

Chapter 22

Gauss's Law

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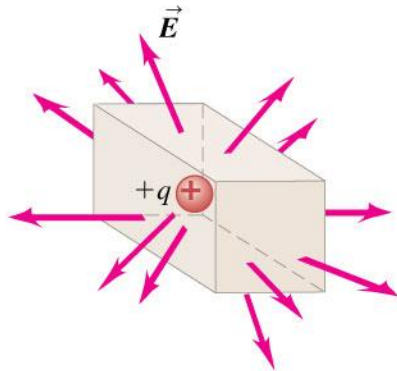
Goals for Chapter 22

- Electric field at a **surface** \rightarrow charge within the surface
 - **Electric flux**
 - **Gauss's law**
 - Charge distribution of a conductor
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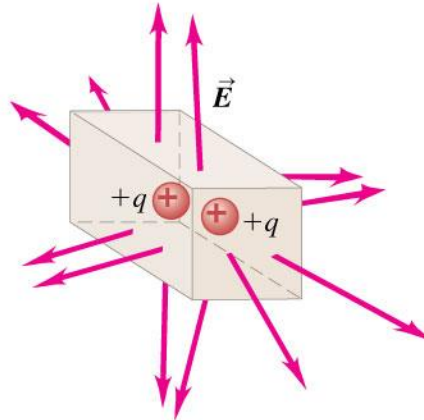
22.1 Charge and electric flux

- **Electric flux** ~ the number of electric field lines passing through a surface
- Positive charge within the box produces outward electric flux through the surface of the box, and negative charge produces inward flux.

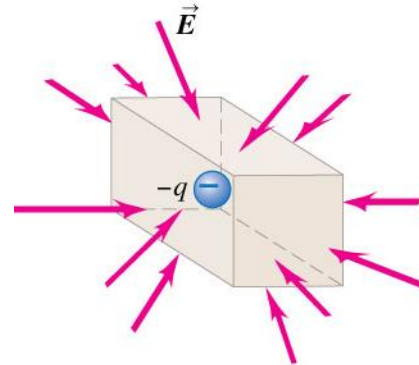
(a) Positive charge inside box, outward flux



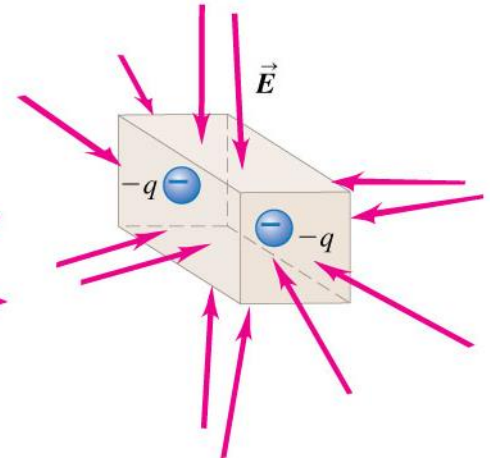
(b) Positive charges inside box, outward flux



(c) Negative charge inside box, inward flux



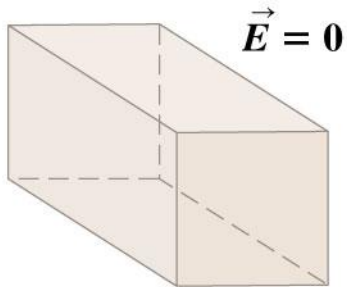
(d) Negative charges inside box, inward flux



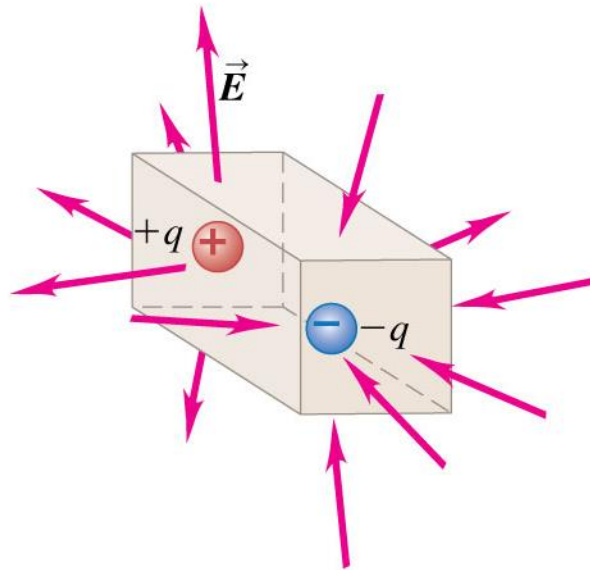
Zero net charge inside a box

- Three cases in which there is **zero net charge** inside a box and **no net electric flux** through the surface of the box.

