Application of the Dirichlet problem
(Upward continuation integral)

$$U(x_0, y_0, z_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, y, z_c)\left(-\partial_z \frac{1}{r}\right) dx\, dy$$

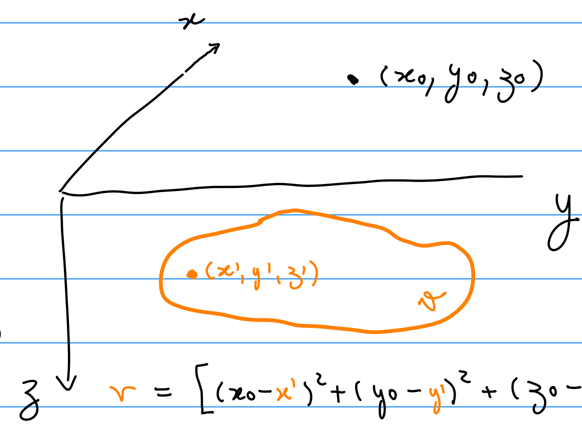
$$\partial_z \frac{1}{r} = \left(-\frac{z_0 - z_c}{r^3}\right)(-1) = -\partial_{z_0} \frac{1}{r}$$

$$U(x_0, y_0, z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \overbrace{\partial_z \frac{1}{r}}^{\text{vertical component of the gravitational attraction produced by a point of mass}} dx\, dy \quad , \quad p(x, y, z_c) = \frac{U(x, y, z_c)}{2\pi}$$

Consider that $U(x_0, y_0, z_0)$ represents the gravity disturbance $\delta g(x_0, y_0, z_0)$

density

$$\delta g(x_0, y_0, z_0) = G \iiint_v \rho(x', y', z')\, \partial_z \frac{1}{r}\, dv$$



derivatives with respect to the coordinates of the observation point

gravity disturbance is harmonic outside the sources

$$\nabla^2 \delta g(x_0, y_0, z_0) = G \iiint_v \rho(x', y', z')\, \partial_z \underbrace{\left(\cancel{\nabla^2 \frac{1}{r}}\right)}_{=0} dv = 0$$

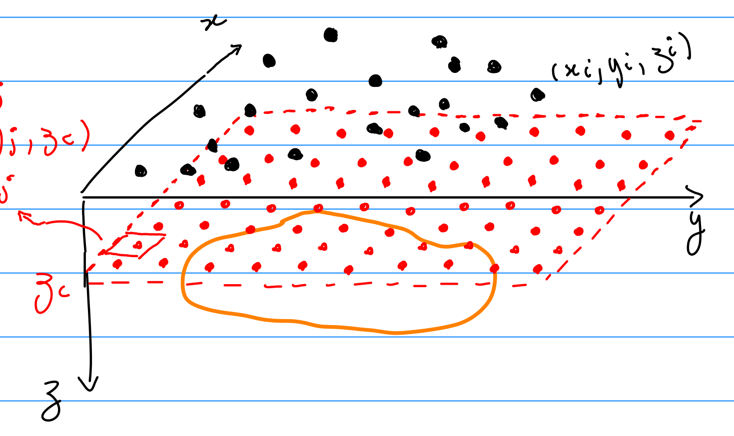
$$r = \left[(x_0 - x')^2 + (y_0 - y')^2 + (z_0 - z')^2\right]^{1/2}$$

Consider now a set of gravity disturbance values at coordinates (x_i, y_i, z_i), $z_i < z_c \ \forall z_i$

$$\delta g_i^o \equiv \delta g^o(x_i, y_i, z_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x, y, z_c)\, \partial_z \frac{1}{r}\, dx\, dy \quad , \quad p(x, y, z_c) = \frac{\delta g(x, y, z_c)}{2\pi}$$

