Application Dirichlet problem
(Upward continuation integral)

$$U(x_0,y_0,z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x,y,z_c)\left(-\partial_z \frac{1}{r}\right) dx\, dy$$

$$\partial_z \frac{1}{r} = \left(-\frac{z_0 - z_c}{r^3}\right)(-1) = -\partial_z \frac{1}{r}$$

vertical component of the gravitational attraction produced by a point of mass

$$U(x_0,y_0,z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,z_c)\,\overbrace{\partial_z \frac{1}{r}}\, dx\, dy \quad , \qquad p(x,y,z_c) = \frac{U(x,y,z_c)}{2\pi}$$

Consider that $U(x_0,y_0,z_0)$ represents the gravity disturbance $\delta g(x_0,y_0,z_0)$

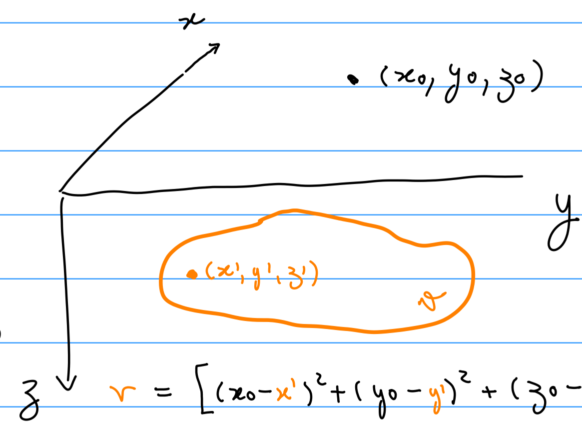
density

$$\delta g(x_0,y_0,z_0) = G \iiint_v \rho(x',y',z')\, \partial_z \frac{1}{r}\, dv$$

derivatives with respect to the coordinates of the observation point

gravity disturbance is harmonic outside the sources

$$\nabla^2 \delta g(x_0,y_0,z_0) = G \iiint_v \rho(x',y',z')\, \partial_z \underbrace{\cancel{\left(\nabla^2 \frac{1}{r}\right)}}_{=0} dv = 0$$



$$r = \left[(x_0-x')^2+(y_0-y')^2+(z_0-z')^2\right]^{\frac{1}{2}}$$

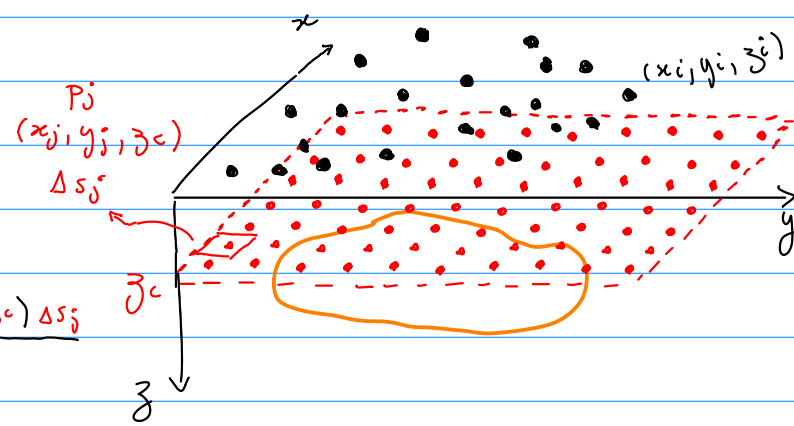
Consider now a set of gravity disturbance values at coordinates (x_i, y_i, z_i), $z_i < z_c \;\; \forall z_i$

$$\delta g_i^o \equiv \delta g^o(x_i,y_i,z_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y,z_c)\,\partial_z \frac{1}{r}\, dx\,dy \quad , \quad p(x,y,z_c) = \frac{\delta g(x,y,z_c)}{2\pi}$$

$$\delta g_i^o \approx \sum_{j=1}^{M} p_j\, h_{ij}$$

$$r_{ij} = \left[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_c)^2\right]^{1/2}$$