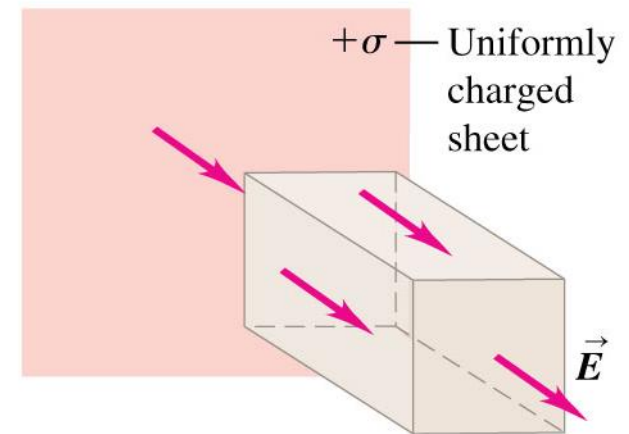
(a) No charge inside box,
zero flux



(b) Zero *net* charge inside box,
inward flux cancels outward flux.

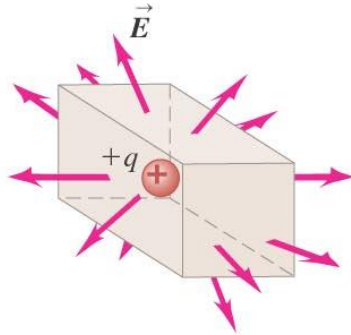


(c) No charge inside box,
inward flux cancels outward flux.

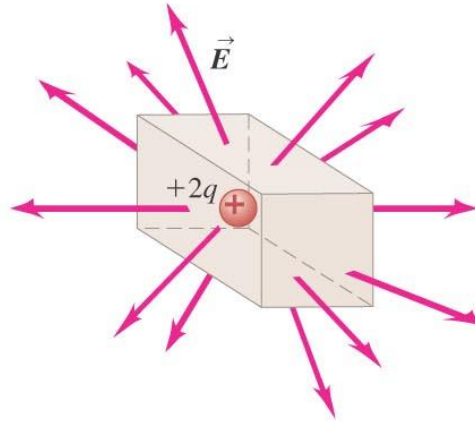


What affects the flux through a box?

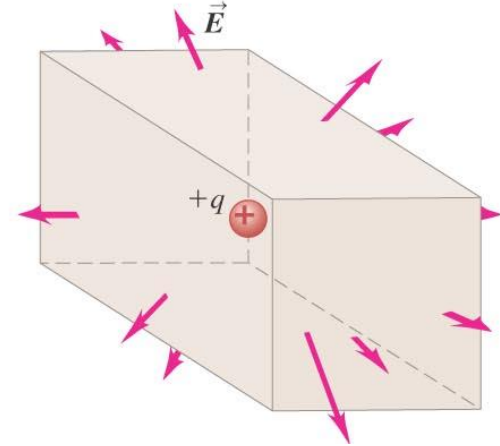
(a) A box containing a charge



(b) Doubling the enclosed charge doubles the flux.



(c) Doubling the box dimensions does not change the flux.

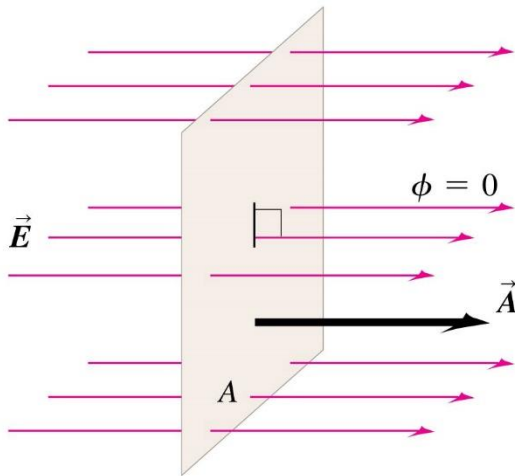


- The net electric flux is **directly proportional** to the net charge within the surface.
- The net electric flux is **independent** of the size of the closed surface.

22.2 Calculating electric flux

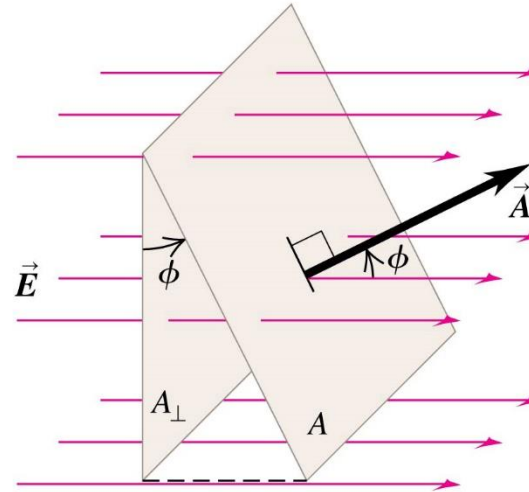
Surface is face-on to electric field:

- \vec{E} and \vec{A} are parallel (the angle between \vec{E} and \vec{A} is $\phi = 0$).
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA$.



Surface is tilted from a face-on orientation by an angle ϕ :

- The angle between \vec{E} and \vec{A} is ϕ .
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$.



- Area (surface) Vector:

$$\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$$

$$\vec{A} = \hat{n}A$$

\hat{n} , unit normal vector.

