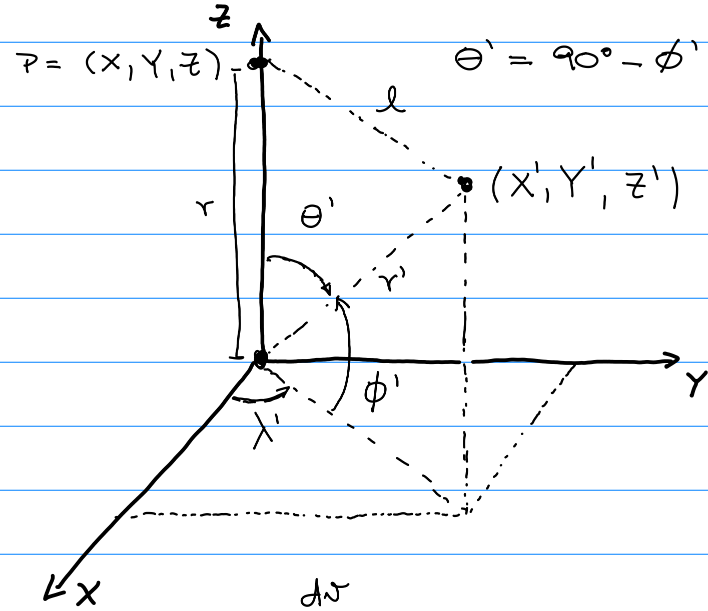
Sphere

$$X' = r'\sin\theta'\cos\lambda'$$
$$Y' = r'\sin\theta'\sin\lambda'$$
$$Z' = r'\cos\theta'$$



$$\theta' = 90° - \phi'$$

$$l = \sqrt{(X-X')^2 + (Y-Y')^2 + (Z-Z')^2}$$

law of cosines $$= \sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}$$

For convenience, consider : $G\rho = 1$

$$U_P = \iiint \frac{1}{l}\, dv$$

radius of the sphere

$$= \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{\sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}}\ \overbrace{r'^2 \sin\theta'\, d\lambda'\, d\theta'\, dr'}^{dv}$$

$$= 2\pi \int_0^R r'^2 \int_0^\pi \frac{\sin\theta'}{\sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}}\, d\theta'\, dr'$$

$$\boxed{\begin{array}{l} t = \cos\theta' \\ dt = -\sin\theta'\, d\theta' \\ \theta' = \pi \rightarrow t = -1 \\ \theta' = 0 \rightarrow t = 1 \end{array}}$$

$$= 2\pi \int_0^R r'^2 \int_{-1}^{1} \frac{1}{\sqrt{r^2 + r'^2 - 2 r r' t}}\, dt\, dr'$$

$$= 2\pi \int_0^R r'^2 \left[- \frac{\sqrt{r^2 + r'^2 - 2 r r' t}}{r r'} \right]_{-1}^{1} dr'$$

$$= 2\pi \int_0^R r'^2 \left[\frac{\sqrt{r^2 + r'^2 + 2 r r'} - \sqrt{r^2 + r'^2 - 2 r r'}}{r r'} \right] dr'$$

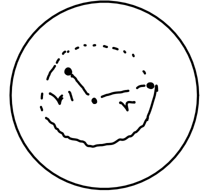
$$= 2\pi \int_0^R r' \left[\frac{|r + r'| - |r - r'|}{r} \right] dr'$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

1) $r > R$, $|r + r'| = r + r'$, $|r - r'| = r - r'$

$$U_{out} = 2\pi \frac{1}{r} \int_0^R r'r + r'^2 - r'r + r'^2\, dr' = \int_0^R 2 r'^2\, dr' =$$

$$= 2\pi \frac{1}{r}\left[2\frac{R^3}{3} \right] = \frac{4\pi}{3} R^3 \frac{1}{r} \quad \boxed{G\rho = 1} \qquad \frac{GM}{r}$$



2) $r < R$

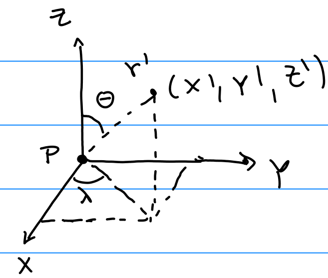
$$U_{IN} = 2\pi \left[\int_0^r \frac{r'}{r} \left(\overbrace{|r+r'|}^{r+r'} - \overbrace{|r-r'|}^{r-r'} \right) dr' + \int_r^R \frac{r'}{r} \left(\overbrace{|r+r'|}^{r+r'} - \overbrace{|r-r'|}^{r'-r} \right) dr' \right]$$

$$= 2\pi \frac{1}{r} \left[\int_0^r 2r'^2 \, dr' + \int_r^R 2r' r \, dr' \right]$$

$$= 2\pi \frac{1}{r} \left[\frac{2r^3}{3} + 2r \left(\frac{R^2}{2} - \frac{r^2}{2} \right) \right]$$

$$= 2\pi \left[\frac{2r^2}{3} + R^2 - r^2 \right] = 2\pi \left(R^2 - \frac{r^2}{3} \right)$$

