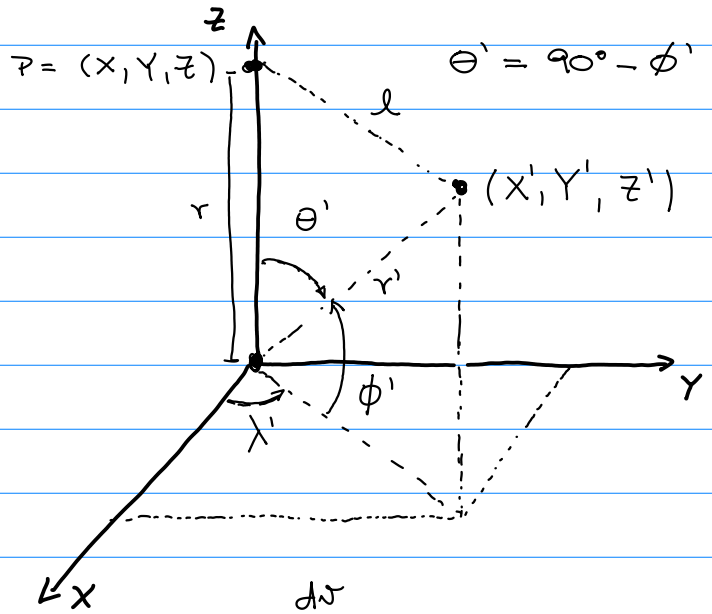
Sphere



$X' = r' \sin\theta' \cos\lambda'$

$Y' = r' \sin\theta' \sin\lambda'$

$Z' = r' \cos\theta'$

$$\ell = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

law of cosines

$$= \sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}$$

For convenience, consider : $G\rho = 1$

$$V_P = \iiint \frac{1}{\ell}\, dv$$

$dv = r'^2 \sin\theta'\, d\lambda' d\theta' dr'$

radius of the sphere

$$= \rho \int_0^R \int_0^{\pi} \int_0^{2\pi} \frac{1}{\sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}}$$

$$= 2\pi \int_0^R r'^2 \int_0^{\pi} \frac{\sin\theta'}{\sqrt{r^2 + r'^2 - 2 r r' \cos\theta'}}\, d\theta' dr'$$

$t = \cos\theta'$

$dt = -\sin\theta' d\theta'$

$\theta' = \pi \rightarrow t = -1$

$\theta' = 0 \rightarrow t = 1$

$$= 2\pi \int_0^R r'^2 \int_{-1}^{1} \frac{1}{\sqrt{r^2 + r'^2 - 2 r r' t}}\, dt\, dr'$$

$$= 2\pi \int_0^R r'^2 \left[-\frac{\sqrt{r^2 + r'^2 - 2 r r' t}}{r r'} \right]_{-1}^{1} dr'$$

$$= 2\pi \int_0^R r'^2 \left[\frac{\sqrt{r^2 + r'^2 + 2 r r'} - \sqrt{r^2 + r'^2 - 2 r r'}}{r r'} \right] dr'$$

$$= 2\pi \int_0^R r' \left[\frac{|r + r'| - |r - r'|}{r} \right] dr'$$

1) $r > R$, $|r + r'| = r + r'$, $|r - r'| = r - r'$

$$V_{out} = 2\pi \frac{1}{r} \int_0^R r' r + r'^2 - r' r + r'^2\, dr' = \int_0^R 2 r'^2\, dr' =$$

$$= 2\pi \frac{1}{r} \left[2 \frac{R^3}{3} \right] = \frac{4\pi}{3} R^3 \frac{1}{r}$$

2) $r < R$

$$V_{in} = 2\pi \left[\int_0^r \frac{r'}{r} \left(\overbrace{|r+r'|}^{r+r'} - \overbrace{|r-r'|}^{r-r'} \right) dr' + \int_r^R \frac{r'}{r} \left(\overbrace{|r+r'|}^{r+r'} - \overbrace{|r-r'|}^{r'-r} \right) dr' \right]$$

$$= 2\pi \frac{1}{r} \left[\int_0^r 2r'^2 \, dr' + \int_r^R 2r'r \, dr' \right]$$

$$= 2\pi \frac{1}{r} \left[2\frac{r^3}{3} + 2r\left(\frac{R^2}{2} - \frac{r^2}{2} \right) \right]$$

