UNIVERSITY OF OSLO

COMPUTATIONAL PHYSICS

# Project 5

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# Abstract

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1

# INTRODUCTION

CHAPTER

# 2 METHOD

The methods introduced in this chapter, are used to study the evolution of bodies, also referred to as particles, in the solar system or in the galaxy. Firstly, the fourth order Runge-Kutta method and the Velocity-Verlet method are introduced for a 2-body problem in 3 dimensions, however, since the aim of this project is to investigate the time evolution of a star cluster consisting of $N$ particles, the code is extended to $N$ bodies by incorporating suitable changes.

To generate the $N$ bodies, a function for generating the position coordinates of the N bodies as uniformly distributed particles within a sphere is introduced as discussed in sec Sec. 2.3, along with a function for generating masses that are randomly distributed by a Gaussian distribution around ten solar masses with a standard deviation of one solar mass.

The source codes for the algorithms described in this chapter can be found in the Github folder `https://?????`. [1]

## 2.1 Transformation between units

When considering at a planetary system or a larger system like a galaxy it is inconvenient to use the SI units for length and time. Instead, to investigate the evolution of astronomical systems, it is an advantage to use days, years (yr) or even longer time periods as the unit of time, and astronomical units (AU) or light years (ly) as the unit of distance. The change in unit system, evidently changes the considered constant, the gravitational constant $G$, which in SI units is given as

$$G = 6.67 \cdot 10^{-11} \frac{\mathrm{Nm}^2}{\mathrm{kg}^2} \tag{2.1}$$

For a planetary system like the Earth-Sun system it is better to consider distances in AU instead of meters, and use days as a measure of time, as the planet doesn't move far on its orbit in a second. Furthermore, it is an advantage to express the masses in the system in units of solar masses. Hence, the constants have to be transformed into these unit systems.

[1] FiXme Note: fix these lines

The gravitational constant G is transformed using

$$1\ \mathrm{AU} = 1.495 \cdot 10^{11}\ \mathrm{m} \qquad \text{and} \qquad 1\ \mathrm{M}_\odot = 1.989 \cdot 10^{30}\ \mathrm{kg} \tag{2.2}$$

giving the gravitational constant

$$G = 2.96 \cdot 10^{-4} \frac{\mathrm{AU}^2}{\mathrm{days}^2 \mathrm{M}_\odot} \tag{2.3}$$

which is convenient when considering a planetary system.

For a star cluster, the distances are greater and the time scales are larger than in the planetary system. Hence, it is more convenient to use years as the unit of time and lightyears as the unit of distance.

$$1\ \mathrm{yr} = 3.1536 \cdot 10^7 \mathrm{s} \qquad \text{and} \qquad c = 2.008 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}} \tag{2.4}$$

in which $c$ is the speed of light. This yields that 1 ly is

$$1\ \mathrm{ly} = 9.45 \cdot 10^1 5\ \mathrm{m} \tag{2.5}$$

Giving the gravitational constant

$$G = 1.536 \cdot 10^{-13} \frac{\mathrm{ly}^2}{\mathrm{yr}^2 \mathrm{M}_\odot} \tag{2.6}$$

## 2.2 Newtonian two-body problem and greater systems in three dimension

The problem of solving the time-evolution of a two-body system in three dimensions can reasonably be considered in two different coordinate systems: one coordinate system with one of the bodies in rest compared to the frame of reference in which the other body is moving, and one coordinate system with both of the bodies moving relative to the frame of reference. Both of these reference systems are depicted in Fig. 2.1.

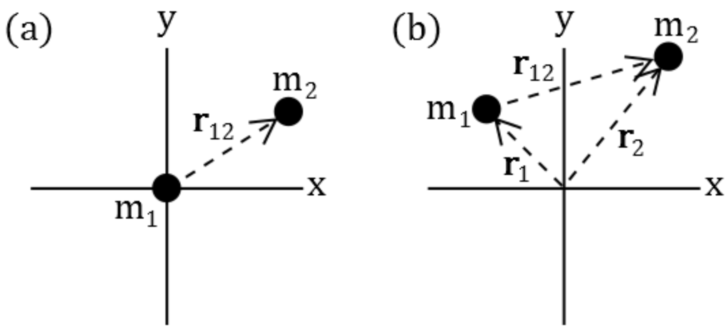



***Figure 2.1.*** Two-dimensional illustration of the three-dimensional problem of determining determining the relative distance and relative velocity between two bodies. In (a) body 1 with mass $m_1$ is considered stationary in position- and velocity-space, whilst body 2 with mass $m_2$ moves relative to body 1. In (b) both body 1 and 2 moves relative to the frame of reference in position and time, yielding that the position vector between body 1 and 2 is given as $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$.

In the codes presented in this section, solving the problem in coordinate system (a) will first be considered for simplicity. Thereafter, the codes will be extended to include the movement of body 1 relative to the coordinate system, since this will be useful when extending the codes to an N body system.

In the problem, $\mathbf{r}(t)$ is the three-dimensional space vector consisting of the coordinated $(x(t), y(t), z(t))$, whilst $\mathbf{v}(t)$ is the three-dimensional velocity vector with coordinates $(v_x(t), v_y(t), v_z(t))$, both of which are dependent on time.

