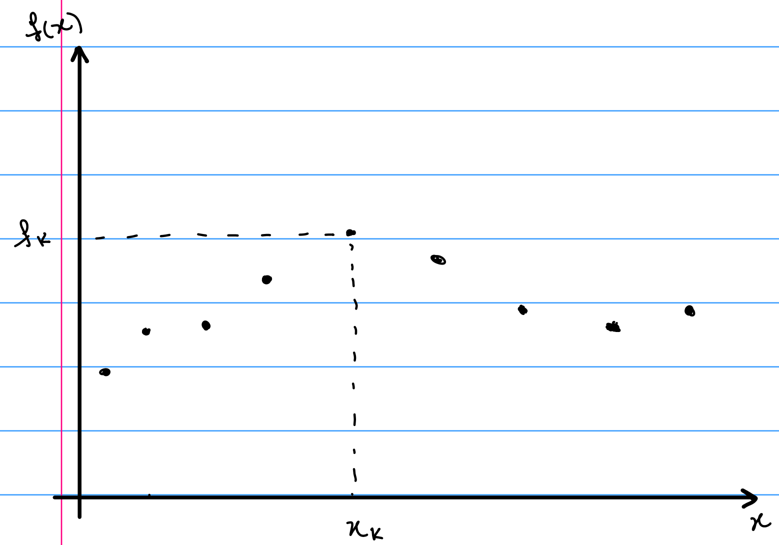
# Interpolation

## Polynomial interpolation




unknown function

$f(x)$ ?

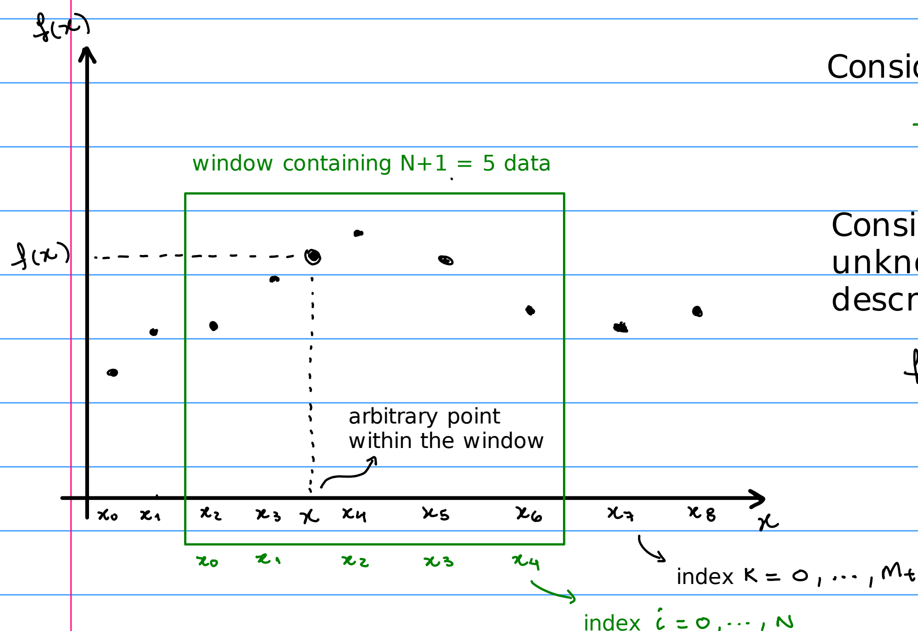
We know this function only at a discrete set of points

$$f_k \equiv f(x_k) \,, \quad k = 0, \dots, M_t \qquad (1)$$

total number of points

In this example, we have

$$M_t = 9$$




Consider a window of N+1 data

$$x_i \,, \quad i = 0, \dots, N \qquad (2)$$

Consider that, at this window, the unknown function can be properly described by a polynomial of degree N:

$$f(x) \approx P_N(x) \qquad (3)$$

In this example, we have N = 4 and:

| | | | | | |
|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | |
| $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | (4) |
| $x_k$ | | $\dots$ | | $x_{k+N}$ | |

| $x$ | $f(x)$ |
|---|---|
| $x_0$ | $f_0$ |
| $\vdots$ | $\vdots$ |
| $x_N$ | $f_N$ |

Determine the unknown function at arbitrary points x within the window by using the points with known values within the window

