medium and $c = \sqrt{\frac{1}{\mu_o \epsilon_o}}$ is the speed of light in vacuum.

Problem 2: The following figure illustrates a wave (in red) travelling in medium n_1, incident on an interface to another medium n_2. The refracted wave (in blue) travels in the second medium.

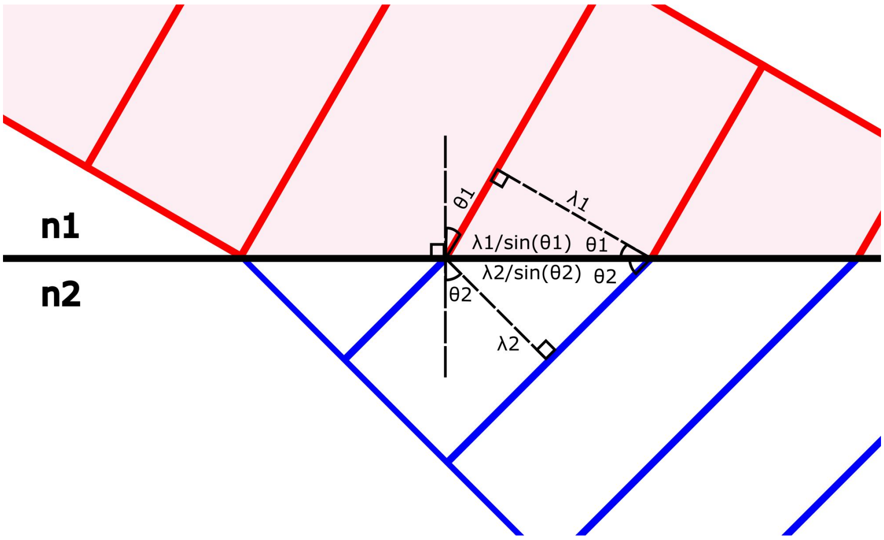

Figure 1: Illustrates refraction of a wave at the interface of two media. The wavefronts are drawn assuming translational invariance along the interface.

Let us denote the free-space wavelength as, λ_o. Therefore, the wavelength:

i. in medium n_1 is $\lambda_1 = \frac{\lambda_o}{n_1}$

ii. in medium n_2 is $\lambda_2 = \frac{\lambda_o}{n_2}$

From the figure, we can see that:

$$\frac{\lambda_1}{\sin\theta_1} = \frac{\lambda_2}{\sin\theta_2}$$
$$\frac{\lambda_o}{n_1\sin\theta_1} = \frac{\lambda_o}{n_2\sin\theta_2}$$

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

This is the expression of Snell's Law.

Problem 3:

a. For a wave incident on the interface of two media, its wave-vectors in the x-plane, k_x, are conserved

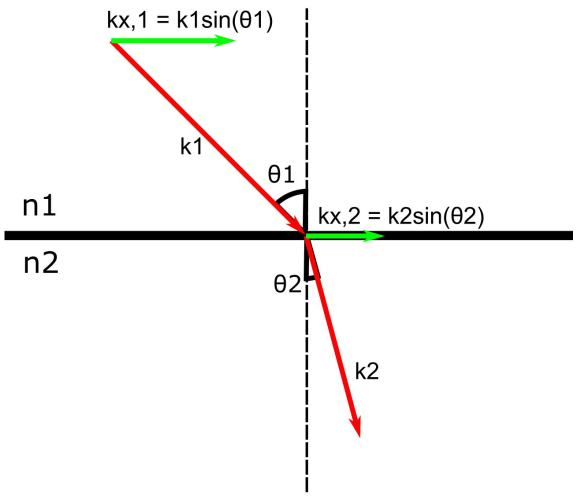

Figure 2: Wave vectors of a wave incident on the interface of two media. The arrows only indicate the direction and not magnitude.

$$
\begin{aligned}
k_{x,1} &= k_{x,2} \\
k_1 \sin\theta_1 &= k_2 \sin\theta_2 \\
\frac{2\pi}{\lambda_1}\sin\theta_1 &= \frac{2\pi}{\lambda_1}\sin\theta_2 \\
\frac{n_1}{\lambda_o}\sin\theta_1 &= \frac{n1}{\lambda_o}\sin\theta_2
\end{aligned}
$$

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

This is the expression of Snell's Law.

b. Using the given expression,

$$
\begin{aligned}
k_{x,2} &= k_{x,1} \ + \ k_{int} \\
\frac{2\pi}{\lambda_1}\sin\theta_2 &= \frac{2\pi}{\lambda_1}\sin\theta_1 + k_{int} \\
\frac{2\pi n_2}{\lambda_o}\sin\theta_2 &= \frac{2\pi n_1}{\lambda_o}\sin\theta_1 + k_{int} \\
n_2\sin\theta_2 &= n_1\sin\theta_1 + \frac{k_{int}}{2\pi/\lambda_o}
\end{aligned}
$$

$$n_2\sin\theta_2 = n_1\sin\theta_1 + \frac{k_{int}}{k_o}$$

where, k_o is the free-space wave-number.
This is the Snell's Law expression for the given interface.