In general, the considered differential equation is

$$\frac{dy}{dt} = f(t,y) \tag{2.7}$$

Which yields that

$$y(t) = \int f(t,y)dt \tag{2.8}$$

[2] For the two bodies in a three dimensional Newtonian gravitational field this corresponds to six coupled differential equations given by the vector equations

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \qquad \text{and} \qquad \frac{d\mathbf{v}}{dt} = -\frac{GM_1M_2}{r^3}\mathbf{r} \tag{2.9}$$

[3] in which $M_1$ and $M_2$ [4] are the masses of the two bodies, respectively, whilst $r$ is the distance between the bodies. The equations in (2.9) are computed by the script given below in which *drdt* corresponds to the derivative of the coordinates of the position, and *dvdt* corresponds to the derivative of the velocity coordinates.

```
void Derivative(double r[3], double v[3], double (&drdt)[3], double (&dvdt)[3], double
    G, double mass){
    drdt[0] = v[0];
    drdt[1] = v[1];
    drdt[2] = v[2];
    double distance_squared = r[0]*r[0] + r[1]*r[1] + r[2]*r[2];
    double newtonian_force = -G*mass/pow(distance_squared,1.5);
    dvdt[0] = newtonian_force*r[0];
    dvdt[1] = newtonian_force*r[1];
    dvdt[2] = newtonian_force*r[2];
}
```

When including movement of both bodies relative to the frame of reference, the *Derivative* function must be slightly modified, since then the relative position of the two bodies will be given as $\mathbf{r}_{12} = \mathbf{r}_2 - \mathbf{r}_1$. For a general case with $N$ particles, the *distance_ squared* between body $i$ and $j$, and the acceleration in $x$, $y$ and $z$ due to the Newtonian force between body $i$ and $j$ can be determined by the following lines of code.

```
for (int j=0; j<number_of_particles; j++)
    {
        if (j!=i)
        {
            distance_squared = 0;
            for (int k=0; k<3; k++)
```

[2]FiXme Note: do we need to write $y_{i+1}$ eq from p. 250 in lecture notes??
[3]FiXme Note: maybe we should divide by mass as on p. 248??
[4]FiXme Note: fix the this with $M_1$ and $M_2$

```
            {
                distance_squared += (r(i,k)-r(j,k))*(r(i,k)-r(j,k));
            }
            force_between_particles = m(j) * pow(distance_squared,-1.5);
            acc_x += G*force_between_particles*(r(j,0)-r(i,0));
            acc_y += G*force_between_particles*(r(j,1)-r(i,1));
            acc_z += G*force_between_particles*(r(j,2)-r(i,2));
        }
    }
```

The if statement in the for loop over all bodies adds up the acceleration in all three dimensions of all particles due to the presence of other particles. Hence, for the two body problem, the argument in the if statement will only be true once for each of the two particles.

### 2.2.1 Velocity-Verlet method

- Remember to write about accuracy of algorithm!!

Consider the Taylor expansion of the vector function $\mathbf{r}(t_i \pm \delta t)$:

$$\mathbf{r}(t_i \pm \delta t) = \mathbf{r}(t_i) \pm \mathbf{v}(t_i)\delta t + \mathbf{a}(t_i)\frac{\delta t^2}{2} \pm \frac{\delta t^3}{6}\frac{d^3\mathbf{r}(t_i)}{dt^3} + \mathscr{O}(\delta t^4) \tag{2.10}$$

Adding the two expressions in Eq. (2.10) gives

$$\mathbf{r}(t_i + \delta t) = 2\mathbf{r}(t_i) - \mathbf{r}(t_i - \delta t) + \mathbf{a}(t_i)\delta t^2 + \mathscr{O}(\delta t^4) \tag{2.11}$$

which has a truncation error that goes as $\mathscr{O}(\delta t^4)$.

$$\mathbf{r}(t_i + \delta t) = \mathbf{r}(t_i) + \mathbf{v}(t_i)\delta t + \frac{1}{2}\mathbf{a}(t_i)\delta t^2 \tag{2.12}$$

$$\mathbf{v}(t + \delta t) = \mathbf{v}(t) + \frac{1}{2}(\mathbf{a}(t) + \mathbf{a}(t + \delta t))\delta t \tag{2.13}$$

The velocity is in the algorithm calculated [5] by first calculating

$$\mathbf{v}_{part1}(t + \delta t) = \mathbf{v}(t) + \frac{1}{2}\mathbf{a}(t)\delta t \tag{2.14}$$

and then use ?? [6] to determine $\mathbf{a}(t + \delta t)$, which is then used to compute the remaining term of Eq. (2.13) as

$$\mathbf{v}_{part2}(t + \delta t) = \frac{1}{2}\mathbf{a}(t + \delta t)\delta t \tag{2.15}$$

The velocity-Verlet method uses the algortihm *Derivative* described in Sec. 2.2, to generate the six differential equations, in the following while-loop that runs until reaching the final time in time steps of length $\delta t = (t_{initial} - t_{final})/(\#timesteps)$.

