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Arithmetic in R

September 9, 2021

What will you learn?

- Perform basic numerical operations
- Translate complex mathematical formulas
- Use logarithms and exponentials
- Brush up on mathematical E-notation
- Know R's special numbers
- Understand logical values and operators

Sources: Some material for this lesson comes from Davies (2016) and Matloff (2020). These and other sources have been important to me in preparing the course. Check them out for a more systematic treatment of R. There is also a more philosophical, personal view on my use of sources in the Wiki for the 2020 version of this course.

What is this? When we say "Arithmetic", we don't mean that we "study" numbers but that we use them to perform computations. After this section, you'll be able to perform any arithmetic operation using R.

We will look at operators first, then at simple but important functions that occur again and again, especially in statistics.

How can you learn better? This presentation consists mostly of text and code chunks. Because this is dry stuff, I urge you (both if you hear this in class, and if you work through this on your own) to open an R session on the side and type along - this will build muscle memory and keep you entertained, too! Another trick, which you will find in Matloff's text, is to make your own little exercises by varying the instructions.

ARITHMETIC OPERATORS

1. Parentheses: `()`
2. Exponentiation: `^` or `**`
3. Multiplication: `*`
4. Division: `/`
5. Addition: `+`
6. Subtraction: `-`

In R, standard mathematical rules apply. The order of operators is as usual - left to right, parentheses, exponents, multiplication, division, addition, subtraction (PEMDAS = Please Excuse My Dear Aunt Sally) mnemonic).

The operators `^` and `**` for exponentiation are identical, though `^` is more common. You can check that in the R console with the `identical` function - the result should be `TRUE` (this is a truth or Boolean value - more on this below) - see figure

```
> 2**3
[1] 8
> 2^3 # same as 2 * 2 * 2
[1] 8
> 2**3 # same as 2 * 2 * 2
[1] 8
> identical(2^3, 2**3)  # checking that these are indeed the same
[1] TRUE
```

Formula translator I

$$24 + 6/3 \times 5 \times 2^3 - 9 \tag{1}$$

- What is the result of this expression?
- Compute in your head first
- Then check in the R console

Challenge: What's the result of the following expression?

$$24 + 6/3 \times 5 \times 2^3 - 9 \tag{1}$$

Compute (1) in your head first, then in the R console!

Let's look at more complicated expressions than this one.

FORMULA TRANSLATOR I

```
2³ = 2^3 = 8
6/3 = 2
2 * 5 * 8 = 80
24 + 80 = 104
104 - 9 = 95
```

You can check this in an R session:

```
>  24 + 6/3*5*8-9
[1] 95
```

- Remember the PEMDAS order
- Instead of `^` you can use `**`

FORMULA TRANSLATOR II

$$10^2 + \frac{3 \times 60}{8} - 3 \tag{2}$$

$$\frac{5^3 \times (6-2)}{61-3+4} \tag{3}$$

$$2^{2+1} - 4 + 64^{-2^{2.25-\frac{1}{4}}} \tag{4}$$

$$\left(\frac{0.44 \times (1-0.44)}{34}\right)^{\frac{1}{2}} \tag{5}$$

- Compute the expressions (2)-(5)
- Use the R console

$$10^2 + \frac{3 \times 60}{8} - 3 \tag{2}$$

$$\frac{5^3 \times (6-2)}{61-3+4} \tag{3}$$

$$2^{2+1} - 4 + 64^{-2^{2.25-\frac{1}{4}}} \tag{4}$$

$$\left(\frac{0.44 \times (1-0.44)}{34}\right)^{\frac{1}{2}} \tag{5}$$

Challenge: compute the expressions in the equations (2) – (5) using R.

Even if you don't code functions yourself, you may need to know these things if you have to check someone else's function, e.g. when the return values seem strange to you.

Formula Translator II

$$10^2 + \frac{3 \times 60}{8} - 3$$

```
R> 10^2+3*60/8-3
[1] 119.5
```

$$\frac{5^3 \times (6-2)}{61-3+4}$$

```
R> 5^3*(6-2)/(61-3+4)
[1] 8.064516
```

$$2^{2+1} - 4 + 64^{-2^{2.25-\frac{1}{4}}}$$

```
R> 2^(2+1)-4+64^((-2)^(2.25-1/4))
[1] 16777220
```

$$\left(\frac{0.44 \times (1-0.44)}{34}\right)^{\frac{1}{2}}$$

```
R> (0.44*(1-0.44)/34)^(1/2)
[1] 0.08512966
```

- You need parentheses in the exponent
- -2 is interpreted as `-1 * 2`
- What does `(-1)^(1/2)` return?

