Application of the Neumann's problem

The scalar magnetic potential produced by a source with volume v and uniform total-magnetization direction is given by:

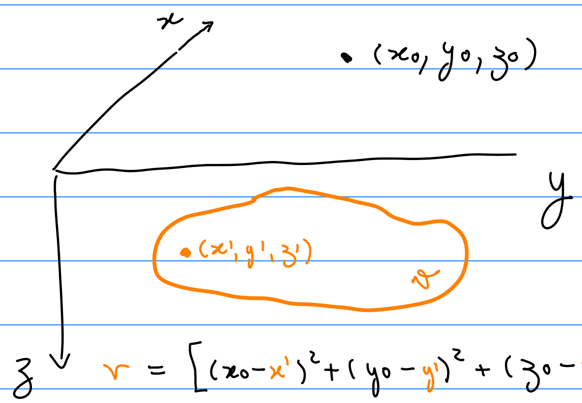
$$\mathbf{h}(x',y',z') = h(x',y',z')\,\hat{\mathbf{h}} \qquad \hat{\mathbf{h}} = \begin{bmatrix} \cos I \cos D \\ \cos I \sin D \\ \sin I \end{bmatrix}$$

$$V(x_0,y_0,z_0) = -\nabla \underbrace{\Theta(x_0,y_0,z_0)}^{T}\,\hat{\mathbf{h}}$$

this function is similar to the gravitational potential produced by a source with positive density

$$\Theta(x_0,y_0,z_0) = c_m \iiint_v h(x',y',z')\,\frac{1}{r}\,dv$$

total-magnetization intensity



$$r = \left[(x_0-x')^2+(y_0-y')^2+(z_0-z')^2\right]^{1/2}$$

Notice that $\Theta(x_0,y_0,z_0)$ does not depend on the unit vectors $\hat{\mathbf{F}}_0$ and $\hat{\mathbf{h}}$!!!

$$\nabla^2\Theta(x_0,y_0,z_0) = c_m \iiint_v h(x',y',z')\,\underbrace{\nabla^2\frac{1}{r}}_{=0}\,dv = 0$$

this function is harmonic at points outside the source

the magnetization intensity is positive at all points inside the source

$$\nabla^2\Theta(x_0,y_0,z_0) = \begin{bmatrix} \partial_{xx}\Theta & \partial_{xy}\Theta & \partial_{xz}\Theta \\ \partial_{xy}\Theta & \partial_{yy}\Theta & \partial_{yz}\Theta \\ \partial_{xz}\Theta & \partial_{yz}\Theta & \partial_{zz}\Theta \end{bmatrix}$$

The magnetic induction and approximated total-field anomaly are given by:

$$\mathbf{B}(x_0,y_0,z_0) = \nabla^2\Theta(x_0,y_0,z_0)\,\hat{\mathbf{h}}$$

$$\Delta\tilde{T}(x_0,y_0,z_0) = \hat{\mathbf{F}}_0^{T}\,\nabla^2\Theta(x_0,y_0,z_0)\,\hat{\mathbf{h}} \qquad \hat{\mathbf{F}}_0 = \begin{bmatrix} \cos I_0 \cos D_0 \\ \cos I_0 \sin D_0 \\ \sin I_0 \end{bmatrix}$$

The solution of the Neumann's problem on a plane is given by:

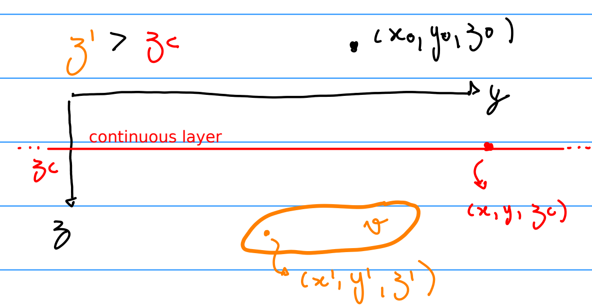
$$U(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,z_c)\,\underbrace{\frac{1}{r}}\,dx\,dy \qquad \begin{cases} p(x,y,z_c) = \dfrac{\partial_z U(x,y,z_c)}{2\pi} \\ r = \left[(x_0-x)^2+(y_0-y)^2+(z_0-z_c)^2\right]^{1/2} \end{cases}$$