[5] FiXme Note: ad to gange
[6] FiXme Note: fix this!

```
while(time<=t_final){
Derivative(r,v,drdt,dvdt,G,mass);
for(int i=0; i<6 ; i++){
r[i] = r[i]+dt*drdt[i] + 0.5 * dt * dt * dvdt[i];
v_partly[i] = drdt[i] + 0.5 * dt * dvdt[i];
dvdt[i] = v_partly[i];
}
Derivative(r,v,drdt,dvdt,G,mass);
for(int i=0; i<n ; i++){
v[i] = v_partly[i] + 0.5 * dt * dvdt[i];
}
time += dt;
}
```

### 2.2.2 Fourth Order Runge-Kutta Method

- Remember to write about accuracy of algorithm!!

The Runge-Kutta method is based on Taylor expansions, with the next function value after a times step $\delta t = t_i - t_{i+1}$ being computed from four more or less improved slopes of the function in the points $t_i$, $t_i + \delta t/2$ and $t_{i+1}$.

The first step of the RK4 method is to compute the slope $k_1$ of the function in $t_i$ by

$$k_1 = \delta t f(t_i, y_i)$$

Then the slope $k_1$ at the midpoint is computed from $k_1$ as

$$k_2 = \delta t f(t_i + \delta t/2, y_i + k_1/2)$$

The slope at the midpoint is then improved from $k_2$ by

$$k_3 = \delta t f(t_i + \delta t/2, y_i + k_2/2)$$

from which the slope $k_4$ at the next step $y_{i+1}$ is predicted to be

$$k_4 = \delta t f(t_i + \delta t, y_i + k_3)$$

From the computed slopes $k_1$, $k_2$, $k_3$ and $k_4$, the function value at $t_i + \delta t$ is computed as

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{2.16}$$

When implementing this for the two-body problem in three dimensions, it boils down to a continuous call of two functions, namely the function *Derivative* given in Sec. 2.2 and the function *updating_dummies* given below.

```
void updating_dummies(double dt, double drdt[3], double dvdt[3], double (&r_dummy)[3],
    double (&v_dummy)[3], double number, double (&kr)[3], double (&kv)[3], double
    r[3], double v[3])
{
```

```
    for (int i = 0; i<3; i++){
        kr[i] = dt * drdt[i];
        kv[i] = dt * dvdt[i];
        r_dummy[i] = r[i] + kr[i]/number;
        v_dummy[i] = v[i] + kv[i]/number;
    }
}
```

The function *updating_dummies* computes the values of $k_1$, $k_2$, $k_3$ and $k_4$ for all three space coordinates and velocity coordinates from the derivatives *drdt* and *dvdt* computed by the *Derivative* function. To compute the next step given by Eq. (2.16), the following succession of function calls are made until the time reaches the final time $t_{final}$ after $(t_{final} - t_{inital})/\delta t$ time steps.

```
while(time<=t_final){
    Derivative(r,v,drdt,dvdt,G,mass);
    updating_dummies(dt,drdt,dvdt,r_dummy,v_dummy,2,k1r,k1v,r,v);
    Derivative(r_dummy,v_dummy,drdt,dvdt,G,mass);
    updating_dummies(dt,drdt,dvdt,r_dummy,v_dummy,2,k2r,k2v,r,v);
    Derivative(r_dummy,v_dummy,drdt,dvdt,G,mass);
    updating_dummies(dt,drdt,dvdt,r_dummy,v_dummy,1,k3r,k3v,r,v);
    Derivative(r_dummy,v_dummy,drdt,dvdt,G,mass);
    for (int i = 0; i<n; i++){
        k4r[i] = dt*drdt[i];
        k4v[i] = dt*dvdt[i];
    }
    for (int i=0; i<n;i++){
        r[i] = r[i] +(1.0/6.0)*(k1r[i]+2*k2r[i]+2*k3r[i]+k4r[i]);
        v[i] = v[i] +(1.0/6.0)*(k1v[i]+2*k2v[i]+2*k3v[i]+k4v[i]);
    }
    time += dt;
    }
```

When including the movement of both bodies relative to the reference system or adding more bodies to the system, **r**'s, **v**'s, **k**'s etc. must be generated for all of the particles, yielding introduction of a for loop over all particles.

## 2.3 Generating Mass and Position for Cluster Particles

When considering the time evolution of an $N$-particle star cluster, in which the mass and position of each single particle is not known from a catalogue or of special significance, it is inappropriate to hard code the values for each single particle. Instead, it is an advantage to use probability distribution functions to generate random positions and masses for each of the particles. It is decided that the mass of the $N$ particles in the cluster should follow a Gaussian distribution around ten solar masses with a standard deviation of one solar mass. For the position, it is decided that the density of particle initially is uniform in a sphere with a radius of twenty solar masses. The is, however, not equivalent to saying that the particles are uniformly distributed in the Cartesian coordinates $x$, $y$ and $z$ and neither in the spherical coordinates $r$, $\phi$ and $\theta$, which complicates the problem a bit. The sections below present the functions for generating random masses an positions of $N$ particles with the desired distributions. Each of the functions are tested for a system with $N = 100,000$.

