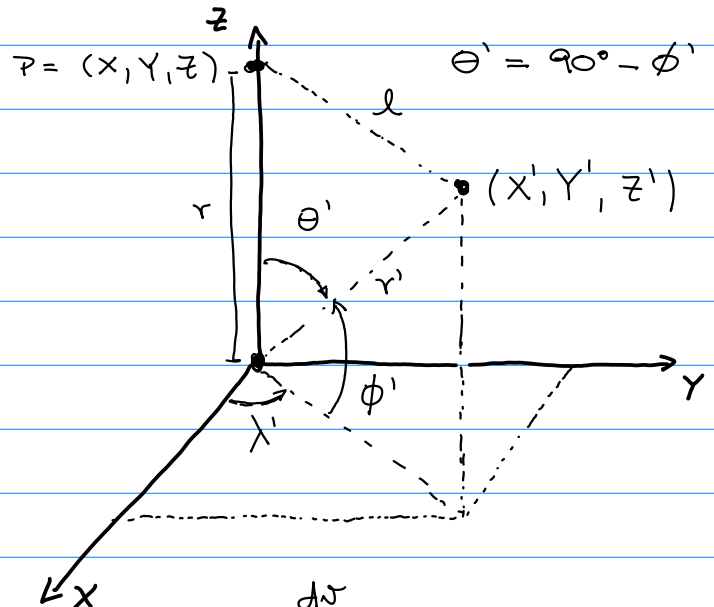


Sphere

$$\begin{aligned} X' &= r' \sin \theta' \cos \lambda' & X &= r \sin \theta \cos \lambda \\ Y' &= r' \sin \theta' \sin \lambda' & Y &= r \sin \theta \sin \lambda \\ Z' &= r' \cos \theta' & Z &= r \cos \theta \end{aligned}$$



$$\begin{aligned} l &= \sqrt{(X-X')^2 + (Y-Y')^2 + (Z-Z')^2} \\ &= \sqrt{r^2 + r'^2 - 2r r' \cos \theta'} \end{aligned}$$

law of cosines

For convenience, consider: $G\rho = 1$

$$\begin{aligned} U_P &= \iiint \frac{1}{l} d\tau \\ &= \int_0^R \int_0^\pi \int_0^{2\pi} \frac{1}{\sqrt{r^2 + r'^2 - 2r r' \cos \theta'}} r'^2 \sin \theta' d\lambda' d\theta' dr' \end{aligned}$$

radius of the sphere

$$\begin{aligned} &= 2\pi \int_0^R r'^2 \int_0^\pi \frac{\sin \theta'}{\sqrt{r^2 + r'^2 - 2r r' \cos \theta'}} d\theta' dr' \\ &= 2\pi \int_0^R r'^2 \int_{-1}^1 \frac{1}{\sqrt{r^2 + r'^2 - 2r r' t}} dt dr' \end{aligned}$$

$$\begin{aligned} t &= \cos \theta' \\ dt &= -\sin \theta' d\theta' \\ \theta' = \pi &\rightarrow t = -1 \\ \theta' = 0 &\rightarrow t = 1 \end{aligned}$$

$$= 2\pi \int_0^R r'^2 \left[-\frac{\sqrt{r^2 + r'^2 - 2r r' t}}{r r'} \right]_{-1}^1 dr'$$

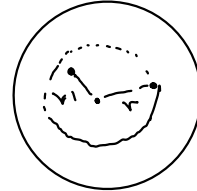
$$= 2\pi \int_0^R r'^2 \left[\frac{\sqrt{r^2 + r'^2 + 2r r'} - \sqrt{r^2 + r'^2 - 2r r'}}{r r'} \right] dr'$$

$$= 2\pi \int_0^R r' \left[\frac{|r+r'| - |r-r'|}{r} \right] dr'$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

1) $r > R$, $|r+r'| = r+r'$, $|r-r'| = r-r'$

$$\begin{aligned} U_{out} &= 2\pi \frac{1}{r} \int_0^R r' r + r'^2 - r' r + r'^2 dr' = \int_0^R 2r'^2 dr' = \\ &= 2\pi \frac{1}{r} \left[\frac{2R^3}{3} \right] = \frac{4\pi R^3}{3} \frac{1}{r} \leftarrow \boxed{G\rho = 1} \quad \left(\frac{GM}{r} \right) \end{aligned}$$



