Week 4

	Lower Bound	Upper Bound	Infimum	Supremum	Bounded (Yes/NO)
(1) A_1	None	None	None	None	No
(2) A_2	π	$\pi+1$	π	$\pi+1$	Yes
(3) A_3	None	$\pi-1$	None	$\pi-1$	No
(4) A_4	None	None	None	None	No
(5) A_5	1	2	1	2	Yes
(6) A_6	$\boxed{\text{LATER}}$				
(7) A_7	0	2	0	2	Yes

Calculation for A 4 :

$$|x^2-3| > 1$$

$$\Updownarrow$$

$$x^2-3>1 \quad \text{or} \quad x^2-3<-1$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x^2>4 \qquad\qquad x^2<2$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$x>2 \text{ or } x<-2 \qquad\qquad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow \quad A_4 = (-\infty,-2)\cup(-\sqrt{2},\sqrt{2})\cup(2,\infty)$$

Note: I have given only one lower bound / upper bound above.
Any number bigger than an upper bound is also an upper bound.
——— smaller ——— a lower bound ——— a lower bound.

(8) Let $\inf(A) = \sup(A) = \alpha$

Since α is a lower bound of A, $\alpha \leq x \ \forall \ x \in A$

Since α is an upper bound of A, $x \leq \alpha \ \forall \ x \in A$

$$\Rightarrow \quad \alpha \leq x \leq \alpha \quad \forall \ x \in A$$

$$\Rightarrow \quad x = \alpha$$

$$\Rightarrow \quad A = \{\alpha\}$$ A has only one element.

(9) Since $\mathbb{N}$ is not bounded above in $\mathbb{R}$ (by the Archimedian property),

no $x \in \mathbb{R}$ is an upper bound of $\mathbb{N}$.

$\Rightarrow$ Given $x \in \mathbb{R}$, $\exists \ n \in \mathbb{N}$ such that $n > x$.

(10) $x > 0 \Rightarrow \frac{1}{x} \in \mathbb{R}$

By the Archimedian property, $\exists \ n \in \mathbb{N}$ such that

$$n > \frac{1}{x}$$ [$\frac{1}{x}$ is not an upper bound of $\mathbb{N}$]

$$\Rightarrow \quad \frac{1}{n} < x.$$

Problems based on Week 4 material

1. By induction on n, prove that for all real numbers $x_1, x_2, \ldots, x_n$, $|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$. — (1)
2. By induction on n, prove that for all real numbers $x_1, x_2, \ldots, x_n$, $|x_1 + x_2 + \cdots + x_n| \geq |x_1| - |x_2| - \cdots - |x_n|$. — (2)

(1) We have triangle inequality for $n = 2$.

Suppose (1) is true for $n-1$. (induction hypothesis). That is, $|x_1 + \cdots + x_{n-1}| \leq |x_1| + \cdots + |x_{n-1}|$.

We prove for n. Consider

$$|x_1 + \cdots + x_n| = |\underbrace{(x_1 + \cdots + x_{n-1})}_{x} + \underset{y}{x_n}| \underset{n=2}{\leq} |x_1 + \cdots + x_{n-1}| + |x_n| \underset{\text{induction hypothesis for } n-1}{\leq} (|x_1| + \cdots + |x_{n-1}|) + |x_n|$$

(2) We proved for $n = 2$ in the class.

Suppose (2) is true for $n-1$. That is, $|x_1 + \cdots + x_{n-1}| \geq |x_1| - |x_2| - \cdots - |x_{n-1}|$.

We prove for n. Consider

$$|x_1 + \cdots + x_n| = |\underbrace{(x_1 + \cdots + x_{n-1})}_{x} + \underset{y}{x_n}| \geq |x_1 + \cdots + x_{n-1}| - |x_n| \underset{\text{induction hypothesis for } n-1}{\geq} (|x_1| - |x_2| - \cdots - |x_{n-1}|) - |x_n|.$$

(W)

Problems based on Week 4 material

3. Consider the sequence $x_n = 1/n$, for $n \in \mathbb{N}$. Take it as a fact that x_n converges to $\overset{x}{0}$. For various values of ϵ give below, produce a cut-off N as in the definition of convergence of a sequence.
   a) $\epsilon = 1/10$.
   b) $\epsilon = 1/100$.
   c) $\epsilon = 1$.
   d) $\epsilon = 2/99$.

a) $|x_n - x| < \varepsilon \quad \forall\, n \geq N$

$\left|\frac{1}{n} - 0\right| < \frac{1}{10} \iff \frac{1}{n} < \frac{1}{10} \iff n > 10$. Take $N = 11$. Infact any number bigger than 10.

b) $\frac{1}{n} < \frac{1}{100} \iff n > 100$. Take $N = 101$.

c) $\frac{1}{n} < 1 \iff n > 1$. Take $N = 2$

d) $\frac{1}{n} < \frac{2}{99} \iff n > \frac{99}{2}$. Take $N = 50$.