(for uniform field and flat surface)

Flux of a nonuniform electric field

- In general, the flux through a surface must be computed using a **surface integral** over the area:

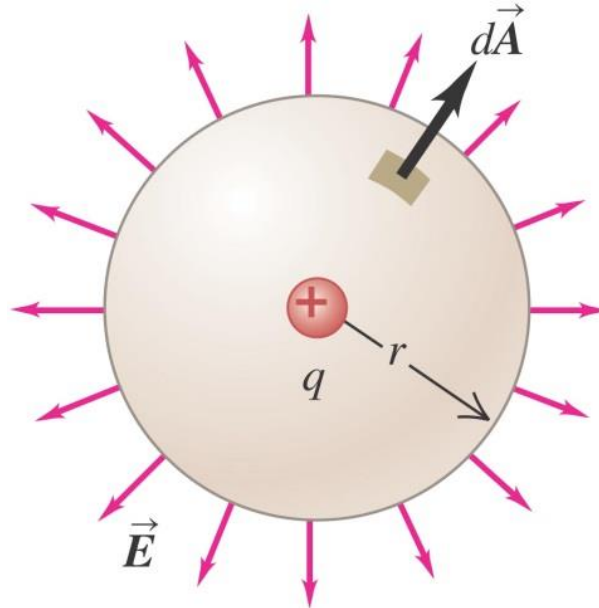
The diagram shows the equation for electric flux through a surface, with labels and arrows pointing to specific parts of the equation:

- Electric flux through a surface**: Points to Φ_E .
- Magnitude of electric field \vec{E}** : Points to E in the first integral.
- Angle between \vec{E} and normal to surface**: Points to $\cos \phi$.
- Element of surface area**: Points to dA in the first integral.
- Component of \vec{E} perpendicular to surface**: Points to E_{\perp} in the second integral.
- Vector element of surface area**: Points to $d\vec{A}$ in the third integral.

$$\Phi_E = \int E \cos \phi \, dA = \int E_{\perp} \, dA = \int \vec{E} \cdot d\vec{A}$$

- SI unit for electric flux: $\text{m}^2 \cdot \text{N/C}$
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Electric flux through a sphere



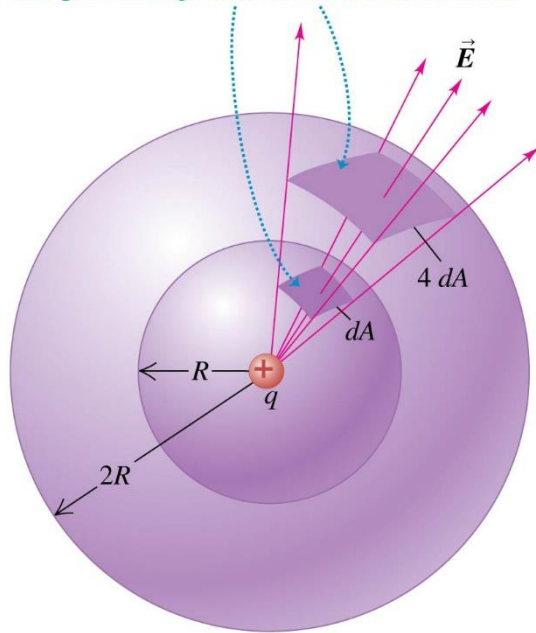
$$\Phi_E = EA = \frac{q}{4\pi\epsilon_0} 4\pi r^2 = \frac{q}{\epsilon_0}$$

independent of $r \rightarrow$ The flux is a conserved quantity!

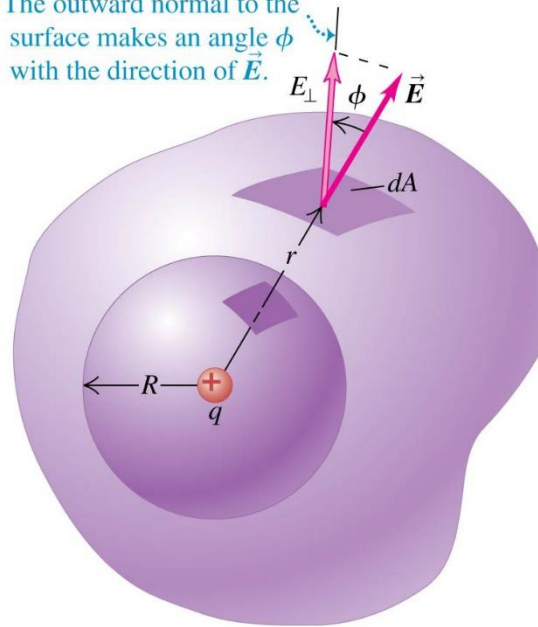
22.3 Gauss's law

- Gauss's law** is completely equivalent to Coulomb's law and provides a different way to express the relationship between electric charge and electric field.

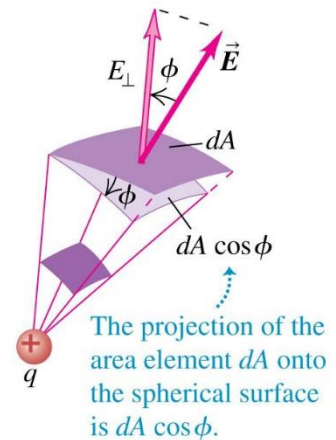
The same number of field lines and the same flux pass through both of these area elements.



(a) The outward normal to the surface makes an angle ϕ with the direction of \vec{E} .