When you use R, you'll often have to translate a formula into code. Consider the formulas (2) – (5), which seem pretty complicated: the only trick here is that you often need to use parentheses, e.g. around calculations in the exponent, or when calculating with negative numbers in eq. (4), because the number -2 e.g. is interpreted by R as the operation `-1 * 2`.

Complex numbers? Last term, Lea S. solved my personal puzzle (thanks!), the "`NaN`" result, which is also "The Math Problem That Broke the Westworld Simulation" (the 2019 AI TV mini-series). Basically, R will hand you a "Not A Number" whenever you try to, e.g. take the square root of a negative number (try `sqrt(-1)` or `(-1)^(1/2)`). We won't need complex numbers in this course, but (of course) there are functions to handle them (see here).

Mathematical functions

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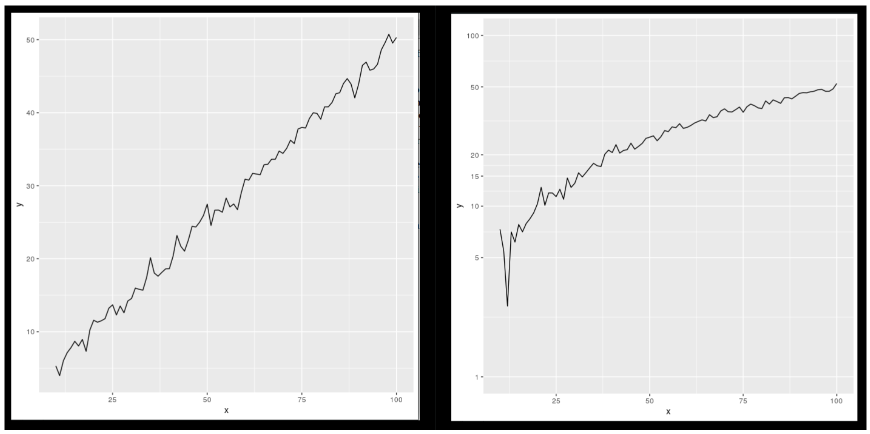
```
?sqrt
?log10
?exp
?pi
```

Logarithmic Transformation



See also: The Economist/Off The Charts 04/20/2021

It is often necessary to transform numerical data, e.g. transforming data using the logarithm leading e.g. from the left to the right graph in the figure. As you can see, this transformation leads to a compression of the `y`-values, so that more of these values can be shown.

The *logarithm* of a number x is always computed using a *base* b. In the diagram, $b = 10$, the numbers on the x axis were transformed using the `log()` function, the logarithm with base 10. The logarithm of $x = 100$ to the base 10 is 2, because $10^2 = 100$. In R, `log(x=100,b=10) = 2` (try this yourself!).

```
log10(1e7)
log10(100) log10(1000) log10(1e3)
log(1) = log10(1) = 0
log(x=100,b=100) = log(4.583,4.583) = 1
log(x=100,b=10) = log(b=10,x=100) = 2
```

Logarithm rules

`./img/rules.gif`

- Argument `x` and base `b` must be positive

- $\forall x$: `log(x,b=x)=1` since only $x^1 = x$
- $\forall b$: `log(x=1,b)=0` since $b^0 = 1$

Logarithm puzzles

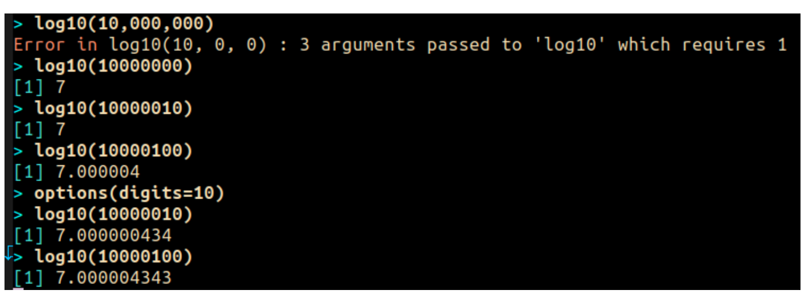
./img/kbd.gif

- Compute $log_{10}(10,000,000)$ in R
- Enter `log10(10,000,000)` in R
- Find the logarithm with base 10 for $10,000,010$.
- Why is the result the same as before?
- Check: enter `log10(10000100)`

