MTH 101: Calculus I

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Problem Set for Week 3

13. Prove that if X is an uncountable set, then $|\mathbb{N}| < |X|$.

Solution: Suppose $f : \mathbb{N} \to X$ be a one-one function. If it is also onto, then $X = \{f(n) \ : \ n \in \mathbb{N}\}$. We have arrived at a contradiction since $\{f(n) \ : \ n \in \mathbb{N}\}$ is a countable set but X is uncountable.

Problems based on Week 3 material

1. Consider the standard order relation $\leq$ on $\mathbb{R}$. Define a relation $\leq$ on $\mathbb{R}^2$ as follows:

$$(x_1, y_1) \leq (x_2, y_2) \text{ if and only if } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

   Prove that this relation on $\mathbb{R}^2$ is a partial order but not a total order.

2. Consider the relation $\leq$ on $\mathbb{N}$ as follows:

$$aRb \text{ if and only if } a \leq 2b.$$

   Is this a partial order on $\mathbb{N}$?

3. Consider $A = \{a, b, c, d\}$ and $X = \mathcal{P}(A)$ with relation $\subseteq$. Let $B = \{\{a, b\}, \{b, c\}, \{a, b, c\}\}$. Identify minimum element of B, maximum element of B, minimal elements of B, maximal elements of B, upper bound of B and lower bound of B.

4. Consider $A = \{a, b, c, d\}$ and $X = \mathcal{P}(A)$ with relation $\subseteq$. Let $C = \{\{a, b\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\}$. Identify minimum element of C, maximum element of C, minimal elements of C, maximal elements of C, upper bound of C and lower bound of C.

Problems based on Week 3 material

5. Use Field Axioms A1-A5 to prove the following:
   i) $-(-x) = x$, for all $x \in \mathbb{R}$.
   ii) $(x^{-1})^{-1} = x$, for all $x \in \mathbb{R}$, $x \neq 0$.
   iii) $-(x - y) = y - x$, for all $x, y \in \mathbb{R}$.
   iv) $x + (-y) = x - y$, for all $x, y \in \mathbb{R}$.
6. Use Axioms A1-A9 to prove the following: If $x < y$, then $xz < yz$, for all $z > 0$, and $xz > yz$, for all $z < 0$.
7. Use Axioms A1-A9 to prove the following:
   $(-1).x = -x$, for all $x \in \mathbb{R}$.
8. Prove the following:
   Let a, b be real numbers such that for all positive real number $x > 0$,
   $$a \leq b + x.$$
   Prove that $a \leq b$.