similar to the gravitational potential produced by a point of mass

Now, consider that $U(x_0,y_0,z_0)$ represents the function $\Theta(x_0,y_0,z_0)$. In this case:

positive at all points on the plane z_c

$$p(x,y,z_c) = \frac{\partial_z\Theta(x,y,z_c)}{2\pi}$$



$$\partial_z\Theta(x,y,z_c) = c_m \iiint_v \frac{h(x',y',z')\,(z'-z_c)\;dx\,dy}{\left[(x-x')^2+(y-y')^2+(z_c-z')^2\right]^{3/2}}$$

this function is equivalent to the vertical component of the gravitational attraction produced by a source with positive density

where are the unit vectors $\hat{\mathbf{F}}_0$ and $\hat{\mathbf{h}}$?

Hence, the scalar magnetic potential can be rewritten as follows:

$$V(x_o, y_o, z_o) = -\nabla \Theta(x_o, y_o, z_o)^T \hat{h}$$

$$= -\Big[(\cos I \cos D) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \partial_x \frac{1}{r}\, dx dy \; +$$

$$+ (\cos I \sin D) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \partial_y \frac{1}{r}\, dx dy$$

$$+ (\sin I) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \partial_x \frac{1}{r}\, dx dy \Big]$$

derivatives with respect to the coordinates of the observation point

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c) \left(\nabla \frac{1}{r}\right)^T \hat{h}\, dx dy$$

Note that this equation represents the magnetic scalar potential produced by a continuous layer of dipoles with total-magnetization direction equal to $\hat{h}$

In this case, the α-component of the magnetic induction can be defined by the following integral:

$$B_\alpha(x_o, y_o, z_o) = -\partial_\alpha V(x_o, y_o, z_o) \qquad \alpha = x, y, z$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c) \underbrace{\left[\partial_{\alpha x} \frac{1}{r} \quad \partial_{\alpha y} \frac{1}{r} \quad \partial_{\alpha z} \frac{1}{r}\right] \hat{h}}_{\alpha\text{-component of the magnetic induction produced by a dipole with magnetization direction } \hat{h}}\, dx dy$$

note that the physical property is the same for any component

By using this integral, we can define the approximated total-field anomaly as follows:

$$\Delta\tilde{T}(x_o, y_o, z_o) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c) \underbrace{\hat{F}_o^T \left(\nabla^2 \frac{1}{r}\right) \hat{h}}_{\text{approximated total-field anomaly produced by a dipole}}\, dx dy$$

Note that this equation represents the approximated total-field anomaly produced by a continuous layer of dipoles with total-magnetization direction equal to $\hat{h}$

Note that the surface integrals defining the scalar magnetic potential, magnetic induction and approximated total-field anomaly were deduced by considering that the continuous layer of dipoles has the same total magnetization direction as that of the true magnetic source.

Equivalent layer!

The dipoles are the equivalent sources!

All these quantities associated with the magnetic field depend on the same physical property!

In this case, the physical property $p(x, y, z_c)$ is a positive function proportional to the vertical derivative of $\Theta(x, y, z_c)$.

It is worth noting that the physical property distribution $p(x, y, z_c)$ is positive at all points (x, y, z_c) on the plane z_c and does not depend on the unit vectors $\hat{\mathbf{F}}_0$ and $\hat{\mathbf{h}}$ defining, respectively, the directions of the main field and total-magnetization direction.

It means that, by determining $p(x, y, z_c)$, it is possible to determine the approximated total-field anomaly, scalar magnetic potential and magnetic induction that would be produced by the source if the main field and/or the total magnetization had different directions.

