Notes on Boundary Representation

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1 Definitions

1.1 Spaces

We define a space $\mathcal{S}$ of n dimensions as a pure manifold of n dimensions

A space $\mathcal{S}$ is a set of points of n dimensions with the following requierments:

- For any point $P_A \in \mathcal{S}$ and an arbitrary small distance d there exists

TODO: replace all of this stuff with a good external definition of a space with the properties we require. Metric space? Manifold?

A space $\mathcal{S}$ in $n > 0$ dimensions can be *segmented* into two parts by a subspace $\mathcal{B} < \mathcal{S}$ of $n-1$ dimensions. A subspace $\mathcal{B} < \mathcal{S}$ has a mapping $\sigma_{\mathcal{B}} : \mathcal{B} \to \mathcal{S}$ that maps any point $P \in \mathcal{B}$ to a point $\sigma_{\mathcal{B}}(P) \in \mathcal{S}$. For any point $P \in \mathcal{S}$ and a subspace $\mathcal{B} < \mathcal{S}$ we say that P can either be on the *inside* of $\mathcal{B}$, on the *outside* of $\mathcal{B}$. A space $\mathcal{S}$ in zero dimensions has a single subspace $\mathcal{B} = \mathcal{S}$. Two distinct spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ are either *disjoint* or they *intersect*. We say that two distinct spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ intersect if there exists at least one point P in $\mathcal{S}_A$ that also exists in $\mathcal{S}_B$. For two spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ that intersect we define their *intersection* $\mathcal{S}_A \cap \mathcal{S}_B$ as the subspace $\mathcal{B}$ such that for each point P in $\mathcal{B}$ a coresponding point exists in both $\mathcal{S}_A$ and $\mathcal{S}_B$. If the two spaces do not intersect we define their intersection as $\mathcal{S}_A \cap \mathcal{S}_B = \varnothing$.

A path P in a space $\mathcal{S}$ is a set of points an TODO: fix this definition. The *endpoints* of a path P TODO: depends on definition of path. A *crossing* $C_{\mathcal{B}}(P)$ of a path P through a subspace $\mathcal{B} < \mathcal{S}$ is a subpath of P that starts with a point on the outside of $\mathcal{B}$ and ends with a point on the inside of $\mathcal{B}$ and where all other points of $C_{\mathcal{B}}(P)$ is in $\mathcal{B}$. The *crossing endpoints* of a crossing $C_{\mathcal{B}}(P)$ are the endpoints of the subpath of $C_{\mathcal{B}}(P)$ that only contains points in $\mathcal{B}$.

A space in zero dimensions is called a *point*, space in one dimension is called a *curve*, while a space in two dimension is called a *surface*. In three dimensions we are usually only interested in the space $\mathbb{R}^3$.

1.2 Regions

A region R of a space $\mathcal{S}$ is defined by a set of faces $F(R)$. We write $R \subset \mathcal{S}$. A *face* $f \in R(R)$ of a region $R \subset \mathcal{S}$ is a region in the subspace $\mathcal{B} < \mathcal{S}$. A region partitions a space into two parts such that a point $P \in \mathcal{S}$ is either *inside* or *outside* R. We say that a path P *crosses* a region R if the path crosses the space $\mathcal{S}$ of R in such a way that at least on of the crossing endpoints of $C_{\mathcal{S}}(P)$ are in R.

Definition 1.1 (Region)**.** A *region* R is a set of faces $F(R)$ that partitions a space $\mathcal{S}$ into two parts such that for any point $P \in \mathcal{S}$, P is either *inside* or *outside* of R. A region has the following invariants:

1. A *face* $f \in F(R)$ of a region $R \subset \mathcal{S}$ is a region in a subspace $\mathcal{B} < \mathcal{S}$.
2. For any two points P_A and P_B and a region R, then $P_A \in R$ and $P_B \notin R$ if and only if all the first and last crossing of all paths between P_A and P_B through the faces of R is an *inside-out* crossing.

TODO: what are the invariants of a region?

If we We say that any path All faces a For all faces f in a region R of the subspace $\mathcal{B}$ we require We write $P \in R$ if a point $P \in \mathcal{S}$ is inside R. We write the *empty region* as $\varnothing$. This region has no faces and contains no points. A *connected region* is a region R such that for any two points $P_a, P_b \in R$ there is a path from P_a to P_b entierly inside of R.

