GravMag

An attempt at organizing self-consistent notes about gravity and magnetic methods

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Chapter 1

General aspects of the gravity field

The resultant of gravitational force and centrifugal force acting on a body at rest on the Earth's surface is called *gravity vector* and its intensity is commonly called *gravity* (Hofmann-Wellenhof and Moritz, 2005). In the case of gravimetry on moving platforms (e.g., airplanes, helicopters, marine vessels), there are additional non-gravitational accelerations due to the vehicle motion, like Coriolis acceleration and high-frequency vibrations (Glennie et al., 2000; Nabighian et al., 2005a; Baumann et al., 2012).

Geophysicists are usually interested in the gravitational component of the observed gravity, which is harmonic and reflects predominantly density variations in the crust and upper mantle (Blakely, 1996). One step of the procedure for isolating the gravitational component of gravity consists in removing the non-gravitational effects due to the vehicle motion and also the time variations of the gravity field (e.g., Earth tides, instrumental drift and barometric pressure changes). If these effects are properly removed, the resultant gravity data can be considered as the sum of a *normal* gravity field and a purely gravitational (and small) disturbing field, which is produced by variations in the Earth's internal density distribution. The isolation of this small disturbing field is the main goal in applied geophysics (Blakely, 1996).

Traditionally, the Earth is approximated by a rigid ellipsoid of revolution which is called *normal Earth*. The normal Earth has the minor axis coincident with the mean Earth's rotation axis, has the same total mass and angular velocity of the Earth and has an undefined internal density distribution (Vaníček and Krakiwsky, 1987; Hofmann-Wellenhof and Moritz, 2005). Similarly to gravity, the resultant of the virtual gravitational and centrifugal forces exerted by the normal Earth on a body at rest on the Earth's surface is called normal gravity vector and its intensity is commonly called *normal gravity*. Notice that, according to this traditional definition, the centrifugal component of the normal gravity field is equal to the centrifugal component of the Earth's gravity field if they are evaluated at the same point. The difference between the observed gravity (corrected from non-gravitational effects due to the vehicle motion) and the normal gravity, at the same point, is called *gravity disturbance* and is a very-well established quantity in geodesy (Hofmann-Wellenhof and Moritz, 2005). Several authors have discussed the differences between *gravity anomaly* and *gravity disturbance*, as well as proposed that the second is more appropriated for geophysical applications (e.g., Li and Götze, 2001; Fairhead et al., 2003; Hackney and Featherstone, 2003; Hinze et al., 2005; Vajda et al., 2006, 2007, 2008). The gravity disturbance approximates the gravitational field produced by contrasts between the actual internal density distribution of the Earth and the unknown internal density distribution of the normal Earth. In applied geophysics, these density differences are generally called *anomalous masses* (e.g., Hammer, 1945; LaFehr, 1965) or *gravity sources* (e.g., Blakely, 1996). Here, we opted for using the second term.

Chapter 2

General aspects of the geomagnetic field

In magnetic exploration methods, the geomagnetic field is commonly split into the *main field*, *crustal field* and *external field*. The *external field* is predominantly produced by electrical currents located in the ionosphere and magnetosphere and is considered "noise" in potential field applications. The *main field* is the strongest component and, according to the most widely accepted theory, is produced by a self-sustaining dynamo process that takes place within the outer core. Finally, the *crustal field* is produced by the magnetized bodies located in the uppermost (and coldest) layers of the Earth (Langel and Hinze, 1998; Hulot et al., 2015). Generally, the crustal field needs to be separated from the remaining components of the geomagnetic field to be interpreted later in applied geophysics. The magnetized bodies located in the subsurface are usually called *magnetic sources* (Blakely, 1996; Nabighian et al., 2005b).

In magnetic surveys, the commonly measured quantity is the resultant of the main, crustal and external fields. By properly removing the external field and also the magnetic field produced by the moving platforms (e.g., aircraft, ships or helicopters) and cultural noise (e.g., pipelines, railroads, bridges and commercial buildings), the remaining field can be considered as a sum of the main field and the crustal field, which is called *internal field* (Hulot et al., 2015) or *total field* (Blakely, 1996). Here, we opted for using the second term. The difference between the magnitude of the total field and the magnitude of a suitable model describing the main field (e.g., the IGRF), at the same point, is called *total-field anomaly* (Blakely, 1996; Nabighian et al., 2005b).

Chapter 3

Vector analysis review

This chapter aims at presenting a brief review on vector analysis. The main goal is discussing the key concepts for defining geophysical quantities in the Terrestrial Reference Systems presented in Chapter 5. Section 3.1 presents a brief review of vector analysis in the Cartesian system and makes extensive use of the concepts found in, for example, Simmonds (1994, chapters I and II), Arfken and Weber (2005, chap. 1) and Lipschutz et al. (2009, chapters 1–5). Section 3.2 is based on Goldstein et al. (1980, sec. 4.4) and Arfken and Weber (2005, sec. 1.2). Section **??** presents a brief review of vector analysis in Orthogonal Curvilinear Coordinate Systems and makes extensive use of the concepts found in, for example, Arfken and Weber (2005, chap. 2), Stratton (2007, sections 1.14–1.16) and Lipschutz et al. (2009, chap. 7).

3.1 Preliminaries

Consider an *arbitrary* Cartesian system with origin O and axes X, Y and Z (Figure 3.1). At the moment, we are not interested in the exact location of O or the specific orientation of axes X, Y and Z, that is why we use the term *arbitrary*[1]. In this system, we may define a *vector*[2] at a given point P in this system as a vertical[3] list of numbers as follows:

$$\mathbf{a} = \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} , \tag{3.1}$$

where a_X, a_Y and a_Z are the (Cartesian) *components* of $\mathbf{a}$ in the X, Y and Z directions. Vectors are geometric objects having *magnitude* and *direction*. Because of that, they are usually represented by an *arrow*. In this case, the magnitude (or Euclidean norm, or 2-norm) of $\mathbf{a}$ is the arrow length is given by:

$$\|\mathbf{a}\| = \sqrt{a_X^2 + a_Y^2 + a_Z^2} \, . \tag{3.2}$$

[1]The Cartesian systems presented in the next chapter will be defined with specific origin and orientation.

[2]A more complete definition of vector will be given in the next subsection.

[3]The convention here is representing a vector as a *column vector*, i.e., as a vertical list of numbers. *Row vectors* (horizontal lists of numbers) are represented with the transposition "$\top$" symbol.

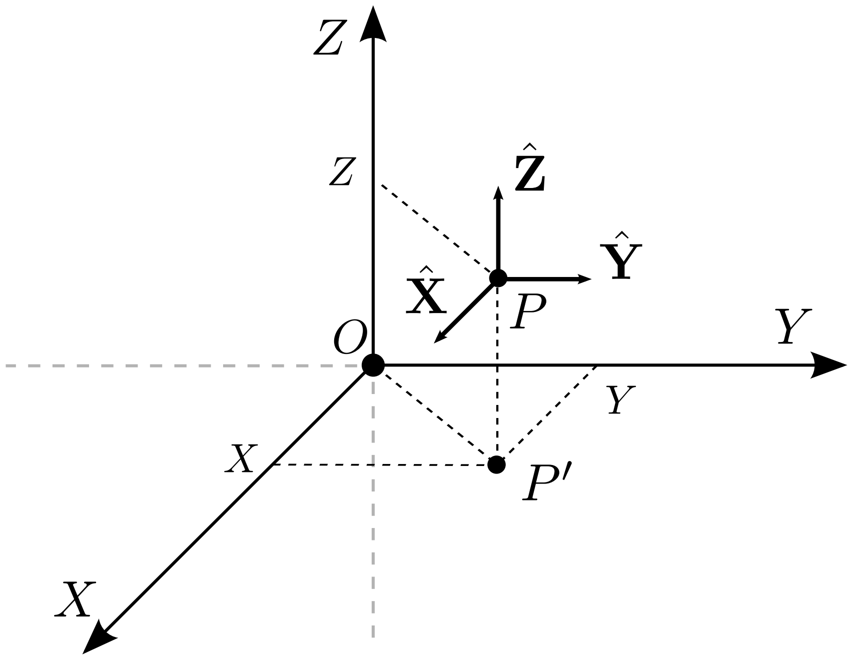

Figure 3.1: Arbitrary Cartesian system with origin O and axes X, Y and Z. In this coordinate system, the position of a point $P = (X, Y, Z)$ is determined by the Cartesian coordinates X, Y and Z. At any point in this system, there are three mutually-orthogonal unit vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$. These unit vectors form the standard basis in the $\mathbb{R}^3$. The point P' is the projection of P onto the XY plane.

Definition 1. Unit vectors

We say that a vector $\mathbf{a}$ (equation 3.1) is a *unit vector* if it has a unit length, i.e., if its magnitude $\|\mathbf{a}\|$ (equation 3.2) is equal to 1.

The vector orientation is defined by the angles α_a, β_a and γ_a counted, respectively, from the positive X, Y and Z axes to the arrow head. These angles also define the components a_X, a_Y and a_Z as follows:

$$a_X = \|\mathbf{a}\| \cos\alpha_a \, , \tag{3.3a}$$
$$a_Y = \|\mathbf{a}\| \cos\beta_a \, , \tag{3.3b}$$
$$a_Z = \|\mathbf{a}\| \cos\gamma_a \, , \tag{3.3c}$$

where

$$\cos^2\alpha_a + \cos^2\beta_a + \cos^2\gamma_a = 1 \, . \tag{3.4}$$

Equations 3.3 show that the components a_X, a_Y and a_Z are *projections* of $\mathbf{a}$ onto the X, Y and Z axes. A projection, in turn, is an operation that is intrinsically related to the *scalar (or dot) product.*

3.1.1 Scalar (or Dot) product

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented in the arbitrary Cartesian coordinate system according to equation 3.1. In this case, the scalar product of $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\mathbf{a}^\top\mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta_{ab} \, , \tag{3.5}$$

where θ_{ab} is the angle between $\mathbf{a}$ and $\mathbf{b}$. The scalar product (equation 3.5) can also be written as follows:

$$\mathbf{a}^\top \mathbf{b} = \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix}^\top \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} = a_X\, b_X + a_Y\, b_Y + a_Z\, b_Z\,. \tag{3.6}$$

We opted for representing the scalar product as a matrix multiplication (with transposition symbol "$^\top$") instead of using the "$\cdot$" notation commonly found in the literature in order to clearly specify that the first vector is transposed, which means that, according to the convention followed here, it is a row vector.

Definition 2. Orthogonal vectors

We say that two vectors are mutually *orthogonal* if the angle between them is 90°. In this case, equation 3.5 shows that their scalar product is null.

Remark 1. Some properties of the scalar product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be real vectors and κ_1 and κ_2 be a real scalars. It can be shown from equations 3.5 and 3.6 that the scalar product satisfies:

$$\mathbf{a}^\top \mathbf{b} = \mathbf{b}^\top \mathbf{a}\,, \tag{3.7a}$$

$$\mathbf{a}^\top (\mathbf{b} + \mathbf{c}) = \mathbf{a}^\top \mathbf{b} + \mathbf{a}^\top \mathbf{c}\,, \tag{3.7b}$$

$$(\kappa_1\, \mathbf{a})^\top (\kappa_2 \mathbf{b}) = \kappa_1\, \kappa_2 \left(\mathbf{a}^\top \mathbf{b}\right)\,, \tag{3.7c}$$

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top \mathbf{a}}\,. \tag{3.7d}$$

By using the scalar product defined by equation 3.5, we may now rewrite the components a_X, a_Y and a_Z (equations 3.3) as projections of the vector $\mathbf{a}$ onto the direction specified by unit vectors aligned with the X, Y and Z axes. These projections are given by:

$$a_X = \|\mathbf{a}\|\, \|\hat{\mathbf{X}}\|\, \cos\alpha_a = \mathbf{a}^\top \hat{\mathbf{X}}\,, \tag{3.8a}$$

$$a_Y = \|\mathbf{a}\|\, \|\hat{\mathbf{Y}}\|\, \cos\beta_a = \mathbf{a}^\top \hat{\mathbf{Y}}\,, \tag{3.8b}$$

$$a_Z = \|\mathbf{a}\|\, \|\hat{\mathbf{Z}}\|\, \cos\gamma_a = \mathbf{a}^\top \hat{\mathbf{Z}}\,, \tag{3.8c}$$

where $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ are the *mutually-orthogonal unit vectors* given below:

$$\hat{\mathbf{X}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\,, \tag{3.9a}$$

$$\hat{\mathbf{Y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\,, \tag{3.9b}$$

$$\hat{\mathbf{Z}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\,. \tag{3.9c}$$

These vectors are parallel to the axes X, Y and Z and represent the standard *basis* (Definition 3) for the Euclidean space $\mathbb{R}^3$ (Figure 3.1). Because these unit vectors define the axes of the Cartesian system and are mutually orthogonal, we say that the Cartesian system is an *orthogonal coordinate system*.

Definition 3. Linear (in)dependence and basis

Vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3 \in \mathbb{R}^3$ are *linearly dependent* if there exists scalars u_1, u_2 and u_3, not all zero, such that

$$u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + u_3\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} .$$

Otherwise, the vectors are *linearly independent*.

