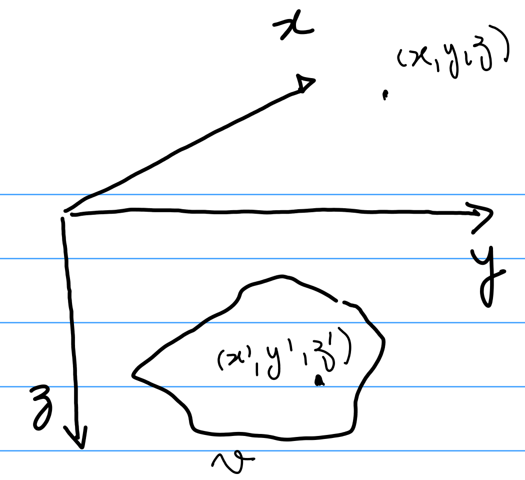
Newtonian potential

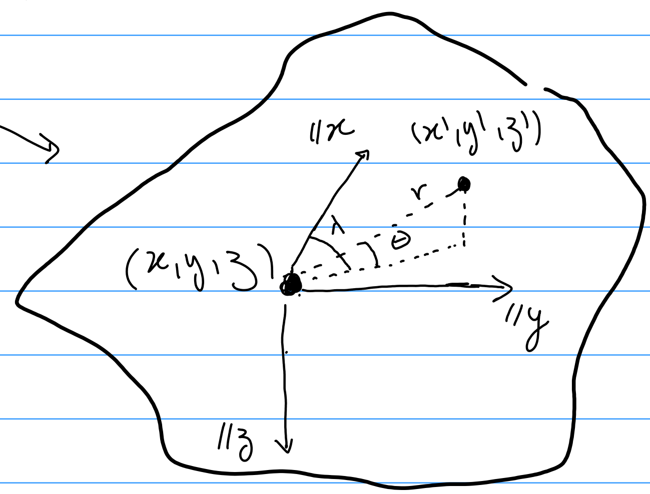


a common assumption is that
this is a piecewise continuous
and bounded function

$$\Omega(x, y, z) = \iiint_{\vartheta} \sigma(x', y', z') \frac{1}{r} \, d\vartheta$$

$$r = \left[(x - x')^2 + (y - y')^2 + (z - z')^2 \right]^{1/2}$$

Consider a point inside de source



$$x' = x + r \cos\theta \cos\lambda$$
$$y' = y + r \cos\theta \sin\lambda$$
$$z' = z + r \sin\theta$$

$$d\vartheta = r^2 \cos\theta \, dr \, d\theta \, d\lambda$$

consider a spherical coordinate system with
origin at (x, y, z)

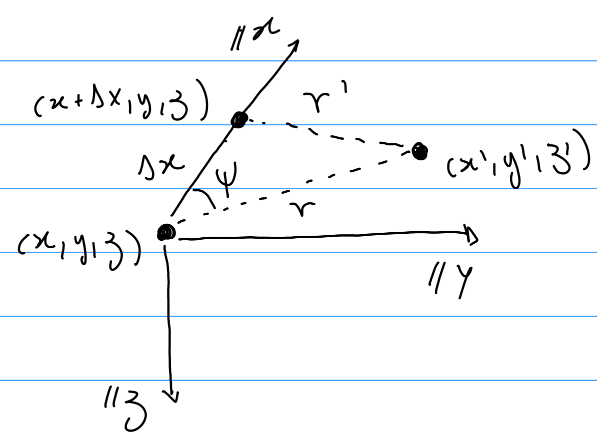
$$\Omega(r, \theta, \lambda) = \iiint_{\vartheta} \sigma(r, \theta, \lambda) \, r \cos\theta \, dr \, d\theta \, d\lambda$$

it is finite!

(Peirce, 1902, p. 32;
MacMillan, 1958, p. 27)

Consider two points inside the source, separated
by a finite distance Δx along the x axis

(Peirce, 1902, p. 33)



law of cosines

$$r'^2 = r^2 + \Delta x^2 - 2 r \Delta x \cos\psi$$

$$\cos\psi = \frac{x' - x}{r}$$

$$\frac{\Delta_x \Omega}{\Delta x} = \frac{\Omega(x+\Delta x, y, z) - \Omega(x, y, z)}{\Delta x} =$$

$$= \frac{1}{\Delta x}\left(\iiint_v \frac{1}{r'}\sigma(x', y', z')\, dv - \iiint_v \frac{1}{r}\sigma(x', y', z')\, dv\right) =$$

$$\frac{r^2 - r'^2}{r'r^2 + rr'^2} = \frac{(r-r')(r+r')}{rr'(r+r')} = \frac{r}{rr'} - \frac{r'}{rr'} = \frac{1}{r'} - \frac{1}{r}$$

$$= \frac{1}{\Delta x}\iiint_v \left(\frac{1}{r'} - \frac{1}{r}\right)\sigma\, dv$$

$$= \frac{1}{\Delta x}\iiint_v \left(\frac{r^2 - r'^2}{r'r^2 + rr'^2}\right)\sigma\, dv$$

$$= \iiint_v \left(\frac{2r\Delta x\cos\psi - \Delta x^2}{r'r^2 + rr'^2}\right)\frac{\sigma}{\Delta x}\, dv$$