  (1) The error in the first line results from the fact that in R functions, the comma separates arguments, so it looks to R as if 3 arguments were provided where only one is required, because, unlike the function `log()`, `log10()` already has a fixed base `b=10`. This is fixed in the next line.

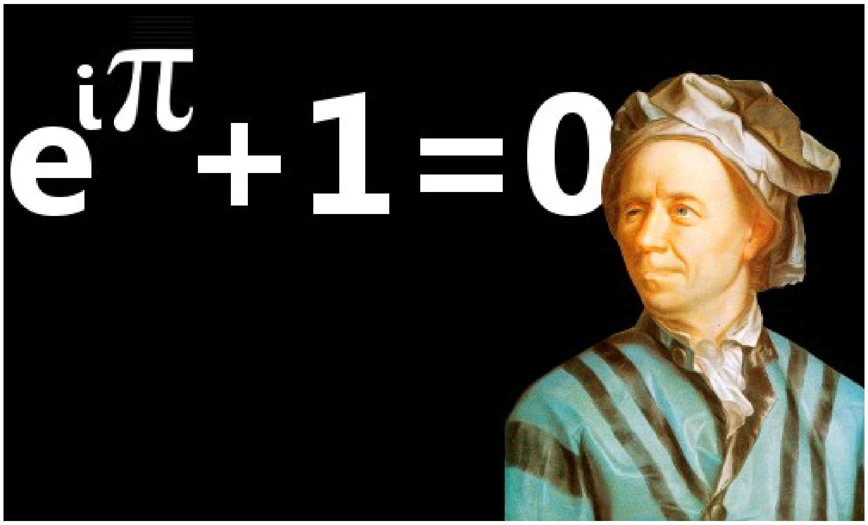
  (2) The trouble with the seemingly identical results of `log10(10000010)` and `log10(10000000)` lies in the suppression of digits. This can be fixed with the `options()` utility function, which we met in an earlier lecture. After setting `options(digits=10)`, the missing numbers appear.

  (3) Typing `log10(10000100)` would have revealed the problem, because this result can be shown with the default number of digits (7).



```
> log10(10,000,000)
Error in log10(10, 0, 0) : 3 arguments passed to 'log10' which requires 1
> log10(10000000)
[1] 7
> log10(10000010)
[1] 7
> log10(10000100)
[1] 7.000004
> options(digits=10)
> log10(10000010)
[1] 7.000000434
> log10(10000100)
[1] 7.000004343
```

Exponential function



- $log(x)$ implies $b = e \approx 2.7182$
- Verify for $x = 10$,$x = 2.718282$,$x = 0$:

$$e^{ln(x)} = ln(e^x) = x \tag{6}$$

In mathematics, the *Euler constant* e is as magical as the other mysterious constants π, 0, 1 and i (the imaginary unit). There are different ways to arrive at its value of approximately 2.718282.

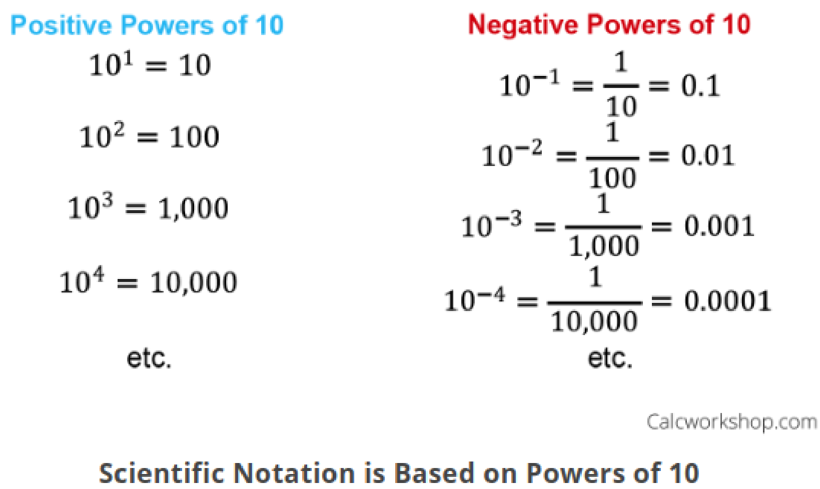
For now, we only care about the fact that e is the base of the natural logarithm, denoted as ln or $log_e(x)$.