Any set of three linearly independent vectors in $\mathbb{R}^3$ forms a *basis* and their elements are called *base vectors*. The base vectors need not be of unit length (Definition 1) nor mutually orthogonal (Definition 2). If the basis is formed by mutually orthogonal vectors, then we say that it is an *orthogonal basis*. If the base vectors are not only mutually orthogonal, but also unit vectors, then we say that the basis is an *orthonormal basis*.

Any vector $\mathbf{u} \in \mathbb{R}^3$ can be uniquely represented as a *linear combination* of the base vectors forming a given base, i.e.,

$$\mathbf{u} = u_1\mathbf{v}_1 + u_2\mathbf{v}_2 + u_3\mathbf{v}_3 .$$

Given three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3 \in \mathbb{R}^3$, how to determine if they form a basis? This question can be answered by computing the *determinant* of the 3×3 matrix

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} .$$

If $\det \mathbf{V} \neq 0$, then the vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis in $\mathbb{R}^3$.

Since $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ (equations 3.9) form a basis in $\mathbb{R}^3$, then they can be used to define any other vector in $\mathbb{R}^3$ as a *linear combination* (Definition 3). For example, vector $\mathbf{a}$ (equation 3.1) can be rewritten as follows:

$$\begin{aligned} \mathbf{a} &= a_X\,\hat{\mathbf{X}} + a_Y\,\hat{\mathbf{Y}} + a_Z\,\hat{\mathbf{Z}} \\ &= \underbrace{\begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}} \end{bmatrix}}_{\mathbf{I}} \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} \\ &= \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} . \end{aligned} \tag{3.10}$$

where $\mathbf{I}$ is the identity matrix, $a_X\,\hat{\mathbf{X}}$, $a_Y\,\hat{\mathbf{Y}}$ and $a_Z\,\hat{\mathbf{Z}}$ are the (Cartesian) *component vectors* and a_X, a_Y and a_Z are the (Cartesian) *components* of $\mathbf{a}$ in the X, Y and Z directions. We are now able to understand that equation 3.1 represents $\mathbf{a}$ by using only its components and omits the base vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$. Note that, according to equation 3.10, $\mathbf{a}$ is defined by premultiplying its components by a matrix. This matrix, in turn, has its columns formed by the standard basis $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ because $\mathbf{a}$ is represented in the arbitrary Cartesian system. Note that, in this case, this matrix is the identity $\mathbf{I}$. In the next section, we will see what happens when we represent $\mathbf{a}$ by using other orthonormal basis (Definition 3).

It is important noting that the new definition of $\mathbf{a}$ as a linear combination (equation 3.10) is totally consistent with all previous equations presented here. Let us illustrate this consistency by using two examples. In both examples we use the orthogonality of the unit vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$. The first example consists in using the scalar product to define the component a_1 according to equation 3.3:

$$\begin{aligned}
\mathbf{a}^\top \hat{\mathbf{X}} &= \left(a_X \, \hat{\mathbf{X}} + a_Y \, \hat{\mathbf{Y}} + a_Z \, \hat{\mathbf{Z}}\right)^\top \hat{\mathbf{X}} \\
&= a_X \underbrace{\hat{\mathbf{X}}^\top \hat{\mathbf{X}}}_{=1} + a_Y \underbrace{\hat{\mathbf{Y}}^\top \hat{\mathbf{X}}}_{=0} + a_Z \underbrace{\hat{\mathbf{Z}}^\top \hat{\mathbf{X}}}_{=0} \\
&= a_X \; .
\end{aligned}$$

The second example consists in computing the scalar product $\mathbf{a}^\top \mathbf{b}$ according to equation 3.6:

$$\begin{aligned}
\mathbf{a}^\top \mathbf{b} &= \left(a_X \, \hat{\mathbf{X}} + a_Y \, \hat{\mathbf{Y}} + a_Z \, \hat{\mathbf{Z}}\right)^\top \left(b_X \, \hat{\mathbf{X}} + b_Y \, \hat{\mathbf{Y}} + b_Z \, \hat{\mathbf{Z}}\right) \\
&= a_X \, b_X \, \hat{\mathbf{X}}^\top \hat{\mathbf{X}} + a_X \, b_Y \, \hat{\mathbf{X}}^\top \hat{\mathbf{Y}} + a_X \, b_Z \, \hat{\mathbf{X}}^\top \hat{\mathbf{Z}} \\
&+ a_Y \, b_X \, \hat{\mathbf{Y}}^\top \hat{\mathbf{X}} + a_Y \, b_Y \, \hat{\mathbf{Y}}^\top \hat{\mathbf{Y}} + a_Y \, b_Z \, \hat{\mathbf{Y}}^\top \hat{\mathbf{Z}} \\
&+ a_Z \, b_X \, \hat{\mathbf{Z}}^\top \hat{\mathbf{X}} + a_Z \, b_Y \, \hat{\mathbf{Z}}^\top \hat{\mathbf{Y}} + a_Z \, b_Z \, \hat{\mathbf{Z}}^\top \hat{\mathbf{Z}} \\
&= a_X \, b_X + a_Y \, b_Y + a_Z \, b_Z \; .
\end{aligned}$$

3.1.2 Cross product

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented in the Cartesian coordinate system according to the preliminary definition given by equation 3.1. In this case, the *cross product* of $\mathbf{a}$ and $\mathbf{b}$ produces a third vector, also in the Cartesian system, given by:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_Y \, b_Z - a_Z \, b_Y \\ a_Z \, b_X - a_X \, b_Z \\ a_X \, b_Y - a_Y \, b_X \end{bmatrix} \; . \tag{3.11}$$

Remark 2. Some properties of the cross product

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be real vectors and κ_1 and κ_2 be a real scalars. It can be shown from equations 3.2, 3.5, 3.6 and 3.11 that:

$$\mathbf{a} \times \mathbf{a} = \mathbf{0}\,, \tag{3.12a}$$
$$\mathbf{a}^\top (\mathbf{a} \times \mathbf{b}) = 0\,, \tag{3.12b}$$
$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}\,, \tag{3.12c}$$
$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})\ , \tag{3.12d}$$
$$(\kappa_1\,\mathbf{a}) \times (\kappa_2 \mathbf{b}) = \kappa_1\,\kappa_2\,(\mathbf{a} \times \mathbf{b})\ , \tag{3.12e}$$
$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}^\top (\mathbf{a} \times \mathbf{c}) - \mathbf{c}^\top (\mathbf{a} \times \mathbf{b})\ , \tag{3.12f}$$
$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) = \{\mathbf{a}^\top (\mathbf{b} \times \mathbf{c})\}\,\mathbf{a}\,, \tag{3.12g}$$
$$(\mathbf{a} \times \mathbf{b})^\top (\mathbf{c} \times \mathbf{d}) = \left(\mathbf{a}^\top\mathbf{c}\right)\left(\mathbf{b}^\top\mathbf{d}\right) - \left(\mathbf{a}^\top\mathbf{d}\right)\left(\mathbf{b}^\top\mathbf{c}\right)\ , \tag{3.12h}$$
$$\|\mathbf{a} \times \mathbf{b}\|^2 + \left(\mathbf{a}^\top\mathbf{b}\right)^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\,, \tag{3.12i}$$
$$\mathbf{a} \times \mathbf{b} = \mathbf{A}\mathbf{b}\,, \quad \mathbf{A} = \begin{bmatrix} 0 & -a_Z & a_Y \\ a_Z & 0 & -a_X \\ -a_Y & a_X & 0 \end{bmatrix}\,. \tag{3.12j}$$

The properties listed in Remark 2 can be deduced by simple substitution and we let as exercise. They show, for example, that vector $\mathbf{a} \times \mathbf{b}$ (equation 3.11) is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ (use 3.12b and c to show this). Other important property of vector $\mathbf{a} \times \mathbf{b}$ that can be computed with those shown in Remark 2 is its magnitude $\|\mathbf{a} \times \mathbf{b}\|$. This can be computed by first substituting the geometric definition of scalar product given by equation 3.5 into 3.12i and rearranging terms as follows:

$$\|\mathbf{a} \times \mathbf{b}\|^2 = \|\mathbf{a}\|^2\,\|\mathbf{b}\|^2\,\left(1 - \cos^2\theta_{ab}\right)\ ,$$

where θ_{ab} is the angle between $\mathbf{a}$ and $\mathbf{b}$. Then, by using the trigonometric identity $\sin^2\theta_{ab} = 1 - \cos^2\theta_{ab}$ and computing the positive square root of both sides, we obtain:

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta_{ab}\,. \tag{3.13}$$

It is also possible to show using the properties of Remark 2 that the unit vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ (equation 3.9) satisfy:

$$\hat{\mathbf{X}} \times \hat{\mathbf{X}} = \mathbf{0}\,, \tag{3.14a}$$
$$\hat{\mathbf{X}} \times \hat{\mathbf{Y}} = \hat{\mathbf{Z}}\,, \tag{3.14b}$$
$$\hat{\mathbf{X}} \times \hat{\mathbf{Z}} = -\hat{\mathbf{Y}}\,, \tag{3.14c}$$
$$\hat{\mathbf{Y}} \times \hat{\mathbf{X}} = -\hat{\mathbf{Z}}\,, \tag{3.14d}$$
$$\hat{\mathbf{Y}} \times \hat{\mathbf{Y}} = \mathbf{0}\,, \tag{3.14e}$$
$$\hat{\mathbf{Y}} \times \hat{\mathbf{Z}} = \hat{\mathbf{X}}\,, \tag{3.14f}$$
$$\hat{\mathbf{Z}} \times \hat{\mathbf{X}} = \hat{\mathbf{Y}}\,, \tag{3.14g}$$
$$\hat{\mathbf{Z}} \times \hat{\mathbf{Y}} = -\hat{\mathbf{X}}\,, \tag{3.14h}$$
$$\hat{\mathbf{Z}} \times \hat{\mathbf{Z}} = \mathbf{0}\,. \tag{3.14i}$$

Equations 3.14 show that the Cartesian system has a *right-hand orientation*. They can be used with equation 3.12d to verify that:

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \left(a_X\,\hat{\mathbf{X}} + a_Y\,\hat{\mathbf{Y}} + a_Z\,\hat{\mathbf{Z}}\right) \times \left(b_X\,\hat{\mathbf{X}} + b_Y\,\hat{\mathbf{Y}} + b_Z\,\hat{\mathbf{Z}}\right) \\
&= (a_Y\,b_Z - a_Z\,b_Y)\,\hat{\mathbf{X}} + (a_Z\,b_X - a_X\,b_Z)\,\hat{\mathbf{Y}} + (a_X\,b_Y - a_Y\,b_Z)\,\hat{\mathbf{Z}} \\
&= \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \left(\begin{bmatrix} a_Y\,b_Z \\ a_Z\,b_X \\ a_X\,b_Y \end{bmatrix} - \begin{bmatrix} a_Z\,b_Y \\ a_X\,b_Z \\ a_Y\,b_X \end{bmatrix} \right) \\
&= \begin{bmatrix} a_Y\,b_Z - a_Z\,b_Y \\ a_Z\,b_X - a_X\,b_Z \\ a_X\,b_Y - a_Y\,b_X \end{bmatrix} \,,
\end{aligned} \tag{3.15}$$

which coincides with the previous definition of cross product (equation 3.11).

3.2 Rotation of the coordinate axes

In the previous section, we present a preliminary definition of vector **a** as a simple vertical list of numbers (equation 3.1). Then we present a second definition of **a** as a linear combination of base vectors (equation 3.10). Those previous definitions may lead you think that a vector is a geometric object that "belongs" to a particular coordinate system. This is not true. Actually, a vector is independent of any particular coordinate system (e.g., Arfken and Weber, 2005, sec. 1.2). Equations 3.1 and 3.10 are only representations of the geometric object **a** in our arbitrary Cartesian system. To understand this, we need two important concepts: *orthogonal matrices* and *proper rotations*.