(b)

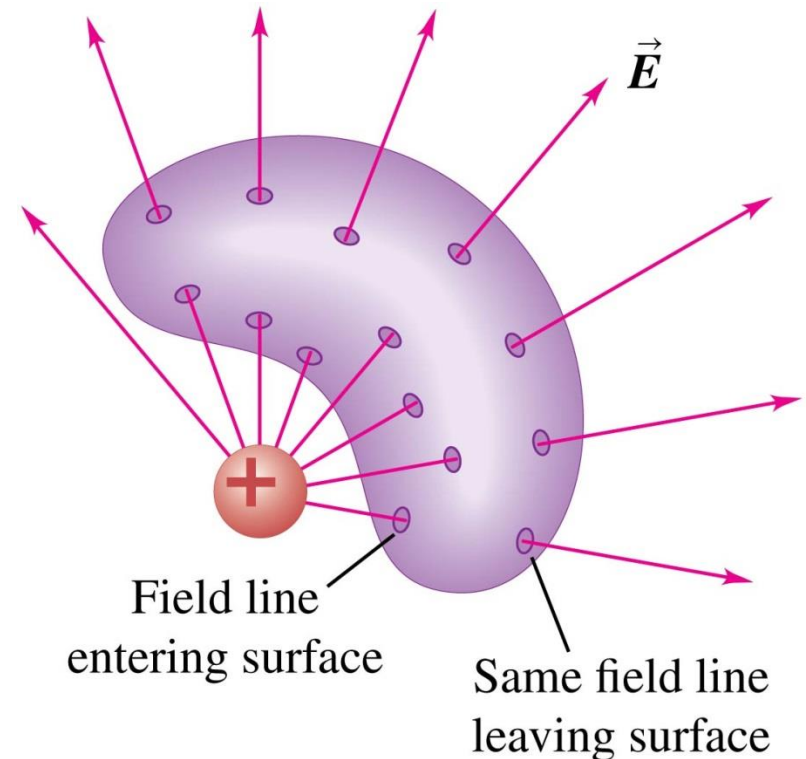


Gauss's law in a vacuum

- For a closed surface enclosing no charge:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = 0$$

- If an electric field line from the external charge enters the surface at one point, **it must leave at another**.



General form of Gauss's law

- **Gauss's law:** the total electric flux through a closed surface is equal to the total (net) electric charge inside the surface, divided by ϵ_0 :

Gauss's law:

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

Electric flux through a closed surface of area A = surface integral of \vec{E}

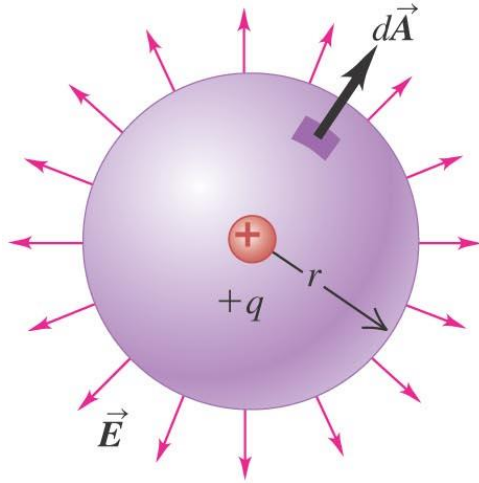
Total charge enclosed by surface

Electric constant

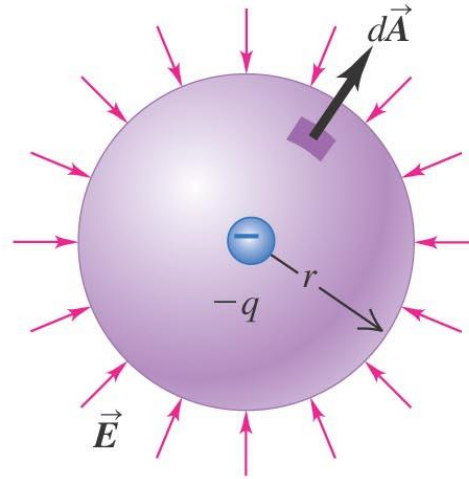
- Gauss's law is very powerful for various **symmetrical charge distributions**.
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Positive and negative flux