# Lagrange's method (theory)

Consider a given set of N+1 adjacent points within a given data window.

$$P_N(x) = l_0(x)\, f_0 + \cdots + l_N(x)\, f_N = \sum_{i=0}^{N} l_i(x)\, f_i \qquad (5)$$

$$l_i(x) = \frac{x-x_0}{x_i-x_0}\,\frac{x-x_1}{x_i-x_1}\cdots\frac{x-x_{i-1}}{x_i-x_{i-1}}\,\frac{x-x_{i+1}}{x_i-x_{i+1}}\cdots\frac{x-x_N}{x_i-x_N} = \prod_{\substack{j=0\\ j\neq i}}^{N} \frac{x-x_j}{x_i-x_j} \qquad (6)$$

cardinal functions

$$\frac{x-x_i}{x_i-x_i}$$ this term is removed!

## Example N = 1

In this case, the unknown function is described by a polynomial of degree N = 1

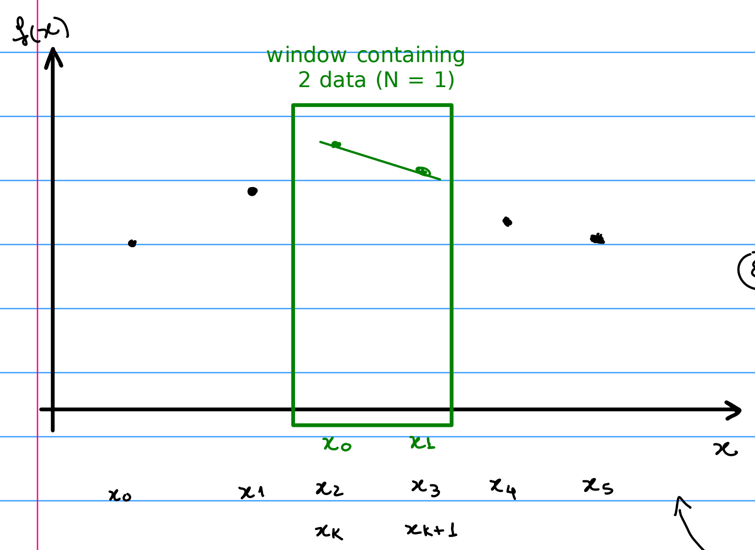
$$P_1(x) = l_0(x)\, f_0 + l_1(x)\, f_1$$

$$= \frac{x-x_1}{x_0-x_1} f_0 + \frac{x-x_0}{x_1-x_0} f_1$$

$$= \frac{(x-x_1)\, f_0 + (x_0-x)\, f_1}{x_0-x_1} \qquad (7)$$

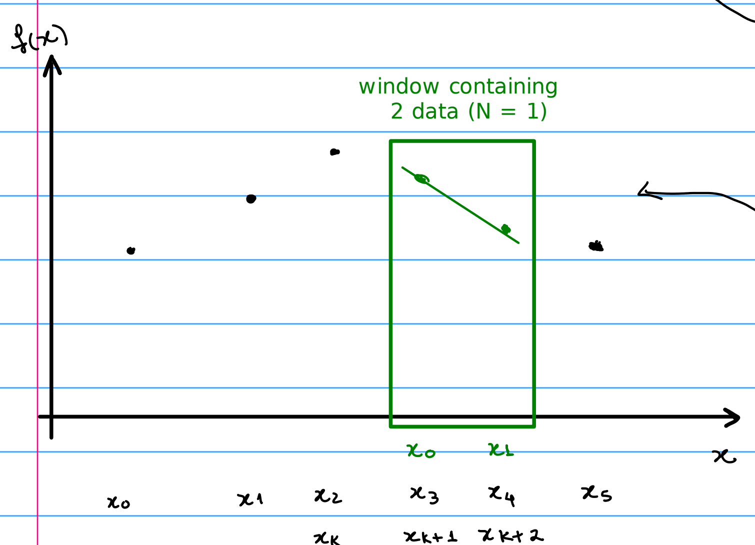
$$(8) \qquad P_1[x_0,x_1] \equiv P_1(x), \quad P_0[x_0] \equiv f_0, \quad P_0[x_1] \equiv f$$

$$P_1[x_0,x_1] = \frac{(x-x_1)\, P_0[x_0] + (x_0-x)\, P_0[x_1]}{x_0-x_1} \qquad (9)$$