3.2.1 Orthogonal matrices

Let $\mathbf{Q}$ be a 3×3 real matrix given by:

$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{bmatrix} . \tag{3.16}$$

We say that $\mathbf{Q}$ is an *orthogonal matrix* if it satisfies the *orthogonality condition*:

$$\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\,\mathbf{Q}^\top = \mathbf{I}\,, \tag{3.17}$$

where $\mathbf{I}$ is the identity. This condition implies that the inverse is equal to the transpose for orthogonal matrices, i.e.,

$$\mathbf{Q}^{-1} = \mathbf{Q}^\top . \tag{3.18}$$

The orthogonality condition (equation 3.17) relates the elements of an orthogonal matrix $\mathbf{Q}$ and the standard base vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ (equations 3.9) as follows:

$$\mathbf{Q}\,\mathbf{Q}^\top = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \implies \mathbf{Q} = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\mathbf{Q} = \mathbf{Q}\begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} , \tag{3.19a}$$

$$\mathbf{Q}^\top\mathbf{Q} = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \implies \mathbf{Q}^\top = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\mathbf{Q}^\top = \mathbf{Q}^\top\begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} . \tag{3.19b}$$

Let us consider two alternative representations of matrix $\mathbf{Q}$ (equation 3.16). In the first, we represent $\mathbf{Q}$ in terms of its columns,

$$\mathbf{Q} = \begin{bmatrix}\hat{\mathbf{q}}_1 & \hat{\mathbf{q}}_2 & \hat{\mathbf{q}}_3\end{bmatrix} , \tag{3.20a}$$

$$\hat{\mathbf{q}}_j = \begin{bmatrix} q_{1j} \\ q_{2j} \\ q_{3j} \end{bmatrix} , \quad j \in \{1, 2, 3\} . \tag{3.20b}$$

The second representation of $\mathbf{Q}$ (equation 3.16) is defined in terms of its rows,

$$\mathbf{Q} = \begin{bmatrix} \hat{\mathbf{p}}_1^\top \\ \hat{\mathbf{p}}_2^\top \\ \hat{\mathbf{p}}_3^\top \end{bmatrix} , \tag{3.21a}$$

$$\hat{\mathbf{p}}_i = \begin{bmatrix} q_{i1} \\ q_{i2} \\ q_{i3} \end{bmatrix} , \quad i \in \{1, 2, 3\} . \tag{3.21b}$$

From equation 3.19a, we can see that the columns $\hat{\mathbf{q}}_j$ (equation 3.20b) of an orthogonal matrix $\mathbf{Q}$ contain the components of unit vectors given by:

$$\hat{\mathbf{q}}_1 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{11} \\ q_{21} \\ q_{31} \end{bmatrix} = \mathbf{Q}\,\hat{\mathbf{X}} , \tag{3.22a}$$

$$\hat{\mathbf{q}}_2 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{12} \\ q_{22} \\ q_{32} \end{bmatrix} = \mathbf{Q}\,\hat{\mathbf{Y}} , \tag{3.22b}$$

$$\hat{\mathbf{q}}_3 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix} = \mathbf{Q}\,\hat{\mathbf{Z}} . \tag{3.22c}$$

Similarly, equation 3.19b shows that the rows $\hat{\mathbf{p}}_i$ (equation 3.21b) of an orthogonal matrix $\mathbf{Q}$ contain the components unit vectors given by:

$$\hat{\mathbf{p}}_1 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = \mathbf{Q}^\top\hat{\mathbf{X}} , \tag{3.23a}$$

$$\hat{\mathbf{p}}_2 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{21} \\ q_{22} \\ q_{23} \end{bmatrix} = \mathbf{Q}^\top\hat{\mathbf{Y}} , \tag{3.23b}$$

$$\hat{\mathbf{p}}_3 = \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix} \begin{bmatrix} q_{31} \\ q_{32} \\ q_{33} \end{bmatrix} = \mathbf{Q}^\top\hat{\mathbf{Z}} . \tag{3.23c}$$

Equations 3.22 and 3.23 show that the columns and rows of an orthogonal matrix $\mathbf{Q}$ are mutually-orthogonal unit vectors (Definition 2) that, consequently, form a basis in $\mathbb{R}^3$

(Definition 3). This orthogonality condition can be formally defined as follows by using the scalar product (Definition 2):

$$\hat{\mathbf{q}}_i^\top \hat{\mathbf{q}}_j = \delta_{ij} \tag{3.24}$$

and

$$\hat{\mathbf{p}}_i^\top \hat{\mathbf{p}}_j = \delta_{ij} \,, \tag{3.25}$$

where δ_{ij} the *Kronecker delta* (Definition 4).

Definition 4. Kronecker delta

The *Kronecker delta* (named after the german mathematician Leopold Kronecker) is a function of two variables, that are defined here as positive integers. The function is 1, if the variables are equal to each other, and 0 otherwise:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \,, \\ 0 & \text{if} \quad i \neq j \,. \end{cases} \tag{3.26}$$

3.2.2 Proper rotations

Rotation[4] is a very familiar movement. We may rotate almost all objects around us. When an object is rotated (e.g., a table), it means that the position of almost all its constituting points is changed with respect to a given reference system (e.g., the walls of a room). We also generally assume that the rotation does not squeeze, stretch or deform the rotated object in any way, i.e., that the position of any constituting point relative to any other forming the object remains unchanged, so that the shape of the object remains unchanged. To explain a rotation mathematically, we will not describe the rotation of general objects, but limit ourselves to vectors. In this case, a rotation is the product of an orthogonal matrix (Subsection 3.2.1) and a vector (the object we want to rotate). An orthogonal matrix $\mathbf{Q}$ defining a rotation is conveniently called *rotation matrix*. Generic rotation matrices have determinant equal to $+1$ or -1. While the first kind leads to *proper rotations*, the second leads to *improper rotations*. Throughout this manuscript, we are only interested in proper rotations.

Rotation matrices (with determinant equal to $+1$) are usually defined in terms of three independent parameters (angles) called *Euler angles*. There are different possible conventions to define the Euler angles and we opted for using the xyz-convention (Goldstein et al., 1980, p. 603), which results in

$$\mathbf{Q} = \mathbf{R}_1(\varepsilon)\, \mathbf{R}_2(\zeta)\, \mathbf{R}_3(\eta) \,, \tag{3.27a}$$

$$\det(\mathbf{Q}) = +1 \,, \tag{3.27b}$$

[4] For details, see Goldstein et al. (1980, sec. 4.4)

where

$$
\mathbf{R}_3(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{3.28a}
$$

$$
\mathbf{R}_2(\psi) = \begin{bmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{bmatrix} , \tag{3.28b}
$$

$$
\mathbf{R}_1(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix} . \tag{3.28c}
$$

Equation 3.27 defines $\mathbf{Q}$ by means of three successive rotations taken around different axes. The first rotation, represented by the rotation matrix $\mathbf{R}_3(\psi)$ (equation 3.28a), is about the vertical axis Z and gives the *heading* or *yaw* angle η. The second, represented by $\mathbf{R}_2(\psi)$ (equation 3.28b), is around the Y axis and is measured by the *pitch* or *attitude* angle ζ. Finally, the third rotation, represented by $\mathbf{R}_1(\psi)$ (equation 3.28c), is about the X axis and is the *roll* or *bank* angle ε.

Remark 3. Rotation matrices $\mathbf{R}_i(\psi)$

These rotation matrices $\mathbf{R}_i(\psi)$ (equations 3.28), $i = 1, 2, 3$, define *counterclockwise* and *clockwise* rotations for, respectively, positive and negative angles ψ. It can be shown that they satisfy the following properties:

$$
\mathbf{R}_i(\psi + \Delta\psi) = \mathbf{R}_i(\psi)\,\mathbf{R}_i(\Delta\psi) = \mathbf{R}_i(\Delta\psi)\,\mathbf{R}_i(\psi) , \tag{3.29a}
$$

$$
\mathbf{R}_i(-\psi) = \mathbf{R}_i(\psi)^\top , \tag{3.29b}
$$

$$
\det\left(\mathbf{R}_i(\psi)\right) = +1 . \tag{3.29c}
$$

Note that, according to equation 3.29b, the transpose $\mathbf{R}_i(\psi)^\top$ produces a rotation contrary to that produced by $\mathbf{R}_i(\psi)$.

For the particular case in which the orthogonal matrix $\mathbf{Q}$ is defined by equation 3.27 and, consequently, represents a proper rotation, the unit vectors forming its columns and rows (equations 3.22 and 3.23) satisfy:

$$
\hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_2 = \hat{\mathbf{q}}_3 , \tag{3.30a}
$$

$$
\hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_3 = \hat{\mathbf{q}}_1 , \tag{3.30b}
$$

$$
\hat{\mathbf{q}}_3 \times \hat{\mathbf{q}}_1 = \hat{\mathbf{q}}_2 , \tag{3.30c}
$$

$$
\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2 = \hat{\mathbf{p}}_3 , \tag{3.30d}
$$

$$
\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_3 = \hat{\mathbf{p}}_1 , \tag{3.30e}
$$

$$
\hat{\mathbf{p}}_3 \times \hat{\mathbf{p}}_1 = \hat{\mathbf{p}}_2 . \tag{3.30f}
$$

These equations show that the columns and rows of an orthogonal matrix defined by equation 3.27 have the same right-rand orientation of the standard basis (equations 3.14).

3.2.3 Rotation of vector components

Let us consider the product of our orthogonal matrix $\mathbf{Q}$ (equation 3.27) and a vector $\mathbf{a}$ defined in our arbitrary Cartesian system:

$$\begin{aligned}\mathbf{Q}\,\mathbf{a} &= \mathbf{Q}\underbrace{\begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}}_{\mathbf{I}}\begin{bmatrix}a_X\\a_Y\\a_Z\end{bmatrix} = \\ &= \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}a'_X\\a'_Y\\a'_Z\end{bmatrix} = \\ &= a'_X\,\hat{\mathbf{X}} + a'_Y\,\hat{\mathbf{Y}} + a'_Z\,\hat{\mathbf{Z}} = \mathbf{a}'\,,\end{aligned} \tag{3.31}$$

where

$$\begin{bmatrix}a'_X\\a'_Y\\a'_Z\end{bmatrix} = \mathbf{Q}\begin{bmatrix}a_X\\a_Y\\a_Z\end{bmatrix}\ . \tag{3.32}$$

Note that equation 3.31 defines a vector $\mathbf{a}'$ in our arbitrary Cartesian system, with components a'_X, a'_Y and a'_Z (equation 3.32). Hence, in this case, $\mathbf{Q}$ acts on the components of $\mathbf{a}$ and transforms it into a new object $\mathbf{a}'$ that is rotated with respect to $\mathbf{a}$ in the same coordinate system. A similar result is obtained by using $\mathbf{Q}^\top$ instead of $\mathbf{Q}$. In this case, however, the rotation has a contrary direction.

We are now able to understand that the rightmost terms in equations 3.22 and 3.23 define the columns of an orthogonal matrix $\mathbf{Q}$ by rotating the standard base vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$.

3.2.4 Rotation of coordinate axes (General definition of vectors)

Now, let us go back to the definition of a vector $\mathbf{a}$ as a linear combination of the standard base vectors (equation 3.10). This definition can be rewritten by using the orthogonality (equation 3.17) of a matrix $\mathbf{Q}$ given by equation 3.27:

$$\begin{aligned}\mathbf{a} &= \begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}a_X\\a_Y\\a_Z\end{bmatrix} \\ &= \mathbf{Q}\,\mathbf{Q}^\top\begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\begin{bmatrix}a_X\\a_Y\\a_Z\end{bmatrix} \\ &= \left(\mathbf{Q}\begin{bmatrix}\hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}}\end{bmatrix}\right)\left(\mathbf{Q}^\top\begin{bmatrix}a_X\\a_Y\\a_Z\end{bmatrix}\right) \\ &= \mathbf{Q}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \\ &= a_1\,\hat{\mathbf{q}}_1 + a_2\,\hat{\mathbf{q}}_2 + a_3\,\hat{\mathbf{q}}_3\,,\end{aligned} \tag{3.33}$$

where

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{Q}^\top \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} , \tag{3.34}$$

with $\hat{\mathbf{q}}_1$, $\hat{\mathbf{q}}_2$ and $\hat{\mathbf{q}}_3$ being the three mutually-orthogonal unit vectors (equations 3.22) forming the columns of an orthogonal matrix $\mathbf{Q}$ given by equation 3.27. In this case, these unit vectors define the axes of a new right-hand orthogonal coordinate system that, according to equation 3.22, are obtained by rotating the standard base vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$. In equation 3.33, the terms $a_1\,\hat{\mathbf{q}}_1$, $a_2\,\hat{\mathbf{q}}_2$ and $a_3\,\hat{\mathbf{q}}_3$ are the *component vectors* and a_1, a_2 and a_3 are the *components* of $\mathbf{a}$ in this new coordinate system.

Equations 3.33 and 3.34 establish the rule to represent the same geometric object (vector $\mathbf{a}$) in two different coordinate systems. Besides, we will see in the next subsections 3.2.5 and 3.2.6 that equations 3.33 and 3.34 guarantee that not only the *magnitude*, but also the *orientation* of $\mathbf{a}$ with respect to the standard base vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ (equations 3.9) does not change in the new orthogonal coordinate system.

It is important noting that, compared to the previous definitions of vector (equations 3.1 and 3.10), this one given by equations 3.33 and 3.34 is more complete because defines a vector as a geometric object in terms of the transformation of its components under rotation of the coordinate axes (e.g., Arfken and Weber, 2005, sec. 1.2). It means that what defines a vector is not a particular orthogonal coordinate system, but how it can be represented at different orthogonal coordinate systems.

3.2.5 Invariance of the scalar product under rotation

The invariance of the scalar product with rotation can be shown by combining equation 3.6, the orthogonality of matrix $\mathbf{Q}$ (equation 3.27) and equation 3.34:

$$\mathbf{a}^\top\mathbf{b} = \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix}^\top \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^\top \mathbf{Q}^\top\,\mathbf{Q} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}^\top \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} . \tag{3.35}$$

Equation 3.35 shows that the scalar product does not depend on the particular orthogonal coordinate system in which the vectors are defined, but only on their components.

Remark 4. Invariance of the magnitude under rotation

By combining equation 3.35 and the definition of magnitude given in by equation 3.7d, we obtain:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^\top\mathbf{a}} = \sqrt{a_X^2 + a_Y^2 + a_Z^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}\,,$$

which shows that the magnitude of a vector $\mathbf{a}$ is also invariant under rotation of the coordinate axes.

