EMA 601/NE 602
Problem set #2
Due 10/29 at 1 pm on Canvas, in pdf format

Guidelines:

- Typed solutions are preferred, but you may scan your handwritten solutions as long as the writing is legible. I find camscanner (available on both Android and IOS mobile platforms) to be a decent application for scanning documents into digital formats, but you may use any software of your preference.
- Please compile your responses into a single pdf document.
- You are welcome to work together on the problems, but you must write your own responses and code.
- Please show your derivations or describe how you arrived at your responses.

1. (15 points) **Derivation of atomic units for size and the magnetic field**
   The atomic unit for the electric field is the field experienced by the proton due to an electron in the ground state of a hydrogen atom. It is denoted as $E_A = \frac{e^2}{a_0}$, where e is the electric charge and a_0 is the Bohr radius. For all responses below, you may ignore numerical factors (factors of 2, etc).
   (a) (4 points) By balancing the energy of the electrostatic potential with the energy of the quantum confinement, derive the expression for the Bohr radius a_0 in terms of the Planck constant $\hbar$, e, and the mass of the electron/proton m_e. *Hint*: you can estimate the confinement energy either using the solution of a particle in a box problem, or by considering the kinetic energy of the electron under the uncertainty relation $\Delta x \Delta p \geq \hbar$.
   (b) (4 points) By assuming a classical orbit for the electron, find the magnetic field of the electron at the site of the proton, B_N. To derive B_N, it is useful to consider the current generated by the electron as it orbits the proton and use the Biot-Savart law in Gaussian units $B = \frac{1}{c} \oint \frac{d\vec{l} \times \hat{r}}{r^2}$.
   (c) (4 points) Find the magnetic field associated with the interaction between the electrostatic potential (which corresponds to a Hartree) with a Bohr magneton $\mu_B = \frac{e\hbar}{2m_e c}$.
   (d) (3 points) Express B_N and B_H in terms of E_A and calculate their values in gauss and tesla. Which of B_N and B_H is more practical to use based on their magnitudes?

2. (15 points) **Scaling properties of hydrogenic atoms**
   Hydrogenic ions are atoms that have one electron bound to a nucleus with atomic number $Z \geq 1$. Determine the scaling factor of Z in each of the following, relative to the values in the case of the hydrogen atom:
   Hint: You should not have to use explicit wave functions for this problem. Please consider the dimensions of the quantities of interest. For example, for length, determine the scaling of the characteristic length a_c relative to Z and the Bohr radius a_0.
   (a) (2 points) Expectation values of r, $1/r$, $1/r^3$, where r is the distance between the electron and the nucleus.
   (b) (2 points) Expectation values of the potential energy (V) and total energy (E).
   (c) (2 points) Characteristic time (t_c), as estimated using the energy-time uncertainty principle, and velocity (v_c).
   (d) (2 points) The probability density of finding the electron at the center of the nucleus.

(e) (2 points) Internal electric (E_c) and magnetic (B_c) fields determined using electrostatic equations.
(f) (2 points) Electric dipole moment ($d_c = qr$, where q is the charge difference and r is the displacement between the charges) and magnetic dipole moment (μ_c, determined by the current $\times$ area$/c$).
(g) (3 points) Evaluate whether a tin ion consisting of a single electron is relativistic. What is the relative magnitude of its magnetic field to that of the hydrogen atom?

3. (5 points) **Effect of the nuclear charge on the fine and hyperfine structure splitting**
For hydrogenic ions with nuclear charge Ze, how do energy splittings associated with the fine and hyperfine interactions scale with Z? Similar to problem 2, you should not have to use explicit wave functions for this problem.

4. (15 points) **Determination of the fine structure constant**
(a) (6 points) Suppose we are able to experimentally determine the following in a gamma-emitting atom:
   i. The mass difference ΔM of two nuclear energy levels in terms of atomic mass units (amu), which is related to the mass in kilograms kg by the Avogadro's number N_A: $mass\ [amu] = mass\ [kg]\ 1000 N_A$
   ii. The wavelength λ (in meters) of the gamma ray emitted when transitioning between the nuclear energy levels.

   By relating the energy of the gamma photon with the mass difference of the atom in grams, show that measurements of $\Delta M\ [amu]$ and $\lambda\ [m]$ can give us the product of two fundamental constants $N_A \hbar$.
(b) (6 points) $N_A \hbar$ is known as the molar Planck constant and can be used to infer the fine constant α. Write α in terms of $N_A \hbar$, the Rydberg constant R_∞, and the electron mass in atomic mass unit M_e.
(c) (3 points) Use the data in https://physics.nist.gov/cuu/Constants/index.html to estimate the uncertainty with which we can determine α.

5. (20 points) **Effect of the finite size of the nucleus**
In our quantum mechanical treatment of hydrogenic atoms, we have assumed the nucleus to be a point charge. In this problem, we will consider the effect the finite size of the nucleus on the Coulomb potential at distances close to the nucleus. We do this by assuming that the total charge Ze in the nucleus is uniformly distributed in a uniform sphere of radius R.
(a) (5 points) Derive the potential produced by the charged sphere by first writing out the charge density $\rho(r)$ of the system and then using Gauss's law.
(b) (5 points) Write the deviation of the potential you derived in (a) from the point source case (what we used in class). Calculate the first order energy change in the ground state of a hydrogenic atom.
(c) (4 points) Calculate the energy and frequency shifts for the $1s$ state of hydrogen. You can assume that the proton has a uniform charge distribution over $R = 0.9$ fm.
(d) (2 points) For an electron in the $1s$ state of hydrogen, calculate the ratio of the mean atomic radius of the unperturbed atom (assuming that the nucleus is a point charge) and R.
(e) (4 points) The size of the proton can be inferred by measuring the energy shift due to the finite nucleus size. When doing $1s - 2s$ spectroscopy of hydrogen (in which we are

measuring the frequency associated with the transition between the $1s$ and $2s$ states), what frequency accuracy is needed in order to determine R with an accuracy of 0.01 fm?