2) $r < R$

$$U_{in} = 2\pi \left[\int_0^r \frac{r'}{r} \left(\overbrace{r+r'}^{r+r'} - \overbrace{r-r'}^{r-r'} \right) dr' + \int_r^R \frac{r'}{r} \left(\overbrace{r+r'}^{r+r'} - \overbrace{r-r'}^{r-r'} \right) dr' \right]$$

$$= 2\pi \frac{1}{r} \left[\int_0^r 2r'^2 dr' + \int_r^R 2r'r dr' \right]$$

$$= 2\pi \frac{1}{r} \left[2 \frac{r^3}{3} + 2r \left(\frac{R^2}{2} - \frac{r^2}{2} \right) \right]$$

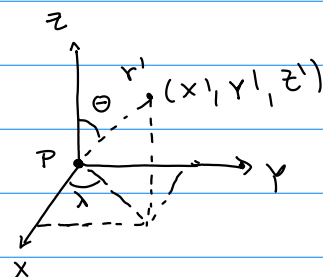
$$= 2\pi \left[\frac{2r^2}{3} + R^2 - r^2 \right] = 2\pi \left(R^2 - \frac{r^2}{3} \right)$$

3) $r = 0$ $\ell = r'$

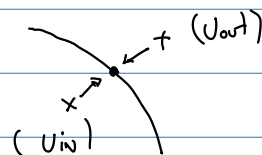
$$U_0 = \iiint \frac{1}{r'} r'^2 \sin\theta d\lambda' d\theta' dr'$$

$$= \int_0^R \int_0^\pi \int_0^{2\pi} r' \sin\theta d\lambda' d\theta' dr'$$

$$= 2\pi \int_0^R r' [-\cos\theta]_0^\pi dr' = 2\pi R^2$$



4) $r = R$, $\lim_{r \rightarrow R^-} U_{in} = \lim_{r \rightarrow R^+} U_{out} = \frac{4\pi R^2}{3}$



$G = 1$

$$U_P = \begin{cases} \frac{4\pi R^3}{3} \frac{1}{r} & , r > R \\ \frac{4\pi R^2}{3} & , r = R \\ 2\pi \left(R^2 - \frac{r^2}{3} \right) & , r < R \\ 2\pi R^2 & , r = 0 \end{cases}$$

(Sansò and Sideris, 2013. p. 10-11)

Ex.: Faça um gráfico do potencial gravitacional produzido por uma esfera sólida em função de r .

Ex.: Defina o Laplaciano, primeira e segunda derivadas radiais do potencial produzido por uma esfera sólida.

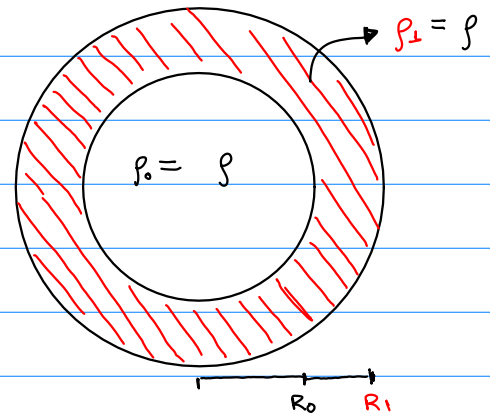
$$\partial_{xx}U + \partial_{yy}U + \partial_{zz}U = \nabla^2 U$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r > R_1$$

$$U_p = G \frac{4}{3} \pi R_1^3 \rho_1 \frac{1}{r} - G \frac{4}{3} \pi R_0^3 \rho_0 \frac{1}{r}$$

$$= G \frac{4}{3} \pi \rho (R_1^3 - R_0^3) \frac{1}{r}$$



$$R_0 < r < R_1$$

$$U_p = G 2\pi \rho_1 \left(R_1^2 - \frac{r^2}{3} \right) - G \frac{4}{3} \pi R_0^3 \rho_0 \frac{1}{r}$$

$$= G 2\pi \rho \left(R_1^2 - \frac{r^2}{3} - \frac{2}{3} \frac{R_0^3}{r} \right)$$

$$r < R_0$$

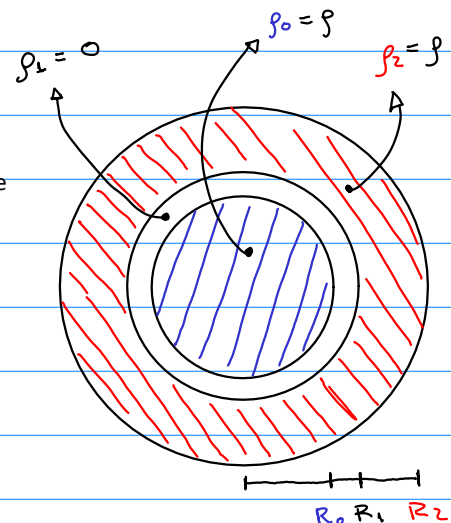
$$U_p = G 2\pi \left(R_1^2 - \frac{r^2}{3} \right) \rho_1 - G 2\pi \left(R_0^2 - \frac{r^2}{3} \right) \rho_0$$

$$= G 2\pi \rho (R_1^2 - R_0^2)$$

Ex.: Faça um gráfico do potencial gravitacional produzido por uma casca esférica em função de r .

