EMA601: Quantum Engineering with Atoms and Photons
University of Wisconsin - Madison
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Practice Set 1

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Question 1: **Planck's Radiation Law**

a. The expression for intensity per wavelength, is given as:

$$\frac{dI}{d\lambda} = \frac{8\pi hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda k_B T)} - 1} \tag{1}$$

The value of λ at which $dI/d\lambda$ maximizes, can be found out by equating the differentiation of equation 1 with respect to λ to 0,

$$\frac{d}{d\lambda}\left(\frac{dI}{d\lambda}\right) = 0$$

To solve this, we can define $x = \frac{hc}{\lambda k_B T}$, hence, we have:

$$\frac{dI}{d\lambda} = \frac{8\pi k_B T}{h^4 c^3}\frac{x^5}{e^x - 1}$$

Now, we use chain rule to eliminate the derivative w.r.t. λ:

$$\frac{d}{d\lambda}\left(\frac{dI}{d\lambda}\right) = \frac{dx}{d\lambda}\frac{d}{dx}\left(\frac{dI}{d\lambda}\right) = 0$$
$$\frac{d}{dx}\left(\frac{dI}{d\lambda}\right) = 0$$

This derivative gives us:

$$\frac{x}{1 - e^{-x}} - 5 = 0$$

A numerical solver was used in MATLAB to find a solution to this equation

```
%Script to find the wavelength at which intensity of blackbody
%radiation maximizes

%Defining parameters
c=2.99792*10^8; %Speed of light
h=6.626*10^(-34); %Planck's constant
kb=1.38*10^(-23); %Boltzmann constant
```

```
syms x

sol_x = vpasolve((x/(1-exp(-x)))-5,x)

syms T %temperature
lambda=(h*c)/(sol_x*kb*T)
```

This gives a solution for λ_{max} as:

$$\lambda_{max} = \frac{hc}{4.9651\ k_B T}$$
$$\lambda_{max} = \frac{0.00289}{T}$$

b. Using the curve fitting tool in MATLAB, the data can be fit as following

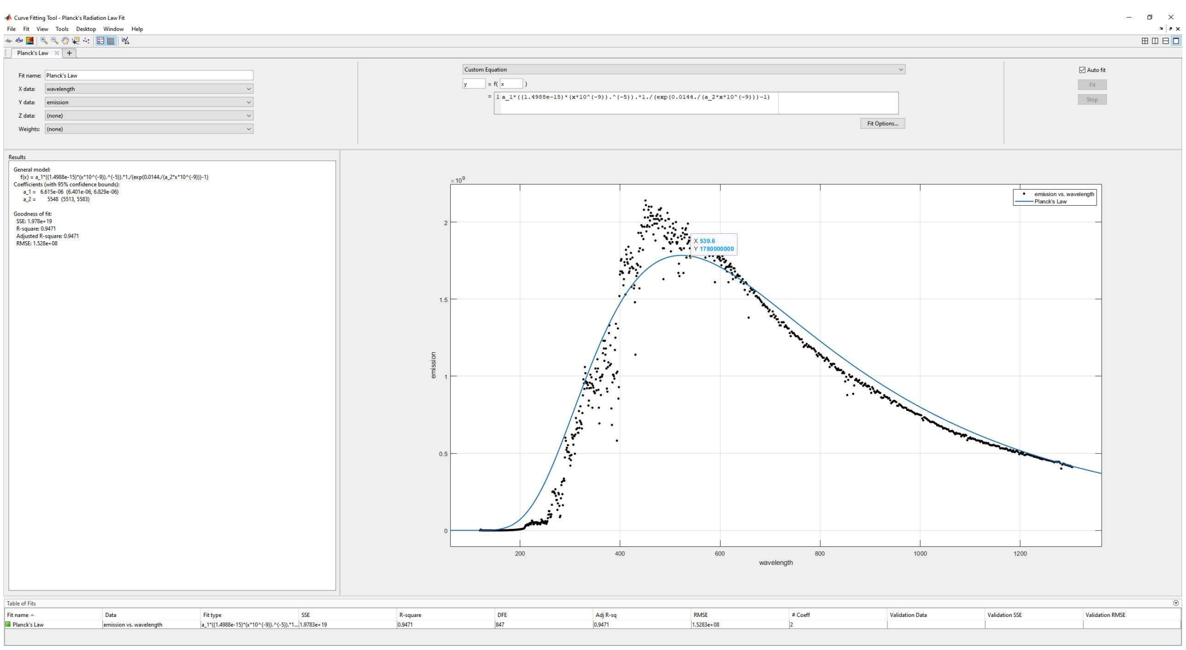


Figure 1: Curve Fitting Tool used to fit the given data to Planck's Radiation Law. a_2 is the coefficient representing the temperature

