Week 3

① Reflexive : $(x_1, y_1) \leq (x_1, y_1)$ since $x_1 \leq x_1$ and $y_1 \leq y_1$.

Anti-symmetric : Let $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_1, y_1)$,

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$x_1 \leq x_2$ and $y_1 \leq y_2$ $\qquad$ $x_2 \leq x_1$ and $y_2 \leq y_1$

$$\left.\begin{array}{l} \text{Now, } x_1 \leq x_2 \text{ and } x_2 \leq x_1 \Rightarrow x_1 = x_2 \\ \text{and} \\ y_1 \leq y_2 \text{ and } y_2 \leq y_1 \Rightarrow y_1 = y_2 \end{array}\right\} \text{ ('$\leq$' is an anti-symmetric relation on } \mathbb{R}.)$$

Hence $(x_1, y_1) = (x_2, y_2)$.

Transitive : Let $(x_1, y_1) \leq (x_2, y_2)$ and $(x_2, y_2) \leq (x_3, y_3)$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$x_1 \leq x_2$ and $y_1 \leq y_2$ $\qquad$ $x_2 \leq x_3$ and $y_2 \leq y_3$

$$\left.\begin{array}{l} \text{Now, } x_1 \leq x_2 \text{ and } x_2 \leq x_3 \Rightarrow x_1 \leq x_3 \\ y_1 \leq y_2 \text{ and } y_2 \leq y_3 \Rightarrow y_1 \leq y_3 \end{array}\right\} \text{ ('$\leq$' is a transitive relation on } \mathbb{R})$$

Hence $(x_1, y_1) \leq (x_3, y_3)$.

Since '$\leq$' is a relation on $\mathbb{R}^2$ which is reflexive, anti-symmetric and transitive, it is a PARTIAL ORDER.

It is not a total order since $(1,2)$ and $(2,1)$ in $\mathbb{R}^2$ are not related, $1 \leq 2$ but $2 \not\leq 1$.

(2) $a R b \iff a \leq 2b$.

R is a reflexive relation since $a \leq 2a$.

It is neither ~~asy~~ anti-symmetric nor transitive.

- Take $a = 6$, $b = 3$, $c = 2$

  aRb since $a \leq 2b$

  bRc since $b \leq 2c$

  but a is not related to c since $a > 2c$.

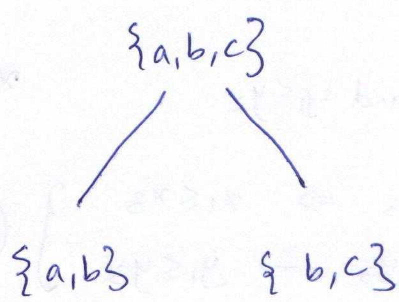
  } Not transitive

- Take $a = 3$, $b = 2$

  then aRb and bRa but $a \neq b$

  } Not anti-symmetric.

(3)



all in B:

Minimum = None

Maximum $\{a,b,c\}$

Minimal $\{a,b\}$ and $\{b,c\}$

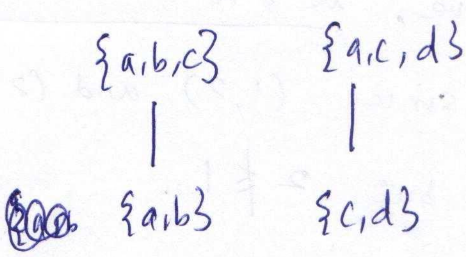
Maximal $\{a,b,c\}$

need not be in B:

Upper Bound $\{a,b,c\}$, A

Lower Bound ϕ, $\{b\}$

(4)



Minimum None

Maximum None

Minimal $\{a,b\}$ and $\{c,d\}$

Maximal $\{a,b,c\}$ and $\{a,c,d\}$

Upper Bd A

Lower Bd ϕ

(5) (i) We need to show that x is the additive inverse of $-x$.

By A4, $x + (-x) = 0$

$\Rightarrow$ x is the additive inverse of $-x$

$\Rightarrow$ $-(-x) = x$.

(ii) We need to show that x is the multiplicative inverse of x^{-1}.

By A5, $x \cdot x^{-1} = 1$

$\Rightarrow$ x is the multiplicative inverse of x^{-1}

$\Rightarrow$ $(x^{-1})^{-1} = x$.

(iii) We need to show that $y-x$ is the additive inverse of $x-y$.

