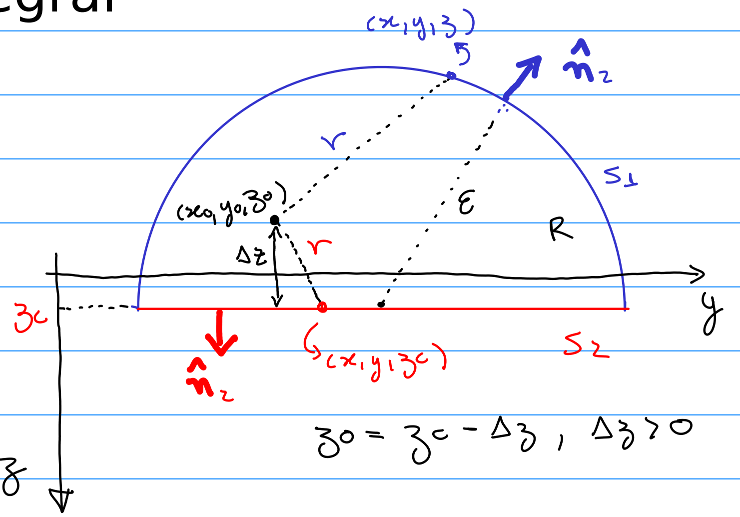
Upward continuation integral



$$z_0 = z_c - \Delta z \,, \quad \Delta z > 0$$

Green's third identity (Kellogg, 1967, p. 219)

$$U_0 = -\frac{1}{4\pi}\iiint_R \frac{1}{r}\nabla^2 U \, dv + \frac{1}{4\pi}\iint_S \frac{1}{r}\partial_n U \, dS - \frac{1}{4\pi}\iint_S U \, \partial_n \frac{1}{r} \, dS$$

Split the surface S into the surfaces S_1 and S_2
and consider that U is harmonic in R.

$$U_0 = \frac{1}{4\pi}\iint_{S_1} \frac{1}{r}\partial_n U \, dS_1 + \frac{1}{4\pi}\iint_{S_2} \frac{1}{r}\partial_n U \, dS_2 - \frac{1}{4\pi}\iint_{S_1} U \, \partial_n \frac{1}{r} \, dS_1 - \frac{1}{4\pi}\iint_{S_2} U \, \partial_n \frac{1}{r} \, dS_2$$

$$\partial_n U = \nabla U^{T} \hat{n}_2$$

$$\partial_n \frac{1}{r} = \nabla \frac{1}{r}^{T} \hat{n}_2$$

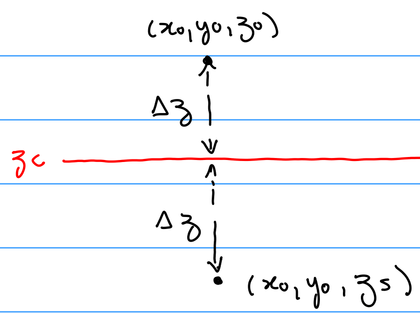
By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 $\overbrace{\text{vanish}}^{\text{We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)}}$ and we obtain:

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r}\partial_z U - U \partial_z \frac{1}{r} \, dx\,dy$$

$$\left\{ \begin{array}{l} \hat{n}_2 = \hat{z} \\ \partial_n \square = \nabla \square^{T} \hat{z} = \partial_z \square \\ dS_2 = dx\,dy \end{array} \right.$$

$$\frac{1}{r} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2\right]^{1/2}}$$

$$\frac{1}{\ell} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_s - z_c)^2\right]^{1/2}}$$



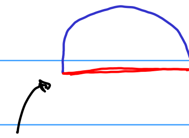
$$z_0 = z_c - \Delta z$$

$$\Delta z > 0$$

$$z_s = z_c + \Delta z$$

Ex: Show that $1/\ell$ is harmonic in R.

Applying the Green's second identity with U and $1/\ell$, we obtain:



$$\iiint_R \frac{1}{\ell}\underbrace{\cancel{\nabla^2 U}}_{=0 \text{ ok?}} - U \underbrace{\cancel{\nabla^2 \frac{1}{\ell}}}_{=0 \text{ ok?}} \, dv = \iint_S \frac{1}{\ell}\,\partial_n U - U\, \partial_n \frac{1}{\ell} \; dS$$

$$\iint_S \frac{1}{\ell}\,\partial_n U \, dS - \iint_S U\, \partial_n \frac{1}{\ell} \, dS = 0$$

We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)

By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 vanish and we obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\ell}\,\partial_z U \, dx\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\, \partial_n \frac{1}{\ell} \, dx\, dy = 0$$

Now, multiply this equation by $1/4\pi$ and subtract or add the result from the previous equation for U_0 :

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{r} \mp \frac{1}{\ell}\right)\partial_z U \, dx\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U \left(\partial_z \frac{1}{r} \mp \partial_z \frac{1}{\ell}\right) dx\, dy$$

Ex: Show that the $\frac{1}{r} = \frac{1}{\ell}$ and $\partial_z \frac{1}{r} = -\partial_z \frac{1}{\ell}$ for points on the surface $z = z_c$