c. For finding the critical angle, $\theta_1 = \theta_c$, we can set the θ_2 to 90^o in the above expression.

$$n_2 \sin 90^o = n_1 \sin \theta_c + \frac{k_{int}}{k_o}$$
$$\sin \theta_c = \frac{1}{n_1}\Big(n_2 - \frac{k_{int}}{k_o}\Big)$$
$$\theta_c = sin^{-1}\Big[\frac{1}{n_1}\Big(n_2 - \frac{k_{int}}{k_o}\Big)\Big]$$

Since it is possible that k_{int} provides a momentum kick to the photons opposite in direction to $k_{x,1}$, it can completely change the direction of the wave such that, $\theta_2 = -90^o$. Therefore, we can derive an expression for the critical angle, θ_c^*, corresponding to this condition.

$$n_2 \sin(-90^o) = n_1 \sin \theta_c^* + \frac{k_{int}}{k_o}$$
$$\sin \theta_c^* = \frac{1}{n_1}\Big(-n_2 - \frac{k_{int}}{k_o}\Big)$$
$$\theta_c^* = sin^{-1}\Big[\frac{1}{n_1}\Big(-n_2 - \frac{k_{int}}{k_o}\Big)\Big]$$

Problem 4:
The following plot of Refractive Index vs Free Space Wavelength of Gold illustrates that the material can have refractive index less than 1, over a range of wavelengths.

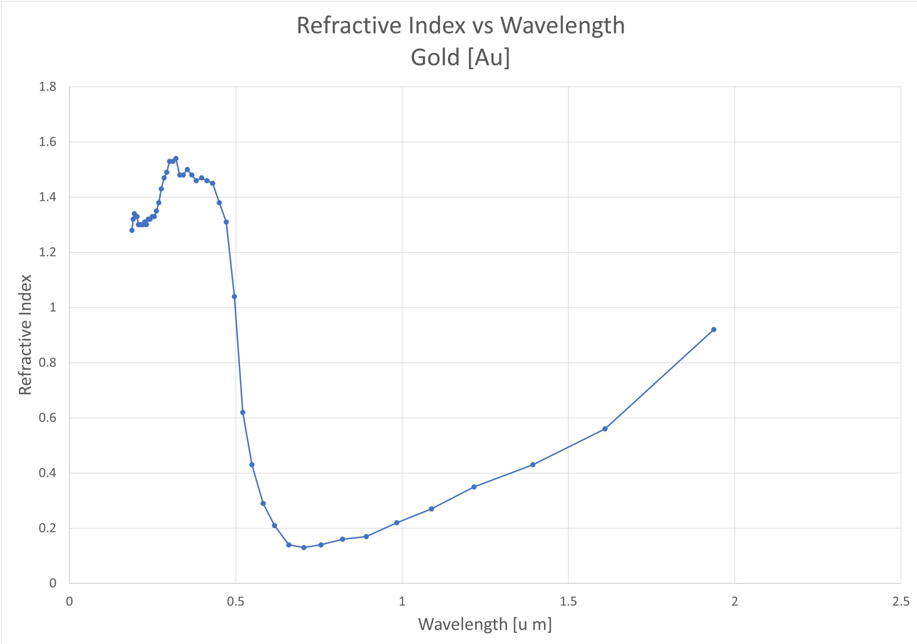

Figure 3: Refractive Index vs Free Space Wavelength of Gold [Au]

Ref: Gold (Au) from Johnson and Christy, 1972. Obtained from

`https://refractiveindex.info/?shelf=main&book=Au&page=Johnson`

Problem 5:

Dispersion relation for a parallel-plate waveguide with PEC walls and $d = 2\mu m$

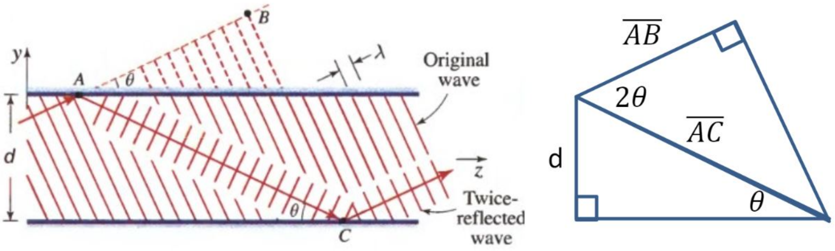

Figure 4: Wave propagating in parallel-plate waveguide. Ref: class notes.