(a) Gaussian surface around positive charge:
positive (outward) flux

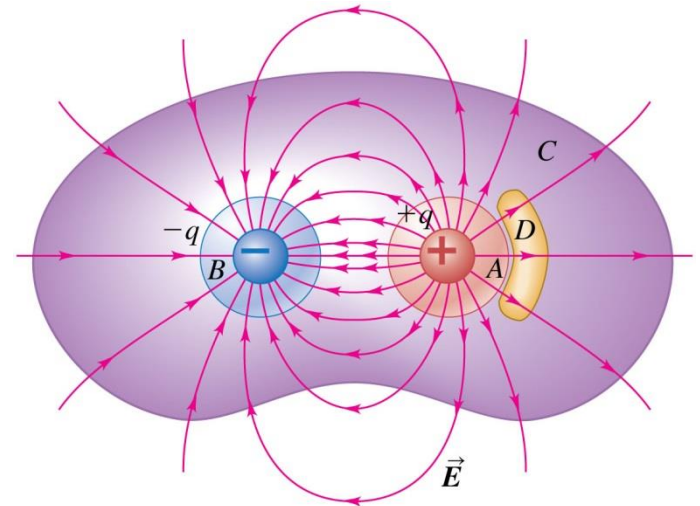


(b) Gaussian surface around negative charge:
negative (inward) flux



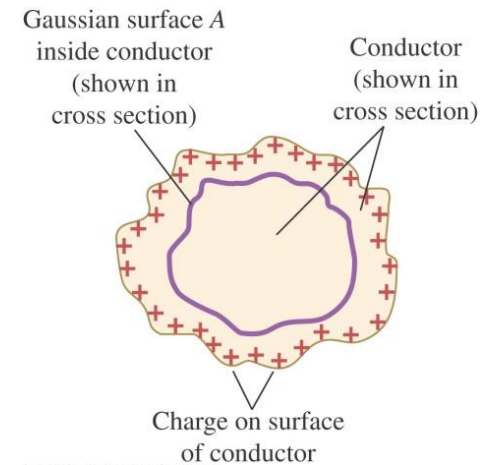
For (a/b)

$$\begin{aligned}\Phi_E &= \oint E_{\perp} dA = \oint \left(\frac{\pm q}{4\pi\epsilon_0 r^2} \right) \\ &= \frac{\pm q}{4\pi\epsilon_0 r^2} \oint dA = \frac{\pm q}{4\pi\epsilon_0 r^2} 4\pi r^2 \\ &= \frac{\pm q}{\epsilon_0}\end{aligned}$$



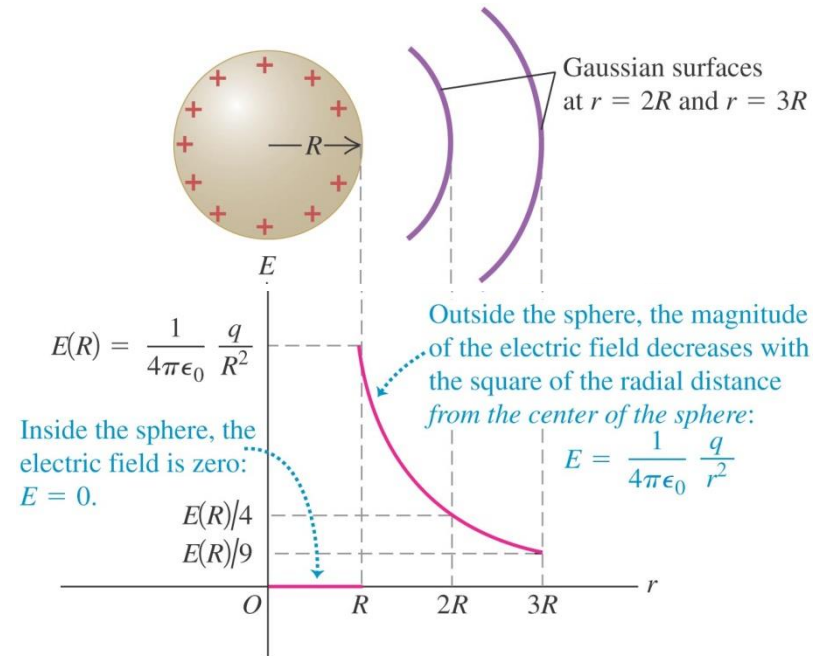
22.4 Applications of Gauss's law

- Under electrostatic conditions, the electric field inside a conductor is 0. If not, free charge within the conductor should move.
- $\Phi_E = 0$ for any Gaussian surface within the conductor.
- Excess charge is 0 within the conductor and resides only on the surface .



We place a total positive charge q on a solid conducting sphere with radius R (Fig. 22.18). Find \vec{E} at any point inside or outside the sphere.

$$E(4\pi r^2) = \frac{q}{\epsilon_0} \Rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$



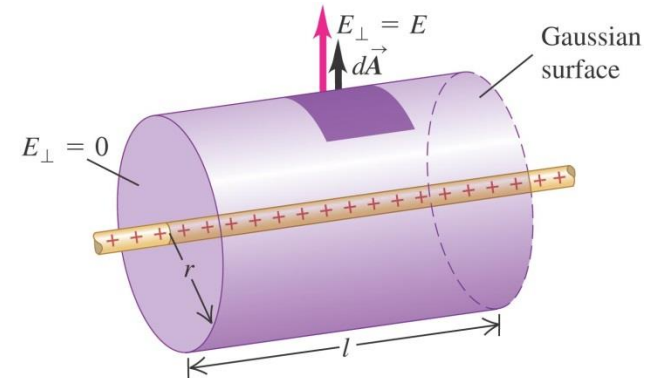
Field of a uniform line charge

Electric charge is distributed uniformly along an infinitely long, thin wire. The charge per unit length is λ (assumed positive). Find the electric field using Gauss's law.