3) $r = 0$ $\qquad \ell = r'$



$$U_0 = \iiint \frac{1}{r'} r'^2 \sin\theta \, d\lambda' d\theta' dr'$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} r' \sin\theta \, d\lambda' d\theta' dr'$$

$$= 2\pi \int_0^R r' \left[-\cos\theta \right]_0^\pi dr' = 2\pi R^2$$

4) $r = R$, $\quad \lim_{r \to R^-} U_{IN} = \lim_{r \to R^+} U_{out} = \frac{4}{3}\pi R^2$

$G\rho = 1$

$$U_P = \begin{cases} \frac{4}{3}\pi R^3 \frac{1}{r}, & r > R \\ \frac{4}{3}\pi R^2, & r = R \\ 2\pi \left(R^2 - \frac{r^2}{3} \right), & r < R \\ 2\pi R^2, & r = 0 \end{cases}$$

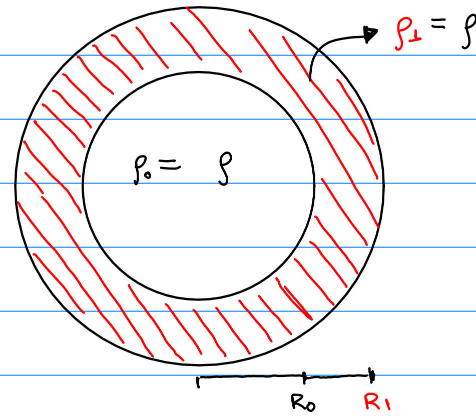
$r = \sqrt{x^2 + y^2 + z^2}$

(Sansò and Sideris, 2013. p. 10-11)

Ex.: Faça um gráfico do potencial gravitacional produzido por uma esfera sólida em função de r.

Ex.: Defina o Laplaciano, primeira e segunda derivadas radiais do potencial produzido por uma esfera sólida.

$$\partial_{xx}U + \partial_{yy}U + \partial_{zz}U = \nabla^2 U$$



$r > R_1$,

$$U_p = G\frac{4}{3}\pi R_1^3 \rho_1 \frac{1}{r} - G\frac{4}{3}\pi R_0^3 \rho_0 \frac{1}{r}$$

$$= G\frac{4}{3}\pi \rho \left(R_1^3 - R_0^3\right)\frac{1}{r}$$

$R_0 < r < R_1$

$$U_p = G\, 2\pi \rho_1 \left(R_1^2 - \frac{r^2}{3}\right) - G\frac{4}{3}\pi R_0^3 \rho_0 \frac{1}{r}$$

$$= G\, 2\pi \rho \left(R_1^2 - \frac{r^2}{3} - \frac{2}{3}\frac{R_0^3}{r}\right)$$

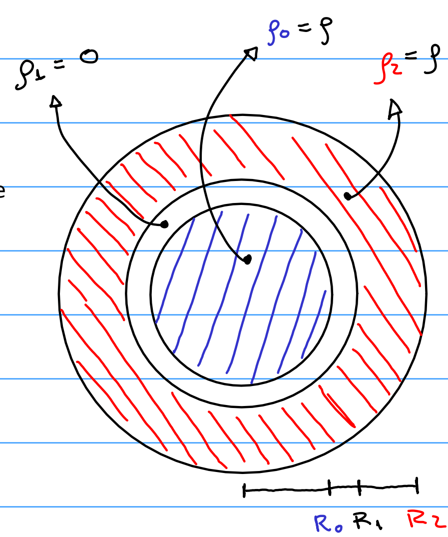
$r < R_0$

$$U_p = G\, 2\pi \left(R_1^2 - \frac{r^2}{3}\right)\rho_1 - G\, 2\pi \left(R_0^2 - \frac{r^2}{3}\right)\rho_0$$

$$= G\, 2\pi \rho \left(R_1^2 - R_0^2\right)$$

Ex.: Faça um gráfico do potencial gravitacional
produzido por uma casca esférica em função de r.

