Quadratic Versus Exponential Population Growth Again

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[This is part of my series on Thomas Malthus' "Essay on the Principle of Population," *first published in 1798. You can find an overview of all my posts here that I will keep updated:* "Synopsis: What's Wrong with the Malthusian Argument?"*]*

In two previous posts, I looked into the growth behavior for actual populations: for the Amish and for various countries in Europe and in Europe as a whole. The conclusion was that a model with quadratic population growth fits the data better than a model with exponential growth. This may be surprising because exponential growth is presumed to be the "natural" model since Thomas Malthus made the claim (and he was not really the first).

The usual argument is that if a population grows at a fixed rate, if its growth is proportional to its size or if the population has a fixed percentage of additional people per time, which all mean the same thing, then it follows that the population has exponential growth. That is true. It is not even a conclusion, it is the definition of the exponential function, so the proof is actually nothing but an example of "begging the question," assuming what you want to conclude.

The underlying logic is that the population will behave always in the same way, independently of its population size. Fertility will always be the same, mortality will always be the same, and then it follows that the growth rate is fixed. Again, that is true, but also a case "begging the question" because fixed fertility and fixed mortality determine population growth, which has to be at a constant rate in this case.

Digging even deeper, it is a confusion that already Thomas Malthus labored under: Populations have no control over their fertility, so whether you have one thousand people or one million, fertility will always be the same. Humans can just not help having as many children as they can. And as there is something of a natural constant here, maximum fertility, this then implies growth at a constant rate if mortality is also fixed.

To prove this, Malthus confounds two different assertions: A population *can have* maximum fertility and it *actually has* maximum fertility. The first does not imply the latter. Someone may have potential fertility of, let's say, ten, but still have only actual fertility of eight, six, four, two or zero. The statement that it *could be* ten does not show that *it is* ten. But if you reject the very idea that humans could have control over their fertility, it may seem innocuous to make this leap. Malthus is certainly willing to jump. Yet, all this does not tell us what actual populations really do. (For a long from of this argument see my post here.)

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Quadratic population growth seems weird because you cannot beg the question as for exponential growth. Hence it may seem as if a better fit for quadatric population growth were just an oddity of some data set and a braindead exercise in fitting a polynomial. However, I still have to find an actual population in its growth phase where exponential growth fits the data better than quadratic growth. I will present another example below where it is not so.

The problem for Malthusians with quadratic population growth is that they have already made up their minds that it can only be exponential growth. I have seen this funny example on the Internet where someone discusses data for a population in a script for a statistics course. He wants to demonstrate how to fit different functional forms. Of course, linear population growth yields a poor fit. Then it is quadratic population growth, and — lo and behold! — the fit is eerily good. But then he moves on and fits an exponential function, which results in a worse fit, and he rejects quadratic growth because it just cannot be so.

Here is an intuitive explanation for why quadratic population growth is actually quite natural:

Suppose a population can only grow to a certain population density. It starts at a center on a two-dimensional plane with that population density. Since the population cannot grow further there, it must expand and settle a circle around the point with the same population density. If the population expands at a fixed speed, then it settles within larger and larger circles over time where the radius grows linearly. The area enclosed grows as a quadratic function. And if the population density is always the same, population size will also grow as a quadratic function. The reason why this is so natural is that we essentially live in two dimensions.

The situation for actual populations is more complicated. Growth might proceed from different centers. As the circles grow together, the expansion will slow down. When half the country is already settled, the expansion may reduce to one dimension, a wavefront that moves over the rest of the country. That would mean only linear growth. And when there is no more space to settle, then growth must come to a halt. So, the general development should be quadratic growth initially that then slows down to linear growth and finally runs out at a *finite* population size. And not exponential growth to *infinity*.

Of course, a population in a settled country cannot settle it again. However, if there is some reason why population density can go up, eg. because productivity in agriculture goes up or more people can live in cities or only a part of a society grows, etc., then this innovation might start at some point from where it expands. And that would lead again to quadratic population growth. While it may seem as quadratic growth is only the dumb next idea after linear population growth does not work, there is actually a good reason to expect it for actual populations. But if you are sold on exponential growth, this is unthinkable.

One reason that leads Malthusians to count quadratic population growth out from the start is also that the implication here is that a population grows to a certain population density and then stops growing. That means the population has probably a lot of control over its fertility. Malthusians find that absurd. Yet, even they would have to consider this possibility because, in a Malthusian model, growth at a location must come to a halt, too, be it only because the local population hits the maximum population density possible and hits the brink of starvation.