In general, $\frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$. Take $n = \left[\frac{1}{\varepsilon}\right] + 1$

greatest integer less than or equal to $\frac{1}{\varepsilon}$.

floor function of $\frac{1}{\varepsilon}$.

Problems based on Week 4 material

4. Consider the sequence $x_n = 1/n^2$, for $n \in \mathbb{N}$. Take it as a fact that x_n converges to 0. For various values of ϵ give below, produce a cut-off N as in the definition of convergence of a sequence.

x

a) $\epsilon = 1/25$.
b) $\epsilon = 1/100$.
c) $\epsilon = 1$.
d) $\epsilon = 2/99$.

$|x_n - x| < \varepsilon$

$\Leftrightarrow \left|\frac{1}{n^2} - 0\right| < \varepsilon$

$\Leftrightarrow \frac{1}{n^2} < \varepsilon$

a) $\frac{1}{n^2} < \frac{1}{25} \Leftrightarrow n^2 > 25 \Leftrightarrow n > 5$. Take $N = 6$

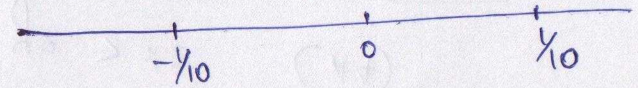
b) $\frac{1}{n^2} < \frac{1}{100} \Leftrightarrow n^2 > 100 \Leftrightarrow n > 10$. Take $N = 11$.

c) $\frac{1}{n^2} < 1 \Leftrightarrow n^2 > 1 \Leftrightarrow n > 1$. Take $N = 2$.

d) $\frac{1}{n^2} < \frac{2}{99} \Leftrightarrow n^2 > \frac{99}{2} \Leftrightarrow n > \sqrt{\frac{99}{2}}$. Take $N =$

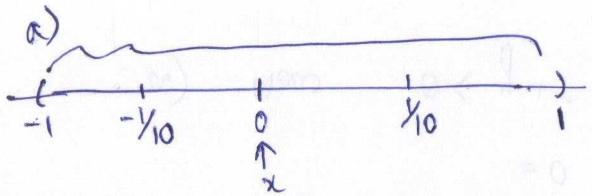
Problems based on Week 4 material

$-\frac{1}{10}, \frac{1}{10}, \frac{-1}{10}, \frac{1}{10}, \ldots$



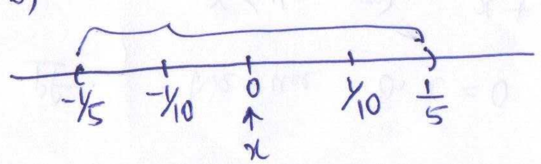
5. Consider the sequence $x_n = \dfrac{(-1)^n}{10}$, for $n \in \mathbb{N}$. List the first 10 terms of the sequence.
For various values of x, ϵ give below, if you can, produce a cut-off N such that $x_n \in (x - \epsilon, x + \epsilon)$.
a) $x = 0$, $\epsilon = 1$.
b) $x = 0$, $\epsilon = 1/5$.
c) $x = 0$, $\epsilon = 1/100$.
d) $x = 1/10$, $\epsilon = 1$.
e) $x = 1/10$, $\epsilon = 1/5$.

a)



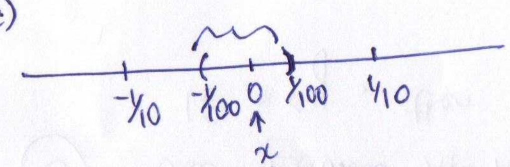
$N=1$

b)



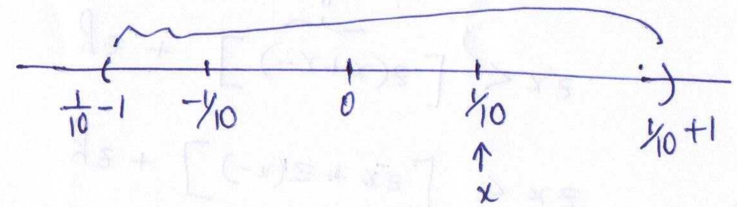
$N=1$

c)



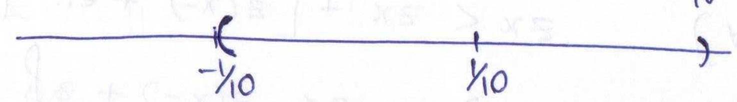
No N exists. $-\frac{1}{10}, \frac{1}{10} \notin \left(-\frac{1}{100}, \frac{1}{100}\right)$

d)



$N=1$

e) $\frac{1}{10} - \frac{1}{5} = -\frac{1}{10}$ $\qquad \frac{1}{10} + \frac{1}{5} = \frac{3}{10}$



No N works!

$-\frac{1}{10} \notin \left(\frac{-1}{10}, \frac{3}{10}\right)$