For a cylindrical Gaussian surface of radius r ,

$$\Phi_E = 2\pi r l E = \frac{\lambda l}{\epsilon_0} \Rightarrow \vec{E} = \hat{r} \frac{\lambda}{2\pi\epsilon_0 r}$$

Cylindrical symmetry

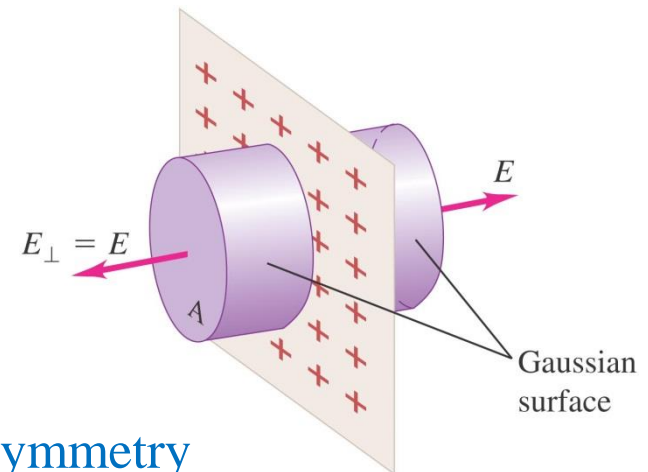


Field of an infinite plane sheet of charge

Use Gauss's law to find the electric field caused by a thin, flat, infinite sheet with a uniform positive surface charge density σ .

For a cylindrical Gaussian surface of crosssection A ,

$$\Phi_E = 2AE = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$



Plane symmetry

Field between two parallel conducting plates

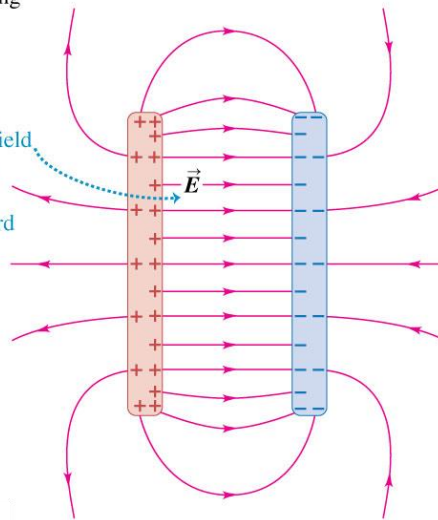
Two large plane parallel conducting plates are given charges of equal magnitude and opposite sign; the surface charge densities are $+\sigma$ and $-\sigma$. Find the electric field in the region between the plates.

For Gaussian surfaces S_1 and S_4 , $\Phi_E = AE = \frac{\sigma A}{\epsilon_0} \Rightarrow E = \frac{\sigma}{\epsilon_0}$

For Gaussian surfaces S_2 and S_3 , $E = 0$.

(a) Realistic drawing

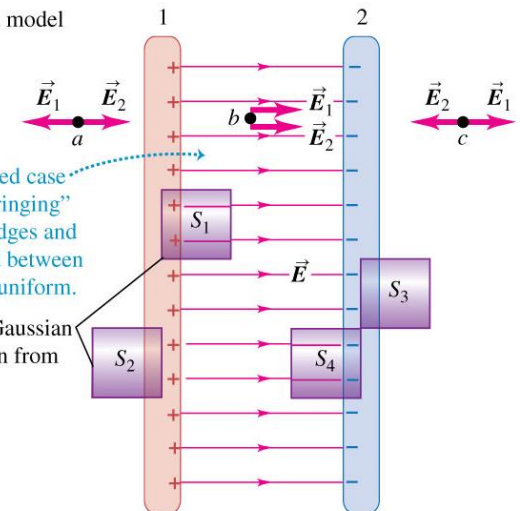
Between the two plates the electric field is nearly uniform, pointing from the positive plate toward the negative one.



(b) Idealized model

In the idealized case we ignore "fringing" at the plate edges and treat the field between the plates as uniform.

Cylindrical Gaussian surfaces (seen from the side)



Superposition principle gives the same results.

A uniformly charged sphere

Positive electric charge Q is distributed uniformly *throughout the volume* of an *insulating* sphere with radius R . Find the magnitude of the electric field at a point P a distance r from the center of the sphere.

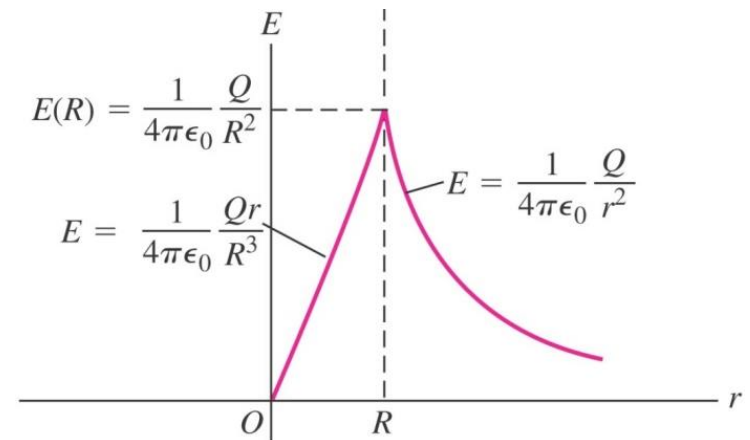
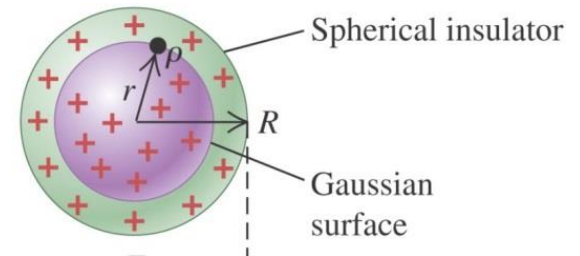
Volume charge density ρ : $\rho = Q / \left(\frac{4}{3} \pi R^3 \right)$

