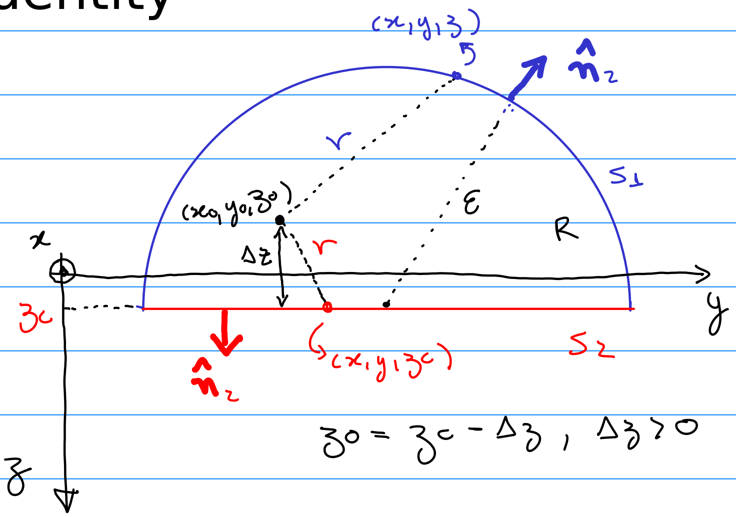
Applications of Green's third identity

Green's third identity (Kellogg, 1967, p. 219)

$$U_0 = -\frac{1}{4\pi}\iiint_R \frac{1}{r}\nabla^2 U \, dv + \frac{1}{4\pi}\iint_S \frac{1}{r}\partial_n U \, dS - \frac{1}{4\pi}\iint_S U \, \partial_n \frac{1}{r} \, dS$$

$\boxed{\text{The integrations and derivations are with respect to the variables x, y and z!}}$



$$z_0 = z_c - \Delta z \, , \quad \Delta z > 0$$

Split the surface S into the surfaces S_1 and S_2 and consider that U is harmonic in R.

$$U_0 = \frac{1}{4\pi}\iint_{S_1} \frac{1}{r}\partial_n U \, dS_1 + \frac{1}{4\pi}\iint_{S_2} \frac{1}{r}\partial_n U \, dS_2 - \frac{1}{4\pi}\iint_{S_1} U \, \partial_n \frac{1}{r} \, dS_1 - \frac{1}{4\pi}\iint_{S_2} U \, \partial_n \frac{1}{r} \, dS_2$$

$$\partial_n U = \nabla U^T \hat{n}_2$$

$$\partial_n \frac{1}{r} = \nabla \frac{1}{r}^T \hat{n}_2$$

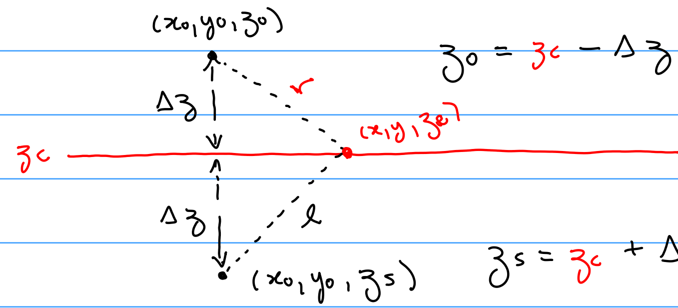
By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 $\overbrace{\text{vanish}}^{\text{We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)}}$ and we obtain:

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r}\partial_z U - U \, \partial_z \frac{1}{r} \, dx\,dy$$

$$\left\{ \begin{aligned} &\hat{n}_2 = \hat{z} \\ &\partial_n \square = \nabla \square^T \hat{z} = \partial_z \square \\ &dS_2 = dx\,dy \end{aligned} \right.$$

$$\frac{1}{r} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2\right]^{1/2}}$$

$$\frac{1}{\ell} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_s - z_c)^2\right]^{1/2}}$$



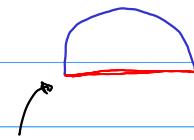
$$z_0 = z_c - \Delta z$$

$$\Delta z > 0$$

$$z_s = z_c + \Delta z$$

Ex: Show that $1/\ell$ is harmonic in R.

