An Essential

Mathematical Analysis

Handbook

Università degli studi di Roma "La Sapienza"
Physics and Astrophysics BSc

Matteo Cheri

Theorems, proofs and some examples for the Physics undergrad

March 4, 2020

Version $\int_{\mathbb{R}} \delta(x)\,\mathrm{d}x$

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An essential handbook on Mathematical Analysis for BSc students majoring in Physics

Written by

Matteo Cheri

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Introduction

The idea of writing this came from a mix of pure laziness and despair in preparing two exams, Analisi Vettoriale (Vector Analysis) and Metodi e Modelli Matematici per la Fisica (Mathematical Models and Methods for Physics, basically just pure Complex Analysis and Functional Analysis), that I have to sustain in my BSc course in Physics here at Sapienza University of Rome.
These two courses are deeply intertwined and it's really difficult to study them apart due to the sheer volume of things that either are done for half in one course and half in the other one, or they simply get generalized in the second, breaking up the logical flow one might get from studying these two seemingly completely different subjects.
There will surely be errors in grammar, typing and probably some mathematical inaccuracies so for any question, inquiry or you just want to say hi to me, don't wait in sending me an email here `cheri.1686219@studenti.uniroma1.it`.
A huge thanks to anyone of you that'll read this, I hope it will be useful for you as it is to me.

Notation

- $\mathbb{F}$ Ordered scalar field, $\mathbb{R}$ or $\mathbb{C}$ when not specified
- a_n, $(a)_n$ Sequence
- $\left((a)_k\right)_n$ Sequence of sequences
- $\mathbf{x}$, x^μ Vector
- $\mathbf{x} \wedge \mathbf{y}$, $\epsilon^\mu_{\nu\gamma} x^\nu y^\gamma$ Cross product of $\mathbf{x}, \mathbf{y}$
- ∇f, $\partial_\mu f$, $\frac{\partial f}{\partial x^\mu}$ Gradient of $\mathbf{f}$
- $\langle \nabla, \mathbf{f} \rangle$, $\mathrm{div}(\mathbf{f})$, $\partial_\mu f^\mu$, $\frac{\partial f^\mu}{\partial x^\mu}$ Divergence of $\mathbf{f}$
- $\nabla \wedge \mathbf{f}$, $\mathrm{rot}(\mathbf{f})$, $\epsilon^\mu_{\nu\gamma} \partial^\nu f^\gamma$ Rotor of $\mathbf{f}$
- $\mathbf{Jf}(\mathbf{x})$, $\partial_\mu f^\nu$, $\frac{\partial f^\mu}{\partial x^\nu}$ Jacobian matrix of $\mathbf{f}$
- $\mathbf{H}_f(\mathbf{x})$, $\partial_\mu \partial_\nu f$, $\partial^2_{\mu\nu} f$, $\frac{\partial^2 f}{\partial x^\mu \partial x^\nu}$ Hessian matrix of the function f
- $\forall^\dagger$ For almost all
- $C^\infty_c(\mathbb{R}) = \mathcal{K}$ Space of test functions, (smooth with compact support)
- $Z^m_f,\ S_f,\ P^m_f$ Sets of zeros of order m, singularities or poles of order m of a function f
- $a_n \to a$ Simple convergence
- $a_n \rightrightarrows a$ Uniform convergence
- $a_n \xrightarrow{\mathrm{A}} a$ Absolute convergence
- $a_n \xrightarrow{\mathrm{T}} a$ Total convergence
- $a_n \rightharpoonup_{\mathrm{w}} a$ Weak convergence
- $a_n \rightharpoonup_{\mathcal{K}} a$ $\mathcal{K}-$convergence
- $a_n \rightharpoonup_{\star} a$ $\mathcal{K}^\star-$convergence

1 Complex Numbers and Functions

§ 1.1 Complex Numbers

Definition 1.1.1 (Complex Numbers)**.** Define with $\mathbb{C}$ the set of *complex numbers*, i.e. the set of numbers $z \in \mathbb{C} : z = (x, y)$ and $x, y \in \mathbb{R}$.
We define the *real and imaginary parts* of z as follows

$$\begin{aligned} \mathfrak{Re}(z) &= x \\ \mathfrak{Im}(z) &= y \end{aligned} \tag{1.1}$$