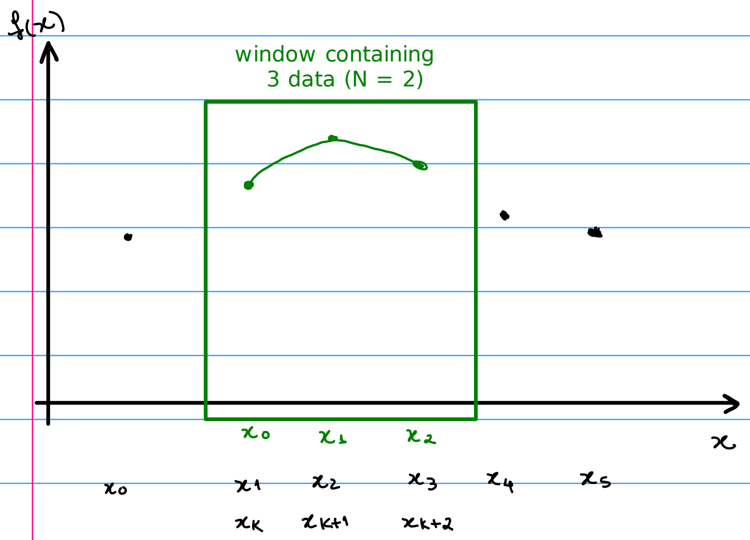

According to eq. 4,

$$P_1[x_K, x_{K+1}] = \frac{(x-x_{K+1})\, P_0[x_K] + (x_K-x)\, P_0[x_{K+1}]}{x_K - x_{K+1}} \qquad (10)$$




$$P_1[x_{K+1}, x_{K+2}] = \frac{(x-x_{K+2})\, P_0[x_{K+1}] + (x_{K+1}-x)\, P_0[x_{K+2}]}{x_{K+1} - x_{K+2}} \qquad (11)$$

# Example N = 2




$$P_2(x) = \ell_0(x) f_0 + \ell_1(x) f_1 + \ell_2(x) f_2$$

$$= \frac{x - x_1}{x_0 - x_1}\,\frac{x - x_2}{x_0 - x_2} f_0 + \frac{x - x_0}{x_1 - x_0}\,\frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_0}{x_2 - x_0}\,\frac{x - x_1}{x_2 - x_1} f_2$$

$$= \frac{x - x_2}{x_0 - x_2}\,\frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_2}{x_1 - x_2}\,\frac{x - x_0}{x_1 - x_0} f_1 + \frac{x_0 - x}{x_0 - x_2}\,\frac{x - x_1}{x_2 - x_1} f_2$$

$$= \frac{x - x_2}{x_0 - x_2}\,\frac{x - x_1}{x_0 - x_1} f_0 + \left[\frac{(x_0 - x_1) + (x_1 - x_2)}{x_0 - x_2}\right] \frac{x - x_2}{x_1 - x_2}\,\frac{x - x_0}{x_1 - x_0} f_1 + \frac{x_0 - x}{x_0 - x_2}\,\frac{x - x_1}{x_2 - x_1} f_2$$

$$= \frac{x - x_2}{x_0 - x_2}\,\frac{x - x_1}{x_0 - x_1} f_0 + \frac{\cancel{x_0 - x_1}}{x_0 - x_2}\,\frac{x - x_2}{x_1 - x_2}\,\frac{x_0 - x}{\cancel{x_0 - x_1}} f_1 + \frac{x_0 - x}{x_0 - x_2}\,\frac{x - x_1}{x_2 - x_1} f_2$$

$$+ \frac{\cancel{x_1 - x_2}}{x_0 - x_2}\,\frac{x - x_2}{\cancel{x_1 - x_2}}\,\frac{x_0 - x}{x_0 - x_1} f_1 +$$

$$= \frac{x - x_2}{x_0 - x_2}\,\frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_2}{x_1 - x_2}\,\frac{x - x_0}{x_2 - x_0} f_1 + \frac{x - x_0}{x_2 - x_0}\,\frac{x - x_1}{x_2 - x_1} f_2$$

