Transformations in Fourier domain

This note is based on Blakely (1996, chapters 11 and 12)

Fourier transform

$$\underset{\text{complex}}{F(k_x, k_y)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underset{\text{real}}{f(x,y)}\, e^{-i(k_x x + k_y y)}\, dx\, dy$$

$i = \sqrt{-1}$

Alternative notations

$F[f]$ or $F[f(x,y)]$

Inverse Fourier transform

$$\underset{\text{real}}{f(x,y)} = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underset{\text{complex}}{F(k_x,k_y)}\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y$$

k_x, k_y

wavenumbers

Some important properties:

Derivative

$$F[\partial_\alpha f] = i k_\alpha F[f]$$

$$F[\partial_{xx} f] = (ik_x)^2 F[f]$$

$$F[\partial_{xy} f] = (ik_x)(ik_y) F[f]$$

Shift

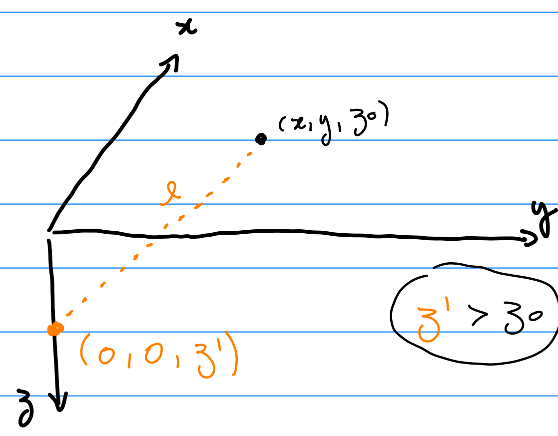
$$F[f(x-x', y-y')] = F[f(x,y)]\, e^{-i(k_x x' + k_y y')}$$

Consider the inverse distance function $1/\ell$ between a point $(0, 0, z')$ and a point (x, y, z_0) on the plane $z = z_0$.

$$\ell = \left[(x-0)^2 + (y-0)^2 + (z_0 - z')^2\right]^{\frac{1}{2}}$$

$$F\left[\frac{1}{\ell}\right] = 2\pi \frac{e^{|k|(z_0 - z')}}{|k|} \quad \begin{cases} z_0 < z' \\ |k| \neq 0 \end{cases}$$

$$|k| = \sqrt{k_x^2 + k_y^2}$$



Important property: $F\left[\partial_z \frac{1}{\ell}\right] = \partial_z F\left[\frac{1}{\ell}\right]$

with respect to x, y — vertical derivative of gravitational potential or vertical component of the gravitational attraction

Exercise: Show that the $\underbrace{\text{Fourier transform}}_{}$ of the gz produced on the horizontal plane $z = z_0$ by a point mass located at (x', y', z'), $z' > z_0$, is given by:

$$F\left[G m \, \partial_z \frac{1}{r} \right] = G m \, 2\pi \, e^{|K|(z_0 - z')} \, e^{-i(k_x x' + k_y y')}$$

with respect to x, y

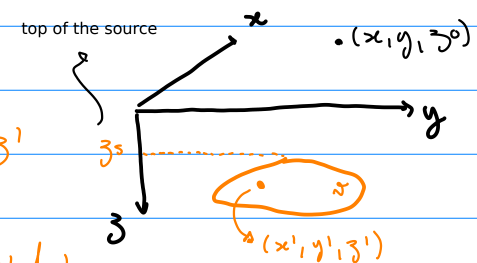
Exercise: Show that the Fourier transform of the approximated total-field anomaly produced on the horizontal plane $z = z_0$ by a dipole located at (x', y', z'), $z' > z_0$, is given by:

$$F\left[C_m h \, \hat{F}_0^{T} \nabla^2 \frac{1}{r} \hat{h} \right] = C_m h \, 2\pi \, \phi_h \, \phi_F \, |K| \, e^{|K|(z_0 - z')}$$

$$\phi_h = \hat{h}_z + i \frac{\hat{h}_x k_x + \hat{h}_y k_y}{|K|}$$

$$\phi_F = \hat{F}_z + i \frac{\hat{F}_x k_x + \hat{F}_y k_y}{|K|}$$

General 3D sources



gravity disturbance

$$\delta g(x, y, z_0) = \overset{\text{constant}}{G} \iiint_v \rho(x', y', z') \, \partial_z \frac{1}{r} \, dx' dy' dz'$$

density zero outside the source(s)

$$= G \int_{z_s}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x', y', z') \, \partial_z \frac{1}{r} \, dx' dy' dz'$$

$$r = \left[(x - x')^2 + (y - y')^2 + (z_0 - z')^2 \right]^{1/2}$$

$$F[\delta g] = G \int_{z_s}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x', y', z') \, F\left[\partial_z \frac{1}{r} \right] dx' dy' dz'$$

$$= G \int_{z_s}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x', y', z') \, F\left[\partial_z \frac{1}{\ell} \right] e^{-i(k_x x' + k_y y')} \, dx' dy' dz'$$

$$= G \int_{z_s}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x', y', z') \, \partial_z F\left[\frac{1}{\ell} \right] e^{-i(k_x x' + k_y y')} \, dx' dy' dz'$$

$$= G \int_{z_s}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x', y', z') \, 2\pi \, e^{|K|(z_0 - z')} \, e^{-i(k_x x' + k_y y')} \, dx' dy' dz'$$

$$\mathcal{F}[\delta g] = 2\pi e^{|k| z_0} G \int_{z_s}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(x', y', z')\, e^{-i(k_x x' + k_y y')}\, dx' dy' \right]$$

with respect to x', y'

$$= 2\pi e^{|k| z_0} G \int_{z_s}^{\infty} \underbrace{\mathcal{F}\left[\rho(x', y', z')\right]}_{\text{Fourier transform of a depth slice located at } z'} e^{-|k| z'}\, dz'$$

