ECE 747 – Nanophotonics
Problem Set 2
Due: October 21, 2020, end of day

Notes:

(a) You are encouraged to discuss the homework with other students in the class, and you can work together. However, all of your code must be your own, all of the figures must be your own, and all of the writeup must be your own.
(b) Your writeup should be typed up, including all necessary equations using a proper math font, and all figures appropriately labeled and captioned. I recommend using either MS Word (with its built-in equation editor) or Latex to prepare your writeups, though you are free to use whatever you prefer.
   - No "one problem exception" like last time: please type up the entire solution set, since this one is shorter
(c) Your homework writeup must be in pdf format, with your code in the text (at the end, or throughout the writeup).
   - Your neat and properly commented working code should also be uploaded as well (e.g., working .m files if written in Matlab)
(d) All plots should be clearly labeled (axis labels, units, etc.), and should be computer generated. Figures should be drawn using some sort of figure-making software (PowerPoint, Illustrator, etc.), though neat touch-screen-drawn sketches may be acceptable.
(e) If you used data from a reference (something from a publication, internet database, etc.), the reference should be given in your writeup.

Problem 1:

This problem will walk through you the derivation of Poynting's theorem, Poynting's vector, and the irradiance.

We begin with the full macroscopic Maxwell's equations, which relate $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{J}$, and ρ, where $\mathbf{J}$ and ρ are the free current density and free charge density, respectively. First, consider the conservation of energy in a small volume element in space. The work done per unit volume by the electromagnetic field is $\mathbf{J} \cdot \mathbf{E}$. This is the vectorial statement of Ohm's law.

(a) (1 pt) Verify that $\mathbf{J} \cdot \mathbf{E}$ has units of power per unit volume, starting with the SI units for $\mathbf{J}$ and $\mathbf{E}$.
(b) (1 pt) Recall that the energy density of the electromagnetic field is $U = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ (**see end of problem set for more discussion). Starting with SI units for all of the fields, show that U has units of energy per volume.

Next, the energy dissipation must be connected with a net decrease of energy density at that spot and power flow out of the volume.

(c) (3 pts) Plug Joule's law into Ampere's law, and manipulate the equation to obtain Poynting's theorem (**see end of problem set for more discussion), as shown in class. Remember that we defined a quantity called the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$. You may need to use some vector identities, and you may find the following "trick" helpful: $\frac{1}{2}\frac{\partial}{\partial t}X^2 = X\frac{\partial X}{\partial t} = \mathbf{X} \cdot \frac{\partial \mathbf{X}}{\partial t}$, where $\mathbf{X}$ is a vector field and $X = |\mathbf{X}|$.

The real, time-dependent electric and magnetic fields for a plane wave at some particular point in space can be written as $\mathbf{E} = Re[\tilde{\mathbf{E}}] = Re[\mathbf{E_0}e^{-i\omega t}]$ and $\mathbf{H} = Re[\tilde{\mathbf{H}}] = Re[\mathbf{H_0}e^{-i\omega t}]$, where $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are the complex fields. For simplicity, let's assume that the wave is linearly polarized and so we can write the field as scalars: $E = Re[E_0 e^{-i\omega t}]$ and $H = Re[H_0 e^{-i\omega t}]$, where E_0 and H_0 can be complex. We know that the Poynting vector is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, so $S = EH$ where $S = |\mathbf{S}|$.

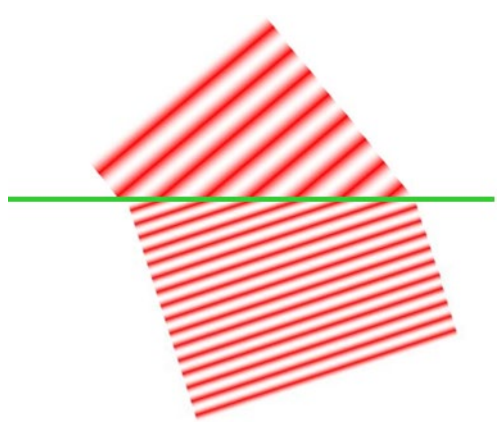
(d) (2 pts) Show that $S \neq Re[\tilde{E}\tilde{H}]$.

(e) (2 pts) Show that the magnitude of the time-average of the Poynting vector $|\langle \mathbf{S} \rangle| = \frac{1}{2}Re[\tilde{E}\tilde{H}^*]$ where $\tilde{H}^*$ is the complex conjugate of $\tilde{H}$.

