Course: Numerical Analysis for Machine Learning
Prof. E. Miglio - September 6th 2023
Duration of the exam: 2.5 hours.

Exercise 1

Load the image of the Karman vortices with the following commands

```
from matplotlib.image import imread
import matplotlib.pyplot as plt
import numpy as np
import os
plt.rcParams['figure.figsize'] = [16,8]

A =imread(os.path.join('.','Karman_vortex.jpg'))
X = np.mean(A,-1); # Convert RGB to grayscale

img = plt.imshow(X)
img.set.cmap('gray')
plt.show()
```

1. Compute the economy SVD.
2. Let $\mathbf{X}$ be the matrix representing the true image and $\tilde{\mathbf{X}}$ the approximation of rank r obtained using the SVD. Compute and plot the relative reconstruction error of the truncated SVD in the Frobenius norm as a function of the rank r. The expression of the relative reconstruction error is given by:

$$\frac{\|\mathbf{X} - \tilde{\mathbf{X}}\|_F}{\|\mathbf{X}\|_F}.$$

3. Square this error (and plot it) to compute the fraction of the missing variance as a function of r.
4. Find the rank $\tilde{r}$ for which the reconstruction captures 99% of the total variance.

Exercise 2

Generate 100 artificial data points (x_i, y_i) where x_i is randomly generated in the interval $[0,1]$ and $y_i = \cos(4\pi x_i) + \epsilon$; ϵ is a random noise in the interval $[-0.2, 0.2]$. Implement the SGD method to solve the regression problem for the data you have generated. Use an initial constant learning rate $\eta = 0.001$ an train a polynomial of the form $h_c = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n$ using your data (n is the maximum degree of the polynomial). Assume that all the initial parameters c_i are randomly generated in $[-0.4, 0.4]$. Try different values of n. Try also different values for α to speed up the learning process. Plot the various results and comment them.

Exercise 3

Consider a sigmoid neuron with 1D input x, weight w, bias b and output $y = \sigma(wx + b)$. The target is the variable z. Consider the cost function $J(w, b) = \frac{1}{2}(y - z)^2$.

- Find $\nabla J(w, b)$ and show that $\|\nabla J\| < \frac{1}{4}\sqrt{1 + x^2}(1 + |z|)$.
- Write the gradient descent iteration for the sequence (w_n, b_n).