Green's identities

The following is heavily based on Kellogg (1967, Chapter VIII).

See Kellogg (1967, Chapter IV) for a rigorous definition of "region"

Let R denote a closed regular region of space, and let U and V be two functions defined in R, and continuous in R together with their partial derivatives of the first order. Moreover, let U have continuous derivatives of the second order in R. Then the divergence theorem holds for R with the field

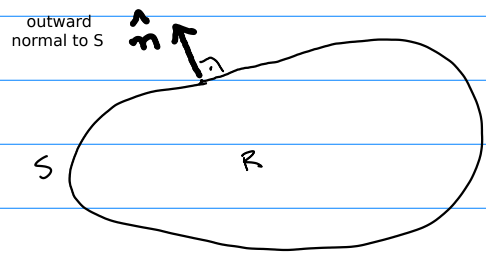
dependence on (x,y,z) is omitted $F_x(x,y,z)$

$$F_x = V\frac{\partial U}{\partial x}, \quad F_y = V\frac{\partial U}{\partial y}, \quad F_z = V\frac{\partial U}{\partial z}, \quad \mathbf{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Green's first identity (Kellogg, 1967, p. 212)

$$\iiint_R V\nabla^2 U + \boldsymbol{\nabla} V^T \boldsymbol{\nabla} U \, dv = \iint_S V \boldsymbol{\nabla} U^T \hat{\mathbf{n}} \, ds$$

Ex: Deduce the Green's first identity by applying the divergence theorem.



Ex: Manipulate the Green's first identity by considering that U is harmonic V = 1 in R and show that the integral of the normal derivative of U over S vanishes (Kellogg, 1967, Theorem I, p. 212).

By considering that V = U and U is harmonic in R, Green's first identity assumes the form:

$$\iiint_R U \underbrace{\cancel{\nabla^2 U}}_{=0} + \boldsymbol{\nabla} U^T \boldsymbol{\nabla} U \, dv = \iint_S U \boldsymbol{\nabla} U^T \hat{\mathbf{n}} \, ds$$

$$\iiint_R \left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial U}{\partial y}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 dv = \iint_S U \boldsymbol{\nabla} U^T \hat{\mathbf{n}} \, ds$$

Ex: What happens if U = 0 on S? (Kellogg, 1967, Theorems II and III, p. 213)
And if the normal derivative of U is zero on S? (Kellogg, 1967, Theorem IV, p. 213)

Suppose that both U and V are continuously differentiable in R and have continuous partial derivatives of the second order in R. We then have the Green's first identity. We can also obtain another identity by simply interchanging U and V. If this new identity is subtracted from the Green's first identity, we obtain:

$$\iiint_R V\nabla^2 U + \cancel{\nabla V^T \nabla U}\, dv - \iiint_R U\nabla^2 V + \cancel{\nabla U^T \nabla V}\, dv =$$

$$= \iint_S V \nabla U^T \hat{n}\, ds - \iint_S U \nabla V^T \hat{n}\, ds$$

$$\iiint_R V\nabla^2 U - U\nabla^2 V\, dv = \iint_S V \overbrace{\nabla U^T \hat{n}}^{\frac{\partial U}{\partial n}} - U \overbrace{\nabla V^T \hat{n}}^{\frac{\partial V}{\partial n}}\, ds$$

Green's second identity (Kellogg, 1967, p. 215)

Ex: What if U and V are harmonic in R? (Kellogg, 1967, Theorem VI, p. 216)

Now, consider that $V = \frac{1}{r}$, $r = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2}$,

where (x_0, y_0, z_0) is a fixed point within R. Then, the Green's second identity becomes:

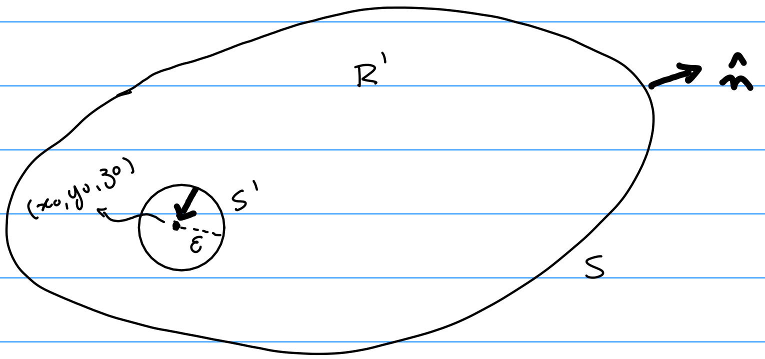
$$\iiint_R \frac{1}{r}\nabla^2 U - U\nabla^2 \frac{1}{r}\, dv = \iint_S \frac{1}{r}\frac{\partial U}{\partial n} - U \frac{\partial}{\partial n}\frac{1}{r}\, ds$$

(The integration and derivatives are computed with respect to the coordinates x, y and z)

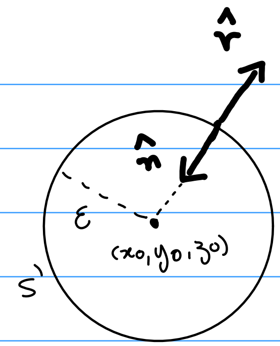
Now, consider a sphere with radius ε and center at (x_0, y_0, z_0)

The region R without the sphere is denoted by R'

S' is the boundary surface of the sphere

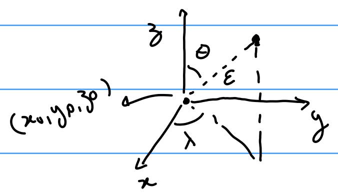


$$\iiint_{R'} \frac{1}{r}\nabla^2 U - \underbrace{\cancel{U\nabla^2 \frac{1}{r}}}_{=0 \text{ ok?}}\, dv = \left(\iint_S \frac{1}{r}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\frac{1}{r}\, ds\right) + \left(\iint_{S'} \frac{1}{r}\frac{\partial U}{\partial n} - U\frac{\partial}{\partial n}\frac{1}{r}\, ds'\right)$$



$$\frac{\partial U}{\partial n} = \nabla U^T \hat{n} = \nabla U^T (-\hat{r}) = -\frac{\partial U}{\partial r}$$

$$\frac{\partial}{\partial n}\frac{1}{r} = -\frac{\partial}{\partial r}\frac{1}{r} = \frac{1}{r^2}, \quad r = \varepsilon$$



$$\iint_{S'} \frac{1}{\varepsilon}\left(-\frac{\partial U}{\partial r}\right) - U\left(\frac{1}{\varepsilon^2}\right)\varepsilon^2 \cos\theta \, d\theta \, d\lambda =$$

$$= -\bar{U}\, 4\pi - \iint_{S'} \varepsilon \frac{\partial U}{\partial r} \cos\theta \, d\theta \, d\lambda$$

U within the sphere

for $\varepsilon \to 0$,

U at (x_0, y_0, z_0)

$$\iiint_R \frac{1}{r} \nabla^2 U \, dv = \iint_S \frac{1}{r}\frac{\partial U}{\partial n} - U \frac{\partial}{\partial n}\frac{1}{r} \, dS - 4\pi U_0$$

$$U_0 = -\frac{1}{4\pi}\iiint_R \frac{1}{r}\nabla^2 U \, dv + \frac{1}{4\pi}\iint_S \frac{1}{r}\frac{\partial U}{\partial n} \, dS - \frac{1}{4\pi}\iint_S U \frac{\partial}{\partial n}\frac{1}{r} \, dS$$

Green's third identity (Kellogg, 1967, p. 219)

Ex: Consider that U is harmonic in R and define the relationship between U0 and the normal derivatives of U and 1/r on S.