From this fit, the temperature is determined to be equal to the coefficient a_2 which is found to be 5548 K

c. From the curve, the peak wavelength seems to be approximately 517.5 nm Using the formula obtained in (a) and temperature in (b), the wavelength can be calculated as:

$$\lambda_{max} = \frac{0.00289}{5548} = 520.9\ nm$$

Hence, there's a 0.65% error.

d. Wavelengths can be determined as:

$$\lambda_{max,human} = 8626.866\ nm$$
$$\lambda_{max,walls} = 9697.986\ nm$$
$$\lambda_{max,space} = 1.059e+06\ nm$$

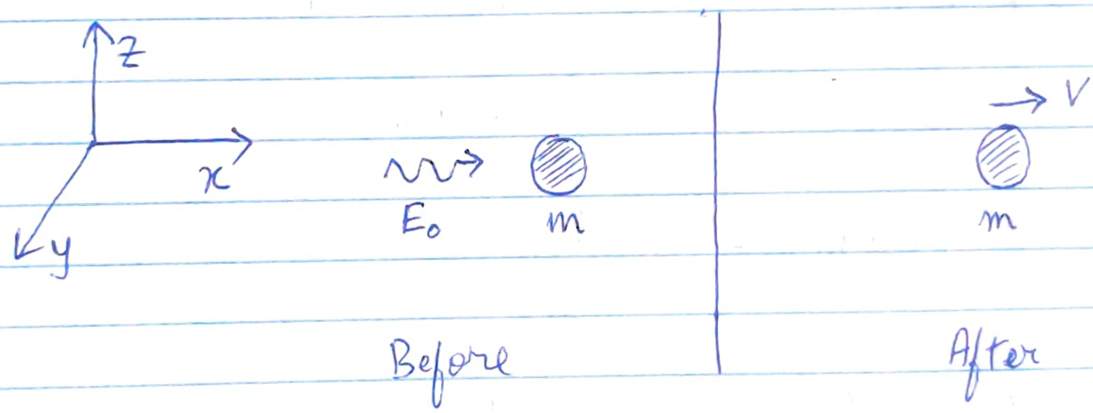
Q2 Energy of incident photon $= E_0 = h\nu_0$

Momentum of incident photon $= \bar{p}_0$

Mass of object $= m$

a Perfectly absorptive surface.

$\therefore$ No photon after the interaction.



~~Since~~ the motion is happening in the x-axis,

~~At~~ Applying conservation of Linear Momentum:

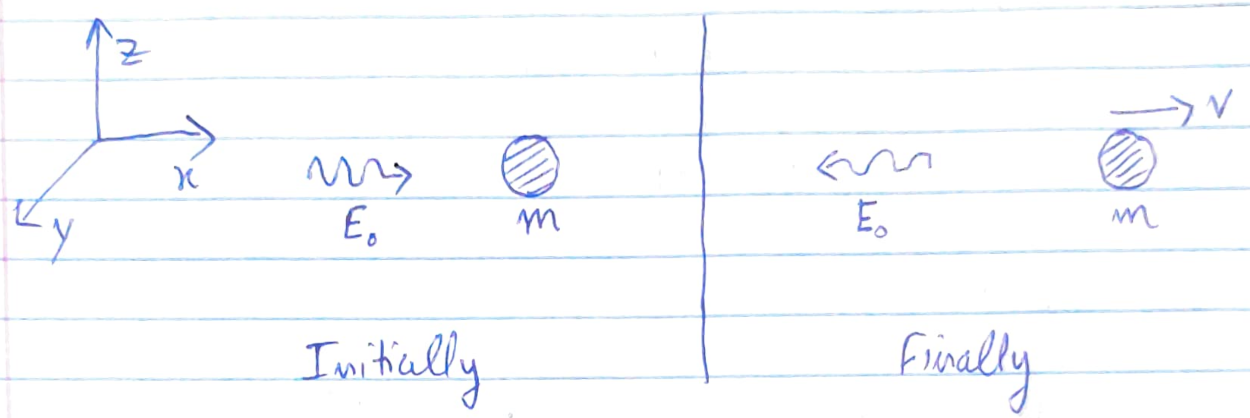
Mom. of photon before + Mom. of mass before = Mom. of photon after + Mom. of mass after.