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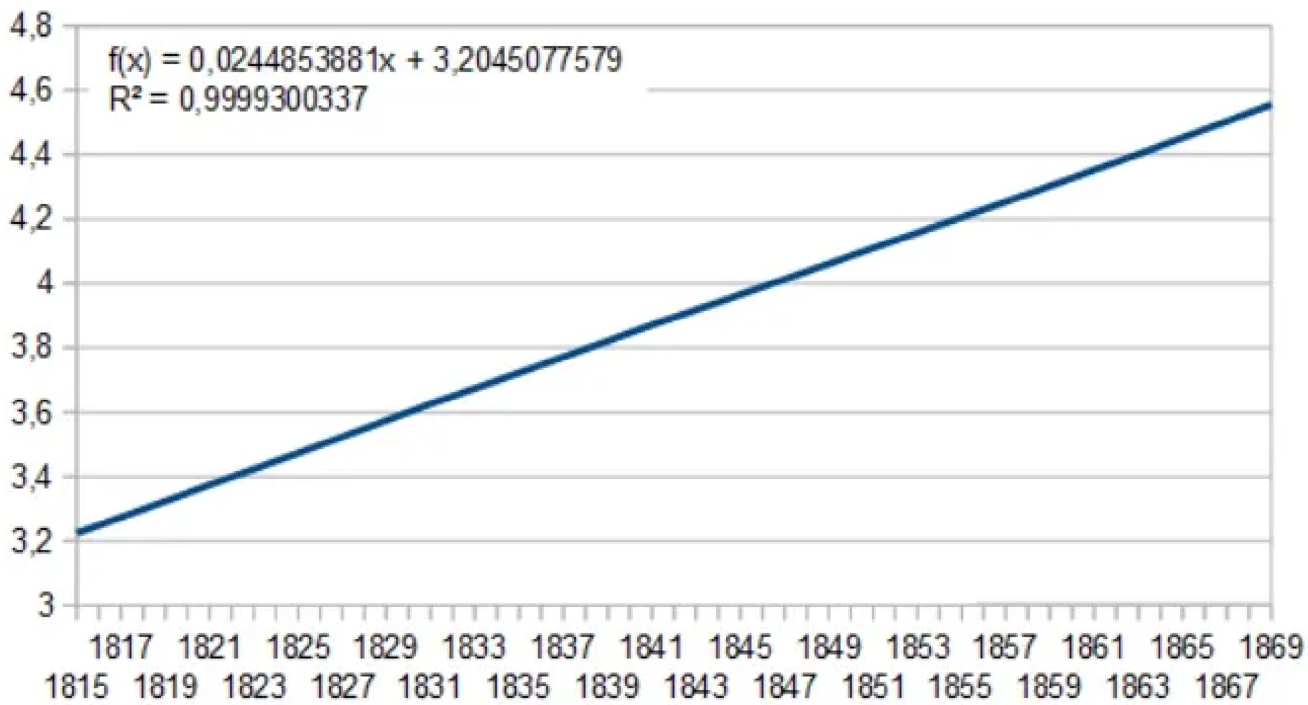
Here is now the concrete example that I announced and which has the additional advantage that I can demonstrate how misleading the fixation on exponential growth can be. It is the development for England from 1815 until 1869. I am grateful that I can use the data compiled by Gregory Clark here. The reason I take this time period is that before 1815 (end of the wars against Napoleon), the population did not grow in a regular fashion. My claim is not that populations always grow quadratically, only that they do when they grow with little perturbation. But this caveat also applies to an exponential model that does not work for the time before 1815 either.