### 2.3.1 Gaussian Distributed Mass

[7]

Pseudo-random numbers, corresponding to masses, randomly distributed by a Gaussian distribution is generated following the Box-Muller transform. The basic form of the Box-Muller transform gives

$$X_1 = \sqrt{-2ln(V_1)}sin(2V_2) = Rcos(\theta)$$
$$X_2 = \sqrt{-2ln(V_1)}cos(2V_2) = Rsin(\theta)$$

where $R^2 = -2ln(V_1)$ and $\theta = 2V_1$. $X_1$ and $X_2$ are random numbers distributed according to a Gaussian distribution of mean 0 and variance 1. In polar form the above two equations becomes

$$X_1 = u\sqrt{\frac{-2ln(s)}{s}}$$
$$X_2 = v\sqrt{\frac{-2ln(s)}{s}}$$

where $s = R^2 = u^2 + v^2$. [8] Here $u$ and $v$ are uniformly distributed in the interval [-1,1] and points only within the unit circle is admitted (see Fig. 2.2). Therefore, only those pairs of $u$ and $v$ which gives a value for $s$ in the interval (0,1) is considered. Value of $s$ is similar to that of $V_1$ and $\theta/2\pi$ is similar to that of $V_2$ in the basic form.

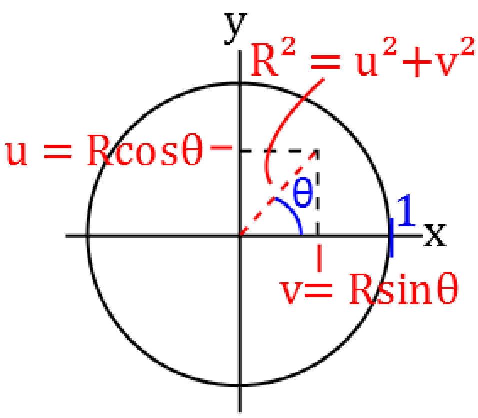



***Figure 2.2.*** ???

The lines of code below shows the implementation of the generation of masses normally distributed with a mean of $10M_\odot$ and a standard deviation of $1M_\odot$.

```
void gaussian_mass_generator(vec (&mass), int number_of_particles)
{
  srand(time(NULL));
  for (int i = 0; i < number_of_particles; i++)
```

[7] FiXme Note: we should probably have a ref. here
[8] FiXme Note: how the hell??

```
  {
  static int iset = 0;
  static double gset;
  double fac, rsq, v1, v2;
    do{
   //generate two random numbers uniformly distributed in the interval [-1,1]
      v1 = 2.*((double) rand() / (RAND_MAX)) -1.0;
      v2 = 2.*((double) rand() / (RAND_MAX)) -1.0;
      //Radius of the numbers (within the unit circle) squared
      rsq = v1*v1+v2*v2;
    } while (rsq >= 1.0 || rsq == 0.);
    //computing the gaussian distributed numbers
    fac = sqrt(-2.*log(rsq)/rsq);
    gset = v1*fac;
    iset = 1;
    mass(i) = v2*fac;
    mass(i) += 10;
  }
}
```

In the function *gaussian_ mass_ generator*, v1 and v2 are random numbers uniformly distributed in the interval [-1,1]. $rsq = v1^2 + v2^2$ corresponds to $s = R^2 = u^2 + v^2$. Using the do-while loop only those pairs of v1 and v2 that produces an s equal to 0 or greater than or equal to 1 is generated so that the points are inside unit circle. Variable gset correspond to X1.

To test whether the generated masses are actually normally distributed around $10M_\odot$ with a standard deviation of $1M_\odot$, 100,000 masses are generated, by the presented code, and plotted in a histogram below.

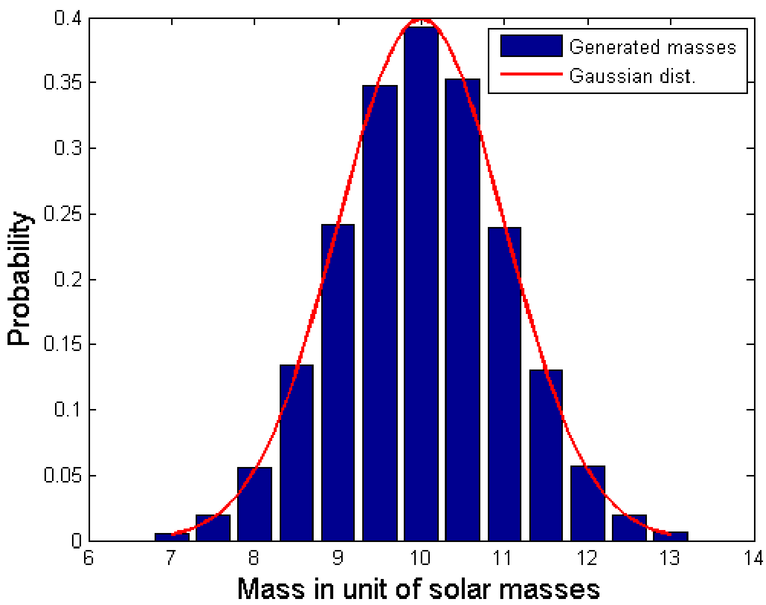



***Figure 2.3.*** Histogram of the mass of 100,000 particles generated by the c++ code introduced above together with a Gaussian distribution of mean $10R_\odot$ and standard deviation $1R_\odot$ generated in MatLab using the *normpdf* function.