Constants

`./img/kbd.gif`

- `pi` ($\pi \approx 3.14$)
- `LETTERS` and `letters`
- `month.name` and `month.abb`
- What about Euler's number e?

E-notation



You already know that the number of digits that is displayed by R can be changed using the `options()` utility function. The default number of digits displayed is 7.

In order to display values with many more digits than that - either very large, or very small numbers, we use the scientific or e-notation. In this notation, any number is expressed as a multiple of 10.

Examples

./img/penguins.gif

$$10\,000 = 10 \times 10 \times 10 \times 10 \times 10 = 1 \times 10^5 = \texttt{1eR+05}$$
$$\texttt{7.45678389e12} = 7.45678389 \times 10^{12} = 745.678389 \times 10^{10}$$
$$e = \texttt{271828182845e-11} = 271828182845 \times 10^{-11}$$

Be the computer!

./img/kbd.gif

- Enter 100 000 000
- Enter 0.00000000000000010
- Enter `exp(1000)` and `(-1)/0`

- Enter `sqrt(-1)`

Let's look at some examples:

$10\,000 = 10 \times 10 \times 10 \times 10 \times 10 = 1 \times 10^5$, shown in R as `1e+05`.

`7.45678389e12` is the same as $7.45678389 \times 10^{12}$ and the same as $745.678389 \times 10^{10}$.

e = `271828182845e-11` $= 271828182845 \times 10^{-11}$

To get from the e-notation with exponent y or $-y$ to the complete number of digits, simply move the decimal point by y places to the right or to the left, resp.

No information is lost even if R hides digits; e-notation is purely to improve readability. Extra bits are stored by R

`Inf`, `-Inf` and `NaN` are special numbers.

Math help in R

`./img/help.gif`

- `?Arithmetic`
- `?Math`
- `?Comparison` etc.

To infinity and beyond

`./img/infinity.gif`

Special numbers

`./img/special.gif`

- `Inf` for positive infinity (∞)
- `-Inf` for negative infinity ($-\infty$)
- `NaN` for "not-a-number" (not displayable)
- `NA` for "not available" (missing value)

`NA` values are especially important when we clean data and must remove missing values. There are Boolean (logical) functions to test for special values.

Missing values can be created easily by doing "forbidden" stuff. An example is trying to compute the square root of a negative number, e.g. `(-2)^(1/2)`. The result is a complex number (in this case the solution to the quadratic equation $x + 1 = 0$, called the imaginary number i). You can also use the function **`is.na`** to test for missing values: compute `is.nan(sqrt(-1))` for example.

Be the computer!

./img/kbd.gif

`Inf+1`	`Inf-1`
`Inf/Inf`	`Inf-Inf`
`NA`	`NA+NA`
`NaN`	`NaN+NaN`

```
> Inf + 1  # adding 1 to infinity won't change much
[1] Inf
> Inf - 1  # same thing
[1] Inf
> Inf - Inf  # can you subtract infinity from itself? (No)
[1] NaN
> Inf/Inf  # using Inf as a number leads to "Not A Number" (NaN)
[1] NaN
> 1/0  # dividing by 0 gives infinity (by definition)
[1] Inf
> -1/0 # this really is 1/0 multiplied by (-1)
[1] -Inf
> NA  # "Not Available" means "missing" - a reserved keyword
[1] NA
> NA + NA  # NA is also not suitable for arithmetic
[1] NA
> NaN  # "Not a Number" is also a reserved keyword
[1] NaN
> NaN + NaN # ... and also not suited for computation
[1] NaN
```

Special functions

./img/penguins.gif

`is.finite(Inf)`	`is.infinite(Inf)`
`is.finite(NA)`	`is.na(NA)`
`is.nan(NaN)`	`is.nan(NA)`

```
> is.finite(NA) # Missing values don't count as 'finite'
[1] FALSE
> is.infinite(Inf)  # Checking infinity is dodgy but works
[1] TRUE
> is.finite(Inf)
[1] FALSE
> is.nan(NaN)  # Checking "Not a Number"
[1] TRUE
> is.na(NA) # Checking missing values "Not Available"
[1] TRUE
> is.nan(NA) # Missing values are not non-numbers!
[1] FALSE
```

Be the computer!