(f) (2 pts) Usually we write the time-averaged Poynting vector as $\langle \mathbf{S} \rangle = \frac{1}{2}Re[\mathbf{E} \times \mathbf{H}^*]$, where we dropped the tilde notation so that $\mathbf{E}$ and $\mathbf{H}$ are complex fields. Derive the irradiance $I = |\langle \mathbf{S} \rangle|$ as a function of the electric field amplitude and the impedance $\eta = \sqrt{\mu/\epsilon}$.

(g) (1 pt) Now assume that the material is non-magnetic ($\mu = \mu_0$). Find the irradiance as a function of the electric field amplitude and the refractive index

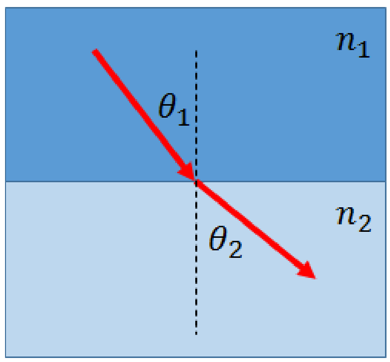
<u>Problem 2</u>

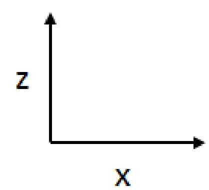


(2 pts) Derive Snell's law from purely geometrical considerations. That is, make the following assumptions only:

1. The wavelength of light in medium with refractive index n is λ_0/n
2. There is translational invariance along the interface. That means that if the instantaneous wavefronts on top and the bottom are connected in one place (as in the diagram above), they are connected in every place; if they are offset by some amount in one place, they are offset by the same amount in every other place.

Problem 3:





Assume you have light coming from medium 1, with index n_1 and incident angle θ_1, refracting in medium 2. Note: the arrow above signifies the direction of propagation only; do not assign any special meaning to its magnitude.

(a) (1 pt) In class, we determined that k_x must be conserved, such that $k_{x,1} = k_{x,2}$. Use this relationship to re-derive Snell's law.

Now assume that the interface has special properties, as follows. When light hits the interface, the interface imparts an additional k-vector component, which we will call K_{int}, which can be positive or negative (this can be accomplished with surface structures referred to as optical metasurfaces, which we will cover at the end of the class). So we can write $k_{x,2} = k_{x,1} + K_{int}$

(b) (2 pt) Derive the new Snell's law in terms of the parameters in the figure above, and K_{int}.

(c) (3 pt) Derive the new critical angle $\theta_1 = \theta_c$ that results in the onset of total internal reflection. If there is more than one such angle, you must give the expression for all of them.

Problem 4:

In class, we discussed the concept of total internal reflection. A similar situation can occur for when light coming at an angle from free space ($n_1 = 1$) encounters an interface with a material with a lower refractive index ($n_2 < 1$), and is then reflected with high efficiency. This is called "total *external* reflection". (Note: if n_2 has an imaginary component, the total external reflection can still approximately occur, as long as the imaginary component is not too large).

(2 pts) Spend some time on refractiveindex.info or another database of optical properties of material, and find a material that one could use to demonstrate total external reflection. You can look at any wavelength range (ultraviolet, visible, mid infrared, etc.). When you find such a material, plot its refractive index vs. free-space wavelength in this region, and provide the appropriate reference.

Problem 5:

(4 pts) In class, we derived the dispersion relation of a parallel-plate waveguide (spacing = $d = 2\ \mu$m) made with PEC walls, such that each reflection resulted in an electric-field phase shift of π. Assume that you have a similar waveguide, but at each reflection the electric field picks up a phase of $\pi/2$. Derive the new dispersion relation (β vs. ω) and plot the four modes with the lowest frequency.

****Additional discussion**

The derivation you are working through here and the Poynting vector presented in class are not very general, as I found out while doing some reading after class. The expression $U = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})$ turns out to only be correct for materials without dispersion or loss. Worse, there is only one "material" that is *actually* without dispersion or loss: vacuum! But the expression is still approximately correct for materials that are transparent and close to dispersionless.