Remark 5. Invariance of the orientation under rotation

To show the invariance of the vector orientation under rotation of the coordinate axes, we may use the orthogonality of matrix $\mathbf{Q}$ (equation 3.27) and substitute equations 3.33 and 3.34 into equation 3.8a:

$$\mathbf{a}^\top \hat{\mathbf{X}} = \left(\mathbf{Q} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right)^\top \hat{\mathbf{X}} = \left(\mathbf{Q}\,\mathbf{Q}^\top \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} \right)^\top \hat{\mathbf{X}} = a_X = \|\mathbf{a}\| \cos \alpha_a \,.$$

The same approach can be used to compute $\mathbf{a}^\top \hat{\mathbf{Y}}$ and $\mathbf{a}^\top \hat{\mathbf{Z}}$. This shows that, no matter if $\mathbf{a}$ is defined in the arbitrary Cartesian system or any other orthogonal system, the angles between $\mathbf{a}$ and the unit vectors $\hat{\mathbf{X}}$, $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ remain unchanged.

3.2.6 Invariance of the cross product under rotation

The invariance of the cross product under rotation of the coordinate axes is more tricky than that of scalar product. Our goal is to show that, given two vectors $\mathbf{a}$ and $\mathbf{b}$ defined in an arbitrary orthogonal coordinate system (See the subsection 3.2.4), their cross-product is equal to that obtained for the case in which they are defined in the arbitrary Cartesian coordinate system (equation 3.15). We start with

$$\begin{bmatrix} a_Y\, b_Z \\ a_Z\, b_X \\ a_X\, b_Y \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} \right) \circ \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} \right) \tag{3.36}$$

and

$$\begin{bmatrix} a_Z\, b_Y \\ a_X\, b_Z \\ a_Y\, b_X \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_X \\ a_Y \\ a_Z \end{bmatrix} \right) \circ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_X \\ b_Y \\ b_Z \end{bmatrix} \right) , \tag{3.37}$$

where "$\circ$" denotes the *Hadamard product* (Definition 5).

Definition 5. Hadamard product

Let $\mathbb{C}$ designate the set of complex numbers. We denote the space of all $N \times M$ complex matrices by $\mathbb{C}^{N\times M}$:

$$\mathbf{A} \in \mathbb{C}^{N\times M} \iff \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} , \quad a_{ij} \in \mathbb{C} \, .$$

The *Hadamard* (*entrywise* or *Schur*) product of $\mathbf{A} \in \mathbb{C}$ and $\mathbf{B} \in \mathbb{C}$ (e.g., Horn and Johnson, 1991, p. 298) is

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} a_{11}\, b_{11} & a_{12}\, b_{12} & a_{13}\, b_{13} \\ a_{21}\, b_{21} & a_{22}\, b_{22} & a_{23}\, b_{23} \\ a_{31}\, b_{31} & a_{32}\, b_{32} & a_{33}\, b_{33} \end{bmatrix} \, .$$

This product is attributed to, and named after, either the french mathematician Jacques Hadamard and the german mathematician Issai Schur.

Then we use the orthogonality (equation 3.17) of matrix $\mathbf{Q}$ (equation 3.27) and equation 3.34 to obtain:

$$\begin{bmatrix} a_Y\, b_Z \\ a_Z\, b_X \\ a_X\, b_Y \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \circ \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \tag{3.38}$$

and

$$\begin{bmatrix} a_Z\, b_Y \\ a_X\, b_Z \\ a_Y\, b_X \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \circ \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \, . \tag{3.39}$$

By subtracting equation 3.39 from 3.38, we obtain

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_Y\, b_Z - a_Z\, b_Y \\ a_Z\, b_X - a_X\, b_Z \\ a_X\, b_Y - a_Y\, b_X \end{bmatrix} = \left(\mathbf{Q} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \times \left(\mathbf{Q} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \, . \tag{3.40}$$

Equation 3.40 shows that the cross product $\mathbf{a} \times \mathbf{b}$ obtained by using $\mathbf{a}$ and $\mathbf{b}$ represented in an arbitrary Cartesian system (equation 3.15) is equal to that obtained with the same vectors represented in an arbitrary orthogonal system (equations 3.33 and 3.34). In other words, equation 3.40 shows that the cross-product is invariant under rotation of the coordinate axes.

Alternatively, we can rewrite equations 3.38 and 3.39 in terms of the orthogonal vectors $\hat{\mathbf{p}}_j$ (equations 3.21 and 3.25):

$$\begin{bmatrix} a_Y\, b_Z \\ a_Z\, b_X \\ a_X\, b_Y \end{bmatrix} = \left(\begin{bmatrix} \hat{\mathbf{p}}_2^\top \\ \hat{\mathbf{p}}_3^\top \\ \hat{\mathbf{p}}_1^\top \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \circ \left(\begin{bmatrix} \hat{\mathbf{p}}_3^\top \\ \hat{\mathbf{p}}_1^\top \\ \hat{\mathbf{p}}_2^\top \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \tag{3.41}$$

and, similarly,

$$\begin{bmatrix} a_Z\, b_Y \\ a_X\, b_Z \\ a_Y\, b_X \end{bmatrix} = \left(\begin{bmatrix} \hat{\mathbf{p}}_3^\top \\ \hat{\mathbf{p}}_1^\top \\ \hat{\mathbf{p}}_2^\top \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \circ \left(\begin{bmatrix} \hat{\mathbf{p}}_2^\top \\ \hat{\mathbf{p}}_3^\top \\ \hat{\mathbf{p}}_1^\top \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) . \tag{3.42}$$

Now we combine equations 3.12h, 3.38 and 3.39 to show that:

$$\begin{aligned} a_Y\, b_Z - a_Z\, b_Y &= \left(\hat{\mathbf{p}}_2^\top \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \left(\hat{\mathbf{p}}_3^\top \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) - \left(\hat{\mathbf{p}}_3^\top \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \left(\hat{\mathbf{p}}_2^\top \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \\ &= (\hat{\mathbf{p}}_2 \times \hat{\mathbf{p}}_3)^\top \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \end{aligned} \tag{3.43}$$

and, similarly,

$$a_Z\, b_X - a_X\, b_Z = (\hat{\mathbf{p}}_3 \times \hat{\mathbf{p}}_1)^\top \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) \tag{3.44}$$

and

$$a_X\, b_Y - a_Y\, b_X = (\hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2)^\top \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) . \tag{3.45}$$

Finally, combining equations 3.40, 3.43–3.45 and the right-hand orientation of unit vectors $\hat{\mathbf{p}}_i$ (equations 3.30d–f), we obtain the following alternative form for the cross product defined in an arbitrary orthogonal system:

$$\mathbf{a} \times \mathbf{b} = \left(\mathbf{Q} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right) \times \left(\mathbf{Q} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \mathbf{Q} \begin{bmatrix} a_2\, b_3 - a_3\, b_2 \\ a_3\, b_1 - a_1\, b_3 \\ a_1\, b_2 - a_2\, b_1 \end{bmatrix} . \tag{3.46}$$

Chapter 4

Orthogonal curvilinear coordinate systems

In the previous chapter 3, there is a brief review on vector analysis in an arbitrary Cartesian system and also a discussion about what happens with some operations when they are performed in a different orthogonal coordinate system, with rotated coordinate axes. In the present section, there is a brief review on vector analysis in *Orthogonal Curvilinear Coordinate Systems*, a concept defined some paragraphs later.

Consider a specific point (X', Y', Z') with fixed coordinates X', Y' and Z' referred to the arbitrary Cartesian system. This point is defined by the intersection of three mutually-orthogonal surfaces formed by all points (X', Y, Z), (X, Y', Z) and (X, Y, Z') with variable coordinates X, Y and Z. Note that we obtain a different set of coordinate curves by choosing a different point (X', Y', Z'). The family of surfaces obtained for all points in this system is called *coordinate surfaces*. The intersection of two coordinate surfaces is a curve called *coordinate curve*. In the arbitrary Cartesian system, the coordinate surfaces are parallel planes and the coordinate curves are parallel straight lines.

Now, consider that through each point (X, Y, Z) referred to our Cartesian system passes a different set of generic coordinate surfaces defined by:

$$q_1 = q_1(X, Y, Z) \;, \tag{4.1a}$$
$$q_2 = q_2(X, Y, Z) \;, \tag{4.1b}$$
$$q_3 = q_3(X, Y, Z) \;, \tag{4.1c}$$

with inverse relations

$$X = X(q_1, q_2, q_3) \;, \tag{4.2a}$$
$$Y = Y(q_1, q_2, q_3) \;, \tag{4.2b}$$
$$Z = Z(q_1, q_2, q_3) \;. \tag{4.2c}$$

Now, through a specific point (X', Y', Z') passes three surfaces (q'_1, q_2, q_3), (q_1, q'_2, q_3) and (q_1, q_2, q'_3), with specific constant values for q'_1, q'_2 and q'_3 and variable q_1, q_2 and q_3. By using these new coordinate surfaces, any point (X, Y, Z) can now be defined by new coordinates (q_1, q_2, q_3), which are called *curviliear coordinates*.

It is important to stress that the generic coordinate surfaces defined by equations 4.1 and 4.2 are not necessarily planar. Consequently, the resultant coordinate curves are not straight lines. Different coordinate systems can be obtained according to the properties of these surfaces. Those where the coordinate surfaces are mutually orthogonal at all points in space are called *Orthogonal Curvilinear Coordinate Systems*. Here, we focus on this kind of system.

4.1 Local basis and scale factors

Let $\mathbf{r}$ be the vector starting at the origin O of the arbitrary Cartesian coordinate system and ending at a given point P. This vector is usually called *position vector*. The point P may be equivalently defined by $P = (X, Y, Z)$ or $P = (q_1, q_2, q_3)$, in terms of the Cartesian coordinates X, Y and Z or curvilinear coordinates q_1, q_2 and q_3. Consequently, the position vector can be explicitly represented by

$$\mathbf{r} = \mathbf{r}(X, Y, Z) = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} , \tag{4.3}$$

in our Cartesian coordinate system, or implicitly by $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$, in a generic curvilinear coordinate system.

In the arbitrary Cartesian system, there are three mutually-orthogonal unit vectors that are tangent to the respective curvilinear curves at P. They are defined by differentiating the position vector $\mathbf{r}$ (equation 4.3) with respect to X, Y and Z as follows:

$$\frac{\partial \mathbf{r}}{\partial X} = \hat{\mathbf{X}} , \tag{4.4a}$$
$$\frac{\partial \mathbf{r}}{\partial Y} = \hat{\mathbf{Y}} , \tag{4.4b}$$
$$\frac{\partial \mathbf{r}}{\partial Z} = \hat{\mathbf{Z}} , \tag{4.4c}$$

which are the base vectors of the standard basis for the Euclidean space $\mathbb{R}^3$ (equations 3.9). Note that these vectors do not depend on the position (X, Y, Z), i.e., the same constant vectors are obtained by differentiating the position vector $\mathbf{r}$ (equation 4.3) at any point (X, Y, Z) of the Cartesian system.

Now consider the three unit vectors that are tangent to the curvilinear curves associated with the curvilinear coordinate system at the same point P. They are obtained by differentiating the position vector $\mathbf{r}$ with respect to the curvilinear coordinates q_1, q_2 and q_3:

$$\hat{\mathbf{q}}_j = \hat{\mathbf{q}}_j(q_1, q_2, q_3) = \frac{1}{h_j} \frac{\partial \mathbf{r}}{\partial q_j} , \quad j \in \{1, 2, 3\} , \tag{4.5}$$

where

$$\frac{\partial \mathbf{r}}{\partial q_j} = \begin{bmatrix} \frac{\partial X}{\partial q_j} \\ \frac{\partial Y}{\partial q_j} \\ \frac{\partial Z}{\partial q_j} \end{bmatrix} , \quad j \in \{1, 2, 3\} , \tag{4.6}$$

and

$$h_j = h_j(q_1, q_2, q_3) = \left\| \frac{\partial \mathbf{r}}{\partial q_j} \right\| , \quad j \in \{1, 2, 3\} , \tag{4.7}$$

are called *scale factors*. Differently from the arbitrary Cartesian system, the unit vectors associated with curvilinear systems (equation 4.5) will generally be different for each point P in space. Consequently, they constitute a *local basis* for all vectors associated with point P. General curvilinear coordinate systems may have local basis formed by vectors that are not mutually orthogonal. Here, we consider the particular case in which the unit vectors $\hat{\mathbf{q}}_j$ (equation 4.5) are mutually orthogonal (equation 3.24) and form the columns of an orthogonal matrix $\mathbf{Q}$ defined in terms of Euler angles ε, ζ and η (equation 3.27). Besides, we consider that the Euler angles vary with position $P = (q_1, q_2, q_3)$ according to the geometry of the curvilinear surfaces and curves associated with the curvilinear system, i.e., $\varepsilon = \varepsilon(P)$, $\zeta = \zeta(P)$ and $\eta = \eta(P)$. Consequently, we consider that the orthogonal matrix $\mathbf{Q}$ and the mutually-orthogonal unit vectors $\hat{\mathbf{p}}_j$ forming its rows (equations 3.23 and 3.25) also vary with P, i.e., $\mathbf{Q} = \mathbf{Q}(P)$ and $\hat{\mathbf{p}}_j = \hat{\mathbf{p}}_j(P)$, $j \in \{1, 2, 3\}$.