Definition 1.1.2 (Operations in $\mathbb{C}$)**.** Take $z_1, z_2 \in \mathbb{C}$, then we define

$$\begin{aligned} z_1 = z_2 &\iff \mathfrak{Re}(z_1) = \mathfrak{Re}(z_2),\ \mathfrak{Im}(z_1) = \mathfrak{Im}(z_2) \\ z_1 + z_2 &= (\mathfrak{Re}(z_1) + \mathfrak{Re}(z_2), \mathfrak{Im}(z_1) + \mathfrak{Im}(z_2)) \\ z_1 z_2 &= (\mathfrak{Re}(z_1)\mathfrak{Re}(z_2) - \mathfrak{Im}(z_1)\mathfrak{Im}(z_2), \mathfrak{Re}(z_1)\mathfrak{Im}(z_2) + \mathfrak{Im}(z_1)\mathfrak{Re}(z_2)) \end{aligned}$$

Theorem 1.1. *With the previous definitions the set $\mathbb{C}$ forms a field.*

Definition 1.1.3 (Imaginary Unit)**.** We define the imaginary unit $i = (0, 1) \in \mathbb{C}$. From this definition and the definition of product of two complex numbers, we have that $i^2 = -1$
With this definition, we have

$$\forall z \in \mathbb{C} \quad z = \mathfrak{Re}(z) + i\mathfrak{Im}(z) \tag{1.2}$$

Definition 1.1.4 (Complex Conjugate)**.** Taken $z \in \mathbb{C}$, we call the *complex conjugate of z* the number w such that

$$w = \mathfrak{Re}(z) - i\mathfrak{Im}(z) \tag{1.3}$$

This number is denoted as $\overline{z}$

Definition 1.1.5 (Complex Module)**.** We define the *module* or *norm* of a complex number, the following operator.

$$\|z\| = \sqrt{z\overline{z}} = \sqrt{\mathfrak{Re}^2(z) + \mathfrak{Im}^2(z)} \tag{1.4}$$

Definition 1.1.6 (Complex Inverse)**.** The inverse of a complex number $z \in \mathbb{C}$ is defined as z^{-1} and it's calculated as follows

$$z^{-1} = \frac{\overline{z}}{\|z\|^2} \tag{1.5}$$

Definition 1.1.7 (Polar Form)**.** Taken a complex number $z \in \mathbb{C}$ one can define it in polar form with its modulus r and its *argument* θ. We have that, if $z = x + iy$

$$\begin{aligned} r &= \sqrt{x^2 + y^2} = \|z\|^2 \\ \tan(\theta) &= \frac{y}{x} \end{aligned} \tag{1.6}$$

Definition 1.1.8 (Principal Argument)**.** Taken $\arg(z) = \theta$ we can define two different arguments, due to the periodicity of the tan function.

1. $\mathrm{Arg}(z) \in (-\pi, \pi]$ called the *principal argument*
2. $\arg(z) = \mathrm{Arg}(z) + 2k\pi,\ k \in \mathbb{Z}$ called the *argument*

As a rule of thumb, using the previous definition of argument of a complex number $z = x + iy$, we have

$$\mathrm{Arg}(z) = \begin{cases} \arctan(y/x) - \pi & x < 0,\ y < 0 \\ \arctan(y/x) & x \geq 0,\ z \neq 0 \\ \arctan(y/x) + \pi & x < 0,\ y \geq 0 \end{cases} \tag{1.7}$$

Definition 1.1.9 ($\arg_+$)**.** Given $z \in \mathbb{C}$ we define the $\arg_+(z)$ as the only value of $\arg(z)$ such that $0 \leq \theta < 2\pi$.
In case we have a polydromic function, in order to specify we're using this argument, there will be a $+$ as index.
I.e. $\log_+(z), [z^a]^+, \sqrt{z}^+, \cdots$ and so on.

Theorem 1.2 (De Moivre Formula)**.** *A complex number $z \in \mathbb{C}$ in polar form can be written with complex exponential and sine and cosine function as follows.*

$$z = \|z\|^2 e^{i \arg z} = \|z\|^2 (\cos(\arg z) + i \sin(\arg z)) \tag{1.8}$$

This formula easily generalizes the calculus of exponentials of complex numbers. With this definition, it's obvious that the $n-$th root of a complex number $\sqrt[n]{z}$ has actually $n-1$ results, given the $2\pi-$periodicity of the $\arg(z)$ function.

§ 1.2 Regions in $\mathbb{C}$

Definition 1.2.1 (Line)**.** A line λ in $\mathbb{C}$, from z_1, z_2 can be written as follows

$$\lambda(t) = z_1 + t(z_2 - z_1) \quad t \in [0, 1] \tag{1.9}$$

If $t \in \mathbb{R}$ this defines the line lying between z_1, z_2. Its non-parametric representation is the following

$$\{\lambda\} := \left\{ z \in \mathbb{C} |\ \mathfrak{Im}\left(\frac{z - z_1}{z_2 - z_1} \right) = 0 \right\} \tag{1.10}$$

Where $z = \lambda(t)$.

