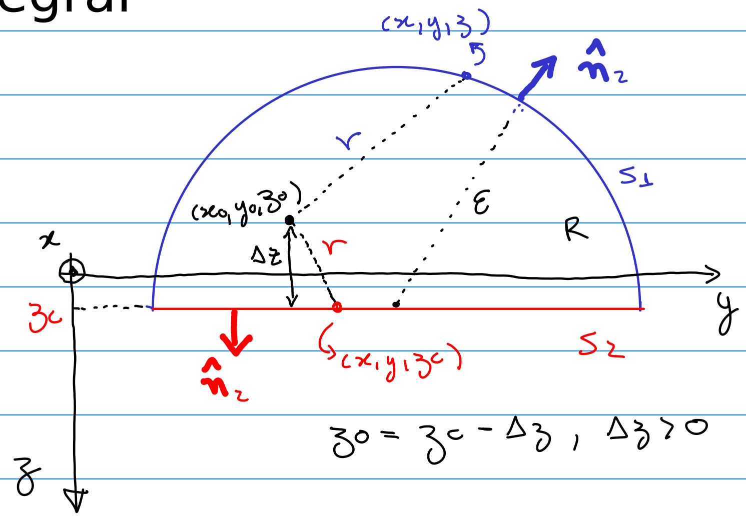
Upward continuation integral

Green's third identity (Kellogg, 1967, p. 219)

$$U_0 = -\frac{1}{4\pi}\iiint\limits_R \frac{1}{r}\nabla^2 U\, dv + \frac{1}{4\pi}\iint\limits_S \frac{1}{r}\,\partial_n U\, dS - \frac{1}{4\pi}\iint\limits_S U\,\partial_n \frac{1}{r}\, dS$$



$$z_0 = z_c - \Delta z \;, \quad \Delta z > 0$$

Split the surface S into the surfaces S_1 and S_2 and consider that U is harmonic in R.

$$U_0 = \frac{1}{4\pi}\iint\limits_{S_1} \frac{1}{r}\,\partial_n U\, dS_1 + \frac{1}{4\pi}\iint\limits_{S_2} \frac{1}{r}\,\partial_n U\, dS_2 - \frac{1}{4\pi}\iint\limits_{S_1} U\,\partial_n \frac{1}{r}\, dS_1 - \frac{1}{4\pi}\iint\limits_{S_2} U\,\partial_n \frac{1}{r}\, dS_2$$

$$\partial_n U = \nabla U^T \hat{n}_2$$

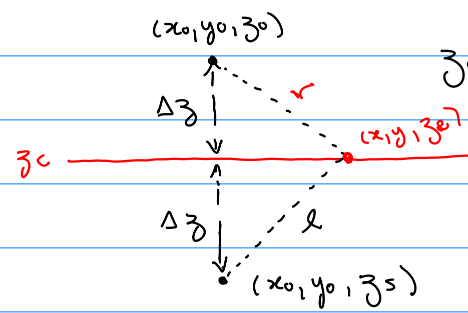
$$\partial_n \frac{1}{r} = \nabla \frac{1}{r}^T \hat{n}_2$$

By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 $\overbrace{\text{vanish}}^{\text{We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)}}$ and we obtain:

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{r}\,\partial_z U - U\,\partial_z \frac{1}{r}\; dx\,dy \qquad \left\{\begin{array}{l} \hat{n}_2 = \hat{z} \\ \partial_n \square = \nabla \square^T \hat{z} = \partial_z \square \\ dS_2 = dx\,dy \end{array}\right.$$

$$\frac{1}{r} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2\right]^{1/2}}$$

$$\frac{1}{\ell} = \frac{1}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_s - z_c)^2\right]^{1/2}}$$



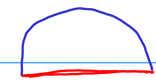
$$z_0 = z_c - \Delta z$$

$$\Delta z > 0$$

$$z_s = z_c + \Delta z$$

Ex: Show that $1/\ell$ is harmonic in R.

Applying the Green's second identity with U and $1/\ell$, we obtain:



$$\iiint_R \frac{1}{\ell}\underbrace{\cancel{\nabla^2 U}}_{=0 \text{ ok?}} - U\underbrace{\cancel{\nabla^2 \frac{1}{\ell}}}_{=0 \text{ ok?}}\, dv = \iint_S \frac{1}{\ell}\,\partial_n U - U\,\partial_n \frac{1}{\ell}\, dS$$

$$\iint_S \frac{1}{\ell}\,\partial_n U\, dS - \iint_S U\,\partial_n \frac{1}{\ell}\, dS = 0$$

We consider that U and its derivatives are regular at infinite (Kellogg, 1967, p. 217)

By letting $\varepsilon \longrightarrow \infty$, the integrals on S_1 vanish and we obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\ell}\,\partial_z U\, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\,\partial_z \frac{1}{\ell}\, dx\,dy = 0$$

Now, multiply this equation by $1/4\pi$ and subtract or add
the result from the previous equation for U_0 :

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{r} \mp \frac{1}{\ell}\right)\partial_z U\, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(\partial_z \frac{1}{r} \mp \partial_z \frac{1}{\ell}\right) dx\,dy$$

Ex: Show that the $\frac{1}{r} = \frac{1}{\ell}$ and $\partial_z \frac{1}{r} = -\partial_z \frac{1}{\ell}$ for points on the surface $z = z_c$

Case 1) Result obtained by <u>subtracting</u>

$$U_0 = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\cancel{\left(\frac{1}{r} - \frac{1}{\ell}\right)}}_{=0}\partial_z U\, dx\,dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(\partial_z \frac{1}{r} - \partial_z \frac{1}{\ell}\right) dx\,dy$$

$$U_0 = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U\left(2\,\partial_z \frac{1}{r}\right) dx\,dy \quad , \quad \partial_z \frac{1}{r} = -\frac{z_0 - z_c}{r^3}$$

$$\boxed{U(x_0, y_0, z_0) = \frac{z_c - z_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{U(x, y, z_c)}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2\right]^{3/2}}\, dx\,dy}$$

upward continuation integral

(Skeels, 1947; Henderson and Zietz, 1949; Henderson, 1960; Roy, 1962; Bhattacharyya, 1967; Henderson, 1970; Blakely, 1996, p. 40)

The upward continuation integral states that the values of a harmonic function $U(x_0, y_0, z_0)$ at any point (x_0, y_0, z_0) , $z_0 < z_c$, can be exactly reproduced by the convolution of its values $U(x, y, z_c)$ and the vertical derivative of the function $1/r$, both evaluated on the horizontal plane $z = z_c$. This equation also shows that any spatial derivative of the harmonic function $U(x_0, y_0, z_0)$ can be obtained by properly differentiating the integrand. Then, by assuming the knowledge of the harmonic function on the horizontal plane $z = z_c$, it is possible to compute not only $U(x_0, y_0, z_0)$, but also any of its spatial derivatives at any point

Case 2) Result obtained by adding

$$U_0 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{r} + \frac{1}{\varrho} \right) \partial_z U \, dx \, dy - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U \left(\overset{=0}{\cancel{\partial_z \frac{1}{r} + \partial_z}} \frac{1}{\varrho} \right) dx \, dy$$

$$U_0 = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 \frac{1}{r} \right) \partial_z U \, dx \, dy$$

$$U(x_0, y_0, z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial_z U(x, y, z_c)}{\left[(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z_c)^2 \right]^{1/2}} \, dx \, dy$$

(Roy, 1962)

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* Henderson, R. G., and I. Zietz, 1949, The upward continuation of anomalies in total magnetic intensity fields: GEOPHYSICS, 14, 517–534. doi: 10.1190/1.1437560

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