4.2 Jacobian matrix, line elements and metric tensor

From equations 4.4a–c, we obtain that a differential change $\mathrm{d}\boldsymbol{\ell}$ in $\mathbf{r}$ due to small displacements $\mathrm{d}X$, $\mathrm{d}Y$ and $\mathrm{d}Z$ along the coordinate curves of the Cartesian system is given by

$$\mathrm{d}\boldsymbol{\ell} = \frac{\partial \mathbf{r}}{\partial X}\,\mathrm{d}X + \frac{\partial \mathbf{r}}{\partial Y}\,\mathrm{d}Y + \frac{\partial \mathbf{r}}{\partial Z}\,\mathrm{d}Z = \begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} . \tag{4.8}$$

The small changes $\mathrm{d}X$, $\mathrm{d}Y$ and $\mathrm{d}Z$, in turn, represent the total differentials:

$$\mathrm{d}X = \frac{\partial X}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial X}{\partial q_2}\,\mathrm{d}q_2 + \frac{\partial X}{\partial q_3}\,\mathrm{d}q_3 \tag{4.9a}$$

$$\mathrm{d}Y = \frac{\partial Y}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial Y}{\partial q_2}\,\mathrm{d}q_2 + \frac{\partial Y}{\partial q_3}\,\mathrm{d}q_3 \tag{4.9b}$$

$$\mathrm{d}Z = \frac{\partial Z}{\partial q_1}\,\mathrm{d}q_1 + \frac{\partial Z}{\partial q_2}\,\mathrm{d}q_2 + \frac{\partial Z}{\partial q_3}\,\mathrm{d}q_3 \, . \tag{4.9c}$$

By using the total differentials $\mathrm{d}X$, $\mathrm{d}Y$ and $\mathrm{d}Z$ (equations 4.9a–c) and the derivatives $\frac{\partial \mathbf{r}}{\partial q_j}$ (equation 4.6), we may rewrite the differential change $\mathrm{d}\boldsymbol{\ell}$ (equation 4.8) in terms of small displacements $\mathrm{d}q_1$, $\mathrm{d}q_2$ and $\mathrm{d}q_3$ along the coordinate curves of the orthogonal curvilinear system as follows:

$$\mathrm{d}\boldsymbol{\ell} = \mathbf{J}_{\mathbf{r}} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} , \tag{4.10}$$

where

$$\mathbf{J}_{\mathbf{r}} = \mathbf{J}_{\mathbf{r}}(q_1, q_2, q_3) = \begin{bmatrix} \frac{\partial \mathbf{r}}{\partial q_1} & \frac{\partial \mathbf{r}}{\partial q_2} & \frac{\partial \mathbf{r}}{\partial q_3} \end{bmatrix} \tag{4.11}$$

is the *Jacobian matrix* of $\mathbf{r}$. Now, by using the relationship between the derivatives $\frac{\partial \mathbf{r}}{\partial q_j}$ and unit vectors $\hat{\mathbf{q}}_j$ (equations 4.5), we obtain the following new expression for $\mathbf{J}_{\mathbf{r}}$ (equation 4.11):

$$\mathbf{J}_{\mathbf{r}} = \mathbf{Q}\,\mathbf{H} \, , \tag{4.12}$$

where $\mathbf{Q}$ is the orthogonal matrix formed by the unit vectors $\hat{\mathbf{q}}_j$ (equation **??**) and $\mathbf{H}$ is the diagonal matrix

$$\mathbf{H} = \mathbf{H}(q_1, q_2, q_3) = \begin{bmatrix} h_1 & & \\ & h_2 & \\ & & h_3 \end{bmatrix} , \tag{4.13}$$

where h_j are the scale factors (equation 4.7). Then, by substituting the Jacobian matrix $\mathbf{J}_\mathbf{r}$ given by equation 4.12 into equation 4.10, we obtain a third expression for the differential change $\mathrm{d}\boldsymbol{\ell}$:

$$\mathrm{d}\boldsymbol{\ell} = \mathbf{Q} \begin{bmatrix} \mathrm{d}\ell_1 \\ \mathrm{d}\ell_2 \\ \mathrm{d}\ell_3 \end{bmatrix} , \tag{4.14}$$

where

$$\mathrm{d}\ell_j = h_j \mathrm{d}q_j \, , \quad j \in \{1, 2, 3\} \, , \tag{4.15}$$

represent the *line elements* in the curvilinear system. Note that equation 4.14 represents the differential change $\mathrm{d}\boldsymbol{\ell}$ in the local basis formed by the unit vectors $\hat{\mathbf{q}}_j$ (equations 4.5). From the line elements $\mathrm{d}\ell_j$ (equation 4.15), we may immediately develop the *area element*

$$\mathrm{d}\mathbf{s} = \mathbf{Q} \begin{bmatrix} \mathrm{d}\ell_2 \, \mathrm{d}\ell_3 \\ \mathrm{d}\ell_1 \, \mathrm{d}\ell_3 \\ \mathrm{d}\ell_1 \, \mathrm{d}\ell_2 \end{bmatrix} = \mathbf{Q} \begin{bmatrix} h_2 h_3 \, \mathrm{d}q_2 \mathrm{d}q_3 \\ h_1 h_3 \, \mathrm{d}q_1 \mathrm{d}q_3 \\ h_1 h_2 \, \mathrm{d}q_1 \mathrm{d}q_2 \end{bmatrix} , \tag{4.16}$$

and *volume element*

$$\mathrm{d}v = \mathrm{d}\ell_1 \, \mathrm{d}\ell_2 \, \mathrm{d}\ell_3 = h_1 h_2 h_3 \, \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 \, . \tag{4.17}$$

It is important noting that the magnitude $\mathrm{d}\ell$ of the differential change $\mathrm{d}\boldsymbol{\ell}$ (equations 4.8, 4.10 and 4.14) represents the distance between two neighboring points in space. The square of this distance, defined in the curvilinear system, may be given by:

$$\begin{aligned} \mathrm{d}\ell^2 &= \mathrm{d}\boldsymbol{\ell}^\top \mathrm{d}\boldsymbol{\ell} = \\ &= \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix}^\top \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} , \end{aligned} \tag{4.18}$$

where the coefficients

$$g_{ij} = g_{ij}(q_1, q_2, q_3) = \frac{\partial \mathbf{r}}{\partial q_i}^\top \frac{\partial \mathbf{r}}{\partial q_j} \, , \quad i, j \in \{1, 2, 3\} \, , \tag{4.19}$$

are called *metric* and they form a matrix called *metric tensor*. Note that, by using equation 4.5 and the orthogonality condition (equation **??**), the metric coefficients g_{ij} (equation 4.19) may be rewritten as follows:

$$g_{ij} = g_{ij}(q_1, q_2, q_3) = \begin{cases} h_i^2 & \text{if} \quad i = j \, , \\ 0 & \text{if} \quad i \neq j \, . \end{cases} \tag{4.20}$$

Consequently, the squared distance $\mathrm{d}\ell^2$ (equation 4.18) assumes the particular form:

$$\mathrm{d}\ell^2 = \mathrm{d}\ell_1^2 + \mathrm{d}\ell_2^2 + \mathrm{d}\ell_3^2 \, . \tag{4.21}$$

4.3 General expressions in orthogonal curvilinear systems

The curvilinear surfaces and curves (equations 4.1 and 4.2), local base vectors $\hat{\mathbf{q}}_j$ (equation 4.5) and scalar factors h_j (equation 4.7) define the general properties of any orthogonal curvilinear coordinate system. In the present section, these quantities are used to define general expressions for some important mathematical stuff in orthogonal curvilinear coordinate systems. These general expressions will be used throughout the manuscript to define geophysical quantities in the Terrestrial Reference Systems presented in Chapter 5. It can be easily verified that the general expressions defined later in this sections reduce to the well-known expressions in the arbitrary Cartesian system for the particular case in which:

$$h_1 = h_2 = h_3 = 1 \tag{4.22}$$

and

$$\hat{\mathbf{q}}_1 = \hat{\mathbf{X}} \, , \tag{4.23a}$$

$$\hat{\mathbf{q}}_2 = \hat{\mathbf{Y}} \, , \tag{4.23b}$$

$$\hat{\mathbf{q}}_3 = \hat{\mathbf{Z}} \, . \tag{4.23c}$$

4.3.1 Scalar and cross products

Due to the invariance of scalar and cross products under rotation of the coordinate axes (Subsections 3.2.5 and 3.2.6), they are the same in orthogonal curvilinear coordinate systems as in any Cartesian system. The only difference is that, in the curvilinear system, the scalar and cross products are computed with the curvilinear coordinates q_1, q_2 and q_3.

4.3.2 Integrals

Let f and $\mathbf{V}$ be scalar and vector fields defined in an orthogonal curvilinear system as follows:

$$f = f(q_1, q_2, q_3) \tag{4.24}$$

and

$$\mathbf{V} = \mathbf{V}(q_1, q_2, q_3) = V_1(q_1, q_2, q_3)\, \hat{\mathbf{q}}_1 + V_2(q_1, q_2, q_3)\, \hat{\mathbf{q}}_2 + V_3(q_1, q_2, q_3)\, \hat{\mathbf{q}}_3 \, . \tag{4.25}$$

By using the line, area and volume elements $\mathrm{d}\ell_i$, $\mathrm{d}\mathbf{s}$ and $\mathrm{d}v$ (equations 4.15, 4.16 and 4.17), respectively, we may write general expressions for line, surface and volume integrals in curvilinear coordinates:

$$\int \mathbf{V}^\top \mathrm{d}\boldsymbol{\ell} = \int V_1 \, h_1 \, \mathrm{d}q_1 + \int V_2 \, h_2 \, \mathrm{d}q_2 + \int V_3 \, h_3 \, \mathrm{d}q_3 \, , \tag{4.26}$$

$$\iint \mathbf{V}^\top \mathrm{d}\mathbf{s} = \iint V_1 \, h_2 h_3 \, \mathrm{d}q_2 \mathrm{d}q_3 + \iint V_2 \, h_1 h_3 \, \mathrm{d}q_1 \mathrm{d}q_3 + \iint V_3 \, h_1 h_2 \, \mathrm{d}q_1 \mathrm{d}q_2 \tag{4.27}$$

and

$$\iiint f \, \mathrm{d}v = \iiint f \, h_1 h_2 h_3 \, \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3 \, . \tag{4.28}$$

4.3.3 Gradient

Consider the total variation $\mathrm{d}f$ of a scalar function $f(X, Y, Z)$ defined in an arbitrary Cartesian coordinate system:

$$\mathrm{d}f = \frac{\partial f}{\partial X}\mathrm{d}X + \frac{\partial f}{\partial Y}\mathrm{d}Y + \frac{\partial f}{\partial Z}\mathrm{d}Z = \begin{bmatrix} \frac{\partial f}{\partial X} \\ \frac{\partial f}{\partial Y} \\ \frac{\partial f}{\partial Z} \end{bmatrix}^{\top} \begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} . \tag{4.29}$$

Similarly, the total variation $\mathrm{d}f$ of the scalar function $f(q_1, q_2, q_3)$ (equation 4.24) in an orthogonal curvilinear coordinate system can be defined by:

$$\mathrm{d}f = \frac{\partial f}{\partial q_1}\mathrm{d}q_1 + \frac{\partial f}{\partial q_2}\mathrm{d}q_2 + \frac{\partial f}{\partial q_3}\mathrm{d}q_3 = \begin{bmatrix} \frac{\partial f}{\partial q_1} \\ \frac{\partial f}{\partial q_2} \\ \frac{\partial f}{\partial q_3} \end{bmatrix}^{\top} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} . \tag{4.30}$$

These two equations must produce the same total variation $\mathrm{d}f$ and are different representations of a more general equation given by:

$$\mathrm{d}f = \boldsymbol{\nabla} f^{\top} \mathrm{d}\boldsymbol{\ell} , \tag{4.31}$$

where $\boldsymbol{\nabla} f$ is the *gradient vector*.