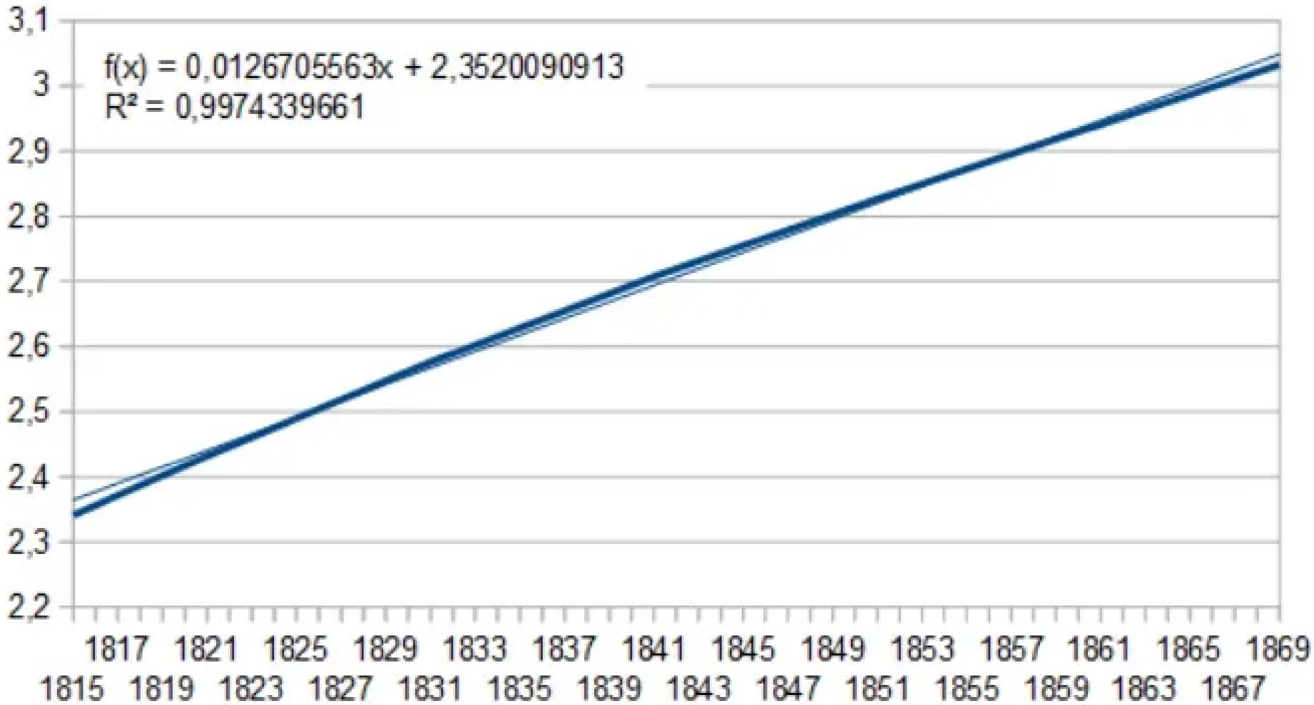
To see whether quadratic or exponential growth is the adequate model, I take the square root of population size (in millions) in the first case and the natural logarithm in the second case. Those are the inverse functions that turn quadratic, resp. exponential growth into linear growth. I then fit a linear function to see how good the fit is.

Here is the case for quadratic growth after a transform with the square root (I have German settings, so commas and dots are the other way around):



The fit is ridiculously good. It is almost impossible to spot the difference between the transformed population size and the linear line fitted to it.

And here is the same exercise for exponential growth after a logarithmic transform:



The fit is also quite good. However, as you can see, the transformed population size bulges a little. It is below the linear line on both ends, and slightly above in the middle. That is how it should be if you transform a quadratic function with the logarithm. The rate of growth (the slope of the transformed curve) is first higher than for an exponential function, but later it goes down.

A common fallacy is to think that the exponential function is always “stronger” than any other function. But that is not true. Initially, the quadratic function grows much faster. The function x^2 grows from 0 to 1 in finite time, while the exponential function takes from minus infinity to 0. Only as time goes to infinity, the exponential function beats the quadratic function and then hands down.

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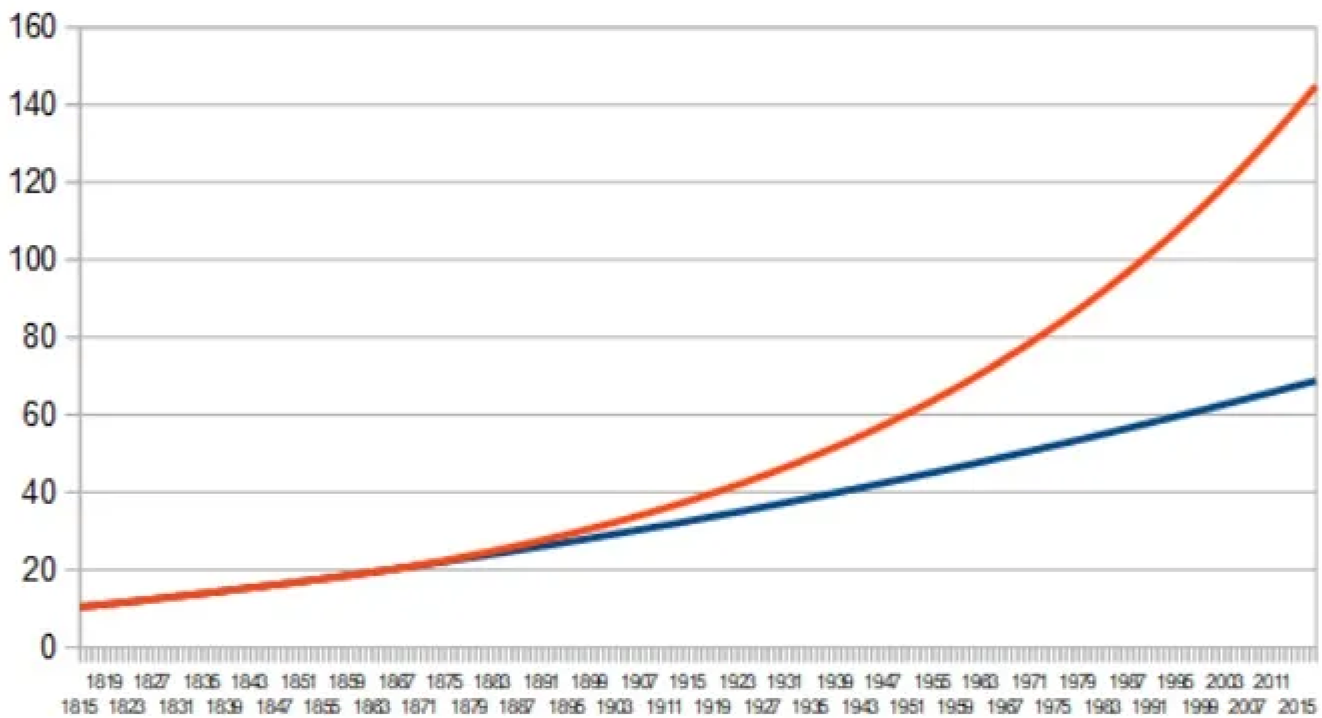
Now, this comparison may seem like mere nitpicking. Both a quadratic and an exponential function can explain the data quite well. Maybe there were some extraordinary circumstances and that's why the exponential function appears to work worse. However, note that it is so for all the examples I had so far. I am not hunting for confirmation. I just look at all the data sets that pass my way. And so far, they all point in the same direction, namely that quadratic growth is a better explanation than exponential growth.