`./img/kbd.gif`

- Enter `10^309`
- Subtract $\sqrt{2}^2$ from 2

  (1) `10^309` is `Inf`. The last number is infinite, because the largest number that can be represented by a 64-bit computer is $1.7976931348623157e+308$.

  (2) Subtract `sqrt(2)^2` from `2`. The answer is: `4.440892e-16`.

Logical values and operators

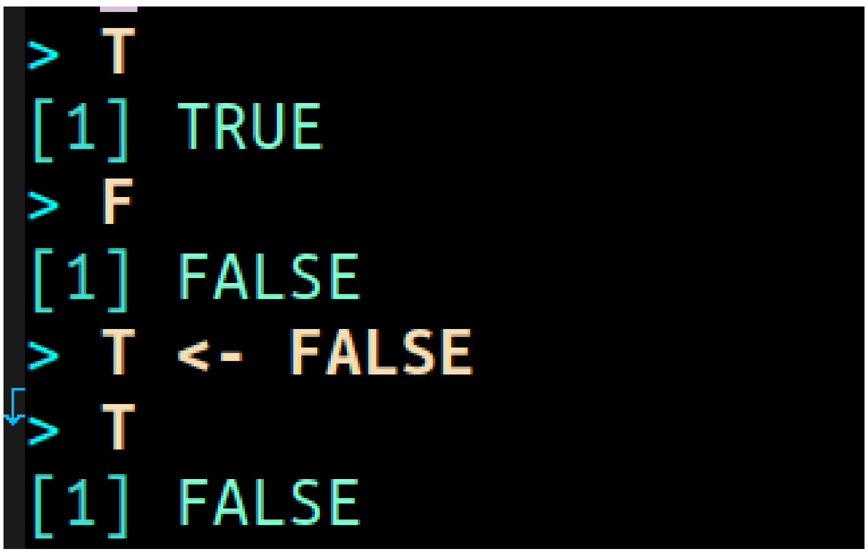
`TRUE` and `FALSE` are reserved in R for logical values, and the variables `T` and `F` are already predefined. This can cause problems, because these variable names are not reserved, i.e. you can redefine them. So better stay away from saving time by using the short versions of these values.

Be the Computer!

`./img/kbd.gif`

```
T             = TRUE
F             = FALSE
T <- FALSE  => ?
F <- TRUE   => ?
```