Consider $(y-x)+(x-y)$. We need to show it equals 0.

$$y + [(-y) + x] \overset{A2}{=} [y + (-y)] + x \overset{A4}{=} 0 + x \overset{A4}{=} x$$

$$\Rightarrow (-y) + x = x - y \quad (A4) \quad \text{—— } (*)$$

Similarly

$$x + [(-x) + y] \overset{A2}{=} [x + (-x)] + y \overset{A4}{=} 0 + y \overset{A4}{=} y$$

$$\Rightarrow (-x) + y = y - x \quad (A4) \quad \text{—— } (**)$$

Consider ~~eq~~ ⊛ and ⊛⊛

$$(y-x)+(x-y) = [(-x)+y]+[(-y)+x]$$

$$\overset{A1,\,A2}{=} [(-x)+x]+[y+(-y)]$$

$$\overset{A4}{=} 0+0$$

$$\overset{A4}{=} 0$$

$$\Rightarrow \quad -(x-y) = y-x.$$

⑥ Use Axioms A1 – A9 to prove:

If $x<y$, then $xz<yz$, $\forall z>0$

and $xz>yz$, $\forall z<0$.

Pf: We use $0 \cdot x = 0 \quad \forall x \in \mathbb{R}$ from the class at some point in this proof.

$$x<y \Rightarrow \underbrace{x+(-x)}_{=0 \text{ by (A4)}} < y+(-x) \text{ by (A7)} \Rightarrow 0<y+(-x).$$

a) Now $0<y+(-x)$ and $z>0$ $\overset{(A8)}{\Rightarrow}$ $(y+(-x))z>0$

$\Rightarrow yz+(-x)z>0$ (A3 and A1)

$\Rightarrow [yz+(-x)z]+xz>xz$ (A7)

$\Rightarrow yz+[(-x)z+xz]>xz$ (A2)

$\Rightarrow yz+[\underbrace{(-x+x)}_{=0}z]>xz$ (A3)

$\Rightarrow yz+0\cdot z>xz$ (A4)

$\Rightarrow yz+0>xz$ $(0\cdot z=z)$

$\Rightarrow yz>xz$ (A4)

~~b) If $z<0$ then $z+(-z)<(-z)+0$ (A7) Use part a) now to get $y(-z)>x(-z)$~~

~~$0<-z$ (A4) Prove that $y(-z)=-yz$ and $x(-z)=-xz$.~~

⑥ b) To prove: If $x < y$ and $z < 0$, then $xz > yz$.

$x < y \Rightarrow 0 < y + (-x)$ see above

Now $z < 0$ and $y + (-x) > 0$

Use part a) to get

$$(y + (-x)) \cdot z < (y + (-x)) \cdot 0$$

$$= 0$$

$$\overset{A3}{=} y \cdot z + (-x) \cdot z$$

$$\Rightarrow \quad y \cdot z + (-x) \cdot z < 0$$

$$\overset{A7}{\Rightarrow} \quad [y \cdot z + (-x) \cdot z] + x \cdot z < x \cdot z$$

$$\overset{A2}{\Rightarrow} \quad y \cdot z + [(-x) \cdot z + x \cdot z] < x \cdot z$$

$$\overset{A3}{\Rightarrow} \quad y \cdot z + [((-x) + x) \cdot z] < x \cdot z$$

$$\overset{A4}{\Rightarrow} \quad y \cdot z + 0 \cdot z < x \cdot z$$

$$\overset{0 \cdot z = 0}{\Rightarrow} \quad y \cdot z + 0 < x \cdot z$$

$$\overset{A4}{\Rightarrow} \quad y \cdot z < x \cdot z.$$

✓

(7) $0 \underset{\uparrow}{=} 0 \cdot x \overset{A4}{=} (1+(-1)) \cdot x \overset{A3}{=} 1 \cdot x + (-1) \cdot x \overset{A5}{=} x + (-1) \cdot x$

proved in the class

$\Rightarrow 0 = x + (-1) \cdot x$

$\Rightarrow (-1) \cdot x$ is the additive inverse of x.

$\Rightarrow (-1) \cdot x = -x$.

(8) Proof by contradiction.

Revised Q8: Prove that:

Let $a, b \in \mathbb{R}$ such that for all positive real numbers $x > 0$,

$$a \leq b + x.$$

Then $a \leq b$.

Proof:

We prove this by contradiction.

Suppose $a > b$.

$\Rightarrow a - b > 0$

$\Rightarrow \frac{a-b}{2} > 0$

Take $x = \frac{a-b}{2} > 0$

By hypothesis, $a \leq b + x = b + \frac{a-b}{2} = \frac{a+b}{2} < \frac{a+a}{2} = a$

Hence $a < a$, which is a contradiction.