Let us now determine the general expression defining the gradient vector in curvilinear coordinates. We start by using equations 4.8, 4.14 and 4.15 to obtain

$$\mathrm{d}\boldsymbol{\ell} = \begin{bmatrix} \mathrm{d}X \\ \mathrm{d}Y \\ \mathrm{d}Z \end{bmatrix} = \mathbf{QH} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} . \tag{4.32}$$

Then we replace the rightmost term of equation 4.32 into the general equation 4.31 for the total variation $\mathrm{d}f$ to obtain:

$$\mathrm{d}f = \boldsymbol{\nabla} f^{\top} \mathbf{QH} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} = \left(\mathbf{HQ}^{\top} \boldsymbol{\nabla} f\right)^{\top} \begin{bmatrix} \mathrm{d}q_1 \\ \mathrm{d}q_2 \\ \mathrm{d}q_3 \end{bmatrix} . \tag{4.33}$$

Finally, by comparing equations 4.30 and 4.33 and using the properties of matrices $\mathbf{Q}$ (equation **??**) and $\mathbf{H}$ (equation 4.13), we deduce the general expression for the gradient vector:

$$\boldsymbol{\nabla} f = \mathbf{Q} \begin{bmatrix} \frac{1}{h_1}\frac{\partial f}{\partial q_1} \\ \frac{1}{h_2}\frac{\partial f}{\partial q_2} \\ \frac{1}{h_3}\frac{\partial f}{\partial q_3} \end{bmatrix} . \tag{4.34}$$

4.3.4 Divergent, Laplacian and Curl

Let f and $\mathbf{V}$ be the scalar and vector fields defined by equations 4.24 and 4.25. The *divergent*, *Laplacian* and *curl* are respectively given by[1]:

[1]See Arfken and Weber (2005, p. 111–113), Stratton (2007, p. 49) and Lipschutz et al. (2009, sec. 7.6).

$$\nabla \cdot \mathbf{V} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (V_1 h_2 h_3) + \frac{\partial}{\partial q_2} (h_1 V_2 h_3) + \frac{\partial}{\partial q_3} (h_1 h_2 V_3) \right] , \tag{4.35}$$

$$\nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial f}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial q_3} \right) \right] \tag{4.36}$$

and

$$\nabla \times \mathbf{V} = \mathbf{Q} \begin{bmatrix} \frac{1}{h_2 h_3} \left(\frac{\partial (h_3 V_3)}{\partial q_2} - \frac{\partial (h_2 V_2)}{\partial q_3} \right) \\ \frac{1}{h_3 h_1} \left(\frac{\partial (h_1 V_1)}{\partial q_3} - \frac{\partial (h_3 V_3)}{\partial q_1} \right) \\ \frac{1}{h_1 h_2} \left(\frac{\partial (h_2 V_2)}{\partial q_1} - \frac{\partial (h_1 V_1)}{\partial q_2} \right) \end{bmatrix} . \tag{4.37}$$

4.4 Vector field conversion between orthogonal systems

Consider the vector field $\mathbf{V}$ defined in an orthogonal curvilinear system with coordinates q_1, q_2 and q_3 and local basis $\hat{\mathbf{q}}_1$, $\hat{\mathbf{q}}_2$ and $\hat{\mathbf{q}}_3$, according to equation 4.25. Equations 3.33 and 3.34 show that, at each point P, $\mathbf{V}$ can be expressed in the arbitrary Cartesian system as follows:

$$\mathbf{V} = \mathbf{Q} \begin{bmatrix} V_1(q_1, q_2, q_3) \\ V_2(q_1, q_2, q_3) \\ V_3(q_1, q_2, q_3) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}} \end{bmatrix} \begin{bmatrix} V_X(X, Y, Z) \\ V_Y(X, Y, Z) \\ V_Z(X, Y, Z) \end{bmatrix} , \tag{4.38}$$

where

$$\begin{bmatrix} V_X(X, Y, Z) \\ V_Y(X, Y, Z) \\ V_Z(X, Y, Z) \end{bmatrix} = \mathbf{Q} \begin{bmatrix} V_1(q_1, q_2, q_3) \\ V_2(q_1, q_2, q_3) \\ V_3(q_1, q_2, q_3) \end{bmatrix} . \tag{4.39}$$

Now, we may use equations 3.33 and 3.34 again to express vector field $\mathbf{V}$ in a different orthogonal curvilinear system with coordinates q_1', q_2' and q_3' and local basis $\hat{\mathbf{q}}_1'$, $\hat{\mathbf{q}}_2'$ and $\hat{\mathbf{q}}_3'$:

$$\mathbf{V} = \begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}} \end{bmatrix} \begin{bmatrix} V_X(X, Y, Z) \\ V_Y(X, Y, Z) \\ V_Z(X, Y, Z) \end{bmatrix} = \mathbf{Q}' \begin{bmatrix} V_1'(q_1', q_2', q_3') \\ V_2'(q_1', q_2', q_3') \\ V_3'(q_1', q_2', q_3') \end{bmatrix} , \tag{4.40}$$

where

$$\begin{bmatrix} V_1'(q_1', q_2', q_3') \\ V_2'(q_1', q_2', q_3') \\ V_3'(q_1', q_2', q_3') \end{bmatrix} = \mathbf{Q}'^{\top} \begin{bmatrix} V_X(X, Y, Z) \\ V_Y(X, Y, Z) \\ V_Z(X, Y, Z) \end{bmatrix} \tag{4.41}$$

and $\mathbf{Q}'$ is the orthogonal matrix

$$\mathbf{Q}' = \begin{bmatrix} \hat{\mathbf{q}}_1' & \hat{\mathbf{q}}_2' & \hat{\mathbf{q}}_3' \end{bmatrix} . \tag{4.42}$$

Then, we combine equations 4.38–4.41 to show that:

$$\begin{bmatrix} V_1'(q_1', q_2', q_3') \\ V_2'(q_1', q_2', q_3') \\ V_3'(q_1', q_2', q_3') \end{bmatrix} = \mathbf{Q}'^{\top} \mathbf{Q} \begin{bmatrix} V_1(q_1, q_2, q_3) \\ V_2(q_1, q_2, q_3) \\ V_3(q_1, q_2, q_3) \end{bmatrix} . \tag{4.43}$$

This equation shows how to convert a vector field $\mathbf{V}$ from a given orthogonal curvilinear system to another.

Chapter 5

Terrestrial Reference Systems (TRS)

For geophysical purposes, there are four important *Terrestrial Reference Systems* (TRS). They rotate with the Earth and are used not only for describing positions and movements of objects, but also potential (gravity and magnetic) fields on and close to the Earth's surface (e.g., Torge and Müller, 2012, p. 28).

5.1 Geocentric Cartesian System (GCS)

Consider a geocentric[1] system of Cartesian coordinates having the Z-axis coincident with the mean rotational axis of the Earth, the X-axis pointing to the Greenwich meridian, and the Y-axis completing a right-handed system (Figure 5.2). This reference system is called by different names in the literature: Mean Terrestrial System (e.g., Soler, 1976, p. 4), Earth-fixed geocentric Cartesian system (e.g., Torge and Müller, 2012, p. 28) or Earth-centered Earth-fixed system (e.g., Bouman et al., 2013), for example. Here, we opted for using the term Geocentric Cartesian System (GCS).

The GCS is equivalent to the arbitrary Cartesian coordinate system used in section 3.1. The difference is that the origin is specifically located at the approximated Earth's center of mass and the axes orientation is also specifically defined.

5.2 Geocentric Spherical System (GSS)

The position of a point in the Geocentric Spherical System (GSS) is defined by the *radial distance* r, the *geocentric latitude* ϕ, and the *longitude* λ (Figure 5.1). The following equations transform spherical coordinates (r, ϕ, λ) referred to the GSS into Cartesian coordinates (X, Y, Z) referred to the GCS (Section 5.1):

$$X = r \cos\phi \cos\lambda \,, \tag{5.1a}$$

$$Y = r \cos\phi \sin\lambda \,, \tag{5.1b}$$

$$Z = r \sin\phi \,. \tag{5.1c}$$

[1]With origin O at the approximated center of mass of the Earth.

The inverse relations to transform (X, Y, Z) into (r, ϕ, λ) are given by:

$$r = \sqrt{X^2 + Y^2 + Z^2} \,, \tag{5.2a}$$

$$\phi = \sin^{-1}\left(\frac{Z}{r}\right) \,, \tag{5.2b}$$

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right) \,. \tag{5.2c}$$

Similarly to equations 4.1 and 4.2 for general curvilinear systems, equations 5.1 and 5.2 define the curvilinear surfaces and curves of the GSS. To obtain the specific scale factors h_j (equation 4.7) for GSS, let us first compute the derivatives of the Cartesian coordinates X, Y and Z (equations 5.1) with respect to the spherical coordinates r, ϕ and λ:

$$\frac{\partial X}{\partial r} = \cos\phi\,\cos\lambda \,, \tag{5.3a}$$

$$\frac{\partial Y}{\partial r} = \cos\phi\,\sin\lambda \,, \tag{5.3b}$$

$$\frac{\partial Z}{\partial r} = \sin\phi \,, \tag{5.3c}$$

$$\frac{\partial X}{\partial \phi} = -r\,\sin\phi\,\cos\lambda \,, \tag{5.4a}$$

$$\frac{\partial Y}{\partial \phi} = -r\,\sin\phi\,\sin\lambda \,, \tag{5.4b}$$

$$\frac{\partial Z}{\partial \phi} = r\,\cos\phi \tag{5.4c}$$

and

$$\frac{\partial X}{\partial \lambda} = -r\,\cos\phi\,\sin\lambda \,, \tag{5.5a}$$

$$\frac{\partial Y}{\partial \lambda} = r\,\cos\phi\,\cos\lambda \,, \tag{5.5b}$$

$$\frac{\partial Z}{\partial \lambda} = 0 \,. \tag{5.5c}$$

Using these derivatives, we compute the scale factors h_j according to equations 4.6 and 4.7 as follows:

$$h_1 = 1 \,, \tag{5.6a}$$

$$h_2 = r \,, \tag{5.6b}$$

$$h_3 = r\,\cos\phi \,. \tag{5.6c}$$

Finally, we use equation 4.5 to compute the unit vectors forming the local basis in GSC:

$$\hat{\mathbf{r}}(\phi, \lambda) = \begin{bmatrix} \cos\phi \, \cos\lambda \\ \cos\phi \, \sin\lambda \\ \sin\phi \end{bmatrix} , \tag{5.7a}$$

$$\hat{\boldsymbol{\phi}}(\phi, \lambda) = \begin{bmatrix} -\sin\phi \, \cos\lambda \\ -\sin\phi \, \sin\lambda \\ \cos\phi \end{bmatrix} , \tag{5.7b}$$

$$\hat{\boldsymbol{\lambda}}(\lambda) = \begin{bmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{bmatrix} . \tag{5.7c}$$

Note that, differently from the global basis associated with Cartesian systems (equations 3.9 and 4.4), the GSS has a local basis that varies with position.

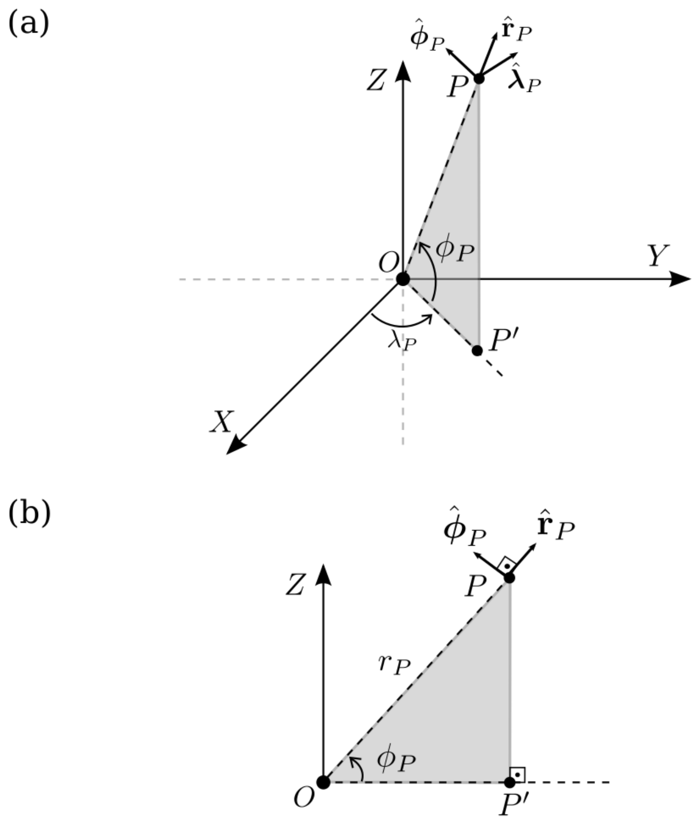

Figure 5.1: (a) and (b) The Geocentric Cartesian System (GCS) and the Geocentric Spherical System (GSS). In this coordinate system, the position of a point is determined by the radial distance r, geocentric latitude ϕ, and longitude λ. O is the Earth's center of mass, P is an observation point, and unit vectors $\hat{\mathbf{r}}_P \equiv \hat{\mathbf{r}}(\phi_P, \lambda_P)$ (equation 5.7a), $\hat{\boldsymbol{\phi}}_P \equiv \hat{\boldsymbol{\phi}}(\phi_P, \lambda_P)$ (equation 5.7b), and $\hat{\boldsymbol{\lambda}}_P \equiv \hat{\boldsymbol{\lambda}}(\lambda_P)$ (equation 5.7c) define mutually orthogonal directions at P.

5.3 Geocentric Geodetic System (GGS)

The Geocentric Geodetic System (GGS) is defined by a reference ellipsoid (Heiskanen and Moritz, 1967; Soler, 1976; Torge and Müller, 2012; Bouman et al., 2013). The position of a point is defined by the *geometric height* h, the *geodetic latitude* φ, and the *longitude* λ (Figure 5.2).

The following equations transform geodetic coordinates (h, φ, λ) referred to the GGS into Cartesian coordinates (X, Y, Z) referred to the GCS (e.g., Hofmann-Wellenhof and Moritz, 2005, p. 193):

$$X = (\mathcal{N} + h)\, \cos\varphi\, \cos\lambda\,, \tag{5.8a}$$
$$Y = (\mathcal{N} + h)\, \cos\varphi\, \sin\lambda\,, \tag{5.8b}$$
$$Z = \left[\mathcal{N}(1 - e^2) + h\right]\, \sin\varphi\,, \tag{5.8c}$$

where

$$\mathcal{N} = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi}} \tag{5.9}$$

is the *principal radius of curvature in the prime vertical plane*,

$$e = \frac{\sqrt{a^2 - b^2}}{a} \tag{5.10}$$

is the *first eccentricity* and the constants a and b are, respectively, the semi-major and semi-minor axes of the reference ellipsoid (Figure 5.2).