$$\lim_{\Delta x \to 0} r' \to r$$

$$\lim_{\Delta x \to 0}\frac{\Delta_x \Omega}{\Delta x} = \iiint_v \left(\frac{2r\cos\psi}{2r^3}\right)\sigma\, dv$$

$$= \iiint_v \left(\frac{x' - x}{r^3}\right)\sigma\, dv = \iiint_v \left(\frac{\partial}{\partial x}\frac{1}{r}\right)\sigma\, dv$$

x-derivative is computed in the same way for points inside or outside the source! The same is valid for y- and z-derivative

consider that spherical coordinate system with origin at a point (x, y, z) inside the source

$$\partial_x \Omega(r, \theta, \lambda) = \iiint_v \left(\frac{\cos\theta\cos\lambda}{r^2}\right)\sigma\, r^2\cos\theta\, dr\, d\theta\, d\lambda$$

(Peirce, 1902, p. 33; MacMillan, 1958, p. 30)

$$= \iiint_v \sigma\cos^2\theta\cos\lambda\, dr\, d\theta\, d\lambda$$

it is also finite!

First derivatives change continuously as the point (x, y, z) moves through the boundaries of the source or any other internal surface separating two regions with continuous physical property distribution (Peirce, 1902, p. 50-52)

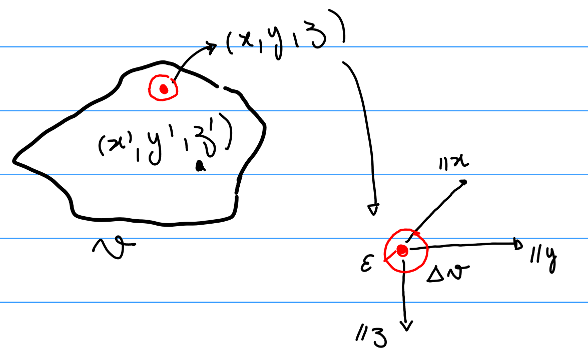
Second derivatives are also finite at points within the source (Peirce, 1902, p. 45-50)

$$\underline{\partial_{\alpha\beta}\Omega(x,y,z) = \iiint_{v} \partial_{\alpha\beta}\frac{1}{r}\, dv} \quad //$$

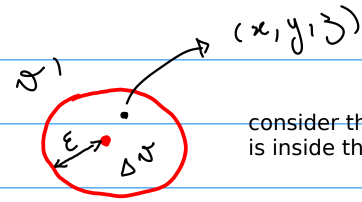
$$\nabla^2\Omega(x,y,z) = 0$$

The potential satisfies the Laplace equation at points (x, y, z) outside the source (ok?)

What about points inside the source?



small sphere with radius ε, volume Δv



consider that (x, y, z) is inside the sphere

$$\nabla^2\Omega(x,y,z) = \nabla^2\left[\underbrace{\cancel{\iiint_{v'}\frac{\sigma}{r}\,dv}}_{=0\ (\text{LAPLACE})} + \iiint_{\Delta v}\frac{\sigma}{r}\,dv\right]$$

$$= \nabla^2\left[\sigma\iiint_{\Delta v}\frac{1}{r}\,dv\right]$$

(Here, we consider that the volume Δv is so small that σ can be considered constant in its interior)

$$= \nabla^2\left[\sigma\, 2\pi\left(\varepsilon^2 - \frac{r^2}{3}\right)\right]$$

(at the computation point (x,y,z))

$$= \underbrace{\left(-\frac{4\pi\sigma}{3}\right)}_{\partial xx} + \underbrace{\left(-\frac{4\pi\sigma}{3}\right)}_{\partial yy} + \underbrace{\left(-\frac{4\pi\sigma}{3}\right)}_{\partial zz} = \underline{-4\pi\sigma} \quad //$$

(Torge and Müller, 2012, p. 58-59)

Torge, W., and Müller, J. (2012). Geodesy, De Gruyter, 4th edition. ISBN: 978-3-11-020718-7

Notice that, for all quantities computed above, we have imposed that the volume containing the computation point inside the source is a sphere. It can be shown, however, that all results presented above can be obtained without restricting the volume containing the computation point within the source. This can be shown for the z-derivative of potential inside the source by using the Cauchy test on pages Kellogg (1967, p. 17-19). A more general approach involves improper integrals and can be summarized by three theorems (Kellogg, 1967, Chapter VI):

Theorem I (Kellogg, 1967, p. 151). The potential U, and the components X, Y, Z of the force, due to a volume distribution of piecewise continuous density in the bounded volume V, exist at the points of V, and are continuous throughout space.

Theorem II (Kellogg, 1967, p. 152). The potential U of the volume distribution of Theorem I is everywhere differentiable, and the equations

$$X = \frac{\partial U}{\partial x} \qquad Y = \frac{\partial U}{\partial y} \qquad Z = \frac{\partial U}{\partial z}$$

hold throughout space.

The mere continuity of the density does not suffice to insure the existence of second derivatives at points inside the source. In this case, we have to impose that σ satisfies the Hölder condition (Kellogg, 1967, p. 152).

Theorem III (Kellogg, 1967, p. 156). Let U be the potential of a distribution with piecewise continuous density σ in a regular region V. Then at any interior point P_0 of V, at which σ satisfies a Hölder condition, the derivatives of second order of U exist and satisfy Poissons' equation

$$\nabla^2 U = -4\pi\sigma(P_0)$$