### 2.3.2 Uniformly Distributed Position

To generate positions uniformly distributed inside a sphere of radius $R_0$, random numbers generated using the *rand* function in c++ are used and converted into coordinates of uniformly distributed particles within the sphere. In order to get a uniform density of particles three variables $v$, $w$ and $u$ corresponding to random numbers uniformly distributed between 0 and 1 are introduced. The spherical coordinates $\theta$, $\phi$ and $r$ are, according to [**?**], then linked to these variables using the equation

$$\theta = cos^{-1}(1-2v)$$
$$\phi = 2\pi w$$
$$r = R_0(u)^{1/3}$$

The following equations are then used to get back to the Cartesian coordinate system.

$$x = rsin(\theta)cos(\phi)$$
$$y = rsin(\theta)sin(\phi)$$
$$z = rcos(\theta)$$

After performing these steps, a uniform distribution of $N$ particles within a sphere of radius $R_0$ is achieved. Below, the code for generating this uniform distribution within a sphere of radius $R_0 = 20$ly is introduced.

```
void uniform_pos_generator(mat (&position), int N)
{
double pi=3.14159, c = 2*pi, R = 20;
vec phi(N), r(N), theta(N), x(N), y(N), v(N);

srand(time(NULL));

for (int i=0;i<N;i++){

       x(i) = ((double) rand() / (RAND_MAX)); //random numbers generated in the
           interval(0,1)
       y(i) = ((double) rand() / (RAND_MAX));
       v(i) = ((double) rand() / (RAND_MAX));
   }
for (int i=0;i<N;i++){
       phi(i)=c*x(i);
       r(i)=R*pow(y(i),1.0/3.0);
       theta(i)=acos(1.0-2.0*v(i));
       position(i,0)=r(i)*sin(theta(i))*cos(phi(i));
       position(i,1)=r(i)*sin(theta(i))*sin(phi(i));
       position(i,2)= r(i)*cos(theta(i));
   }
}
```

To test whether the generated positions within the sphere of radius 20 ly, the density of particles in the cross-sectional area of each $x$-value is determined and plotted as a histogram in Fig. 2.5 for $100,000$ particles with position generated by the introduced lines of code. The density of particles in the cross-sectional area of each $x$-value is found by dividing the total number of particles with that $x$-value with the cross-sectional area of the sphere in that $x$-value (see Fig. 2.4).

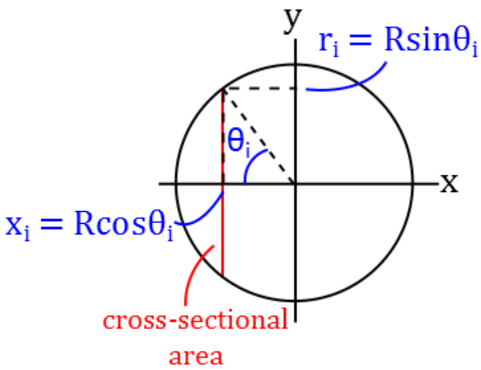



***Figure 2.4.*** Two-dimensional illustration of the three-dimensional problem of determining the density of particles in each $x$-value.

The cross-sectional area of the sphere in a specific area is found from a little trigonometry, by first considering that the radius of the circle that makes of the cross-sectional area in a point $x_i$ is given by $r_i = 20sin\theta_i$ ly. This yields that the area $A_i$ of the cross-sectional area, in ly, is given as

$$A_i = 400\pi sin^2\theta_i = 400\pi(1 - cos^2\theta) \tag{2.17}$$

in which the last equal sign stems from $1 = cos^2\theta + sin^2\theta$. But $x_i = 20cos\theta_i$ ly, giving

$$A_i = \pi(400 - x_i^2) \tag{2.18}$$

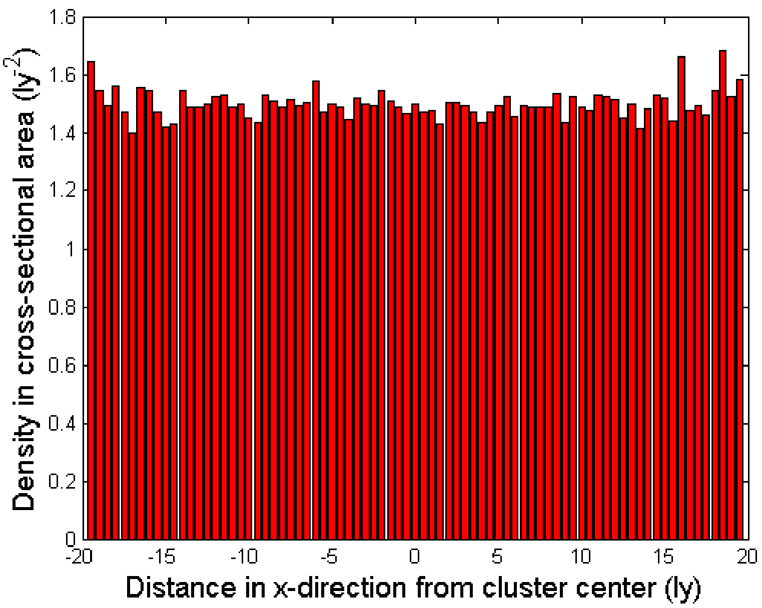



***Figure 2.5.*** Histogram of density of 100,000 particles with position generated by the code introduced in [9] as a function of the $x$-coordinate of the particles. The histogram is made with bins in the interval $[-19.5; 19.5]$ and a bin-size of 0.5. The distance $x = \pm 20$ from the cluster center is not considered, since the cross-sectional area in that point is zero.