The inverse procedure to transform (X, Y, Z) into (h, φ, λ) is frequently performed iteratively by using the Hirvonen-Moritz algorithm (e.g., Hofmann-Wellenhof and Moritz, 2005, p. 195):

Algorithm 1 Hirvonen-Moritz algorithm

Require: X, Y, Z, a, b, k_{max}
- $\lambda \leftarrow \tan^{-1}\left(\frac{Y}{X}\right)$
- $e \leftarrow \frac{\sqrt{a^2-b^2}}{a}$
- $p \leftarrow \sqrt{X^2 + Y^2}$
- $\varphi_0 \leftarrow \tan^{-1}\left(\frac{Z}{p(1-e^2)}\right)$
- $\mathcal{N}_0 \leftarrow \frac{a}{\sqrt{1-e^2\sin^2\varphi_0}}$
- $h_0 \leftarrow \frac{p}{\cos\varphi_0} - \mathcal{N}_0$
- **for** $k \leftarrow 1, k_{max}$ **do**
  - $\varphi_0 \leftarrow \tan^{-1}\left[\frac{Z}{p}\left(1 - e^2\frac{\mathcal{N}_0}{\mathcal{N}_0 + h_0}\right)^{-1}\right]$
  - $\mathcal{N} \leftarrow \frac{a}{\sqrt{1-e^2\sin^2\varphi}}$
  - $h \leftarrow \frac{p}{\cos\varphi} - \mathcal{N}$
  - $\varphi_0 \leftarrow \varphi$
  - $\mathcal{N}_0 \leftarrow \mathcal{N}$
  - $h_0 \leftarrow h$
- **end for**

Let us now compute the derivatives of Cartesian coordinates X, Y and Z (equations 5.8) with respect to geodetic coordinates h, φ and λ:

$$\frac{\partial X}{\partial h} = \cos\varphi\,\cos\lambda\,, \tag{5.11a}$$
$$\frac{\partial Y}{\partial h} = \cos\varphi\,\sin\lambda\,, \tag{5.11b}$$
$$\frac{\partial Z}{\partial h} = \sin\varphi\,, \tag{5.11c}$$

$$\frac{\partial X}{\partial \varphi} = -(\mathcal{M}+h)\,\sin\varphi\,\cos\lambda\,, \tag{5.12a}$$
$$\frac{\partial Y}{\partial \varphi} = -(\mathcal{M}+h)\,\sin\varphi\,\sin\lambda\,, \tag{5.12b}$$
$$\frac{\partial Z}{\partial \varphi} = (\mathcal{M}+h)\,\cos\varphi\,, \tag{5.12c}$$

and

$$\frac{\partial X}{\partial \lambda} = -(\mathcal{N}+h)\,\cos\phi\,\sin\lambda\,, \tag{5.13a}$$
$$\frac{\partial Y}{\partial \lambda} = (\mathcal{N}+h)\,\cos\phi\,\cos\lambda\,, \tag{5.13b}$$
$$\frac{\partial Z}{\partial \lambda} = 0\,, \tag{5.13c}$$

where

$$\mathcal{M} = \frac{a(1-e^2)}{\left(1-e^2\sin^2\varphi\right)^{\frac{3}{2}}} \tag{5.14}$$

is the *principal radius of curvature in the meridian plane*. From these derivatives, we obtain the scale factors (equation 4.7) of the GGS:

$$h_1 = 1\,, \tag{5.15a}$$
$$h_2 = (\mathcal{M}+h)\,, \tag{5.15b}$$
$$h_3 = (\mathcal{N}+h)\,\cos\phi\,. \tag{5.15c}$$

Finally, we use the derivatives defined by equations 5.11–5.13 and scale factors given by equation 5.15 to compute the local basis (equation 4.5) for the GGS:

$$\hat{\mathbf{h}}(\varphi,\lambda) = \begin{bmatrix}\cos\varphi\,\cos\lambda\\ \cos\varphi\,\sin\lambda\\ \sin\varphi\end{bmatrix}\,, \tag{5.16a}$$

$$\hat{\boldsymbol{\varphi}}(\varphi,\lambda) = \begin{bmatrix}-\sin\varphi\,\cos\lambda\\ -\sin\varphi\,\sin\lambda\\ \cos\varphi\end{bmatrix} \tag{5.16b}$$

and the same unit vector $\hat{\boldsymbol{\lambda}}(\lambda)$ (equation 5.7c) defined for the GSS. Note that the GGS also has a local basis that varies with position.

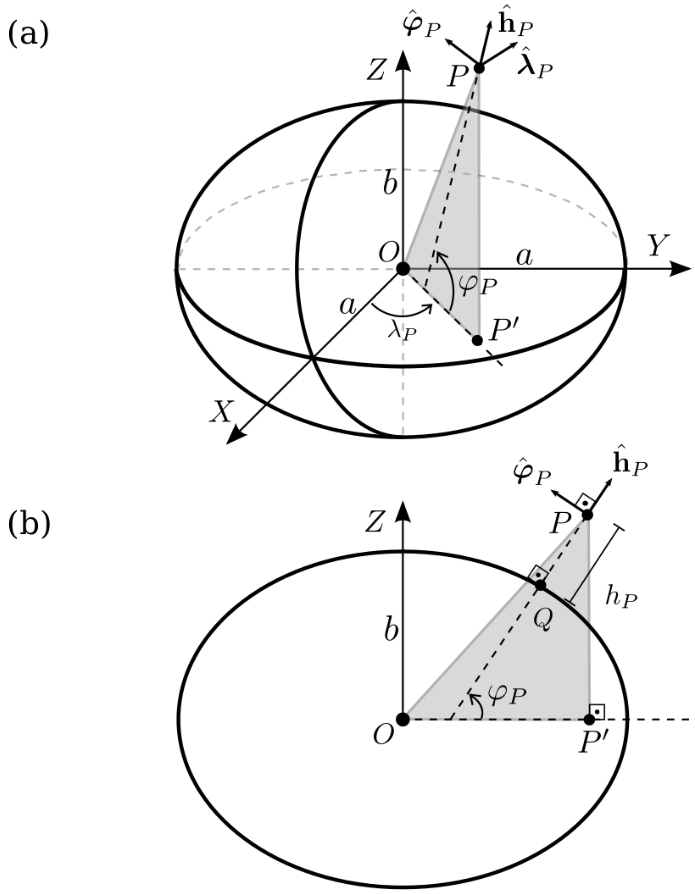

Figure 5.2: (a) and (b) The Geocentric Cartesian System (GCS) and the Geocentric Geodetic System (GGS). The GGS is defined by an oblate ellipsoid with semi-minor axis b and semi-major axis a. In this coordinate system, the position of a point is determined by the geometric height h, geodetic latitude φ, and longitude λ. O is the Earth's center of mass, P is an observation point, and unit vectors $\hat{\mathbf{h}}_P \equiv \hat{\mathbf{h}}(\varphi_P, \lambda_P)$ (equation 5.16a), $\hat{\boldsymbol{\varphi}}_P \equiv \hat{\boldsymbol{\varphi}}(\varphi_P, \lambda_P)$ (equation 5.16b), and $\hat{\boldsymbol{\lambda}}_P \equiv \hat{\boldsymbol{\lambda}}(\lambda_P)$ (equation 5.7c) define mutually orthogonal directions at P. In (b), Q is the projection of P onto the reference ellipsoid at the same latitude φ_P and longitude λ_P.

5.4 Topocentric Cartesian System (TCS)

The Topocentric[2] Cartesian System (TCS) is commonly used in local or regional scale geophysical studies. Its origin is located at a point P_0 on (or close to) the Earth's surface and each point is defined by Cartesian coordinates x, y and z. Because TCS is a Cartesian system, its scale factors are $h_1 = h_2 = h_3 = 1$. Besides, it also has a global basis, i.e., a basis that does not vary with position. The global basis of TCS may be defined by using the GSS (equations 5.7a–c) or GGS (equations 5.16a and b), according to Table 5.1.

[2]With origin O at or close to the surface of the Earth.

TCS	GGS	GSS
$\hat{\mathbf{x}}$	$\hat{\boldsymbol{\varphi}}_0$	$\hat{\boldsymbol{\phi}}_0$
$\hat{\mathbf{y}}$	$\hat{\boldsymbol{\lambda}}_0$	$\hat{\boldsymbol{\lambda}}_0$
$\hat{\mathbf{z}}$	$-\hat{\mathbf{h}}_0$	$-\hat{\mathbf{r}}_0$

Table 5.1: Unit vectors defining the axes of the TCS. The subscript "0" indicates that the unit vectors defining the origin P_0 are computed at the coordinates $(h_0, \varphi_0, \lambda_0)$ (equation 5.16) or (r_0, ϕ_0, λ_0) (equation 5.7), depending if the TCS is defined in terms of the GGS or GSS.

The following equation transforms Cartesian coordinates (x, y, z) referred to the TCS into Cartesian coordinates (X, Y, Z) referred to the GCS:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \end{bmatrix} + \mathbf{R}_{TC} \begin{bmatrix} x \\ y \\ z \end{bmatrix} , \tag{5.17}$$

where (X_0, Y_0, Z_0) are the coordinates defining the origin P_0 at the GCS and

$$\mathbf{R}_{TC} = \begin{bmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \end{bmatrix} \tag{5.18}$$

is an orthogonal matrix defined by the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ (Table 5.1). The inverse procedure to transform (X, Y, Z) into (x, y, z) is given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}_{TC}^{\top} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} . \tag{5.19}$$

Chapter 6

Moving coordinate systems

6.1 Position, velocity and acceleration vectors

Consider an arbitrary orthogonal system with axes q_1, q_2 and q_3 that is moving with respect to an arbitrary Cartesian system with axes X, Y and Z (Figure 6.1).

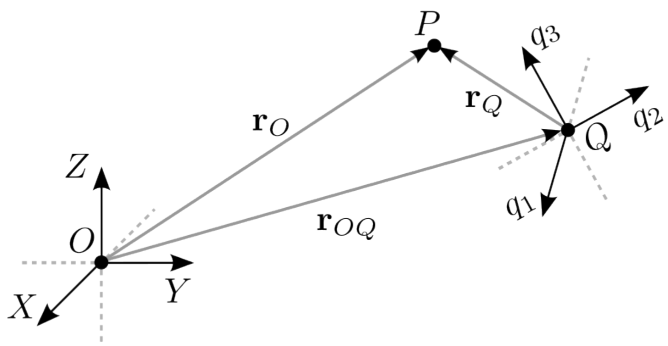

Figure 6.1: Arbitrary orthogonal system with origin at Q and axes q_1, q_2 and q_3 moving with respect to an arbitrary Cartesian system having origin at O and axes X, Y and Z. The position vector of a point P referred to the Cartesian system is $\mathbf{r}_O$. The position vector of P referred to the moving system is $\mathbf{r}_Q$. The relative position of Q with respect to O is $\mathbf{r}_{OQ}$.