Exercise: By following a similar approach, show that the Fourier transform of the approximated total-field anomaly produced on the plane z_0 by a 3D source with constant magnetization direction is given by:

$$\mathcal{F}[\Delta \tilde{T}] = 2\pi e^{|k| z_0} \phi_h \phi_F C_m \int_{z_s} \mathcal{F}\left[h(x', y', z')\right] e^{-|k| z'}\, dz'$$

$$\phi_h = \hat{h}_z + i \frac{\hat{h}_x k_x + \hat{h}_y k_y}{|k|}$$

$$\phi_F = \hat{F}_z + i \frac{\hat{F}_x k_x + \hat{F}_y k_y}{|k|}$$

Upward/downward continuation

arbitrary harmonic function

constant plane

potential-field data

$$U(x, y, z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x', y', z_c)\, \partial_z \frac{1}{r}\, dx' dy'$$

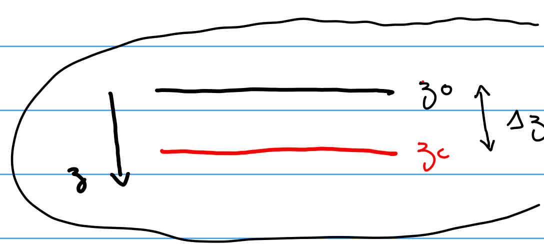
$$\mathcal{F}\left[U(x, y, z_0)\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x', y', z_c)\, \underbrace{\mathcal{F}\left[\partial_z \frac{1}{r}\right]}\, dx' dy'$$

$$= \cancel{\frac{1}{2\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x', y', z_c)\, \cancel{2\pi}\, e^{|k|(z_0 - z_c)}\, e^{-i(k_x x' + k_y y')}\, dx' dy'$$

upward continuation

$$\mathcal{F}\left[U(x, y, z_0)\right] = e^{-|k| \Delta z}\, \mathcal{F}\left[U(x', y', z_c)\right]$$

$$\Delta z = z_c - z_0$$



downward continuation

$$\mathcal{F}\left[U(x', y', z_c)\right] = e^{|k| \Delta z}\, \mathcal{F}\left[U(x, y, z_0)\right]$$

Reduction to the pole

$$F[\Delta\tilde{T}] = 2\pi e^{|k|z_0} \phi_h \phi_F C_m \int_{z_s}^{\infty} F[h(x', y', z')] e^{-|k|z'} dz'$$

$$\phi_h = \hat{h}_z + i \frac{\hat{h}_x k_x + \hat{h}_y k_y}{|k|} \qquad \hat{h} = \begin{bmatrix} \hat{h}_x \\ \hat{h}_y \\ \hat{h}_z \end{bmatrix} = \begin{bmatrix} \cos I \cos D \\ \cos I \sin D \\ \sin I \end{bmatrix}$$

$$\phi_F = \hat{F}_z + i \frac{\hat{F}_x k_x + \hat{F}_y k_y}{|k|} \qquad \hat{F}_0 = \begin{bmatrix} \hat{F}_x \\ \hat{F}_y \\ \hat{F}_z \end{bmatrix} = \begin{bmatrix} \cos I_0 \cos D_0 \\ \cos I_0 \sin D_0 \\ \sin I_0 \end{bmatrix}$$

$$F[\Delta\tilde{T}_{RTP}] = \frac{1}{\phi_h \phi_F} F[\Delta\tilde{T}]$$

We conveniently define the functions ϕ_h and ϕ_F in polar coordinates by using the relations

$$k_x = |k| \cos\theta$$

$$k_y = |k| \sin\theta$$

Then, we obtain:

$$\frac{1}{\phi_h \phi_F} = \frac{1}{[i \cos I_0 \cos(D_0 - \theta) + \sin I_0][i \cos I \cos(D - \theta) + \sin I]}$$

For $I_0 \approx 0$ (equator) $\longrightarrow$ fictitious trend along D_0 in space domain

" $I \approx 0$ $\longrightarrow$ " " " D " " "

Complementary references:

* GUNN, P.J. (1975), LINEAR TRANSFORMATIONS OF GRAVITY AND MAGNETIC FIELDS*. Geophysical Prospecting, 23: 300-312. doi: 10.1111/j.1365-2478.1975.tb01530.x

* Silva, J. B. C. (1986), Reduction to the pole as an inverse problem and its application to low-latitude anomalies. Geophysics, 51(2), 369-382, doi: 10.1190/1.1442096