Definition 1.2.2 (Circumference)**.** A circumference γ centered in a point $z_0 \in \mathbb{C}$ with radius R is defined as follows

$$\gamma(\theta) = z_0 + Re^{i\theta} \quad \theta \in [0, 2\pi] \tag{1.11}$$

Non parametrically, it can be defined as follows

$$\{\gamma\} := \{ z \in \mathbb{C} |\ \|z - z_0\| = R\} \tag{1.12}$$

§§ 1.2.1 Extended Complex Plane $\hat{\mathbb{C}}$

Definition 1.2.3 (Extended Complex Plane)**.** We define the *extended complex plane* $\hat{\mathbb{C}}$ as follows

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \tag{1.13}$$

This can be imagined by projecting $\mathbb{C}$ into the Riemann sphere centered in the origin.

Definition 1.2.4 (Points in $\hat{\mathbb{C}}$)**.** Given a point $z \in \mathbb{C},\ z = x + iy$ we can find its coordinates with the following transformation

$$\hat{z} = (xt, yt, 1 - t) \in \hat{\mathbb{C}} \tag{1.14}$$

Where the condition $\|\hat{z}\| = 1$ must hold, defining the value of $t \in \mathbb{R}$ Inversely, given $\hat{z} = (x_1, x_2, x_3) \in \hat{\mathbb{C}}$ one finds

$$z = \frac{x_1 + ix_2}{1 - x_3} \tag{1.15}$$

§ 1.3 Elementary Functions

Definition 1.3.1 (Exponential)**.** The exponential function $z \mapsto e^z$ with $z \in \mathbb{C}$ is defined as follows

$$e^z = e^{\mathfrak{Re}(z) + i\mathfrak{Im}(z)} = e^{\mathfrak{Re}\, z} \left(\cos(\mathfrak{Im}(z)) + i \sin(\mathfrak{Im}(z))\right) \tag{1.16}$$

This gives

$$\begin{aligned} \|e^z\| &= \left| e^{\mathfrak{Re}(z)} \right| \\ \arg(e^z) &= \mathfrak{Im}(z) + 2\pi k \quad k \in \mathbb{Z} \end{aligned} \tag{1.17}$$

We have therefore, for $z, w \in \mathbb{C}$

$$\begin{aligned} e^z e^w &= e^{z+w} \\ \frac{e^z}{e^w} &= e^{z-w} \end{aligned} \tag{1.18}$$

Definition 1.3.2 (Logarithm)**.** We define the logarithm function $z \mapsto \log z$ as follows

$$\log(z) = \log \|z\| + i \arg(z) \tag{1.19}$$

It's evident how this function has multiple values for the same z value, and therefore is known as a *polydromic function*, like the square root. We also define the principal branch of the logarithm as $\mathrm{Log}(z)$

$$\mathrm{Log}(z) = \log \|z\| + i\, \mathrm{Arg}(z) \tag{1.20}$$

Lastly we define the $\log_+(z)$ as follows

$$\log_+(z) = \log(\|z\|) + i \arg_+(z) \tag{1.21}$$

Definition 1.3.3 (Branch of the Logarithm)**.** A general branch of the log function is defined as the function $f(z) : D \subset \mathbb{C} \longrightarrow \mathbb{C}$ such that

$$e^{f(z)} = z \tag{1.22}$$

§§ 1.3.1 Complex Exponentiation

Definition 1.3.4 (Complex Exponential)**.** Taken $s, z \in \mathbb{C}$, we define the complex exponential a follows, taken z a variable

$$z^s = e^{s \log(z)} \quad z \neq 0 \tag{1.23}$$

Its derivative has the following value

$$\frac{\mathrm{d}}{\mathrm{d}z} z^s = s e^{(s-1)\log(z)} = s z^{s-1} \tag{1.24}$$

Alternatively, we define

$$s^z = e^{z \log(s)} \tag{1.25}$$

§§ 1.3.2 Properties of Trigonometric Functions

Definition 1.3.5 (Trigonometric Functions)**.** Using De Moivre's formula, we define

$$\begin{aligned} \sin(z) &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\ \cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \end{aligned} \tag{1.26}$$

Definition 1.3.6 (Hyperbolic Functions)**.** We define the hyperbolic functions as follows, given $z = iy$

$$\begin{aligned} \sinh(y) &= -i \sin(iy) \\ \cosh(y) &= \cos(iy) \end{aligned} \tag{1.27}$$

For a general value of z, we define

$$\begin{aligned} \sinh(z) &= \frac{1}{2}\left(e^{z} - e^{-z}\right) \\ \cosh(z) &= \frac{1}{2}\left(e^{z} + e^{-z}\right) \end{aligned} \tag{1.28}$$