Applying the Green's second identity with U and $1/\ell$, we obtain:



$$\iiint_R \frac{1}{\ell}\underbrace{\cancel{\nabla^2 U}}_{=0 \text{ ok?}} - U \underbrace{\cancel{\nabla^2 \frac{1}{\ell}}}_{=0 \text{ ok?}} \, dv = \iint_S \frac{1}{\ell}\partial_n U - U \partial_n \frac{1}{\ell} \, dS$$

$$\iint_S \frac{1}{\ell}\partial_n U \, dS - \iint_S U \partial_n \frac{1}{\ell} \, dS = 0$$

We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)

By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 vanish and we obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\ell}\partial_z U \, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \partial_z \frac{1}{\ell} \, dx\,dy = 0$$

Now, multiply this equation by $1/4\pi$ and subtract or add the result from the previous equation for U_0 :

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{r} \mp \frac{1}{\ell}\right)\partial_z U \, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \left(\partial_z \frac{1}{r} \mp \partial_z \frac{1}{\ell}\right) dx\,dy$$

Ex: Show that the $\frac{1}{r} = \frac{1}{\ell}$ and $\partial_z \frac{1}{r} = -\partial_z \frac{1}{\ell}$ for points on the surface $z = z_c$

Case 1) Result obtained by <u>subtracting</u>

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\cancel{\left(\frac{1}{r} - \frac{1}{\ell}\right)}}_{=0}\partial_z U \, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \left(\partial_z \frac{1}{r} - \partial_z \frac{1}{\ell}\right) dx\,dy$$

$$U_0 = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \left(2\,\partial_z \frac{1}{r}\right) dx\,dy \quad , \quad \partial_z \frac{1}{r} = -\frac{z_0 - z_c}{r^3}(-1)$$

$$U(x_0, y_0, z_0) = \frac{z_c - z_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{U(x, y, z_c)}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2\right]^{3/2}} \, dx\,dy$$

upward continuation integral

(Skeels, 1947; Henderson and Zietz, 1949; Henderson, 1960; Roy, 1962; Bhattacharyya, 1967; Henderson, 1970; Blakely, 1996, p. 40)

The upward continuation integral states that the values of a harmonic function $U(x_0, y_0, z_0)$ at any point (x_0, y_0, z_0), $z_0 < z_c$, can be exactly reproduced by the convolution of its values $U(x, y, z_c)$ and the vertical derivative of the function $1/r$, both evaluated on the horizontal plane $z = z_c$. This equation also shows that any spatial derivative of the harmonic function $U(x_0, y_0, z_0)$ can be obtained by properly differentiating the integrand. Then, by assuming the knowledge of the harmonic function on the horizontal plane $z = z_c$, it is possible to compute not only $U(x_0, y_0, z_0)$, but also any of its spatial derivatives at any point (x_0, y_c, z_0)

Case 2) Result obtained by adding

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{1}{r}+\frac{1}{\ell}\right)\partial_z U \, dx\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \left(\cancelto{=0}{\partial_z \frac{1}{r} + \partial_z \frac{1}{\ell}}\right) dx\, dy$$

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(2\,\frac{1}{r}\right)\partial_z U \, dx\, dy$$

$$U(x_0, y_0, z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial_z U(x, y, z_c)}{\left[(x_0-x)^2+(y_0-y)^2+(z_0-z_c)^2\right]^{1/2}}\, dx\, dy$$

(Roy, 1962)

* Skeels, D. C., 1947, Ambiguity in gravity interpretation: GEOPHYSICS, 12, 43–56. doi: 10.1190/1.1437295