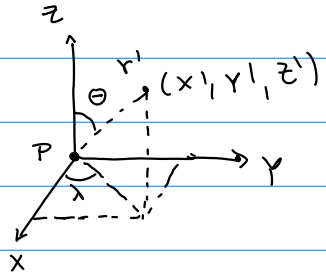
$$= 2\pi \left[\frac{2r^2}{3} + R^2 - r^2 \right] = 2\pi \left(R^2 - \frac{r^2}{3} \right)$$

3) $r = 0 \qquad l = r'$

$$V_0 = \iiint \frac{1}{r'} r'^2 \sin\theta \, d\lambda' d\theta' dr'$$

$$= \int_0^R \int_0^{\pi} \int_0^{2\pi} r' \sin\theta \, d\lambda' d\theta' dr'$$

$$= 2\pi \int_0^R r' \left[-\cos\theta \right]_0^{\pi} dr' = 2\pi R^2$$

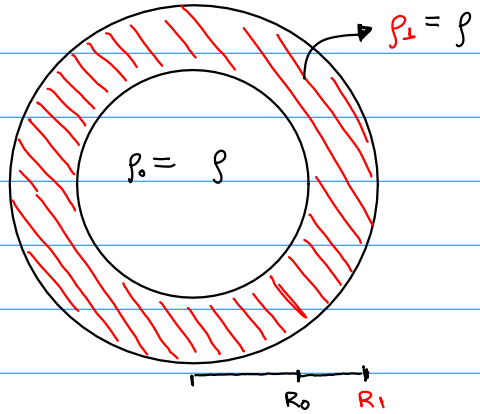


4) $r = R$, $\lim_{r \to R^-} V_{in} = \lim_{r \to R^+} V_{out} = 2\pi R^2$

$$V_p = \begin{cases} \frac{4}{3}\pi R^3 \frac{1}{r}, & r > R \\ \frac{4}{3}\pi R^2, & r = R \\ 2\pi \left(R^2 - \frac{r^2}{3} \right), & r < R \\ 2\pi R^2, & r = 0 \end{cases}$$

(Sansò and Sideris, 2013. p. 10-11)

Ex.: Defina o Laplaciano, primeira e segunda derivadas radiais do potencial produzido por uma esfera sólida.



$r > R_1$,

$$V_p = G\frac{4}{3}\pi R_1^3 \rho_1 \frac{1}{r} - G\frac{4}{3}\pi R_0^3 \rho_0 \frac{1}{r}$$

$$= G\frac{4}{3}\pi\rho\left(R_1^3 - R_0^3\right)\frac{1}{r}$$

$R_0 < r < R_1$

$$V_p = G\,2\pi\rho_1\left(R_1^2 - \frac{r^2}{3}\right) - G\frac{4}{3}\pi R_0^3 \rho_0 \frac{1}{r}$$

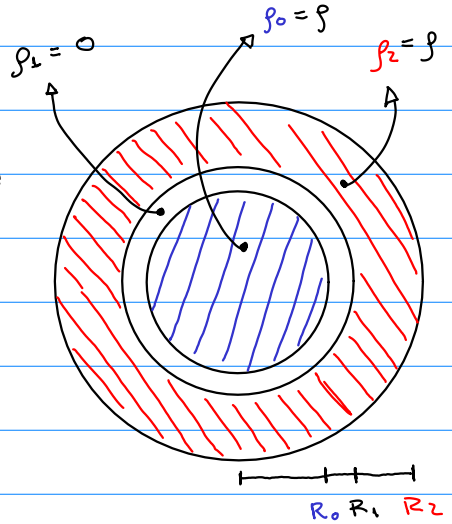
$$= G\,2\pi\rho\left(R_1^2 - \frac{r^2}{3} - \frac{2}{3}\frac{R_0^3}{r}\right)$$

$r < R_0$

$$V_p = G\,2\pi\left(R_1^2 - \frac{r^2}{3}\right)\rho_1 - G\,2\pi\left(R_0^2 - \frac{r^2}{3}\right)\rho_0$$

$$= G\,2\pi\rho\left(R_1^2 - R_0^2\right)$$

Ex.: Defina o Laplaciano, primeira e segunda derivadas radiais do potential produzido por uma cascas esférica.



Ex.: Defina o potencial, Laplaciano, primeira e segunda derivadas radiais produzidos pelo modelo de esféricas concêntricas.

Ex.: Usando a relação de Poisson, defina o potencial magnético escalar e a indução magnética produzida pelos 3 modelos definidos acima (esfera sólida, casca esférica e esferas concêntricas)