Theorem 1.3 (Trigonometric Identities). *Given $z, z_1, z_2 \in \mathbb{C}$ we have*

$$\begin{aligned}
\sin^2(z) + \cos^2(z) = 1 \\
\sin(z_1 \pm z_2) &= \sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2) \\
\cos(z_1 \pm z_2) &= \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2) \\
\sin(z) &= \sin\left(\mathfrak{Re}(z)\right)\cosh\left(\mathfrak{Im}(z)\right) + i\cos\left(\mathfrak{Re}(z)\right)\sinh\left(\mathfrak{Im}(y)\right) \\
\cos(z) &= \cos\left(\mathfrak{Re}(z)\right)\cosh\left(\mathfrak{Im}(z)\right) - i\sin\left(\mathfrak{Re}(z)\right)\sinh\left(\mathfrak{Im}(y)\right) \\
\|\sin(z)\|^2 &= \sin^2\left(\mathfrak{Re}(x)\right) + \sinh^2\left(\mathfrak{Im}(y)\right) \\
\|\cos(z)\|^2 &= \cos^2\left(\mathfrak{Re}(x)\right) + \sinh^2\left(\mathfrak{Im}(y)\right) \\
\cosh^2(z) - \sinh^2(z) = 1 \\
\sinh(z_1 \pm z_2) &= \sinh(z_1)\cosh(z_2) \pm \cosh(z_1)\sinh(z_2) \\
\cos(z_1 \pm z_2) &= \cosh(z_1)\cosh(z_2) \pm \sinh(z_1)\sinh(z_2) \\
\sinh(z) &= \sinh\left(\mathfrak{Re}(z)\right)\cos\left(\mathfrak{Im}(z)\right) + i\cosh\left(\mathfrak{Re}(z)\right)\sin\left(\mathfrak{Im}(y)\right) \\
\cos(z) &= \cosh\left(\mathfrak{Re}(z)\right)\cos\left(\mathfrak{Im}(z)\right) + i\sinh\left(\mathfrak{Re}(z)\right)\sin\left(\mathfrak{Im}(y)\right) \\
\|\sin(z)\|^2 &= \sinh^2\left(\mathfrak{Re}(x)\right) + \sin^2\left(\mathfrak{Im}(y)\right) \\
\|\cos(z)\|^2 &= \cosh^2\left(\mathfrak{Re}(x)\right) + \sin^2\left(\mathfrak{Im}(y)\right)
\end{aligned} \tag{1.29}$$

Definition 1.3.7 (Inverse Trigonometric Functions). Given $z \in \mathbb{C}$ we define

$$\begin{aligned}
\arcsin(z) &= -i\log\left(iz + \sqrt{1 - z^2}\right) \\
\arccos(z) &= -i\log\left(z + i\sqrt{1 - z^2}\right) \\
\arctan(z) &= -\frac{i}{2}\log\left(\frac{i - z}{i + z}\right)
\end{aligned} \tag{1.30}$$

Definition 1.3.8 (Inverse Hyperbolic Functions). Given $z \in \mathbb{C}$ we define

$$\begin{aligned}
\operatorname{asinh}(z) &= \log\left(z + \sqrt{z^2 + 1}\right) \\
\arccos(z) &= \log\left(z + \sqrt{z^2 - 1}\right) \\
\operatorname{atanh}(z) &= \frac{1}{2}\log\left(\frac{1 + z}{1 - z}\right)
\end{aligned} \tag{1.31}$$

2 Abstract Spaces

§ 2.1 Metric Spaces

§§ 2.1.1 Topology

Definition 2.1.1 (Metric Space)**.** Let X be a non-empty set equipped with an application d, defined as follows

$$\begin{aligned} d\,:X\times X &\longrightarrow \mathbb{F} \\ (x,y) &\to d(x,y) \end{aligned} \tag{2.1}$$

Where $\mathbb{F}$ is an ordered field.
The couple (X,d) is said to be a *metric space*, if and only if $\forall x,y,z\in X$ the application d satisfies the following properties

1. $d(x,y)\geq 0$
2. $d(x,x)=0$
3. $d(x,y)=d(y,x)$
4. $d(x,y)\leq d(x,z)+d(z,y)$

Definition 2.1.2 (Ball)**.** Let (X,d) be a metric space. We then define the *open ball of radius* r, centered in x in X (B_r^X), and the *closed ball of radius* r centered in x ($\overline{B_r^X}$) as follows

$$\begin{aligned} B_r^X(x) &:= \{\, u\in X|\; d(u,x)<r\} \\ \overline{B_r^X}(x) &:= \{\, u\in X|\; d(u,x)\leq r\} \end{aligned} \tag{2.2}$$

When there won't be doubts on on where the ball is defined, the superscript indicating the set of reference will be omitted.