$$\Rightarrow p_0 \hat{x} + 0 = 0 + \bar{P}_{mass,f}$$

$$\therefore \boxed{P_{mass,f} = \bar{p}_0}$$

Final momentum of ~~that~~ the object

b Perfectly Reflective Surface



Applying conservation of linear momentum:

$$p_0 \hat{x} + 0 = -p_0 \hat{x} + \bar{p}_{mass,f}$$

$$\Rightarrow \boxed{\bar{p}_{mass,f} = 2p_0 \hat{x}}$$

Note that the reflected photon will have the same momentum, but in opposite direction, because the surface is completely (perfectly) reflective.

(i) Energy of reflected photon = Energy of incident photon.

(ii) Since $p = \frac{E}{c}$, $|\bar{p}_{photon,f}| = |\bar{p}_{photon,i}|$

c The velocity of the object after the interaction can be given by :

$$\bar{v}_{mass} = \frac{\bar{p}_{mass,f}}{m} = \frac{2\bar{p_0}}{m}$$

The observer is now on the object of mass 'm'

Conserving the linear momentum :

$$p_i = p_f$$

$$\Rightarrow \frac{E_\nu}{\underbrace{\text{obs. velocity of photon}}_{=c}} = \frac{E_\nu'}{\underbrace{\text{obs. velocity of photon}}_{=c+V_{mass}}}$$

$\because$ both are moving away.

$$\therefore \frac{h\nu_0}{c} = \frac{h\nu'}{c+V_{mass}}$$

$$\therefore \boxed{\nu' = \nu_0\left(1 + \frac{V_{mass}}{c}\right)}$$

Question 2(d): The following MATLAB script was used to find the radiation pressure:

```
%Q2(d)

spectral_data=readmatrix('Problem 1.1 (Spectral data from the sun) - Sheet1.csv');
wavelength=spectral_data(:,1);
emission=spectral_data(:,2);
I=wavelength.*emission;
I_net=sum(I);
P=I_net/(3*10^8)
```

This gives us the radiation pressure as $P = 23mW$

This is a linear system (single dimension)

Q3 a

Let the rest mass of the free electron be denoted as m

The initial energy of the system can be defined as:

$$E_i = E_\nu + \sqrt{(mc^2)^2 + (p_1 c)^2}$$

where p_1 is the momentum of electron moving with velocity V_1

And, initial momentum of the system:

$$p_i = \frac{E_\nu}{c} + p_1$$

The final energy of the system can be defined as:

$$E_f = \sqrt{(mc^2)^2 + (p_2 c)^2}$$

where p_2 is momentum of electron moving with velocity V_2

And, final momentum of the system:

$$p_f = p_2$$

If linear momentum & energy were conserved, we could write:

$$E_f = E_i \quad \text{and}$$

$$p_f = p_i$$

$$\Rightarrow p_2 = \frac{E_\nu}{c} + p_1$$

$$\Rightarrow p_2 c = E_\nu + p_1 c \quad \text{—— (1)}$$

from, $E_f = E_i$

$$\Rightarrow \sqrt{(mc^2)^2 + (p_2 c)^2} = E_\nu + \sqrt{(mc^2)^2 + (p_1 c)^2}$$

$$\Rightarrow (mc^2)^2 + (p_2 c)^2 = E_\nu^2 + (mc^2)^2 + (p_1 c)^2 + 2E_\nu\sqrt{(mc^2)^2 + (p_1 c)^2}$$

$\Rightarrow$ Using (1) and cancelling $(mc^2)^2$ terms on both sides:

$$\Rightarrow (E_\nu + p_1 c)^2 = E_\nu^2 + (p_1 c)^2 + 2E_\nu\sqrt{(mc^2)^2 + (p_1 c)^2}$$

$$\Rightarrow E_\nu^2 + (p_1 c)^2 + 2E_\nu p_1 c = E_\nu^2 + (p_1 c)^2 + 2E_\nu\sqrt{(mc^2)^2 + (p_1 c)^2}$$

