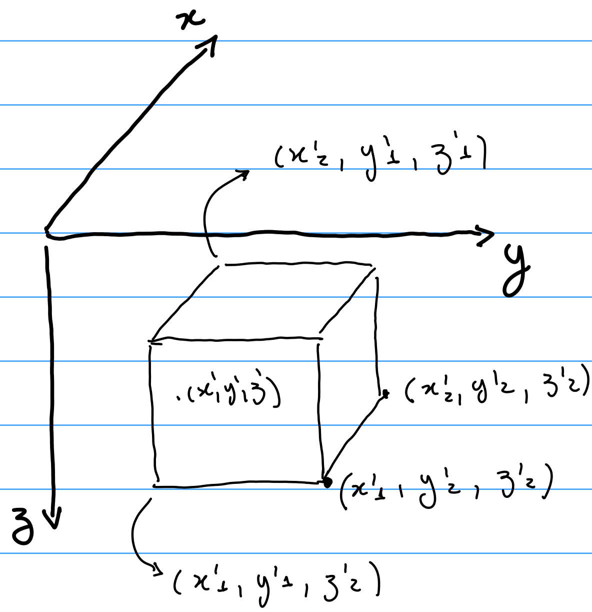
Rectangular prism

Consider a rectangular prism with constant density and sides aligned with a topocentric Cartesian coordinate system having x-, y-, and z-axis pointing to north, east and down, respectively. The gravitational potential produced by this prism at a point P = (x, y, z) is given by:



$$U_P = G\rho \int_{x'_1}^{x'_2} \int_{y'_1}^{y'_2} \int_{z'_1}^{z'_2} \frac{1}{\ell}\, dv$$

$$\ell = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$i = 1,2 \qquad j = 1,2 \qquad k = 1,2$$

A solution for this integral is given by (Nagy et al., 2000, 2002):

$$X_1 = x'_1 - x \qquad Y_1 = y'_1 - y \qquad Z_1 = z'_1 - z$$

$$X_2 = x'_2 - x \qquad Y_2 = y'_2 - y \qquad Z_2 = z'_2 - z$$

$$\ell_{ijk} = \sqrt{X_i^2 + Y_j^2 + Z_k^2}$$

$$i = 1,2 \qquad j = 1,2 \qquad k = 1,2$$

$$f(i,j,k) = X_i Y_j \ln(Z_k + \ell_{ijk}) + Y_j Z_k \ln(X_i + \ell_{ijk}) + X_i Z_k \ln(Y_j + \ell_{ijk}) + {} - \frac{X_i^2}{2} \tan^{-1}\left(\frac{Y_j Z_k}{X_i \ell_{ijk}}\right) - \frac{Y_j^2}{2} \tan^{-1}\left(\frac{X_i Z_k}{Y_j \ell_{ijk}}\right) - \frac{Z_k^2}{2} \tan^{-1}\left(\frac{X_i Y_j}{Z_k \ell_{ijk}}\right)$$

$$U_P = G\rho \left|\left|\left| f(i,j,k) \right|_{X_1}^{X_2} \right|_{Y_1}^{Y_2} \right|_{Z_1}^{Z_2}$$

$$= G\rho \left|\left| f(2,j,k) - f(1,j,k) \right|_{Y_1}^{Y_2} \right|_{Z_1}^{Z_2}$$

$$= G\rho \left| f(2,2,k) - f(2,1,k) - f(1,2,k) + f(1,1,k) \right|_{Z_1}^{Z_2}$$

$$= G\rho \left[f(2,2,2) - f(2,2,1) - f(2,1,2) + f(2,1,1) - f(1,2,2) + f(1,2,1) + f(1,1,2) - f(1,1,1) \right]$$

$$= G\rho \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} (-1)^{(i+j+k)} f(i,j,k)$$

Nagy, D., Papp, G., and Benedek, J. (2000). The gravitational potential and its derivatives for the prism: Journal of Geodesy, 74, 552–560, doi: 10.1007/s001900000116.
Nagy, D., Papp, G., and Benedek, J. (2002). Corrections to "The gravitational potential and its derivatives for the prism": Journal of Geodesy, 76, 475, doi: 10.1007/s00190-002-0264-7

Ex.: Implement the potential, its first and second derivatives (see the notebook prism.ipynb)