For a spherical Gaussian surface with radius $r < R$,

$$Q_{\text{encl}}(r) = \rho V_{\text{encl}}(r) = Q \frac{r^3}{R^3}$$

From Gauss's law,

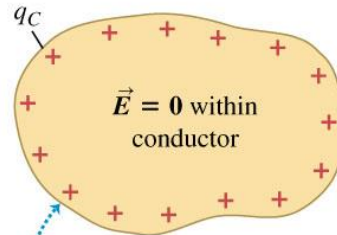
$$\Phi_E(r) = 4\pi r^2 E(r) = \frac{Q_{\text{encl}}(r)}{\epsilon_0} \Rightarrow E(r) = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3}$$



22.5 Charges on conductors

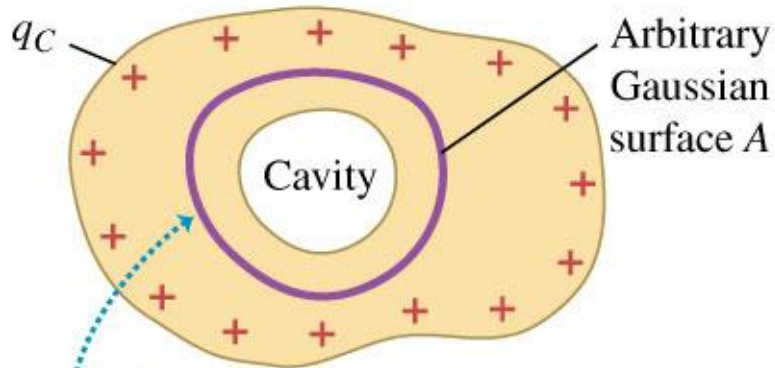
- Charges in a conductor having a cavity within it.

(a) Solid conductor with charge q_C



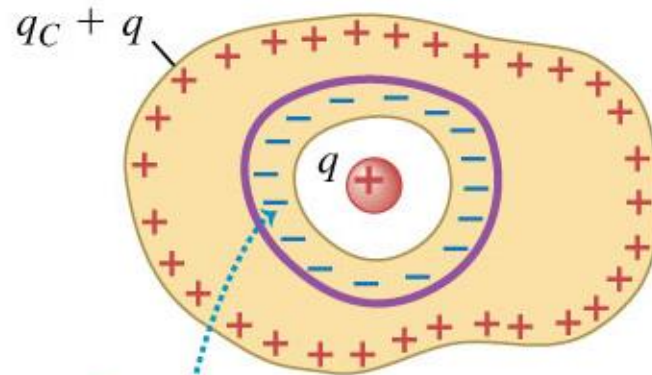
The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = 0$ within the conductor.

(b) The same conductor with an internal cavity



Because $\vec{E} = 0$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

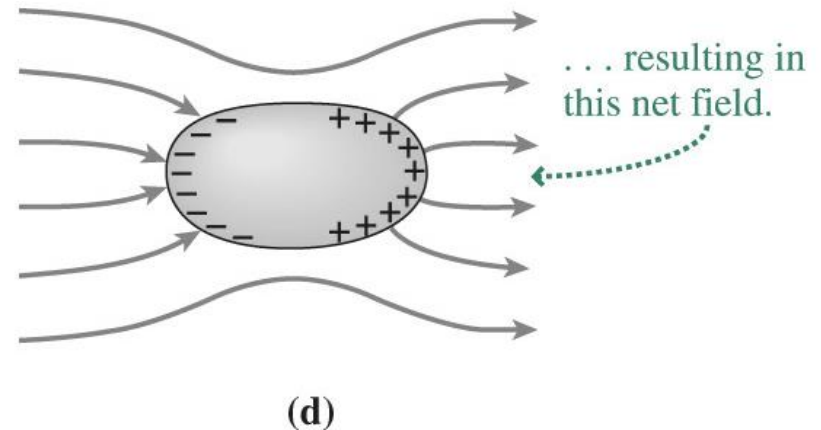
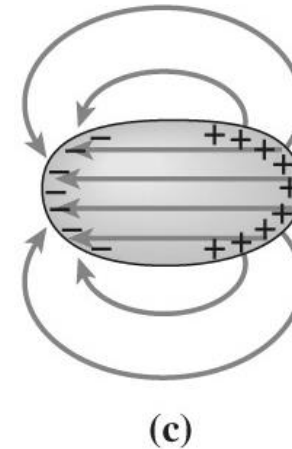
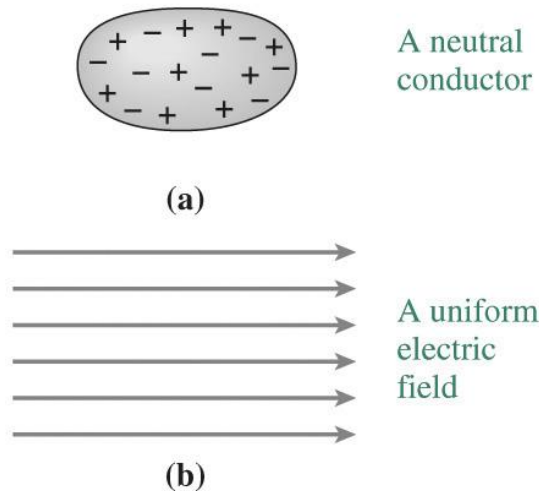
(c) An isolated charge q placed in the cavity