## 2.4 Computing the Energy

In order to test whether the energy is conserved, the initial energy of the system can be calculated and printed together with the final energy after a specific time interval. According to the conservation of energy, these are equal.

The total energy $E_{tot}$ of the system is found by summing up the potential energy $E_{pot}$ and kinetic energy $E_{kin}$ of the $N$ bodies that constitutes the system. The total potential energy is calculated as

$$E_{pot} = \sum_{1=0}^{N} \sum_{j \neq i} \frac{m_i m_j}{r_{ij}} \tag{2.19}$$

in which $m_i$ and $m_j$ are the masses of the $i$'th and $j$'th body, respectively, $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between the two bodies, and $G$ is the gravitational constant. The total kinetic energy of the system is calculated as

$$E_{kin} = \frac{1}{2} \sum_{1=0}^{N} m_i v_i^2 \tag{2.20}$$

with $v_i$ being the speed of the $i$'th particle calculated as $v_i = \sqrt{v_{ix}^2 + v_{iy}^2 + v_{iz}^2}$, and $m_i$ is the corresponding mass of that body.

The c++ code for computing the total energy of the system is given here below. When the kinetic energy is calculated, $v_i$ is not explicitly calculated. Instead $v_i^2$ is calculated to reduce the number of floating point operations.

```
for (int i=0; i<number_of_particles; i++){
   for (int k=0; k<3; k++){
      kin_en(i) += v(i,k)*v(i,k);
    }
    kin_en(i) = 0.5*m(i)*kin_en(i);
    for (int j=0; j<number_of_particles; j++){
        if (j != i){
         pot_en(i) += pow(distance_between_particles(i,j),-1.0)*m(j);
        }
    }
    pot_en(i) = pot_en(i)*G*m(i);
    tot_en(i) = kin_en(i)+pot_en(i);
}
```

CHAPTER

# 3 RESULTS AND DISCUSSION

The starting point is solving the two body system, where Earth- and Sun-like forms the two bodies with masses $1\mathrm{M}_\odot$ and $3 \cdot 10^{-6}\mathrm{M}_\odot$, respectively, for which $\mathrm{M}_\odot$ is the solar mass. With the help of the Runge-Kutta method and the Velocity-Verlet method introduced in Sec. 2.2, the problem is solved both with a stationary Sun-like body relative to the frame of reference, and with both Earth and Sun moving relative to the coordinate system, with an initial velocity (0,0,0) and initial position an (1,1,1) for the Sun. For earth the initial position is assigned to be (2,1,1) and initial velocity (0,0.017,0).

For an $N$ body system, the movement of the bodies with the evolution of time is estimated with the Velocity-Verlet method. From the result analysis the behaviour of the system is unfold. [1]

The results from running the codes described in Chap. 2 for computing the blah blah blah ?? can be found in the GitHub folder `https:/??`, together with the MatLab scripts for the plots presented in this chapter. [2]

## 3.1 Stability of the 2-body System using Runge-Kutta and Velocity-Verlet

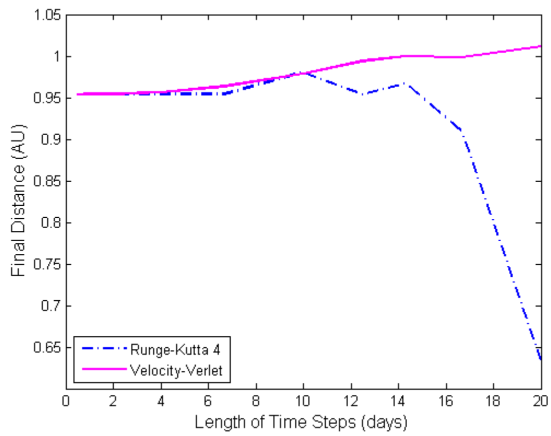
The considered two-body system consists of an Earth-like body that orbits a Sun-like body at an initial distance of 1 AU. The initial velocity is chosen so that the orbital period of the Earth-like body is 365 days, and due to the choice of the initial position of the Earth-like body relative to the Sun-like body and the initial direction of the velocity of the Earth-like body, the orbit of the Earth-like body is purely in the $x-y$-plane. Two cases are considered: the situation when the Sun is at rest compared to the frame of reference at all times, and the situation with both the Sun and the Earth moving relative to the frame of reference. The chosen position and velocity vectors for the Earth and Sun in these two situations are shown in Tab. 3.3.