$$h_{ij} = \partial_z \frac{1}{r_{ij}} = \frac{z_c - z_i}{r_{ij}^3} \qquad p_j \approx \frac{\delta g(x_j,y_j,z_c)\,\Delta s_j}{2\pi}$$



$$\delta g_1^o \approx p_1 h_{11} + \cdots + p_m h_{1m}$$

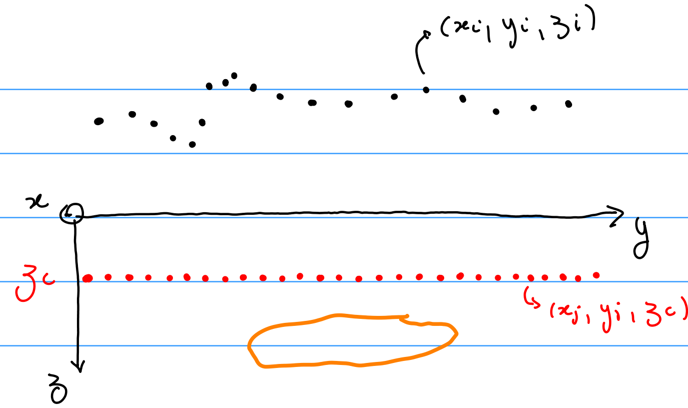
$$\vdots \qquad \vdots \qquad \vdots$$

$$\delta g_N^o \approx p_1 h_{N1} + \cdots + p_m h_{Nm}$$

$$\underset{N\times 1}{\delta g^o} \approx \underset{N\times m}{H} \; \underset{m\times 1}{p} \rightarrow p_j$$

equivalent layer

equivalent source



solve this linear system for estimating p

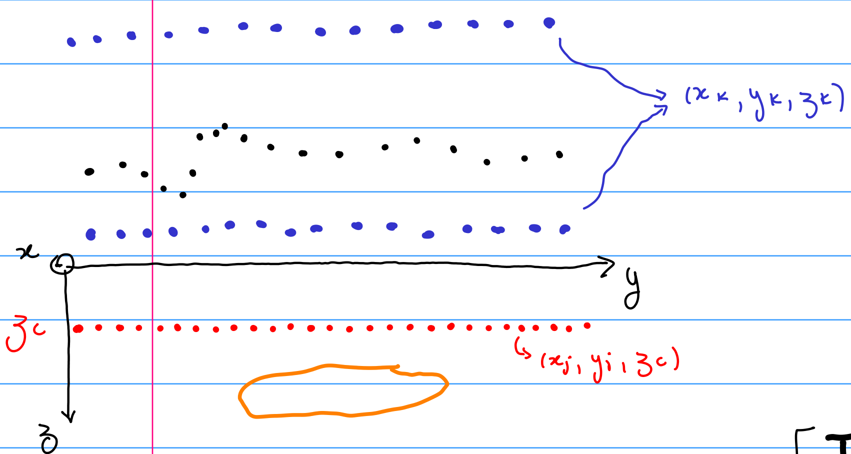
estimated gravity disturbance on the layer

$$\underset{m\times 1}{\delta g} = 2\pi \, \underset{m\times m}{S^{-1}} \, p^*_{m\times 1}$$

estimated parameter vector

$$S = \begin{bmatrix} \Delta s_1 & & \\ & \ddots & \\ & & \Delta s_m \end{bmatrix}_{m\times m}$$

By estimating the parameter vector p^* we are actually computing a downward continuation of the harmonic function (in this case, the gravity disturbance)



transformed data

$$\underset{L\times 1}{t} \approx \underset{L\times m}{T} \; \underset{m\times 1}{p^*}$$

transformation matrix

upward/downward continuation $\quad [T]_{kj} = \dfrac{z_c - z_k}{r_{kj}^3} \quad z_k$ above/below z_i

vertical derivative

$$t_k = \partial_z \delta g_k \qquad [T]_{kj} = \frac{3(z_c - z_k)^2}{r_{kj}^5} - \frac{1}{r_{kj}^3}$$

This application of the upward continuation integral with the gravity disturbance can be made with any harmonic function (i.e., gravitational potential and its derivatives, components of magnetic induction, approximated total-field anomaly, etc.)

References:

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