For example, if we determine $p(x, y, z_c)$, we can compute the approximated total-field anomaly that would be produced by the source if both $\hat{\mathbf{F}}_0$ and $\hat{\mathbf{h}}$ were vertical, i.e.:

$$\hat{\mathbf{F}}_0 = \hat{\mathbf{h}} = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \end{bmatrix} .$$

In this case, the approximated total-field anomaly assumes the form:

$$\Delta \tilde{T}_{RTP}(x_0, y_0, z_0) = \partial_{zz} \Theta (x_0, y_0, z_0) ,$$

which is similar to the zz-derivative of the gravitational potential produced by a source with positive density. Transforming an observed total-field anomaly into this simpler anomaly is called "reduction to the pole". This trasformation was proposed by Baranov (1957) and is valid for the case in which the sources have a uniform magnetization direction.

It is possible to determine a continuous layer of dipoles that (i) has a uniform magnetization direction different from that of the true sources and (ii) exactly reproduces the observed total-field anomaly. In this case, however, there is no guarantee that the physical property distribution is positive at all points on the layer and it is not possible to compute the reduction to the pole.

References:

* Baranov, V., 1957, A new method for interpretation of aeromagnetic maps: Pseudo-gravimetric anomalies: Geophysics, 22, 359–382. doi: 10.1190/1.1438369

* Silva, J. B. C., 1986, Reduction to the pole as an inverse problem and its application to low-latitude anomalies: GEOPHYSICS, 51, 369–382. doi: 10.1190/1.1442096

* Li, Y., M. Nabighian, and D. W. Oldenburg, 2014, Using an equivalent source with positivity for low-latitude reduction to the pole without striation: GEOPHYSICS, 79, J81–J90. doi: 10.1190/geo2014-0134.1

* André L. A. Reis, Vanderlei C. Oliveira Jr., and Valéria C. F. Barbosa, (2020). Generalized positivity constraint on magnetic equivalent layers. Geophysics, 85(6), 1-45. doi:10.1190/geo2019-0706.1

Consider now a set of total-field anomaly values at coordinates (x_i, y_i, z_i), $z_i < z_c \;\; \forall z_i$

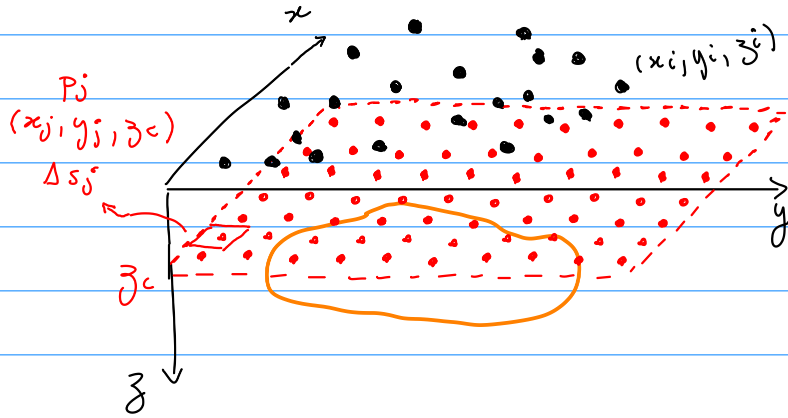
equal to that of the true sources!