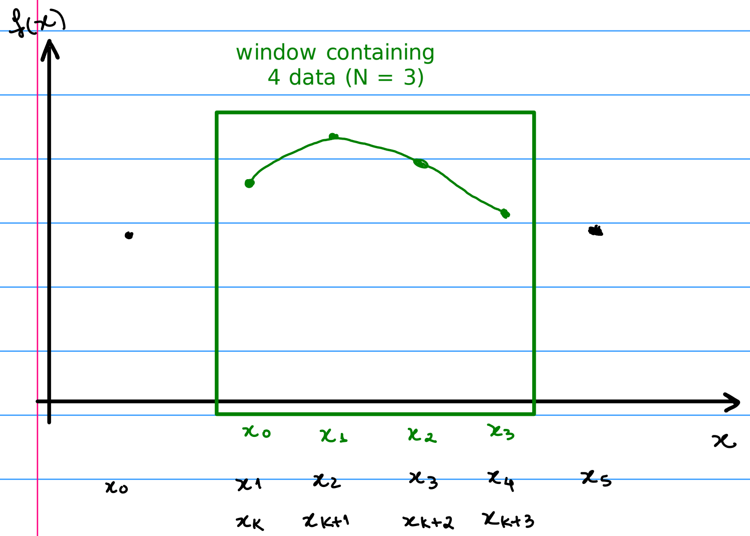
$$+ \frac{x - x_2}{x_0 - x_2}\,\frac{x - x_0}{x_1 - x_0} f_1 +$$

$$= \frac{x - x_2}{x_0 - x_2} \underbrace{\left( \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \right)}_{\substack{\text{eq. 10} \\ P_1[x_k,\, x_{k+1}]}} + \frac{x - x_0}{x_2 - x_0} \underbrace{\left( \frac{x - x_2}{x_1 - x_2} f_1 + \frac{x - x_1}{x_2 - x_1} f_2 \right)}_{\substack{\text{eq. 11} \\ P_1[x_{k+1},\, x_{k+2}]}} \qquad (12)$$

$$P_2[\underbrace{x_0, x_1, x_2}_{x_k,\, x_{k+1},\, x_{k+2}}] \equiv P_2(x) \qquad (13)$$

$$P_2[x_k, x_{k+1}, x_{k+2}] = \frac{(x - x_{k+2})\, P_1[x_k, x_{k+1}] + (x_k - x)\, P_1[x_{k+1}, x_{k+2}]}{x_k - x_{k+2}} \qquad (14)$$

Example N = 3




$$P_3(x) = \ell_0(x) f_0 + \ell_1(x) f_1 + \ell_2(x) f_2 + \ell_3(x) f_3$$

$$= \frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2}\,\frac{x-x_3}{x_0-x_3}\, f_0 + \frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\, f_1 +$$

$$+ \frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\, f_2 + \frac{x-x_0}{x_3-x_0}\,\frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\, f_3$$

$$= \frac{x-x_3}{x_0-x_3}\left(\frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2}\, f_0\right) + \frac{(x_0-x_1)+(x_1-x_3)}{x_0-x_3}\,\frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\, f_1 +$$

$$+ \frac{(x_0-x_2)+(x_2-x_3)}{x_0-x_3}\,\frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\, f_2 + \frac{x_0-x}{x_0-x_3}\left(\frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\, f_3\right)$$

$$= \frac{x-x_3}{x_0-x_3}\left(\frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2}\, f_0\right) + \frac{x_0-x_1}{x_0-x_3}\,\frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\, f_1 +$$

$$+ \frac{x_1-x_3}{x_0-x_3}\,\frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\, f_1 + \frac{x_0-x_2}{x_0-x_3}\,\frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\, f_2 +$$

$$+ \frac{x_2-x_3}{x_0-x_3}\,\frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\, f_2 + \frac{x_0-x}{x_0-x_3}\left(\frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\, f_3\right)$$

$$= \frac{x-x_3}{x_0-x_3}\left(\frac{x-x_1}{x_0-x_1}\,\frac{x-x_2}{x_0-x_2}\, f_0\right) + \frac{\cancel{x_1-x_0}}{x_0-x_3}\,\frac{x_0-x}{\cancel{x_1-x_0}}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{x_1-x_3}\, f_1 +$$