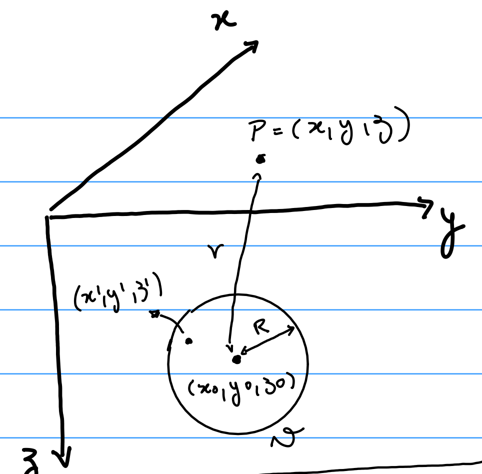
Ex.: Defina o Laplaciano, primeira e
segunda derivadas radiais do potential
produzido por uma cascas esférica.



Ex.: Defina o potencial, Laplaciano, primeira e
segunda derivadas radiais produzidos pelo
modelo de esféricas concêntricas.

Ex.: Faça um gráfico do potencial gravitacional
produzido pelo modelo de esferas concêntricas em função de r.

Now, consider a solid sphere with radius R, volume v and center at the coordinates (x_0, y_0, z_0) referred to a topocentric Catesian system.



$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

$$\ell = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

Consider also that there is a constant magnetic intensity field $\mathbf{H}_0$

The magnetic intensity produced by this sphere at a point P (inside or outside) is given by:

derivatives with respect to the coordinates x, y, z

$$\mathbf{H}_P = \mathbf{H}_0 - \nabla \Gamma_P$$

$$\Gamma_P = \frac{1}{4\pi} \iiint_v \mathbf{h}(x', y', z')^T \, \nabla \frac{1}{\ell} \, dv$$

resultant magnetic intensity field inside the sphere

Consider a uniform magnetization given by $\mathbf{h} = \boldsymbol{\chi} \mathbf{H}^*$

$$\Gamma_P = \frac{1}{4\pi} \iiint_v (\boldsymbol{\chi} \mathbf{H}^*)^T \, \nabla \frac{1}{\ell} \, dv = \nabla \Psi_P^T \boldsymbol{\chi} \mathbf{H}^* \, , \qquad \Psi_P = \frac{1}{4\pi} \iiint_v \frac{1}{\ell} \, dv$$

$$\nabla \Gamma_P = \nabla^2 \Psi_P \, \boldsymbol{\chi} \mathbf{H}^* \, , \qquad \mathbf{H}_P = \mathbf{H}_0 - \nabla^2 \Psi_P \, \boldsymbol{\chi} \mathbf{H}^*$$

Consider a point P inside the sphere $(\mathbf{H}_P = \mathbf{H}^*)$

$$\mathbf{H}_P + \nabla^2 \Psi_P \, \boldsymbol{\chi} \mathbf{H}_P = \mathbf{H}_0$$

$$\mathbf{H}_P = (\mathbf{I} + \nabla^2 \Psi_P \, \boldsymbol{\chi})^{-1} \mathbf{H}_0 \rightarrow \mathbf{h} = \boldsymbol{\chi} (\mathbf{I} + \nabla^2 \Psi_P \, \boldsymbol{\chi})^{-1} \mathbf{H}_0 = (\mathbf{I} + \boldsymbol{\chi} \nabla^2 \Psi_P)^{-1} \boldsymbol{\chi} \mathbf{H}_0$$

(Searle, 1982, p. 151)

(see the potential U_{in}) $\rightarrow$ $\Psi_P = \left(\frac{1}{4\pi}\right) 2\pi \left(R^2 - \frac{r^2}{3}\right)$,

$$\partial_x \Psi_P = -\left(\frac{1}{4\pi}\right) \frac{4\pi}{3} (x - x_0)$$

$$\partial_{xx} \Psi_P = -\left(\frac{1}{4\pi}\right) \frac{4\pi}{3}$$

$$\partial_{xy} \Psi_P = 0$$

$$\nabla^2 \Psi_P = -\frac{1}{3} \mathbf{I}$$

Consider a constant isotropic susceptibility $\boldsymbol{\chi} = \chi \mathbf{I}$

$$\mathbf{h} = \left(\mathbf{I} - \frac{1}{3} \chi \mathbf{I}\right)^{-1} \chi \mathbf{H}_0$$

Searle, S. R., (1982). Matrix Algebra Useful for Statistics, Wiley-Interscience, ISBN:. 0-471-86681-4