Ex.: Defina o Laplaciano, primeira e segunda derivadas radiais do potencial produzido por uma cascas esférica.

Ex.: Defina o potencial, Laplaciano, primeira e segunda derivadas radiais produzidos pelo modelo de esferas concêntricas.



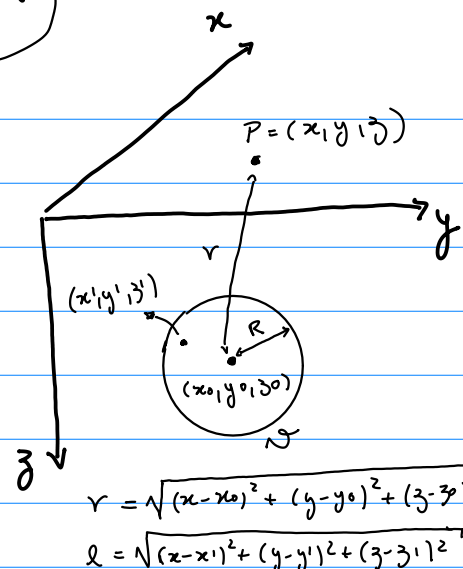
Ex.: Faça um gráfico do potencial gravitacional produzido pelo modelo de esferas concêntricas em função de r .

$$\frac{B}{\mu_0} = H$$

Now, consider a solid sphere with radius R , volume v and center at the coordinates (x_0, y_0, z_0) referred to a topocentric Cartesian system.

Consider also that there is a constant magnetic intensity field H_0

The magnetic intensity produced by this sphere at a point P is given by:



derivatives with respect to the coordinates x, y, z

$$H_P = H_0 - \nabla \Gamma_P$$

(Stratton, 2007, p. 228; Reitz and Milford, 1960, p. 189)

$$\Gamma_P = -\frac{1}{4\pi} \iiint_V h(x', y', z')^T \nabla \frac{1}{l} dV$$

resultant magnetic intensity field inside the sphere

Consider a uniform magnetization given by

$$h = \chi H^*$$

$$\Gamma_P = -\frac{1}{4\pi} \iiint_V (\chi H^*)^T \nabla \frac{1}{l} dV = -\nabla \Psi_P^T \chi H^*, \quad \Psi_P = \frac{1}{4\pi} \iiint_V \frac{1}{l} dV$$

$$\nabla \Gamma_P = -\nabla^2 \Psi_P \chi H^*, \quad H_P = H_0 + \nabla^2 \Psi_P \chi H^*$$

Consider a point P inside the sphere ($H^* = H_P$)

$$H_P - \nabla^2 \Psi_P \chi H_P = H_0 \quad \rightarrow \quad h = \chi (\mathbf{I} - \nabla^2 \Psi_P \chi)^{-1} H_0$$

$$H_P = (\mathbf{I} - \nabla^2 \Psi_P \chi)^{-1} H_0 = (\mathbf{I} - \chi \nabla^2 \Psi_P)^{-1} \chi H_0$$

(Searle, 1982, p. 151)

$$\left(\begin{smallmatrix} \text{see the} \\ \text{potential} \\ \text{in} \end{smallmatrix} \right) \rightarrow \Psi_P = \left(\frac{1}{4\pi} \right) 2\pi \left(R^2 - \frac{r^2}{3} \right),$$

$$\partial_x \Psi_P = -\left(\frac{1}{4\pi} \right) \frac{4\pi}{3} (x - x_0)$$

$$\partial_{xx} \Psi_P = -\left(\frac{1}{4\pi} \right) \frac{4\pi}{3}$$

$$\nabla^2 \Psi_P = -\frac{1}{3} \mathbf{I}$$

Consider a constant isotropic susceptibility $\chi = \chi \mathbf{I}$

$$\partial_{xy} \Psi_P = 0$$

$$h = \left(\mathbf{I} + \frac{1}{3} \chi \mathbf{I} \right)^{-1} \chi H_0$$

$$p / \chi \ll 1 \quad \tilde{h} = \chi H_0$$

See the notebooks sphere.ipynb and demagnetization_sphere.ipynb

Ex:

$$\Delta h = h - \tilde{h}$$

$$\| \Delta h \|_2 \propto \chi \rightarrow 10^{-5} - 1$$