$$\therefore p_1 c = \sqrt{(mc^2)^2 + (p_1 c)^2}$$

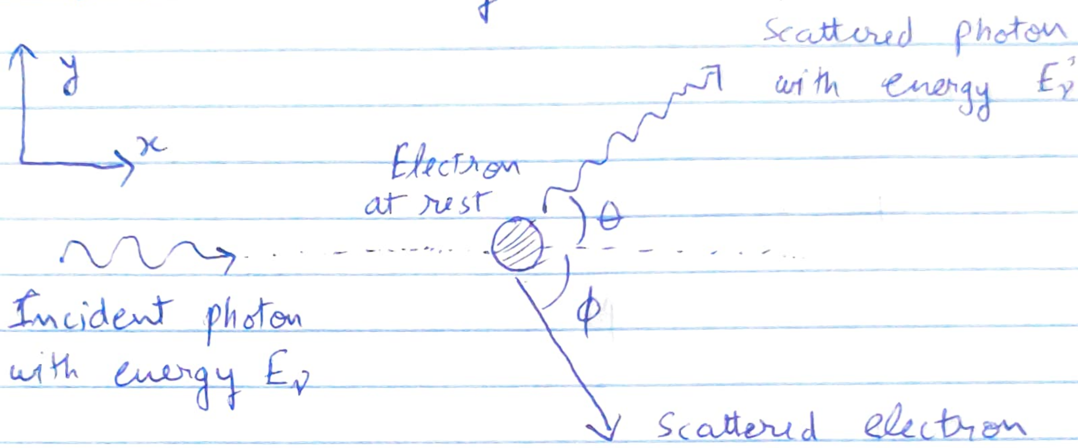
This can only be true if $mc^2 = 0$, which is not possible, since:

(i) Rest mass of electron cannot be 0

(ii) speed of light cannot be 0

$$\boxed{\text{Hence, Energy \& Linear momentum are NOT conserved}}$$

Q3 b In Compton scattering, the photon isn't absorbed by the electron, but instead undergoes elastic scattering.



The initial energy of the system can be written as:

$$E_i = E_\nu + m_0c^2$$

where m_0 is rest mass of the electron

The final energy of the system can be written as:

$$E_f = E_\nu' + \sqrt{(m_0c^2)^2 + (p_ec)^2}$$

where p_e is momentum of electron

Conservation of Energy can be applied as:

$$E_i = E_f$$

$$E_\nu + m_0c^2 = E_\nu' + \sqrt{(m_0c^2)^2 + (p_ec)^2} \quad (1)$$

The initial momentum can be written along x-axis as:

$p_{x,i} = \frac{E_\nu}{c}$ & along y-axis, $p_{y,i} = 0$

The final momentum can be written as:

Along x-axis: $p_{x,f} = \frac{E_\nu'}{c} \cos\theta + p_e \cos\phi$

Along y-axis: $p_{y,f} = \frac{E_\nu'}{c} \sin\theta - p_e \sin\phi$

Linear momentum can be conserved in both axes as,

$p_{x,i} = p_{x,f}$

$$\Rightarrow \frac{E_\nu}{c} = \frac{E_\nu'}{c} \cos\theta + p_e \cos\phi \quad \text{—— } (2)$$

and

$p_{y,i} = p_{y,f}$

$$\Rightarrow 0 = \frac{E_\nu'}{c} \sin\theta - p_e \sin\phi \quad \text{—— } (3)$$

Hence, equations 1, 2, 3 are results of conservation of Energy & Linear momentums in Compton Scattering.

Q3 c Atomic electrons are ~~eab~~ capable of absorbing photons.
They do so by absorbing the photon energy and rising to a higher energy state.
Like in the case of the ~~Bohr~~ Bohr Atom, the energy states of electrons are quantised.

Therefore, electrons can absorb photons if ~~ther~~ their energy matches the energy required to get excited to a higher, quantised state.

In the case of the Bohr atom, ~~to~~ the electron states are quantised as,

$$E_n = \frac{-E_0}{n^2}$$

where E_0 is a physical constant and n is the principle atomic number.

Hence, for an electron to absorb a photon and move ~~to~~ from n to $n+1$, the photon energy, $\hbar\omega$ should be:

$$\hbar\omega = \frac{-E_0}{(n+1)^2} - \left(-\frac{E_0}{n^2}\right)$$

Moreover, in the case of the photoelectric effect, the incident photon can knock off an electron from its atomic state if $\hbar\omega > \phi$, where ϕ is the work function of the material/atom.