$$+ \frac{\cancel{x_1-x_3}}{x_0-x_3}\,\frac{x-x_0}{x_1-x_0}\,\frac{x-x_2}{x_1-x_2}\,\frac{x-x_3}{\cancel{x_1-x_3}}\, f_1 + \frac{\cancel{x_2-x_0}}{x_0-x_3}\,\frac{x_0-x}{\cancel{x_2-x_0}}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{x_2-x_3}\, f_2 +$$

$$+ \frac{\cancel{x_2-x_3}}{x_0-x_3}\,\frac{x-x_0}{x_2-x_0}\,\frac{x-x_1}{x_2-x_1}\,\frac{x-x_3}{\cancel{x_2-x_3}}\, f_2 + \frac{x_0-x}{x_0-x_3}\left(\frac{x-x_1}{x_3-x_1}\,\frac{x-x_2}{x_3-x_2}\, f_3\right)$$

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$$P_3(x) = \frac{x-x_3}{x_0-x_3}\underbrace{\left(\frac{x-x_1}{x_0-x_1}\frac{x-x_2}{x_0-x_2}f_0 + \frac{x-x_0}{x_1-x_0}\cdot\frac{x-x_2}{x_1-x_2}f_1 + \frac{x-x_0}{x_2-x_0}\frac{x-x_1}{x_2-x_1}f_2\right)}_{P_2[x_k,\, x_{k+1},\, x_{k+2}] \text{ (eq 14)}} +$$

$$\frac{x_0-x}{x_0-x_3}\underbrace{\left(\frac{x-x_2}{x_1-x_2}\frac{x-x_3}{x_1-x_3}f_1 + \frac{x-x_1}{x_2-x_1}\frac{x-x_3}{x_2-x_3}f_2 + \frac{x-x_1}{x_3-x_1}\frac{x-x_2}{x_3-x_2}f_3\right)}_{P_2[x_{k+1},\, x_{k+2},\, x_{k+3}]} \quad (16)$$

$$P_3[\underbrace{x_0, x_1, x_2, x_3}_{x_k,\, x_{k+1},\, x_{k+2},\, x_{k+3}}] \equiv P_3(x) \quad (17)$$

$$P_3[x_k, x_{k+1}, x_{k+2}, x_{k+3}] = \frac{(x - x_{k+3})\, P_2[x_k, x_{k+1}, x_{k+2}] + (x_k - x)\, P_2[x_{k+1}, x_{k+2}, x_{k+3}]}{x_k - x_{k+3}} \quad (18)$$

$$P_N[x_k, \dots, x_{k+N}] = \frac{(x - x_{k+N})\, P_{N-1}[x_k, \dots, x_{k+N-1}] + (x_k - x)\, P_{N-1}[x_{k+1}, \dots, x_{k+N}]}{x_k - x_{k+N}} \quad (19)$$

# Lagrange's method (computational implementation)

## Example N = 3

$\bar{x} = [x_0, x_1, x_2, x_3]$ $\quad \bar{f} = [f_0, f_1, f_2, f_3]$ $\quad \bar{\ell} = [0, 0, 0, 0]$

$j = 0$ $\quad$ mask = [False, True, True, True]

$\bar{x}[\text{mask}] = [x_1, x_2, x_3]$

$$\bar{\ell}[0] = \frac{\text{prod}(x_c - \bar{x}[\text{mask}])}{\text{prod}(x_0 - \bar{x}[\text{mask}])}$$

$\bar{a} = [a_0, a_1, a_2]$

$\text{prod}(\bar{a}) = a_0 \cdot a_1 \cdot a_2$

$j = 1$ $\quad$ mask = roll(mask, 1)

$$\bar{\ell}[1] = \frac{\text{prod}(x_c - \bar{x}[\text{mask}])}{\text{prod}(x_0 - \bar{x}[\text{mask}])}$$

...

$j = 3$ $\quad$ mask = roll(mask, 1)

$$\bar{\ell}[3] = \frac{\text{prod}(x_c - \bar{x}[\text{mask}])}{\text{prod}(x_0 - \bar{x}[\text{mask}])}$$

$$P_3[x_0, \dots, x_3] \equiv P_3[x_c] = \text{sum}(\bar{f} \circ \bar{\ell})$$