We're now ready to define the *topology* on a metric space

Definition 2.1.3 (Open Set)**.** Let (X,d) be a metric space, and $A\subseteq X$ a subset. A is said to be an *open set* if and only if

$$\forall x\in X\ \exists B_r^X(x)\subset A \tag{2.3}$$

Definition 2.1.4 (Complementary Set)**.** Let A be a generic set, then the set A^c is defined as follows

$$A^c := \{a \notin A\} \tag{2.4}$$

This set is said to be the *complementary set* of A.
It's also obvious that $A \cap A^c = \{\}$

Definition 2.1.5 (Closed Set)**.** Alternatively to the notion of open set, we can say that $E \subseteq X$ is a *closed set*, if and only if

$$\forall x \in E^c \cap X \ \exists B_r^X(x) \subset E^c \cap X \tag{2.5}$$

Remark. A set isn't necessarily open nor closed!

Proposition 1.

1. The set $B_r^X(x)$ is open
2. The set $\overline{B_r^X}(x)$ is closed

Proof. Let $A = B_r^X(x)$. If A is open, we have therefore, applying the definition of open set, that

$$\forall x \in A \ \exists \epsilon > 0 \ : \ B_\epsilon^X(x) \subset A$$

So

$$x_0 \in A \implies d(x, x_0) < r$$
$$\therefore \epsilon = r - d(x, x_0) > 0$$

Then, by definition of open ball we have

$$y \in B_\epsilon^X(x) \implies d(x, y) < \epsilon$$

Then, we can say that

$$d(y, x_0) \leq d(y, x) + d(x, x_0) < \epsilon + d(x, x_0) = r$$
$$\therefore y \in B_\epsilon^X(x) \implies y \in B_r^X(x_0) \subset A$$

The demonstration of the second point is exactly the same, whereby we take E as our closed ball and $A = E^c$ □

Proposition 2. Let (X, d)

1. The sets $\{\}, X$ are open
2. The sets $\{\}, X$ are closed
3. If $\{A_i\}_{i=1}^n$ is a collection of open sets, then $A = \bigcap_{i=1}^n A_i$ is open
4. If $\{C_i\}_{i=1}^n$ is a collection of closed sets, then $C = \bigcup_{i=1}^n C_i$ is closed
5. Let $I \subset \mathbb{N}$ be an index set, then
   (a) If $\{A_\alpha\}_{\alpha \in I}$ is a collection of open sets, then $B = \bigcup_{\alpha \in I} A_\alpha$ is open
   (b) If $\{C_\alpha\}_{\alpha \in I}$ is a collection of closed sets, then $D = \bigcap_{\alpha \in I} C_\alpha$ is closed

Proof. The first two statements are of easy proof. Let $B_\epsilon^X \subset \{\}$. This means that B_ϵ^X is empty and therefore $B_\epsilon^X = \{\}$, which makes it open by definition. Therefore we have that $\{\}^\mathrm{c} = X$, and X must be closed, but if we reason a bit, we can say that $\forall x \in X \; B_\epsilon^X(x) \subset X$, which means that X is open, thus $X^\mathrm{c} = \{\}$ must be closed.
Since we gave a proof for $\{\}$ and X being open, we have that these two sets are both open and closed. These two sets are said to be *clopen*.
For the other statements we use the De Morgan laws on set calculus, therefore we have

$$x \in \bigcap_{i=1}^n A_i \implies x \in A_i$$
$$\therefore \exists \epsilon_i \,:\, B_{\epsilon_i}^X(x) \subset A_i$$

Taking $\epsilon = \min_{i \in I} \epsilon_i$ we have

$$B_\epsilon^X(x) \subset B_\epsilon^X(x) \implies B_\epsilon^X(x) \subset \bigcap_{i=1}^n A_i = A$$

And A is open
If we let $C = A^\mathrm{c}$ we have that

$$C = A^\mathrm{c} = \left(\bigcap_{i=1}^n A_i\right)^\mathrm{c} = \bigcup_{i=1}^n A_i^\mathrm{c}$$
$$\therefore C \text{ is closed}$$

For the last two we proceed as follows

$$x \in A_\alpha \implies \exists \alpha_0 \in I \,:\, x \in A_{\alpha_0}$$
$$\therefore \exists \epsilon > 0 \,:\, B_\epsilon^X(x) \subset A_{\alpha_0} \subset \bigcup_{\alpha \in I} = B$$

For the last one, we use the De Morgan laws and the proposition is demonstrated □

Definition 2.1.6 (Internal Points, Closure, Border)**.** Let (X, d) be a metric space and $A \subset X$ a subset.
We define the following sets from A

1. $A^\circ = \bigcup_{\alpha \in I} G_\alpha$ is the set of internal points of A, where I is an index set and $G_\alpha \subset A$ are open
2. $\overline{A} = \bigcap_{\beta \in J} F_\beta$ is the closure of A, where J is another index set and $F_\beta \subset A$ are closed
3. $\partial A = \overline{A} \setminus A^\circ = \overline{A} \cup (A^\circ)^\mathrm{c}$ is the border of A

Proposition 3.