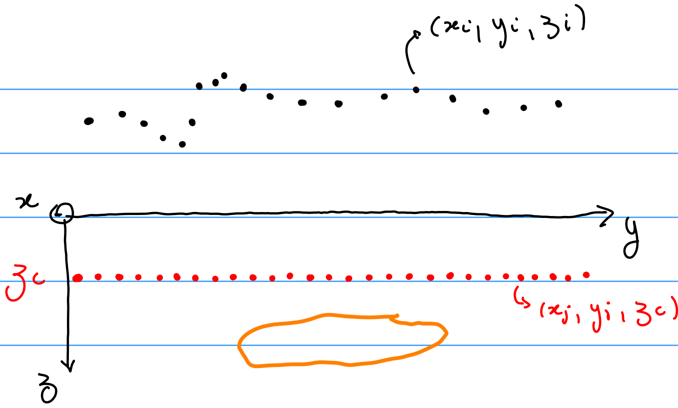
$$\delta g_i^o \approx \sum_{j=1}^{M} p_j h_{ij}$$

p_j
(x_j, y_j, z_c)
Δs_j



$$r_{ij} = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_c)^2\right]^{1/2}$$

$$h_{ij} = \partial_z \frac{1}{r_{ij}} = \frac{z_c - z_i}{r_{ij}^3} \qquad p_j \approx \frac{\delta g(x_j, y_j, z_c)\, \Delta s_j}{2\pi}$$



$$\delta g_1^o \approx p_1 h_{11} + \cdots + p_M h_{1M}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\delta g_N^o \approx p_1 h_{N1} + \cdots + p_M h_{NM}$$

$$\underset{N \times 1}{\boldsymbol{\delta g}^o} \approx \underset{N \times M}{\mathbf{H}} \; \underset{M \times 1}{\mathbf{p}} \rightarrow p_j$$

equivalent layer

equivalent source

solve this linear system for estimating $\mathbf{p}$

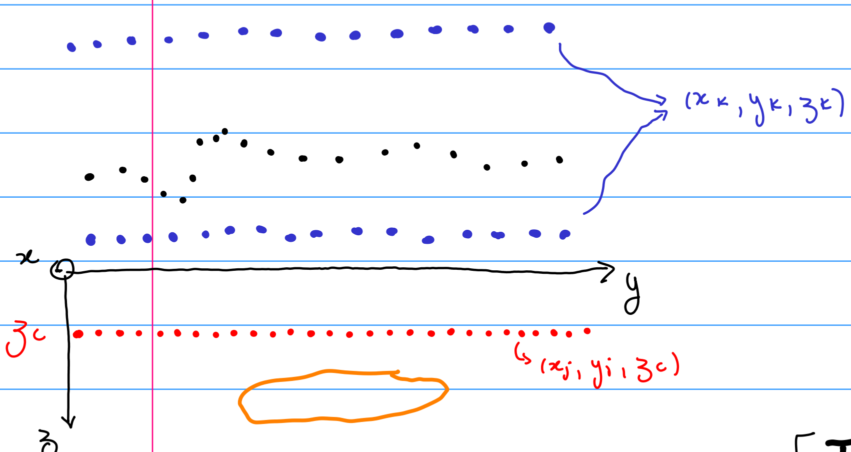
estimated gravity disturbance on the layer

estimated parameter vector

$$\underset{M \times 1}{\boldsymbol{\delta g}} = 2\pi \, \underset{M \times M}{\mathbf{S}^{-1}} \, \underset{M \times 1}{\mathbf{p}^*}$$

$$\mathbf{S} = \begin{bmatrix} \Delta S_1 & & \\ & \ddots & \\ & & \Delta S_M \end{bmatrix}_{M \times M}$$

By estimating the parameter vector $\mathbf{p}^*$ we are actually computing a downward continuation of the harmonic function (in this case, the gravity disturbance)



transformed data

transformation matrix

$$\underset{L \times 1}{\mathbf{t}} \approx \underset{L \times M}{\mathbf{T}} \; \underset{M \times 1}{\mathbf{p}^*}$$

upward/downward continuation

$$[\mathbf{T}]_{kj} = \frac{z_c - z_k}{r_{kj}^3} \qquad z_k \text{ above/below } z_i$$

vertical derivative

$$t_k = \partial_z \delta g_k \qquad [\mathbf{T}]_{kj} = \frac{3(z_c - z_k)^2}{r_{kj}^5} - \frac{1}{r_{kj}^3}$$

This application of the upward continuation integral with the gravity disturbance can be made with any harmonic function (i.e., gravitational potential and its derivatives, components of magnetic induction, approximated total-field anomaly, etc.)

References:

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