MTH 101: Calculus I

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Comparing cardinality

For given sets X and Y (finite or infinite). We write

- $|X| = |Y|$, if X and Y have the same cardinality. Otherwise, $|X| \neq |Y|$.
- $|X| \leq |Y|$, if either X is an empty set, or there is a one-one map $f : X \to Y$.
- $|X| < |Y|$, if $|X| \leq |Y|$ and $|X| \neq |Y|$. That is, no one-one map $f : X \to Y$ is onto.

Examples

- If X is a finite set, then $|X| < |\mathbb{N}|$.
- If X is an infinite set, then $|\mathbb{N}| \leq |X|$.
- If X is an uncountable set, then $|\mathbb{N}| < |X|$.
- Let $\mathcal{P}(X)$ denotes the power set of X, the collection of all subsets of X, then $|X| < |\mathcal{P}(X)|$.
  This implies that there is no largest set.

These will be discussed in the tutorial.

Order Relations

Relations on Sets

- Let X be a nonempty set.
- Let $X \times X = \{(x_1, x_2) \ : \ x_1, x_2 \in X\}$ (collection of ordered pairs of elements of X).

Relation on X

A relation R on X is a subset of $X \times X$.
For every pair $(x_1, x_2) \in R$, we say that x_1 is related to x_2 under R, denoted as $x_1 R x_2$.

Examples

- Let X be the collection of all human beings on the earth. A human being $x_1 \in X$ is related to $x_2 \in X$ is x_1 is a child of x_2. Then
$$R = \{(x_1, x_2) \ : \ x_1 \text{ is a child of } x_2\},$$
is the relation described above, which is a subset of $X \times X$.
- Let $X = \mathbb{R}$, the set of real numbers. Suppose x_1 is related to x_2 if $x_1 \leq x_2$. Then
$$R = \{(x_1, x_2) \ : \ x_1 \leq x_2\},$$
is the relation described above, which is a subset of $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.
Note that $(2, 1) \notin R$.

Relations on Sets

- Let X, Y be nonempty sets.
- Let $X \times Y = \{(x, y) \ : \ x \in X, y \in Y\}$ (collection of ordered pairs of elements of X and Y).

Relation between X and Y

A relation R between X and Y is a subset of $X \times Y$.
For every pair $(x, y) \in R$, we say that $x \in X$ is related to $y \in Y$ under R, denoted as xRy.

Examples

- Let X be the collection of all students of IISER Bhopal, and $Y = \mathbb{N}$. Let

  $$R = \{(x, y) \ : \ x \text{ is a BS first year student whose roll number is } y\}.$$

  Then R is a relation between X and Y.
- Let $X = \{1, 2, 3\}$ and $Y = \{a, b\}$. Let

  $$R = \{(1, a), (1, b), (3, a)\},$$

  is the relation between X and Y, which is a subset of $X \times Y$.

Functions and Relations

- Let X, Y be nonempty sets.

- (Function gives a relation) Let $f : X \to Y$ be a function. Then

$$R = \{(x, f(x)) \ : \ x \in X\},$$

is a relation between X and Y.

Note: R is the graph of the function f in $X \times Y$.

- (Relation may not give a function) A relation R between X and Y may not give a function $f : X \to Y$ such that R is of the form

$$R = \{(x, f(x)) \ : \ x \in X\},$$

That is, R is the graph of the function f.

For example, let $X = Y = \mathbb{R}$, $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} \ : \ x^2 + y^2 = 1\}$.
R is a unit circle in $\mathbb{R}^2$, which cannot be a graph of any function, since for each $-1 < x < 1$, both $(x, -\sqrt{1 - x^2})$ and $(x, \sqrt{1 - x^2})$ belong to R, so we cannot define a function f.

Types of relations, Partial order and Total order

Let R be a relation on a nonempty set X. Then R is said to be

- Reflexive: For all $x \in X$, xRx. That is, each $x \in X$ is related to itself.
- Symmetric: for each pair $x, y \in X$, if xRy, then yRx. That is, if x is related to y, then y is related to x.
- Anti-symmetric: for each pair $x, y \in X$, if xRy and yRx, then $x = y$. That is, if x is related to y and y is related to x, then x and y are the same elements.
- Transitive: for $x, y, z \in X$, if xRy and yRz, then xRz. That is, if x is related to y and y is related to z, then x is related to z.

Partial Order

A relation R on a nonempty set X is said to be a partial order if it is reflexive, anti-symmetric, and transitive.

Notation: In case of a partial order, we generally write $x \leq y$ in place of xRy.
The pair $(X, \leq)$ is called a partially ordered set.

Total Order

A partial order R on a nonempty set X is said to be a total order if for each pair $x, y \in X$, either $x \leq y$ or $y \leq x$. The pair $(X, \leq)$ is called a totally ordered set.

Example 1

- $(\mathbb{R}, \leq)$ is a totally ordered set with usual relation $\leq$.

  Reflexive: $x \leq x$, for all $x \in \mathbb{R}$.
  Anti-symmetric: For $x, y \in \mathbb{R}$, $x \leq y$ and $y \leq x$ implies $x = y$.
  Transitive: For $x, y, z \in \mathbb{R}$, $x \leq y$ and $y \leq z$ implies $x \leq z$.

  Hence $(\mathbb{R}, \leq)$ is a partially ordered set.