1. A is an open set iff $A = A^\circ$
2. A is closed iff $A = \overline{A}$
3. $A^\circ = \overline{(A^\circ)}^\mathrm{c}$
4. $\overline{A} = \left[(A^\mathrm{c})^\circ\right]^\mathrm{c}$

5. $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

6. $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof. Let $\mathcal{O}(A)$ be a collection of open sets, such that $\forall G \in \mathcal{O}(A) \implies G \subset A$, then

$$A = A^{\circ} \implies A = \bigcup_{G \in \mathcal{O}(A)} G$$

Therefore, being a union of a finite number of open sets, A is open.
For the same reason as before and the previous proposition, we have that $\overline{A}$ is closed
For the third proposition, we have

$$\left(\overline{A^{c}}\right)^{c} = \left(\bigcap_{A^{c} \subset F} F\right)^{c} = \bigcup_{A^{c} \subset F} F^{c} = \bigcup_{G \in \mathcal{O}(A)} G = A^{\circ}$$

The others are easily demonstrated throw this process, iteratively $\square$

Proposition 4. Let (X, d) be a metric space, and $A \subset X$, $x \in X$

1. $x \in A \iff \exists \epsilon > 0 \, : \, B_{\epsilon}(x) \subset A$

2. $x \in \overline{A} \iff \forall \epsilon > 0 \; B_{\epsilon}(x) \cap A \neq \{\}$

3. $x \in \partial A \iff \forall \epsilon > 0 \; B_{\epsilon}(x) \cap A \neq \{\} \wedge B_{\epsilon}(x) \cap \overline{A} \neq \{\}$

Proof. $\boxed{1}$ Let $I(A) := \{x \in X | \, \exists \epsilon > 0 \, : \, B_{\epsilon}(x) \subset A$.

$$x \in I(A) \implies \exists \epsilon > 0 \, : \, B_{\epsilon}(x) \subset A, \; \therefore x \in \bigcup_{G \subset A} G$$

But

$$\begin{aligned} &x \in A^{\circ} \implies \exists G \subset X \text{ open} : x \in G \implies \exists \epsilon > 0 \, : \, B_{\epsilon}(x) \subset G \subset A \\ &\therefore A^{\circ} \subset I(A) \ni x, \; I(A) \subset A \text{ by definition}, \; \therefore I(A) = A^{\circ} \end{aligned}$$

$\boxed{2}$ For the second proposition, we have

$$\begin{aligned} &\overline{A} = \left[(A^{c})^{\circ}\right]^{c} \implies x \in A \iff x \in (A^{c})^{\circ} \implies \forall \epsilon > 0 \; B_{\epsilon}(x) \not\subset A^{c} \\ &\therefore \forall \epsilon > 0 \; B_{\epsilon}(x) \cap A \neq \{\} \end{aligned}$$

$\boxed{3}$ For the last one, we have, taking into account the first two proofs

$$\begin{aligned} &x \in \partial A \iff x \in \overline{A} \setminus A^{\circ} \implies x \in \overline{A} \wedge x \notin A^{\circ} \\ &\boxed{1} \wedge \boxed{2} \implies x \in \overline{A} \iff \forall \epsilon > 0 \; B_{\epsilon}(x) \cap A \neq \{\} \\ &\therefore x \notin A^{\circ} \iff \forall \epsilon > 0 \; B_{\epsilon}(x) \cap A^{c} \neq \{\} \end{aligned}$$

$\square$

Definition 2.1.7 (Isometry)**.** Let $(X, d), (Y, \rho)$ be two metric spaces and f an application, defined as follows

$$f : (X, d) \to (Y, d)$$

f is said to be an *isometry* iff

$$\forall x_1, x_2 \in X,\ \rho(f(x_1), f(x_2)) = d(x_1, x_2)$$

Remark. If f is an isometry, then f is injective, but it's not necessarily surjective

Example 2.1.1. Let $X = [0, 1]$ and $Y = [0, 2]$, therefore

$$\begin{aligned} f :& [0, 1] \to [0, 2] \\ & x \to f(x) = x \end{aligned}$$

f is obviously an isometry, since, for $x, y \in [0, 1]$

$$d(f(x), f(y)) = d(x, y)$$

But it's obviously not surjective.

