# MTH 101: Calculus I

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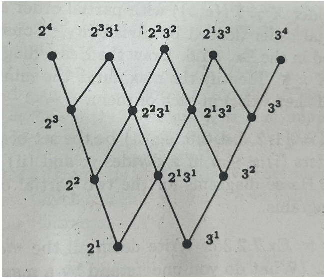
# Example 2

Let $X = \{2^m 3^n \;:\; m, n \in \mathbb{N} \cup \{0\},\ 1 \leq m+n \leq 4\}$.
Consider the relation $\leq$ on $X$ as follows:

$$x, y \in X,\ x \leq y \text{ if and only if } y \text{ is a multiple of } x.$$




Then $(X, \leq)$ is a partially ordered set, but is not totally ordered since $2^4, 3^4$ are not related.
Then $Y$ neither has a minimum element not a maximum element.
$Y$ has two minimal elements $2^1$ and $3^1$, since no element is smaller than these two.
The maximal elements of $Y$ are $2^4, 2^3 3, 2^2 3^2, 2^1 3^3, 3^4$ since no other element is greater than these two.

# Well-Ordering Principle

### Well-Ordered Set

Let $(X, \leq)$ be a totally ordered set.
We say that $\leq$ is a well-ordering on $X$ if every nonempty subset of $X$ has a minimum.
In that case, we say that $(X, \leq)$ is a well-ordered set.

### Well-Ordering Principle

There is a well-ordering on any nonempty set.

### Well-Ordering Principle for countable sets

a) If $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set. Define a relation $\leq$ on $X$ as

$$x_i \leq x_j \text{ if and only if } i \leq j.$$

Then $(X, \leq)$ is a well-ordered set with $x_1 \in X$ as the minimum.
b) If $X$ is countably infinite, let $f : X \to \mathbb{N}$ be a bijection.
Define a relation $\leq$ on $X$ as

$$x \leq y \text{ if and only if } f(x) \leq f(y).$$

Then $(X, \leq)$ is a well-ordered set with $f^{-1}(1) \in X$ as the minimum.

# Equivalents of the Axiom of choice

### Zorn's Lemma

Let $X$ be a partially ordered set. Assume further that the partial order is such that every chain in $X$ has an upper bound in $X$. Then there exists a maximal element in $X$.

### Application of Zorn's Lemma

Any vector space has a basis.

### A non-example

Consider the set $X = \{(0, t) \ : \ 0 < t < 1\}$ with relation $\subseteq$ (inclusion).
Then $(X, \subseteq)$ is a totally ordered set and is thus a chain.

It has no upper bound in $X$, and $X$ has no maximal element:
For if $(0, t_0)$ is an upper bound of $X$, then $(0, t) \subseteq (0, t_0)$, for all $(0, t) \in X$. However $(0, (t_0 + 1)/2) \in X$ but is not a subset of $(0, t_0)$. Thus $(0, t_0)$ is not an upper bound of $X$.

# Axiom of Choice

### Axiom of Choice

Let $I$ be an index set (countable or uncountable). Let $\{X_i \ : \ i \in I\}$ be a family of nonempty sets. Then there exists a set $A$ which has exactly one element from each of the sets $X_i$, $i \in I$.

Remark: All of the above three principles (Well-ordering Principle, Zorn's lemma and Axiom of Choice are equivalent. They are all assumed to be true.)

Real Number System

- Natural numbers $\mathbb{N}$: $1, 2, 3, 4, \ldots$
- Integers $\mathbb{Z}$: $0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$
- Rational numbers $\mathbb{Q}$: $p/q$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

  **Note**: $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$.
- There are gaps in the number line after placing all the rational numbers. Why?
  - Take a square whose sides have length 1.
  - What is the length of its diagonal? It is a number whose square is 2.
  - Let us denote it by $\sqrt{2}$.
  - It is not a rational number (proved earlier in this course).
- Such numbers are called irrational numbers. *We will eventually construct all of them using rational numbers.*

## Axioms defining the Real numbers

Assume that there is a nonempty set $\mathbb{R}$ of elements which satisfy the 10 axioms (A1-A10) which we will now discuss.

The elements of $\mathbb{R}$ are called *real numbers*.

## Field Axioms

Along with $\mathbb{R}$, there are two operations: addition $+$ and multiplication .

A1. Commutative Laws: $x + y = y + x$ and $xy = yx$, $\forall x, y \in \mathbb{R}$.

A2. Associative Laws:

$$x + (y + z) = (x + y) + z, \quad x(yz) = (xy)z, \quad \forall x, y, z \in \mathbb{R}.$$

A3. Distributive Law: $x(y + z) = xy + xz$, $\forall x, y, z \in \mathbb{R}$.

A4. Given $x, y \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $x + z = y$.
This $z$ is denoted by $y - x$, and
$x - x$ is denoted as $0$ and is independent of $x$.
$0 - x$ is denoted as $-x$, and is called the *negative or additive inverse* of $x$.

A5. There exists at least one real number $x \in \mathbb{R}$ such that $x \neq 0$.
Given $x, y \in \mathbb{R}$ with $x \neq 0$, there exists $z \in \mathbb{R}$ such that $xz = y$.
This $z$ is denoted by $y/x$, and $x/x$ is denoted as $1$ and is independent of $x$.
$1/x$ is denoted as $x^{-1}$, and is called the *multiplicative inverse* of $x$.

# $1 \neq 0$

i) Let $x \in \mathbb{R}$.
Then $x.0 = x.(0+0) = x.0 + x.0$ (A4 and A3).
Hence $x.0 = 0$ (A4).

ii) Suppose on the contrary that $1 = 0$.
Let $x \in \mathbb{R}$ be such that $x \neq 0$ (A5).
Then

$$\begin{aligned} x + x &= x.1 + x.1 \ (A5) \\ &= x.1 + x.0 \ (1 = 0) \\ &= x.(1+0) \ (A3) \\ &= x.1 \ (A4) \\ &= x \ (A5). \end{aligned}$$

Hence, by (A4), $x = 0$, which is a contradiction. $\square$

**Remark**

We thus have at least two elements in $\mathbb{R}$, given by 0 and 1.

## Order Axioms – will be discussed in the next lecture

Along with $\mathbb{R}$, there exists an operation $<$ which establishes an ordering among real numbers. Note $x < y$ is same as $y > x$.

A6. Law of Trichotomy: Given $x, y \in \mathbb{R}$, exactly one of the following hold:

$$x = y, \quad x < y, \quad x > y.$$

A7. If $x < y$, then $x + z < y + z$, for all $z \in \mathbb{R}$.

A8. If $x > 0$ and $y > 0$, then $xy > 0$.

A9. If $x > y$ and $y > z$, then $x > z$.

**Remark**

We say that $x \in \mathbb{R}$ is positive if $x > 0$ and is negative if $x < 0$.
$x \leq y$ means that either $x = y$ or $x < y$.
Similarly, $x \geq y$ means that either $x = y$ or $x > y$.
Sometimes to write two inequalities $x < y$ and $y < z$ together, we write $x < y < z$.