* Henderson, R. G., and I. Zietz, 1949, The upward continuation of anomalies in total magnetic intensity fields: GEOPHYSICS, 14, 517–534. doi: 10.1190/1.1437560

* Henderson, R. G., 1960, A comprehensive system of automatic computation in magnetic and gravity interpretation: GEOPHYSICS, 25, 569–585. doi: 10.1190/1.1438736

* Roy, A., 1962, Ambiguity in geophysical interpretation: GEOPHYSICS, 27, 90–99. doi: 10.1190/1.1438985

* Bhattacharyya, B. K., 1967, Some general properties of potential fields in space and frequency domain: a review: Geoexploration, 5, 127–143. doi: 10.1016/0016-7142(67)90021-X

* Henderson, R. G., 1970, On the validity of the use of the upward continuation integral for total magnetic intensity data: GEOPHYSICS, 35, 916–919. doi: 10.1190/1.1440137

Interpretation of deduced integrals in terms of Green's functions

By considering that U is harmonic in the Green's third identity, we obtain:

$$U(x_0, y_0, z_0) = \frac{1}{4\pi} \iint_S \frac{1}{r} \partial n U(x, y, z) - U(x, y, z) \partial n \frac{1}{r} \, dx dy$$

$$\frac{1}{r} = \frac{1}{[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2]^{1/2}}$$

(x_0, y_0, z_0) point in R

(x, y, z) point on S

Let V be an arbitrary harmonic function depending on the the variables (x_0, y_0, z_0) and (x, y, z)

$$V \equiv V(x_0, y_0, z_0, x, y, z)$$

By using the Green's second identity, we obtain

$$\frac{1}{4\pi} \iint_S V \partial n U(x, y, z) - U(x, y, z) \partial n V \, ds = 0$$

This equation can be subtracted or added to the previous one as follows:

$$U(x_0, y_0, z_0) = \frac{1}{4\pi} \iint_S \left(\frac{1}{r} \mp V \right) \partial n U(x, y, z) - U(x, y, z) \left(\partial n \frac{1}{r} \mp \partial n V \right) ds$$

Now, consider a harmonic function

$$G \equiv G(x_0, y_0, z_0, x, y, z) = \frac{1}{r} + V$$

that vanishes at all point on S. Is this function existis, the normal derivative of U can be eliminated in the integrand by considering the case in which 1/r and V are added. In this case, the integral can be rewritten as follows:

$$U(x_0, y_0, z_0) = -\frac{1}{4\pi} \iint_S U(x, y, z) \, \partial n G \, ds ,$$

where G is known as the Green's function and the differentiation and integration are with respect to the coordinates x, y and z (Kellogg, 1967, p. 237). Notice that, for the case in which S is a plane,

$$V = -\frac{1}{\ell} \quad , \quad G = \frac{1}{r} - \frac{1}{\ell}$$

as we can see in the upward continuation integral.

Actually, the upward continuation integral represents the solution of the Dirichlet's problem for the particular case in which S is a plane. This Dirichlet's problem can be stated as follows (Kellogg, 1967, p. 236 and 244-245):

The problem of finding a function, harmonic in a closed the region R, and having values equal to a given function on the boundary surface S is known as Dirichlet's problem, or the first boundary value problem of potential theory, and the theorem asserting the existence of a solution of this problem is known as the first fundamental existence theorem of potential theory. It is historically the oldest problem of existence of potential theory

The problem of finding a function, harmonic in the region R, and having normal derivatives equal to a given function on the boundary S is known as Neumann's problem, or the second boundary value problem of potential theory, and the theorem asserting the existence of a solution of this problem is known as the second fundamental existence theorem of potential theory (Kellogg, 1967, p. 245-247. Notice that the second integral obtained previously by removing the term depending on U and keeping the term depending on the normal derivative of 1/r is o solution of the Neumann's problem for a plane.