For \vec{E} to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.

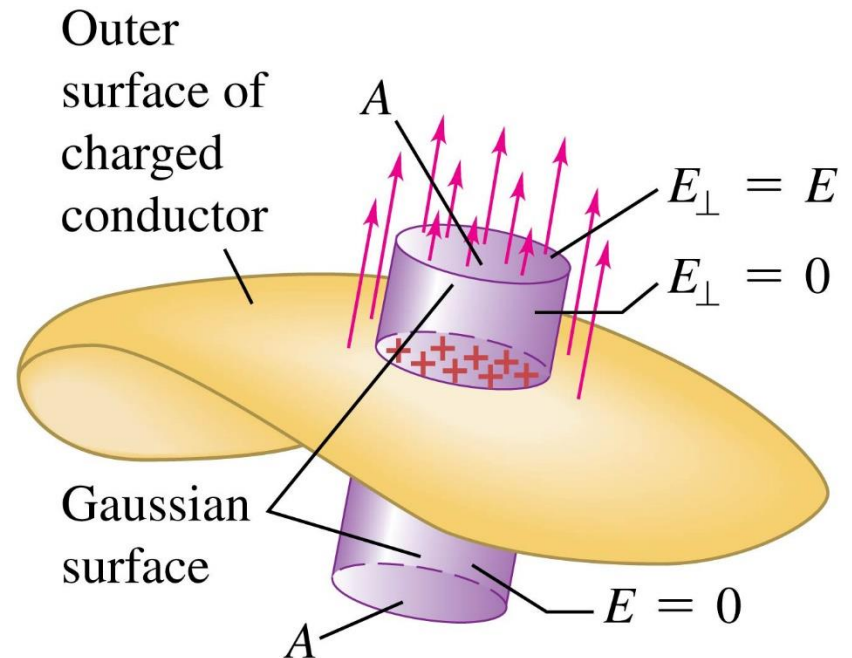
Charges on conductors in an external electric field

- Charges in conductors are free to move, and respond to an applied electric field until they become stationary with zero internal field
- Electrostatic Equilibrium**



Field at the surface of a conductor

- Gauss's law can be used to show that the **direction** of the electric field at the surface of any conductor is always **perpendicular to the surface**.
- The **magnitude** of the electric field just outside a charged conductor is proportional to the surface charge density σ .

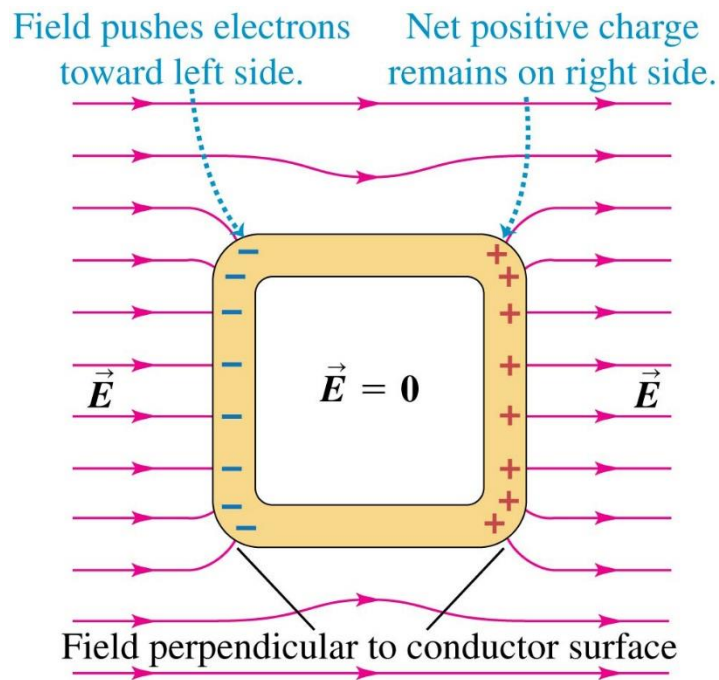


Electric field at surface of a conductor, \vec{E} perpendicular to surface $\rightarrow E_{\perp} = \frac{\sigma}{\epsilon_0}$

σ Surface charge density
 ϵ_0 Electric constant

Electrostatic shielding

- A conducting box is immersed in an external electric field.
- The field of the induced charges on the box combines with the external field to give **zero total field inside the box**.



22 Summary

- **Electric flux**
 - **Gauss's law**
 - **Various symmetric charge distributions**
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