From 4.3

$$\mathbf{r}_O = \mathbf{r}_{OQ} + \mathbf{r}_Q \tag{6.1}$$

$$\mathbf{r}_O = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{6.2}$$

$$\mathbf{r}_{OQ} = \begin{bmatrix} X_Q \\ Y_Q \\ Z_Q \end{bmatrix} \tag{6.3}$$

$$\mathbf{r}_Q = \begin{bmatrix} X - X_Q \\ Y - Y_Q \\ Z - Z_Q \end{bmatrix} \tag{6.4}$$

$$\mathbf{r}_O = \mathbf{Q} \begin{bmatrix} q_1^O \\ q_2^O \\ q_3^O \end{bmatrix} \tag{6.5a}$$

$$\begin{bmatrix} q_1^O \\ q_2^O \\ q_3^O \end{bmatrix} = \mathbf{Q}^\top \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{6.5b}$$

$$\mathbf{r}_{OQ} = \mathbf{Q} \begin{bmatrix} q_1^{OQ} \\ q_2^{OQ} \\ q_3^{OQ} \end{bmatrix} \tag{6.6a}$$

$$\begin{bmatrix} q_1^{OQ} \\ q_2^{OQ} \\ q_3^{OQ} \end{bmatrix} = \mathbf{Q}^\top \begin{bmatrix} X_Q \\ Y_Q \\ Z_Q \end{bmatrix} \tag{6.6b}$$

$$\mathbf{r}_Q = \mathbf{Q} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{6.7a}$$

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \mathbf{Q}^\top \begin{bmatrix} X - X_Q \\ Y - Y_Q \\ Z - Z_Q \end{bmatrix} \tag{6.7b}$$

$$\dot{\mathbf{r}}_O = \dot{\mathbf{r}}_{OQ} + \dot{\mathbf{r}}_Q \tag{6.8}$$

$$\dot{\mathbf{r}}_O = \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} \tag{6.9}$$

$$\dot{\mathbf{r}}_{OQ} = \begin{bmatrix} \dot{X}_Q \\ \dot{Y}_Q \\ \dot{Z}_Q \end{bmatrix} \tag{6.10}$$

$$\dot{\mathbf{r}}_Q = \begin{bmatrix} \dot{X} - \dot{X}_Q \\ \dot{Y} - \dot{Y}_Q \\ \dot{Z} - \dot{Z}_Q \end{bmatrix} \tag{6.11a}$$

$$= \dot{\mathbf{Q}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + \mathbf{Q} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} \tag{6.11b}$$

$$\ddot{\mathbf{r}}_O = \ddot{\mathbf{r}}_{OQ} + \ddot{\mathbf{r}}_Q \tag{6.12}$$

$$\ddot{\mathbf{r}}_O = \begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix} \tag{6.13}$$

$$\ddot{\mathbf{r}}_{OQ} = \begin{bmatrix} \ddot{X}_Q \\ \ddot{Y}_Q \\ \ddot{Z}_Q \end{bmatrix} \tag{6.14}$$

$$\ddot{\mathbf{r}}_Q = \begin{bmatrix} \ddot{X} - \ddot{X}_Q \\ \ddot{Y} - \ddot{Y}_Q \\ \ddot{Z} - \ddot{Z}_Q \end{bmatrix} \tag{6.15a}$$

$$= \ddot{\mathbf{Q}} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} + 2\,\dot{\mathbf{Q}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} + \mathbf{Q} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} \tag{6.15b}$$

6.2 Time derivatives of a rotation matrix Q

Let $\mathbf{Q}$ be a rotation matrix defined by equation 3.27, but with Euler angles that vary with time. In this case, the time derivative of the product $\mathbf{Q}\mathbf{Q}^\top$ (equation 3.17) at a given instant (of time) t satisfies

$$\dot{\mathbf{Q}}\mathbf{Q}^\top + \mathbf{Q}\dot{\mathbf{Q}}^\top = \mathbf{0}\,, \tag{6.16}$$

where $\mathbf{0}$ is a 3×3 matrix of zeros. By rearranging equation 6.16, we get

$$\dot{\mathbf{Q}}\mathbf{Q}^\top = -\left(\dot{\mathbf{Q}}\mathbf{Q}^\top\right)^\top\,, \tag{6.17}$$

which means that $\dot{\mathbf{Q}}\mathbf{Q}^\top$ is a *skew-symmetric* matrix $\mathbf{\Omega}$ (Definition 6) and, consequently,

$$\dot{\mathbf{Q}} = \mathbf{\Omega}\mathbf{Q}\,. \tag{6.18}$$

Definition 6. Skew-symmetric matrix

A *skew-symmetric* (or *antisymmetric*) matrix is a square matrix whose transpose equals its negative. That is, a matrix

$$\boldsymbol{\Omega} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{bmatrix} \tag{6.19}$$

that satisfies the condition

$$\boldsymbol{\Omega} = -\boldsymbol{\Omega}^{T} \, . \tag{6.20}$$

It means that the elements ω_{ij} (equation 6.19) of $\boldsymbol{\Omega}$ satisfies the condition

$$\omega_{ij} = \begin{cases} 0 & \text{if} \quad i = j \, , \\ -\omega_{ji} & \text{if} \quad i \neq j \, . \end{cases} \tag{6.21}$$

Now we redefine the elements of $\boldsymbol{\Omega}$ (equation 6.19) in terms of those forming $\mathbf{Q}$. Due to the orthogonality condition defined by equation 3.24, we obtain

$$\dot{\hat{\mathbf{p}}}_i^{\top} \hat{\mathbf{p}}_j = \begin{cases} 0 & \text{if} \quad i = j \, , \\ -\hat{\mathbf{p}}_i^{\top} \dot{\hat{\mathbf{p}}}_j & \text{if} \quad i \neq j \, . \end{cases} \tag{6.22}$$

By using the cross-product identity defined by equation 3.12j, we rewrite $\boldsymbol{\Omega}$ (equation 6.19) as follows:

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & -\omega_Z & \omega_Y \\ \omega_Z & 0 & -\omega_X \\ -\omega_Y & \omega_X & 0 \end{bmatrix} \, , \tag{6.23}$$

where

$$\omega_X = \dot{\hat{\mathbf{p}}}_3^{\top} \hat{\mathbf{p}}_2 \, , \tag{6.24a}$$

$$\omega_Y = \dot{\hat{\mathbf{p}}}_1^{\top} \hat{\mathbf{p}}_3 \, , \tag{6.24b}$$

$$\omega_Z = \dot{\hat{\mathbf{p}}}_2^{\top} \hat{\mathbf{p}}_1 \, , \tag{6.24c}$$

so that

$$\boldsymbol{\Omega} \, \boldsymbol{\upsilon} = \boldsymbol{\omega} \times \boldsymbol{\upsilon} \, , \tag{6.25}$$

where $\boldsymbol{\upsilon}$ is an arbitrary vector and

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_X \\ \omega_Y \\ \omega_Z \end{bmatrix} \tag{6.26}$$

is the instantaneous *angular velocity* (e.g., Goldstein et al., 1980, p. 172). From the physical point of view, the time derivative $\dot{\mathbf{Q}}$ (equation 6.18) defines an infinitesimal rotation that is represented by the skew-symmetric matrix $\boldsymbol{\Omega}$ (Definition 6 and equation 6.24), occurs between the instants t and $t + dt$ (dt being an infinitesimal time step) and has a rotation axis defined by $\boldsymbol{\omega}$ (equation 6.26). Then, equation 6.18 shows that $\dot{\mathbf{Q}}$ is a matrix whose columns are obtained by applying the infinitesimal rotation $\boldsymbol{\Omega}$ to the original columns of $\mathbf{Q}$ at the instant t. From equations 6.18 and 6.25, we get

$$\dot{\mathbf{Q}} = \begin{bmatrix} \dot{\hat{\mathbf{q}}}_1 & \dot{\hat{\mathbf{q}}}_2 & \dot{\hat{\mathbf{q}}}_3 \end{bmatrix} , \tag{6.27}$$

where

$$\dot{\hat{\mathbf{q}}}_j = \boldsymbol{\Omega}\,\hat{\mathbf{q}}_j = \boldsymbol{\omega} \times \hat{\mathbf{q}}_j \, , \quad j \in \{1, 2, 3\} \, . \tag{6.28}$$

It is important to bear in mind that, similarly to $\mathbf{Q}$, we consider that $\dot{\mathbf{Q}}$, $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$ (equations 6.18, 6.23 and 6.26) also vary with time. In this case, the time derivative of $\dot{\mathbf{Q}}$ (equation 6.18) can be written as follows:

$$\ddot{\mathbf{Q}} = \dot{\boldsymbol{\Omega}}\mathbf{Q} + \boldsymbol{\Omega}\dot{\mathbf{Q}} \, , \tag{6.29a}$$

$$= \dot{\boldsymbol{\Omega}}\mathbf{Q} + \boldsymbol{\Omega}\boldsymbol{\Omega}\mathbf{Q} \, , \tag{6.29b}$$

where the time derivative of $\boldsymbol{\Omega}$ (equation 6.23) is given by

$$\dot{\boldsymbol{\Omega}} = \begin{bmatrix} 0 & -\dot{\omega}_Z & \dot{\omega}_Y \\ \dot{\omega}_Z & 0 & -\dot{\omega}_X \\ -\dot{\omega}_Y & \dot{\omega}_X & 0 \end{bmatrix} , \tag{6.30}$$

with

$$\dot{\omega}_X = \ddot{\hat{\mathbf{p}}}_3^\top \hat{\mathbf{p}}_2 + \dot{\hat{\mathbf{p}}}_3^\top \dot{\hat{\mathbf{p}}}_2 \, , \tag{6.31a}$$

$$\dot{\omega}_Y = \ddot{\hat{\mathbf{p}}}_1^\top \hat{\mathbf{p}}_3 + \dot{\hat{\mathbf{p}}}_1^\top \dot{\hat{\mathbf{p}}}_3 \, , \tag{6.31b}$$

$$\dot{\omega}_Z = \ddot{\hat{\mathbf{p}}}_2^\top \hat{\mathbf{p}}_1 + \dot{\hat{\mathbf{p}}}_2^\top \dot{\hat{\mathbf{p}}}_1 \, , \tag{6.31c}$$

so that

$$\dot{\boldsymbol{\Omega}}\,\boldsymbol{v} = \dot{\boldsymbol{\omega}} \times \boldsymbol{v} \, , \tag{6.32}$$

where $\boldsymbol{v}$ is an arbitrary vector and

$$\dot{\boldsymbol{\omega}} = \begin{bmatrix} \dot{\omega}_X \\ \dot{\omega}_Y \\ \dot{\omega}_Z \end{bmatrix} \tag{6.33}$$

is the time derivative of the instantaneous angular velocity $\boldsymbol{\omega}$ (equation 6.26).

Note that equation 6.29b defines the second derivative $\ddot{\mathbf{Q}}$ in terms of $\mathbf{Q}$. This representation is particularly useful for the case in which we have a product $\ddot{\mathbf{Q}}\mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector. In this case, we use the identities 6.25 and 6.32 to obtain

$$\ddot{\mathbf{Q}}\mathbf{v} = \dot{\boldsymbol{\omega}} \times (\mathbf{Q}\mathbf{v}) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{Q}\mathbf{v})] \, . \tag{6.34}$$

6.3 3D equation of motion

Chapter 7

Inverse distance function

Geophysicists are commonly interested in determining the density distribution producing a given set of gravity disturbance data (Chapter 1) and/or the magnetization distribution producing a given set of crustal field data (Chapter 2). The estimated density distribution explaining the gravity disturbance data is then interpreted with the purpose of determining parameters associated with the true gravity sources. Similarly, the estimated magnetization distribution explaining the crustal field data are interpreted to obtain parameters associated with the true magnetic sources. While the gravity disturbance represents the gravitational field produced by the gravity sources, the crustal field is the magnetic induction produced by the magnetic sources. The mathematical-physical description of gravitational and magnetic induction fields depends on the *inverse distance function* and its spatial derivatives. In this chapter, this function and some of its properties are defined for all Terrestrial Reference Systems presented in Chapter 5.

7.1 Inverse distance function in the GCS

Definition

Let $P = (X, Y, Z)$ and $P' = (X', Y', Z')$ be two points defined in the Geocentric Cartesian System (GCS, Section 5.1). The position vectors $\mathbf{r}$ and $\mathbf{r}'$ (equation 4.3) associated with these points are given by:

$$\mathbf{r} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{7.1}$$

and

$$\mathbf{r}' = \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} . \tag{7.2}$$

Now, let $\boldsymbol{\ell}$ be the vector that starts at P' and ends at P,

$$\boldsymbol{\ell} = \mathbf{r} - \mathbf{r}' = \begin{bmatrix} X - X' \\ Y - Y' \\ Z - Z' \end{bmatrix} , \tag{7.3}$$

with magnitude (equations 3.2 and 3.7d) given by

$$\ell = \|\mathbf{r} - \mathbf{r}'\| = \sqrt{(X - X')^2 + (Y - Y')^2 + (Z - Z')^2} \, . \tag{7.4}$$

The magnitude ℓ defines the distance from P' to P, so that the *inverse distance function* is given by:

$$\frac{1}{\ell} = \frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{1}{\sqrt{(X - X')^2 + (Y - Y')^2 + (Z - Z')^2}} \, , \quad \ell \neq 0 \, , \tag{7.5}$$

in terms of the coordinates of P and P'. Note that the inverse distance function is not defined for $\ell = 0$ (equation 7.4) or, equivalently, for the case in which $\mathbf{r} = \mathbf{r}'$ (equations 7.1 and 7.2).

First derivatives and gradient vector

Using equations 4.22, 4.23 and 4.34, we define the gradient vector $\boldsymbol{\nabla}\frac{1}{\ell}$ of the inverse distance function (equation 7.5) with respect to the coordinates of P:

$$\boldsymbol{\nabla}\frac{1}{\ell} = \begin{bmatrix} \frac{\partial}{\partial X}\frac{1}{\ell} \\ \frac{\partial}{\partial Y}\frac{1}{\ell} \\ \frac{\partial}{\partial Z}\frac{1}{\ell} \end{bmatrix} \, , \quad \ell \neq 0 \, , \tag{7.6}$$

where

$$\frac{\partial}{\partial \alpha}\frac{1}{\ell} = -\frac{\alpha - \alpha'}{\ell^3} \, , \quad \alpha = X, Y, Z \, . \tag{7.7}$$

By substituting equation 7.7 into 4.34, we obtain:

$$\boldsymbol{\nabla}\frac{1}{\ell} = -\frac{\boldsymbol{\ell}}{\ell^3} = -\frac{1}{\ell^2}\,\hat{\boldsymbol{\ell}} \, , \quad \ell \neq 0 \, , \tag{7.8}$$

where $\hat{\boldsymbol{\ell}}$ is the unit vector

$$\hat{\boldsymbol{\ell}} = \frac{\boldsymbol{\ell}}{\ell} \, , \quad \ell \neq 0 \, , \tag{7.9}$$

with $\boldsymbol{\ell}$ and ℓ given by equations 7.3 and 7.4, respectively.

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