Here is why this matters. Quadratic growth means that there is a fixed acceleration (second derivative). The population size rises and that speeds up. But for exponential growth it is much more: Not only the first derivative (speed) goes up as the population grows, but also the second derivative (acceleration). And so do the third, fourth, fifth derivative to infinity. That's why the exponential function has such strong growth as time goes to infinity and beats all lamer functions like the linear, quadratic, cubic funtions and so forth.

Malthusians love this property of the exponential function because it defeats any efforts to keep up with linear (or quadratic or cubic, etc.) growth for food production The argument is then: *If the population keeps growing exponentially*, then there will be a problem sooner or later. Since the assumption is that a population just can't stop growing exponentially unless it is stopped the hard way, all the dismal conclusions follow. However, all this relies on the assumption I have highlighted: *If the premise is true and a population grows indeed exponentially!* If it does not and the population has control over its fertility, then its size need not increase ever more frantically.

The typical Malthusian doom scenario is what happens when the population keeps growing exponentially. The behavior in the past is extrapolated for this purpose. And, of course, exponential growth is extremely fast as time goes to infinity. Yet, if quadratic growth is also in the running, then we could also extrapolate with that. Note that the derivation I have given above does not imply that the population must keep growing quadratically forever. Growth should slow down later to linear growth, and then come to an end. So, the population would eventually grow to a *finite* size and stay there. Hence we can call the nightmarish scenario off that there must be inevitable growth to *infinity*. Food production could, of course, increase enough to feed a stable population when going to infinity is not on the table.

Here is what you get if you extrapolate the data from 1815 to 1869 for England with an exponential and a quadratic function until 2015:



The red line is for exponential extrapolation and the blue line for quadratic extrapolation. Let me give you some additional information: The population of England stood at 54.8 million people in 2015.

As you can see the exponential extrapolation is vastly off with 145 million people. But the quadratic extrapolation is actually quite decent with 69 million people if you take into account that this is an extrapolation based on about half a century of data almost 150 years ago. Of course, there should be all kinds of other effects here: emigration, immigration, other fundamental changes that affect population growth. As noted above, my claim is not that quadratic growth will go on forever. It should also slow down from some point on, and even come to a halt if the underlying derivation is correct. So an overestimation for the actual population size by about 14 million people should not be a such a surprise.

Think of the agitated Malthusian of 1869 with his exponential extrapolation. He would fret about how England could ever host 145 million people and how that would mean inevitable doom and gloom. Roughly a tripling over one and half century might have also seemed like a challenge, but would have been far less scary.

I am sure you have seen many examples before where people have extrapolated population sizes with an exponential function and also over a long period of time. And then they were shocked how fast population grows and felt overwhelmed. However, if they maybe just used the wrong model, exponential and not quadratic population growth, their predictions might be just as far off as the predictions in 1869 would have been: by a whopping 150%!

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Correction

In the original version I thought that the data I was working with were for England and Wales, but they were only for England. That doesn’t make a difference for the question of whether quadratic or exponential growth is a better fit, and also not for the extrapolation. But I have to compare the forecast for 2015 with a lower figure. The population of Wales was about 3 million then. Since my point is only qualitative, the correction does not lead to a different conclusion: Quadratic extrapolation is roughly in the right ballpart whereas exponential extrapolation is way off.