From the geometry, we can write,

$$AC\ \sin\theta = d$$
$$AC = \frac{d}{\sin\theta}\ \ \&$$

$$AC\ \cos 2\theta = AB$$
$$AB = \frac{d\cos 2\theta}{\sin\theta}$$
$$AB = \frac{d}{\sin\theta} - 2d\sin\theta$$

The phase of the wave at point B should be equal to the twice-reflected wave at point C.

$$k\ AB = \frac{\pi}{2} + k\ AC + \frac{\pi}{2} + 2\pi g; \quad g \in \mathbf{Z}$$
$$k(AB - AC) = \pi(1 + 2g)$$
$$-2d\ k\ \sin\theta = \pi(2g+1)$$
$$-2d\,\frac{2\pi}{\lambda}\ \sin\theta = \pi(2g+1)$$
$$\sin\theta = \frac{-\lambda}{4d}(2g+1)$$
$$\sin(\theta_m) = \frac{\lambda}{4d}m \quad m \in \{... -5, -3, -1, 1, 3, 5...\}$$

Using this expression, we can derive the dispersion relation

$$k_y = k \sin \theta_m$$
$$k_y = \frac{2\pi}{\lambda} \frac{\lambda}{4d} m$$
$$k_y = \frac{\pi m}{2\ d}$$

$$\beta_m = k_z = \sqrt{k^2 - k_y^2}$$
$$\beta_m = \sqrt{\frac{\omega^2}{c^2} - \frac{\pi^2 m^2}{4d^2}}$$
$$\omega = c\sqrt{\beta^2 + \frac{\pi^2 m^2}{4d^2}}$$

From looking at the equation, we can determine that the lowest frequencies are generated for the lowest modulus values of m. Hence, the following plot indicates the dispersion relation for $m = 1, 3, 5, 7$

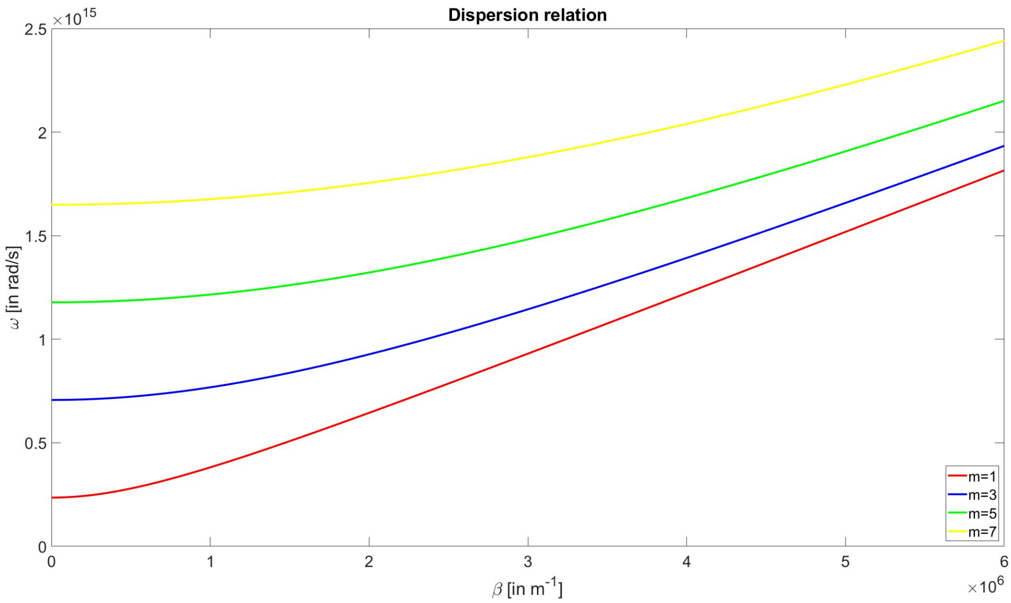

Figure 5: Dispersion relation for a wave propagating in the given waveguide.

MATLAB Scripts

Problem 5:

```
%MATLAB code for ECE747 HW2 Q5 - to plot the dispersion relation in a
%parallel-plate waveguide

m_range=1:2:7; %range of lowest m-values
d=2e-6;%given that the separation of plates = 2 um
c=3e+8;%speed of light
beta=linspace(0,6e+6,1000);%range of wave-guide wave-numbers


w_1=c*sqrt(beta.^2+(pi()^2*m_range(1)^2)/(4*d^2));%omega for m=1
w_2=c*sqrt(beta.^2+(pi()^2*m_range(2)^2)/(4*d^2));%omega for m=3
w_3=c*sqrt(beta.^2+(pi()^2*m_range(3)^2)/(4*d^2));%omega for m=5
w_4=c*sqrt(beta.^2+(pi()^2*m_range(4)^2)/(4*d^2));%omega for m=7

plot(beta,w_1,'Color','r','Linewidth',3)
hold on
plot(beta,w_2,'Color','b','Linewidth',3)
plot(beta,w_3,'Color','g','Linewidth',3)
plot(beta,w_4,'Color','y','Linewidth',3)
hold off
legend({'m=1','m=3','m=5','m=7'},'Location','southeast')
title('Dispersion relation')
xlabel('\beta_m [in m^{-1}]')
ylabel('\omega [in rad/s]')
ax = gca;
ax.FontSize = 25;
```