$$\Delta \tilde{T}_i \equiv \Delta \tilde{T}(x_i, y_i, z_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \hat{\mathbf{F}}_0^{T} \left(\nabla^2 \frac{1}{r}\right) \hat{\mathbf{h}} \, dx\,dy \, , \quad p(x, y, z_c) = \frac{\partial_z \Theta(x, y, z_c)}{2\pi}$$

$$\Delta \tilde{T}_i \approx \sum_{j=1}^{M} p_j h_{ij}$$

$$h_{ij} = \hat{\mathbf{F}}_0^{T} \left(\nabla^2 \frac{1}{r_{ij}}\right) \hat{\mathbf{h}}$$

$$r_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_c)^2\right]^{1/2}$$

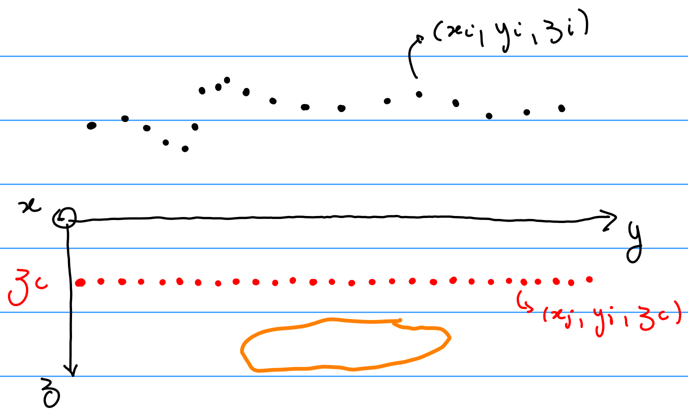


$$\Delta \tilde{T}_1 \approx p_1 h_{11} + \cdots + p_M h_{1M}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\Delta \tilde{T}_N \approx p_1 h_{N1} + \cdots + p_M h_{NM}$$

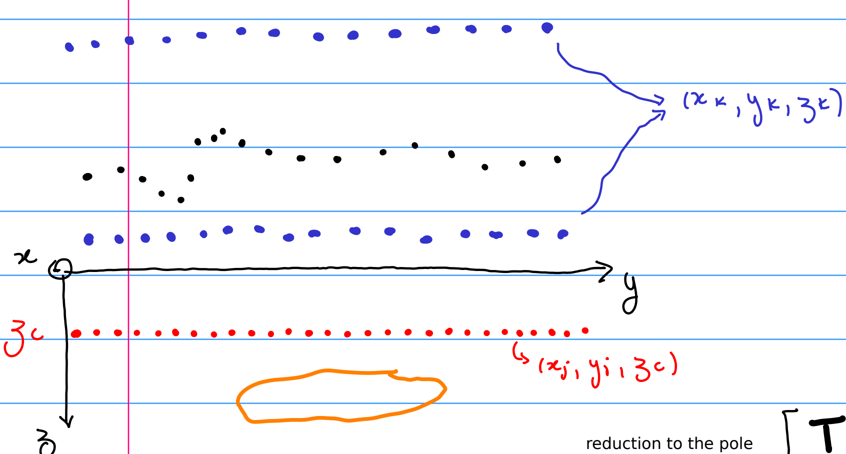
$$\underset{N \times 1}{\mathbf{\Delta \tilde{T}}} \approx \underset{N \times M}{\mathbf{H}} \; \underset{M \times 1}{\mathbf{p}} \longrightarrow p_j$$

equivalent layer

equivalent source



solve this linear system for estimating $\mathbf{p}$



transformed data

transformation matrix

$$\underset{L \times 1}{\mathbf{t}} \approx \underset{L \times M}{\mathbf{T}} \; \underset{M \times 1}{\mathbf{p}^*}$$

reduction to the pole $\quad [\mathbf{T}]_{kj} = \partial_{zz} \frac{1}{r_{kj}}$

z_k above/below z_i

upward/downward continuation $\quad [\mathbf{T}]_{kj} = \hat{\mathbf{F}}_0^{T} \left(\nabla^2 \frac{1}{r_{kj}}\right) \hat{\mathbf{h}}$

spatial derivative $\quad [\mathbf{T}]_{kj} = \hat{\mathbf{F}}_0^{T} \partial_\alpha \left(\nabla^2 \frac{1}{r_{kj}}\right) \hat{\mathbf{h}} \quad \alpha = x, y, z$