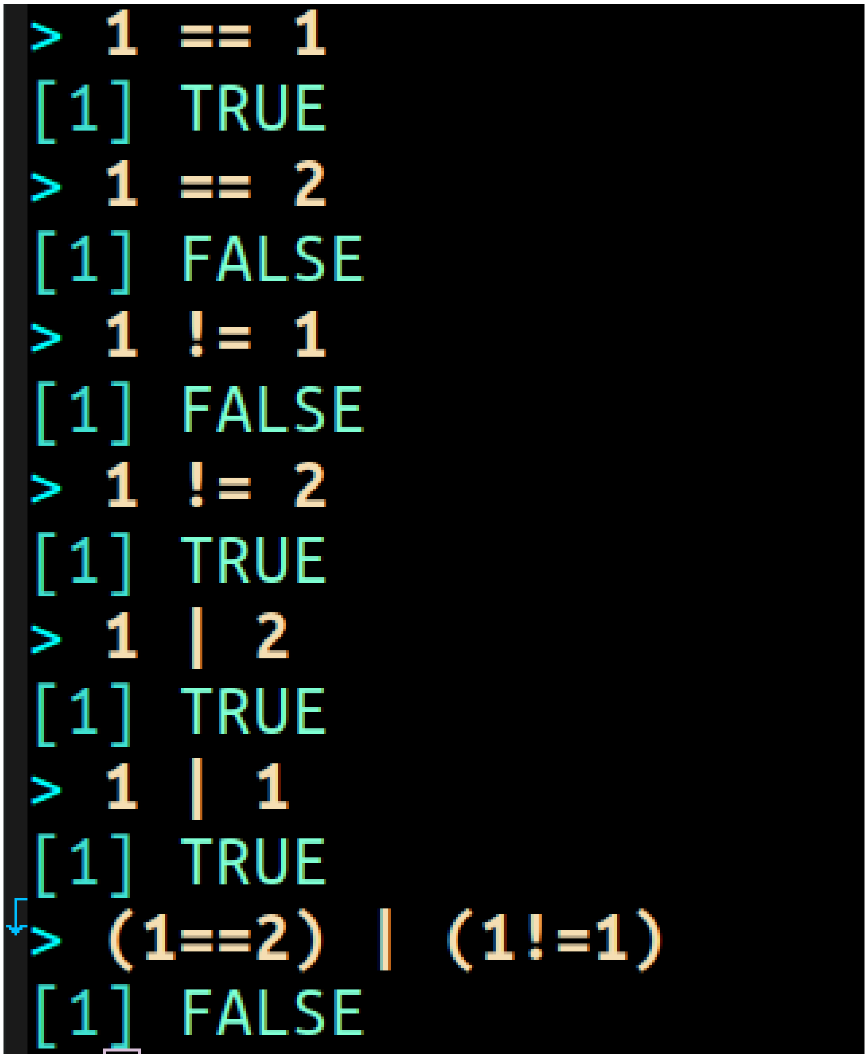
Cotton (2013) calls R's logic "Troolean" logic, because besides the so-called Boolean values `TRUE` and `FALSE`, R also has a third logical value, the "missing" value, `NA`



Logical operators

There are three logical operators in R:

`!` for "*not*":	`1 != 1`
`&` for "*and*":	~(1==1)&(1==2)
| for "*or*":	(1==2)|(1!=1)



In the last command, we generated a `FALSE` value by comparing two `FALSE` values, which is the only way to make an `|` statement `FALSE`.

Be the Computer!

./img/kbd.gif

```
sqrt(2)^2
sqrt(2)^2 == 2
all.equal(sqrt(2)^2, 2)
identical(sqrt(2)^2, 2)
```

Comparing non-integers is iffy, because non-integers (floating-point numbers) are only an approximation of the "pure", real numbers - how accurate they are depends on the architecture of your computer. In practice, this means that rounding errors can creep in your calculations, leading to wildly wrong answers. The R FAQ has an own entry about it. The figure shows a simple example: `sqrt(2)^2` and `2` should be the same, but they aren't as far as R is concerned - a logical comparison with `==` gives `FALSE`. To test near equality (bar rounding errors), you can use the function `all.equal`. To test for exact equality, use `identical`:

`../../img/3/floating.png`

CHALLENGE: (1) Check the help pages `?all.equal` and `?identical`. (2) Which of these numbers are infinite? `0`, `Inf`, `-Inf`, `NaN`, `NA`, `10^308`, `10^309`. (3) How small is the rounding error in the example in the figure actually?

Concept summary

- In R mathematical expressions are evaluated according to the PEMDAS rule.
- The natural logarithm $ln(x)$ is the inverse of the exponential function e^x.
- In the scientific or e-notation, numbers are expressed as positive or negative multiples of 10.
- Each positive or negative multiple shifts the digital point to the right or left, respectively.
- Infinity `Inf`, not-a-number `NaN`, and not available numbers `NA` are special values in R.

Code summary I

CODE	DESCRIPTION
`log(x=,b=)`	logarithm of `x`, base `b`
`exp(x)`	e^x, exp[onential] of x
`is.finite(x)`	tests for finiteness of `x`
`is.infinite(x)`	tests for infiniteness of `x`
`is.nan(x)`	checks if `x` is not-a-number
`is.na(x)`	checks if `x` is not available

Code summary II

CODE	DESCRIPTION
`all.equal(x,y)`	tests near equality
`identical(x,y)`	tests exact equality
`1e2`, `1e-2`	$10^2 = 100$, $10^{-2} = \frac{1}{100}$

Thank you! Questions?

`./img/waterfall.gif`

REFERENCES

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