[1] FiXme Note: ok, is this what we want to do?
[2] FiXme Note: fix these lines

***Table 3.1.*** Initial position and velocity for Earth and Sun in the Sun-Earth-like two-body system. F1 refers to the frame of reference at which the Sun is at rest in origo at all times, whilst F2 refers to the frame of reference in which both the Sun and the Earth moves relative to the coordinate axis. The mass of the Earth is given in solar masses, that is $M_E = 3.0 \times 10^{-6} M_\odot$, and the gravitational constant is given is $2.96 \cdot 10^{-4} \frac{AU^2}{days^2 M_\odot}$ (see Sec. 2.1).

| | $\mathbf{r}_{initial}$ [AU] | $\mathbf{v}_{initial}$ [AU/day] |
|---|---|---|
| Earth (F1) | $(1.0, 0.0, 0.0)$ | $(0.0, 0.017, 0.0)$ |
| Sun (F2) | $(1.0, 1.0, 1.0)$ | $(0.0, 0.0, 0.0)$ |
| Earth (F2) | $(2.0, 1.0, 1.0)$ | $(0.0, 0.017, 0.0)$ |

Fig. 3.3 shows the final distance between the two bodies as a function of time step length for the two situations: one in which sun is stationary relative to frame of reference and another in which sun is moving relative to the frame of reference. In both cases the final distance after 100 years is calculated between the Sun- and the Earth-like bodies and plotted as a function of length of time steps using both the fourth order Runga Kutta method and the Velocity-Verlet method.



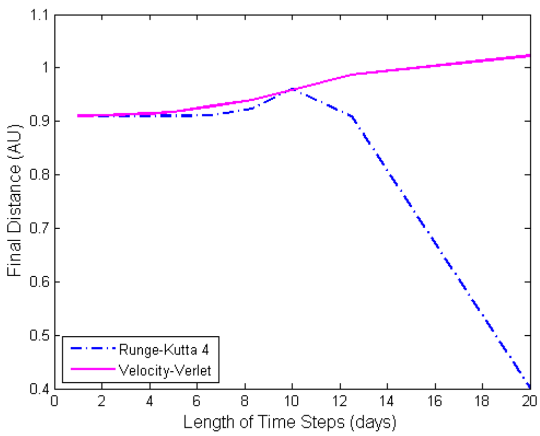




***Figure 3.1.*** Distance between bodies after 100 years as a function of time step length for the Earth-Sun-like two-body system using both the forth order Runge-Kutta method and the Velocity-Verlet method. The leftmost plot do not allow for Sun motion relative to the frame of reference, whilst the rightmost allows for movement of both the Earth and the Sun relative to the frame of reference.

The result using the Velocity-Verlet method shows a gradual increase in the final distance between the two bodies after 100 years, as the length of time steps increases, whereas the final distance after 100 years gained by the fourth order Runga-Kutta method shows fluctuations for shorter time step lengths, which is slightly more when sun is stationary relative to the frame of reference. As time step length reaches about 10 days, the distance, determined by the Runge-Kutta method, between the two bodies decreases, and the final distance after 100 years between the Earth-like body and the Sun-like body starts varying a lot with a change in time step length, meaning that the Runge-Kutta method is very unstable for large time steps, greater than approximately 10 days for this situation. However, both the Velocity-Verlet and forth order Runge-Kutta method seems to have stabilized for time steps smaller than or equal to 5 days, both for the situation with a stationary Sun and for the situation with both the Sun and Earth moving relative to the coordinate system. Hence, the time step length of 5 days is used to study the stability of the two methods for long time periods in Fig. 3.2.

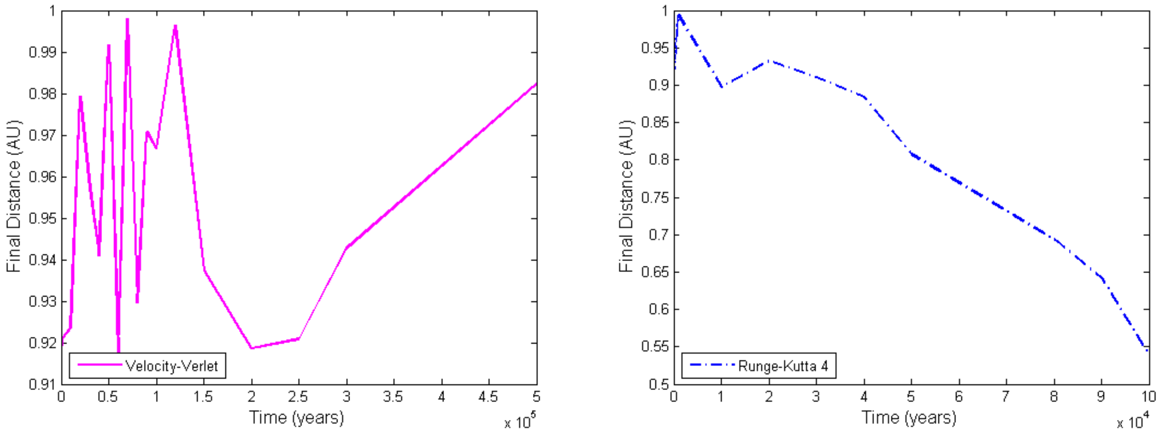



***Figure 3.2.*** The final distance as a function of time with a time step length of 5 days for both the Velocity-Verlet and Runge-Kutta method with both the Earth and Sun moving relative to the frame of reference. After $2.5 \times 10^4$ years, the Earth continuously moves towards the Sun, in the Runge-Kutta method, whilst the distance between the Earth and Sun still fluctuates between 0.92 AU and 1 AU for the Velocity-Verlet method after $5 \times 10^5$ years.

