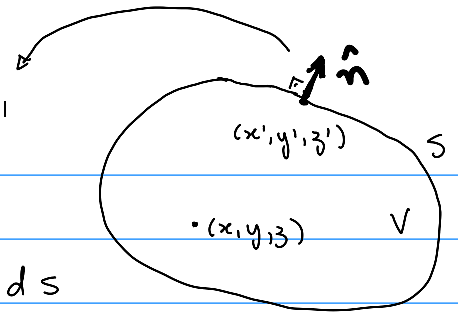
Divergence theorem

outward normal
to surface S



$$\iiint_V \nabla \cdot \mathbf{F}(x,y,z)\, dv = \iint_S \mathbf{F}(x',y',z')^T \hat{\mathbf{n}}(x',y',z')\, ds$$

Divergence theorem:
The integral of the divergence of a vector field over a region of space is equal to the integral over the surface of that region of the component of the field in the direction of the outward directed normal to the surface (Kellogg, 1967, p. 39).

This surface does not have restriction as to size, position, or general shape, but it must have a definite normal nearly everywhere.

(vector field) $$\mathbf{F}(x,y,z) = \begin{bmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{bmatrix}$$

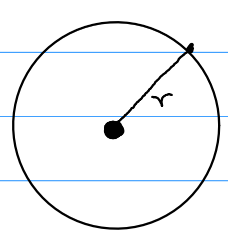
(divergent of $\mathbf{F}(x,y,z)$) $$\nabla \cdot \mathbf{F}(x,y,z) = \frac{\partial F_x}{\partial x}(x,y,z) + \frac{\partial F_y}{\partial y}(x,y,z) + \frac{\partial F_z}{\partial z}(x,y,z)$$

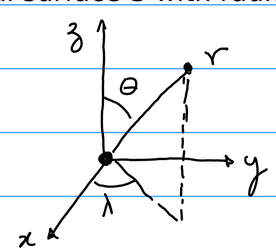
The divergence theorem requires that not only the vector field, but also the boundary surface satisfies some conditions. The condition imposed to field is that its components and their partial derivatives of the first order are continuous within V and on the boundary surface S. The condition imposed to the surface is that it must be a normal region. For an in-depth discussion about the divergence theorem, see Kellogg (1967, Chapter IV).

Exercises:
1) Compute the divergent of a vector field defined by the gradient of a Newtonian potential at points outside the sources.
2) Consider a closed region outside the sources (at free space). Compute the integral of the component of the vector field in the direction of the outward normal to the boundary surface of this region.

Gauss theorem

Consider a small sphere with mass m located at the center of a spherical surface S with radius r.





The (Newtonian) potential produced by this sphere at the surface S is given by

$$\Omega = \frac{m}{r}, \quad m = \frac{4}{3}\pi R^3 \rho$$

$$\iint_S \partial_r \Omega\, ds = m \int_0^{2\pi}\int_0^{\pi} \left(-\frac{1}{r^2}\right) r^2 \sin\theta\, d\theta\, d\lambda$$

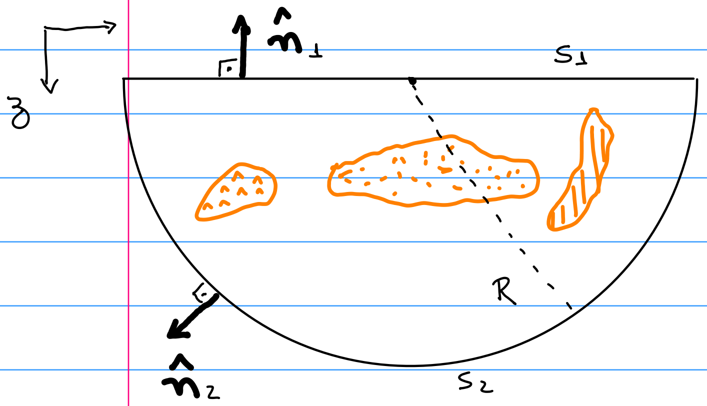
$$= m \int_0^{2\pi} \cos\theta \Big|_0^{\pi} d\lambda = -4\pi m$$

It can be shown that this result is the same for the case in which the sphere is not at the center of the spherical surface. Actually, the result is the same for any surface that contains the small sphere and satisfies the conditions for applying the divergence theorem. In this case, if S meets no sources, the following equation is valid:

$$\iint_S \nabla \Omega^T \hat{\mathbf{n}}\, ds = -4\pi M$$

where M is the total mass inside S (Kellogg, 1967, p. 43). This is integral defines the Gauss' theorem: the flux outward across the surface bounding a region is equal to -4π times the total mass in the region, provided the bounding surface meets no masses. Under certain conditions, this theorem can be extended to the case in which S passes through sources (Kellogg, 1967, p. 43-44).

Consider a set of sources enclosed by a surface formed by a hemisphere of radius R and a plane:



Exercise:
Compute the integral of the vector field defined by the gradient of the Newtonian potential produced by the sources at the surface for the case in which R tends to infinite. Hint: use the Gauss theorem and split the integral as follows Blakely (1996, p. 41-43):

$$\iint_S \nabla\Omega^T \hat{n}\, ds = \iint_{S_1} \nabla\Omega^T \hat{n}_1\, ds + \iint_{S_2} \nabla\Omega^T \hat{n}_2\, ds$$