  Further, any pair $x, y \in \mathbb{R}$ are related: $x \leq y$ or $y \leq x$. Hence $(\mathbb{R}, \leq)$ is a totally ordered set.

Example 2

- Let A be a nonempty set and let $X = \mathcal{P}(A)$, collection of all subsets of A. Consider $(X, \subseteq)$.

  Reflexive: $B \subseteq B$, for all $B \in X$.
  Anti-symmetric: For $B, C \in X$, $B \subseteq C$ and $C \subseteq B$ implies $B = C$.
  Transitive: For $B, C, D \in X$, $B \subseteq C$ and $C \subseteq D$ implies $B \subseteq D$.

  Hence $(X, \subseteq)$ is a partially ordered set.

  Is it a totally ordered set? **NO**!

  Let $A = \{1, 2\}$. Let $B = \{1\} \in X = \mathcal{P}(A)$ and $C = \{2\} \in \mathcal{P}(A)$ be subsets of A.

  Then B and C are not related; neither is contained in the other.

Chain

Chain

Let $(X, \leq)$ be a partially ordered set.
A nonempty subset Y of X is known as a chain in X if $(Y, \leq)$ is a totally ordered set.

Examples

- Let $A = \{1, 2\}$.
  Let $X = \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, A\}$.
  $(X, \subseteq)$ is a partially ordered set.
  $Y = \{\emptyset, \{1\}, A\}$ is a chain in X, but $\{\emptyset, \{1\}, \{2\}\}$ is not a chain in X, since $\{1\}$ and $\{2\}$ are not related.

- Any subset of a totally ordered set is a chain.

- (Example of an uncountable chain)
  Let $A = [0, 1]$ be an interval and $X = \mathcal{P}(A)$.
  $(X, \subseteq)$ is a partially ordered set.
  Then the set $Y = \{[0, t] \ : \ 0 \leq t \leq 1\}$ is a chain in X, since if $[0, t_1], [0, t_2] \in Y$ with $t_1 \leq t_2$, then $[0, t_1] \subseteq [0, t_2]$.

Bounds and Maximal Elements

Let $(X, \leq)$ be a partially ordered set. Let Y be a nonempty subset of X.

- An element $u \in X$ is called an upper bound of Y if $x \leq u$, for all $x \in Y$.
  Note that there can be many upper bounds.

- An element $\ell \in X$ is called a lower bound of Y if $\ell \leq x$, for all $x \in Y$.
  Note that there can be many lower bounds.

- An element $a \in Y$ is said to be a maximum of Y if $x \leq a$, for all $x \in Y$.
  (all the elements of Y are smaller than $a \in Y$)

- An element $b \in Y$ is said to be a minimum of Y if $b \leq x$, for all $x \in Y$.
  (all the elements of Y are greater than $b \in Y$)

- An element $a_0 \in Y$ is said to be a maximal element of Y if $x \in Y$ and $a_0 \leq x$ implies $x = a_0$.
  (no element of Y is greater than a_0)

- An element $b_0 \in Y$ is said to be a minimal element of Y if $x \in Y$ and $x \leq b_0$ implies $x = b_0$.
  (no element of Y is smaller than b_0)

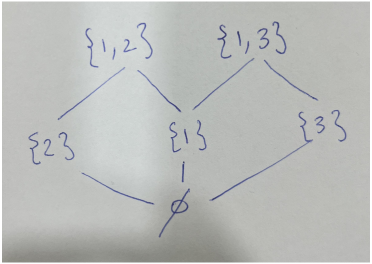
Remark

- Let $(X, \leq)$ be a partially ordered set. Let Y be a nonempty subset of X.
- Upper bounds of Y and Lower bounds of Y are elements of X (need not be in Y).
- Maximum of Y and minimum of Y are elements of Y.
- Maximal element of Y and minimal element of Y are elements of Y.

Example 1

Let $A = \{1, 2, 3\}$ and $X = \mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$, with relation $\subseteq$.
Consider the following subset of X:

$$Y = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}\}.$$

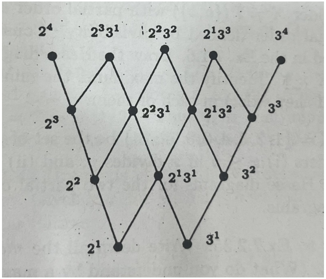


$\emptyset$ is the only minimum of Y and Y has no maximum.
Y has only one minimal element, which is $\emptyset$. The maximal elements of Y are $\{1, 2\}$ and $\{1, 3\}$ since no other element is greater than these two.

Example 2 (will be discussed in the next lecture)

Let $X = \{2^m 3^n \ : \ m, n \in \mathbb{N}, \ 1 \leq m+n \leq 4\}$.
Consider the relation $\leq$ on X as follows:

$$x, y \in X, \ x \leq y \text{ if and only if } y \text{ is a multiple of } x.$$



Then $(X, \leq)$ is a partially ordered set, but is not totally ordered since $2^4, 3^4$ are not related.
Then Y neither has a minimum element not a maximum element.
Y has two minimal elements 2^1 and 3^1, since no element is smaller than these two.
The maximal elements of Y are $2^4, 2^3 3, 2^2 3^2, 2^1 3^3, 3^4$ since no other element is greater than these two.