When allowing for motion of both the Earth and the Sun relative to the frame of reference, both the Velocity-Verlet method and the forth order Runge-Kutta method show that the distance between the Earth-like body and the Sun-like body will fluctuate with time, which seems reasonable from the fact that Earth's orbit around the Sun is not circular but elliptical. From the Runge-Kutta method it is found that after approximately $2.5 \times 10^4$ years the Earth will start moving rapidly towards the Sun, and after $10^5$ years the distance between the Sun-like object and the Earth-like object is only around 0.55 AU. This is, however, not seen in the Velocity-Verlet method. In the real solar system, the orbit of the Earth is in addition to the gravitational pull from the Sun also affected by the presence of the other planets. In the absence of these remaining planets and other objects in the solar system, the movement of the Earth-like object considered in this two-body system will, obviously, be different from the orbit known from astrophysics. However, it is estimated that the absence of other objects in the solar system, will not cause this rapid motion of Earth and Sun towards each other after only in the order of $10,000$ years, and hence it is concluded that the rapid motion seen in the rightmost figure of Fig. 3.2 is due to instabilities in the fourth order Runge-Kutta method presented in Sec. 2.2.2, yielding a greater stability in the Velocity-Verlet method than in the Runge-Kutta method for solving this two-body problem.

The table below shows the respective computational times for the fourth order Runge-Kutta method and the Velocity-Verlet method for computing the final position after 1 year using different time step length.

***Table 3.2.*** Computational time for the fourth order Runge-Kutta method and the Velocity-Verlet method for different time steps during 1 year.

| # time steps | Comp. time RK4 | Comp. time VV |
|---|---|---|
| 1 | 6 | 2 |
| 10 | 25 | 8 |
| $10^4$ | $8.4 \times 10^3$ | $4.6 \times 10^3$ |
| $10^6$ | $8.0 \times 10^5$ | $3.6 \times 10^5$ |

From the table it is evident that the computational time of both the fourth order Runge-Kutta method and the Velocity-Verlet method is more or less proportional to the number of time steps. Furthermore, the Velocity-Verlet method seems to be faster than the Runge-Kutta mthos for all investigated number of time steps. Together with the grater stability of the Velocity-Verlet method than the Runge-Kutta method, this yields that it is an advantage to use the Velocity-Verlet method to study this two-body system.

## 3.2 Testing Runge-Kutta and Velocity-Verlet for Sun-Earth-Mars System

***Table 3.3.*** Mass, initial position and initial velocity of Sun, Earth and Mars when running the Runge-Kutta 4 algorithm for this three-body problem. The Earth and Mars are set to orbit in the $x-y$-plane at $z = 1$ AU with the distance 1 AU and 1.5 AU to the Sun, respectively, which is not physically true. However, this initialization of position and velocity is reasonable to illustrate the validity of the Runge-Kutta method and Velocity-Verlet method presented in Sec. 2.2.

| | mass [$M_\odot$] | $\mathbf{r}_{initial}$ [AU] | $\mathbf{v}_{initial}$ [AU/day] |
|---|---|---|---|
| Sun | 1.0 | $(1.0, 1.0, 1.0)$ | $(0.0, 0.0, 0.0)$ |
| Earth | $3.0 \times 10^{-6}$ | $(2.0, 1.0, 1.0)$ | $(0.0, 0.017, 0.0)$ |
| Mars | $3.2 \times 10^{-7}$ | $(-0.5, 1.0, 1.0)$ | $(0.0, 0.014, 0.0)$ |

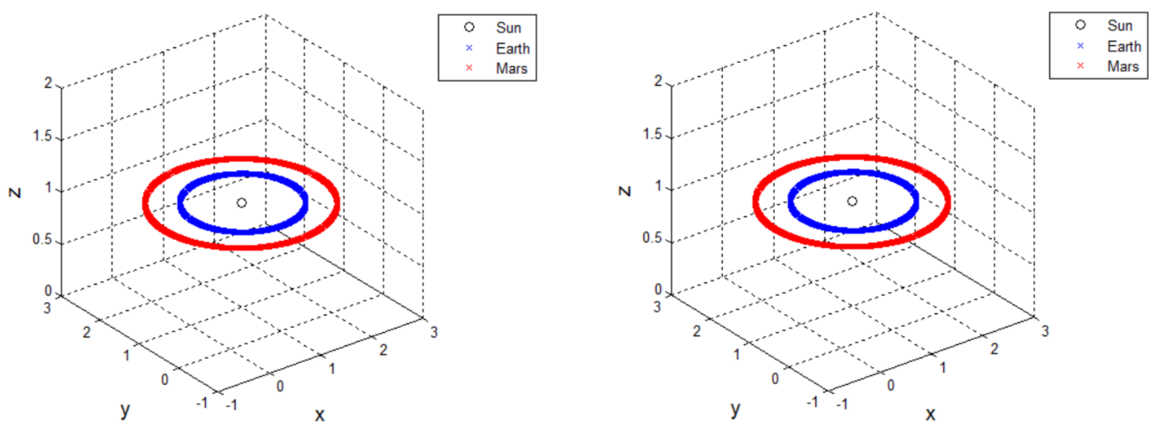



***Figure 3.3.*** Time evolution of the simplified system of Sun-Earth-Mars over a time period of 20 years using Runge-Kutta (leftmost) and Velocity-Verlet (rightmost) method with a time step length of 1 day. The masses, initial positions, and initial velocities of the three objects are given in Tab. 3.3.

CHAPTER

4

# CONCLUSION

# Bibliography