I spent some time trying to find a derivation of Poynting's theorem that is valid for materials with dispersion and loss that is simple enough to present in class, but this turns out to be quite messy. The famous textbook, *Electrodynamics* by Landau and Lifshitz has a derivation under "The field energy in dispersive media", but it is complex and also uses a low-loss approximation. A number of papers over the last 20 years derive fully correct expressions for the energy density and Poynting's theorem, though they are also beyond the scope of this class. Nevertheless, here are some if you would like to take a look:

- Ruppin, "Electromagnetic energy density in a dispersive and absorptive material", *Physics Letters A* 299, 309–312 (2002), https://www.sciencedirect.com/science/article/abs/pii/S0375960101008386

- Shin, Raman, and Fan, "Instantaneous electric energy and electric power dissipation in dispersive media", *JOSA B* 29, 1048 (2012), https://www.osapublishing.org/josab/abstract.cfm?uri=josab-29-5-1048

Neither of the above are show complete derivations of everything, but you can follow the references to learn more. Personally, I ran out of energy to work through it this time around.

So, give the above you might ask: what is the point of presenting this derivation of Poynting's theorem that mostly only applies to nonphysical situations? It turns out that the reason for this is that you still get the right answer for the Poynting vector! There is a clever way to show this, that I found in Landau and Lifshitz (note that they are using Gaussian units rather than SI, hence the weird prefactor):

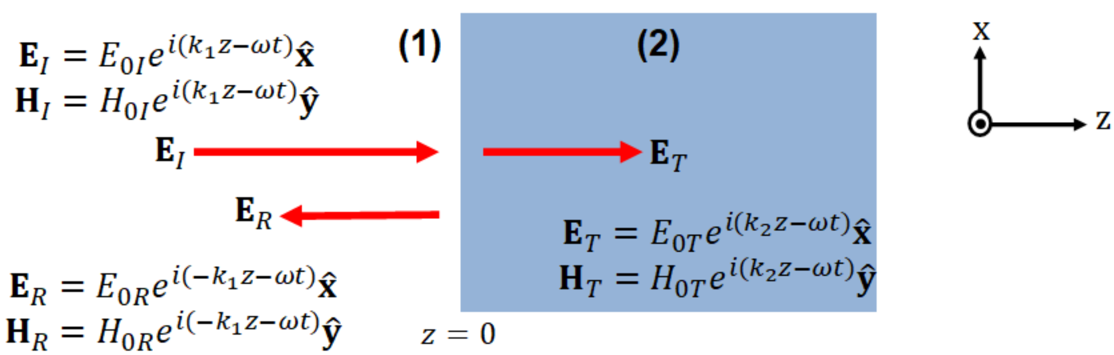
§80. The field energy in dispersive media

The formula

$$\mathbf{S} = c\mathbf{E}\times\mathbf{H}/4\pi \tag{80.1}$$

for the energy flux density remains valid in variable electromagnetic fields, even if dispersion is present. This is evident from the arguments given at the end of §30: on account of the continuity of the tangential components of **E** and **H**, formula (80.1) follows from the condition that the normal component of **S** be continuous at the boundary of the body and the validity of the same formula in the vacuum outside the body.

Here's how this argument works: take the problem from class, where a plane wave is normally incident from free space (1) onto a flat surface of a lossy, dispersive material (2):



Whether you pick this or any other polarization, all E and H fields are parallel to the interface, and therefore the total fields are continuous across the interface. That is, at $z = 0$, $\mathbf{E}_I + \mathbf{E}_R = \mathbf{E}_T$ and $\mathbf{H}_I + \mathbf{H}_R = \mathbf{H}_T$. The total Poynting vector describing the energy flow on the left (including both the right- and left-moving waves) just to the left of the surface is

$$\mathbf{S}_1 = (\mathbf{E}_I + \mathbf{E}_R) \times (\mathbf{H}_I + \mathbf{H}_R)$$

And we know that this definition for the Poynting vector is correct, because the Poynting theorem we derived assumed a dispersionless, lossless medium—and vacuum fits the bill exactly.

Assuming no energy is gained or lost *exactly* at the interface, $\mathbf{S}_1$ has to be equal to the to $\mathbf{S}_2$ on the other side, which we know ahead of time should be some function of the fields, but perhaps we don't know what function exactly, since our derivation is not valid. However, because we know that $\mathbf{S}_1 = \mathbf{S}_2$ and $\mathbf{E}_I + \mathbf{E}_R = \mathbf{E}_T$ and $\mathbf{H}_I + \mathbf{H}_R = \mathbf{H}_T$, it becomes obvious that we should define $\mathbf{S}_2$ exactly the same way as we defined $\mathbf{S}_1$:

$$\mathbf{S}_2 = \mathbf{E}_T \times \mathbf{H}_T$$

Therefore, we conclude that even in a dispersive, lossy medium, we should define our Poynting vector as $\mathbf{S} = \mathbf{E} \times \mathbf{H}$.