Definition 2.1.8 (Diameter of a Set)**.** Let A be a set and the couple (A, d) be a metric space. We define the *diameter* of A as follows

$$\operatorname{diam}(A) = \sup_{x,y\in A} (d(x, y))$$

§ 2.2 Convergence and Compactness

Definition 2.2.1 (Convergence)**.** Let (X, d) be a metric space and $x \in X$. A sequence $(x_k)_{k\geq 0}$ in X is said to converge in X and it's indicated as $x_k \to x \in X$, iff

$$\forall \epsilon > 0\ \exists N > 0 : \forall k \geq N,\ d(x_k, x) < \epsilon\ \therefore \lim_{k\to\infty} x_k = x$$

Theorem 2.1 (Unicity of the Limit)**.** *Let (X, d) be a metric space and $(x_k)_{k\geq 0}$ a sequence in X. If $x_k \to x \wedge x_k \to y$, then $x = y$*

Definition 2.2.2 (Adherent point)**.** Let (X, d) be a metric space and $A \subset X$. $x \in X$ is said to be an *adherent point* of A if $\exists (x_k)_{k\geq 0} \in A : x_k \to x \in X$. The set of all adherent points of A is called $\operatorname{ad}(A)$

Definition 2.2.3 (Accumulation point)**.** Let (X, d) be a metric space and $A \subset X$. $x \in X$ is an *accumulation point* of A, or also *limit point* of A if $\exists (x_k)_{k\geq 0} : x_k \neq x \wedge x_k \to x \in \operatorname{ad}(A)$

Proposition 5. Let (X, d) be a metric space and $A \subset X$, then $\overline{A} = \operatorname{ad}(A)$

Proof. Let $Y = \operatorname{ad}(A)$, then

$$\begin{aligned} & x \in \overline{A} \implies \forall \epsilon > 0\ B_\epsilon(x) \cap A \neq \{\} \\ & \therefore \forall n \in \mathbb{N}\ B_{\frac{1}{n}}(x) \cap A \neq \{\} \implies \forall n \in \mathbb{N}\ \exists x_n \in B_{\frac{1}{n}}(x) \end{aligned}$$

But $d(x, x_n) < n^{-1}$, therefore $x \in Y \implies x \in \mathrm{ad}(A)$, and by definition

$$\exists (x_n)_{n\geq 0} \,:\, \forall \epsilon > 0 \; \exists N \in \mathbb{N} \,:\, \forall k \geq N \; d(x_k, x) < \epsilon \implies x_N \in B_\epsilon(x) \;\; \therefore x_N \in A$$
$$\therefore \forall \epsilon > 0 \; x_N \in B_\epsilon(x) \cap A \neq \{\} \implies x \in \overline{A} \implies Y \subset \overline{A}, \; \therefore Y = \mathrm{ad}(A) = \overline{A}$$

□

Proposition 6. Let (X, d) be a metric space and $A \subset X$. Then A is closed iff $\exists (x_k)_{k\geq 0} \in A \,:\, x_k \to x \in \overline{A} \implies \mathrm{ad}(A) \subset A$

Definition 2.2.4 (Dense Set)**.** Let (X, d) be a metric space and $A, B \subset X$. A is said to be dense in B iff $B \subset \overline{A}$, therefore $\forall \epsilon > 0 \; \exists y \in A \,:\, d(x, y) < \epsilon$. One example for this is $\mathbb{Q} \subset \mathbb{R}$, with the usual euclidean distance defined through the modulus.

Definition 2.2.5. Let (X, d) be a metric space and $(x_k)_{k\geq 0} \in X$. The sequence x_k is said to be a *Cauchy sequence* iff

$$\forall \epsilon > 0 \; \exists N > 0 \,:\, \forall k, n \geq N \; d(x_k, x_n) < \epsilon$$

Proposition 7. Let (X, d) be a metric space and $(x_k)_{k\geq 0} \in X$ a sequence. Then, if $x_k \to x$, x_k is a Cauchy sequence

Definition 2.2.6 (Complete Space)**.** Let (X, d) be a metric space. (X, d) is said to be *complete* iff $\forall (x_k)_{k\geq 0} \in X$ Cauchy sequences, we have $x_k \to x \in X$

Theorem 2.2 (Completeness)**.** *Let (X, d) be a metric space and $Y \subset X$. (Y, d) is complete iff $Y = \overline{Y}$ in X*

Proof. Let (Y, d) be a complete space, then

$$(x_k) \in Y \text{ Cauchy sequence } \implies \exists y \in Y \,:\, x_k \to y$$