The *intersection* $R_A \cap R_B$ of two regions R_A and R_B is the region such that

$$\forall P \in R_A.\ P \in R_B \iff P \in (R_A \cap R_B).$$

If we have $R_A \cap R_B \neq \varnothing$ then we say that the regions R_A and R_B *intersect*.

Lemma 1.1. Given two regions R_A and R_B of the same space $\mathcal{S}$, the intersection $R_A \cap R_B$ is a region of the space $\mathcal{S}$.

Proof. TODO $\square$

Lemma 1.2. Given two regions R_A and R_B of the two spaces $\mathcal{S}_A$ and $\mathcal{S}_B$, respectivly, the intersection $R_A \cap R_B$ is a region of the space $\mathcal{S}_B \cap \mathcal{S}_B$.

Proof. TODO $\square$

Theorem 1.1. The intersection $R_A \cap R_B$ of the two regions

We say that two regions R_A and R_B *connect*

Given two regions R_A and R_B they can either be *disjoint*, they can *intersect* or they can *connect*. Given two regions R_A and R_B we define $R_A \cap R_B$ as their *intersection*. For two regions R_A and R_B of the same space $\mathcal{S}$ we define their intersection $R_A \cap R_B$ as a the set of *disjoint* regions of $\mathcal{S}$ such that $\forall P \in R_A.\ P \in R_B \iff (\exists R_C \in R_A \cap R_B.\ P \in R_C)$. For two regions R_A and R_A of seperate spaces $\mathcal{S}_A$ and $\mathcal{S}_B$ we define the intersection $R_A \cap R_B$ as a region in the subspace $\mathcal{S}_A \cap \mathcal{S}_B$. The region $R_A \cap R_B$ We say that two regions R_A and R_B intersect if $R_A \cap R_B \neq \varnothing$ We say that two regions R_A and R_B are disjoint if $R_A \cap R_B = \varnothing$ and $\forall f_A \in F(R_A), f_B \in F(R_B).\ f_A \cap f_B = \varnothing$. We say that two regions R_A and R_B connect if $R_A \cap R_B = \varnothing$, for all faces $f_A \in F(R_A)$ there exists at most one face $f_B \in F(R_B)$ such that $f_A = f_B$ and no other faces of R_B intersect with f_A, and there exists at least one pair of faces $f_A \in F(R_A)$ and $f_B \in F(R_B)$ such that $f_A = f_B$.

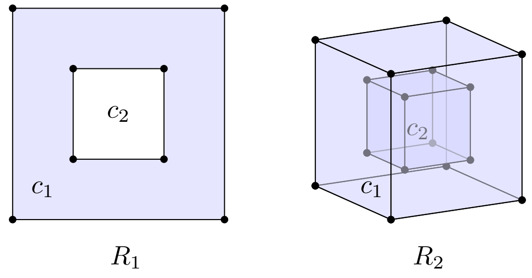

Figure 1: A drawing of a 2 dimensional region R_1 and a 3 dimensional region R_2, both with two cells, c_1 and c_2.

A face $f \in F(R)$ is a region in a subspace $\mathcal{B} < \mathcal{S}$. For a region R we require that $\forall f_a, f_b \in F(R). f_a \cap f_b = \varnothing$.

A region R in a space $\mathcal{S}$ is defined by a set of cells $C(R)$. A point P is said to be *inside* a region R if it is inside an odd number of cells of R.

A cell c consists of a set of faces $F(c)$. A face f is a region R_f in a subspace of $\mathcal{S}$. We call the faces of the cells in R_f *edges*. A cell has two constraints: The first is that each edge e of each face f_a is shared with exactly one other face f_b. The second is that for each face f_a there exists no other face f_b such that the subspaces of f_a and f_b has an intersection that *crosses* both f_a and f_b. We say that an intersection I crosses a region R if it intersects with any edges in R. Alternativly we also say that an intersection I crosses a region R

Given a cell c of a region R in a space $\mathcal{S}$ and an intersection of $\mathcal{S}$, I, we can compute the intersection of R and I, known as R_I.

A face f is a region R_f in an subspace $\mathcal{B}$ of $\mathcal{S}$. The edges of a face f is the

A cell c also segments a space in the same way as its coresponding region, but it has some additional constraints. A cell consists of a set of with the requirment that for each edge of a face

2 Topological regions

3 Planar geometry

3.1 Representation of lines and planes

3.2 Intersections of lines and planes

4 NURBS geometry