Q4 (a)

$$\lambda_{dB} = \frac{h}{mv} = \frac{h}{p}$$

mass of neutron $= m = 1.675 \times 10^{-27}$ kg

$h = 6.626 \times 10^{-34}$ m^2 kg/s

given, $\lambda_{dB} = 1$ nm $= 1 \times 10^{-9}$ m

$$\therefore v = \frac{h}{m \lambda_{dB}} = \frac{3.956 \times 10^{-34}}{10^{-27} \times 10^{-9}} = \underline{\underline{395.6 \text{ m/s}}}$$

$$K.E. = \frac{p^2}{2m} = \frac{\left(\frac{h}{\lambda_{dB}}\right)^2}{2m} = \frac{h^2}{2m\lambda_{dB}} = \underline{\underline{1.31 \times 10^{-22} \text{ J}}}$$

(b) At the above velocity, time taken to travel 100 m is,

$$t = \frac{s}{v} = \frac{100}{395.6} = 0.253 \text{ s}$$

Given, $t_{1/2} = 15 \times 60$ s

Using the half-life equation,

$$N(t) = N_0 \left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

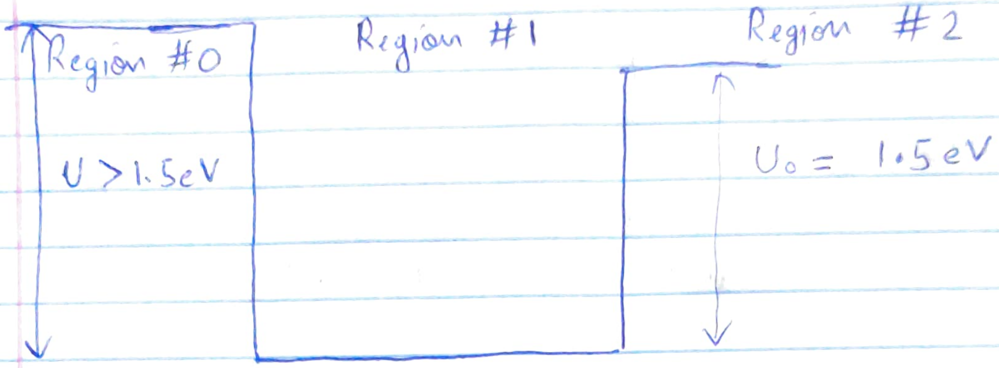
Fraction of Neutrons remaining & have not decayed

$$\therefore \frac{N(t=0.253)}{N_0} = \left(\frac{1}{2}\right)^{\frac{0.253}{900}}$$

$$\therefore \text{Fraction decayed} = 1 - \left(\frac{1}{2}\right)^{\frac{0.253}{900}} = 2.811e-4$$

The well can be represented as:

35



The time variant schrodinger wave can be given as:

$$\frac{d^2\Psi}{dx^2} = \frac{2m}{\hbar^2}(U-E)\Psi$$

In the barrier in RHS, we can define,

$$K^2 = \frac{2m}{\hbar^2}(U_0 - E)$$

Hence, the S.W.E can be written as:

$$\frac{d^2\Psi_2}{dx^2} = K^2\Psi_2$$

$$\therefore \Psi_2 = A\exp(-Kx) + \cancel{B(Kx)}\; B\exp(Kx)$$

This is the general sol^n in RHS barrier.

Inside the well, we can define:

$$k^2 = \frac{2m}{\hbar^2} E$$

Hence, the S.W.E can be written as:

$$\frac{d^2 \psi_1}{dx^2} = -k^2 \psi_1$$

$$\therefore \boxed{\psi_1 = C \sin(kx) + D \cos(kx)}$$

This is the general solⁿ inside the well.

Applying Boundary condⁿ:

① when $x \to \infty$, ψ should be finite.
$\therefore$ In ψ_2, B shoul be set to 0.

$$\therefore \boxed{\psi_2 = A \exp(-Kx)}$$

② The ψ should be continuous at the boundary and so should the derivative.

③ The ψ when normalised in the entire region = 1

Since the electron has an eigenstate at $E = 1.3\,eV$

$$k = \sqrt{\frac{2m}{\hbar^2} E} = \sqrt{\frac{2 \times 9.11e{-31} \times 1.3}{6.58e{-16}}} = \sqrt{3.599e{-15}}$$

$$= 5.999 \times 10^{-8}\ m$$

$$K = \sqrt{\frac{2m}{\hbar^2}(1.5 - 1.3)} = 2.353 \times 10^{-8}\ m$$