Let $z \in \mathrm{ad}(A)$ and η_k a subsequence of x_k, then

$$\exists (\eta_k) \in Y \,:\, \eta_k \to z \implies \exists y \in Y \,:\, \eta_k \to y \;\; \therefore z = y \implies \mathrm{ad}(Y) \subset Y$$

Going the opposite way we have that $\mathrm{ad}(Y) = Y$ and therefore $Y = \overline{Y}$ □

Definition 2.2.7 (Compact Space)**.** A metric space (X, d) is said to be *compact* or *sequentially compact* if

$$\forall (x_k) \in X \; x_k \to x \in X, \exists (y_k) \text{ Subsequence } : \, y_k \to y \in X$$

Theorem 2.3. *Let (X, d) be a compact space. Then (X, d) is also complete*

Proof. (X, d) is compact, therefore

$$\forall (x_k) \in X \text{ Cauchy sequence } \implies x_k \to x \in X$$

Taken $(x_{n_k})_k \in X$ a subsequence, we have

$$x_k \to x \implies x_{n_k} \to x \in X$$

□

Definition 2.2.8 (Completely Bounded)**.** Let (X, d) be a metric space. X is *totally bounded* iff

$$\exists Y \subset X \, : \, \forall \epsilon > 0, \forall x \in Y \; X = \bigcup_{i=1}^{n} B_\epsilon(x)$$

Definition 2.2.9 (Poligonal Chain)**.** Let $z, w \in \mathbb{C}$. We define a *polygonal* $[z, w]$ as follows

$$[z, w] := \{\, z, w \in \mathbb{C} |\; z + t(w - z),\;\; t \in [0, 1] \subset \mathbb{R}\}$$

A *polygonal chain* will be indicated as follows $P_{z,w}$ and it's defined as follows

$$P_{z,w} = \bigcup_{k=1}^{n-1} [z_k, z_{k+1}] = [z, z_1, \cdots, z_{n-1}, w]$$

It can also be defined analoguously for every metric space $(X, d) \neq (\mathbb{C}, \|\cdot\|)$, where $\|\cdot\| : \mathbb{C} \to \mathbb{R}$ is the usual complex norm $\|z\| = \sqrt{z\overline{z}} = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

Definition 2.2.10 (Connected Space)**.** Let (G, d) be a metric space, G is *connected* if

$$\forall z, w \in G \; \exists P_{z,w} \subset G$$

Definition 2.2.11 (Contraction Mapping)**.** Let (X, d) be a complete metric space. Let $T : X \longrightarrow X$. T is said to be a *contraction mapping* or *contractor* if

$$\forall x, y \in X \; \exists q \in [0, 1) \, : \, d(T(x), T(y)) \leq q d(x, y) \tag{2.6}$$

Note that a contractor is necessarily continuous.

Theorem 2.4 (Banach Fixed Point)**.** *Let (X, d) be a complete metric space, with $X \neq \{\}$ and equipped with a contractor $T : X \longrightarrow X$. Then*

$$\exists! x^\star \in X \, : \, T(x^\star) = x^\star \tag{2.7}$$

Proof. Take $x_0 \in X$ and a sequence $x_n : \mathbb{N} \longrightarrow X$, where

$$x_n = T(x_{n-1}), \quad \forall n \in \mathbb{N}$$

It's obvious that

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq q d(x_n, x_{n-1}) \leq q^n d(x_1, x_0)$$

We need to prove that x_n is a Cauchy sequence. Let $m, n \in \mathbb{N} \, : \, m > n$, then

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \leq q^{m-1} d(x_1, x_0) + \cdots + q^n d(x_1, x_0)$$

Regrouping, we have

$$d(x_m, x_n) \leq q^n d(x_1, x_0) \sum_{k=0}^{m-n-1} q^k \leq q^n d(x_1, x_0) \sum_{k=0}^{\infty} q^k = q^n d(x_1, x_0) \left(\frac{1}{1-q} \right)$$

By definition of convergence, we have then

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} : \forall n > N\ d(s_n, s) < \epsilon$$

Then

$$\frac{q^n d(x_1, x_0)}{1-q} < \epsilon \implies q^n < \frac{\epsilon(1-q)}{d(x_1, x_0)}, \quad \forall n > N$$

Therefore, after taking $m > n > N$, we have

$$d(x_m, x_n) < \epsilon$$

Therefore x_n is a Cauchy sequence. Since (X, d) is a complete metric space, this sequence must have a limit $x_n \to x^\star \in X$, but, by definition of convergence and limit, we have that by continuity

$$x^\star = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1}) = T\left(\lim_{n\to\infty} x_{n-1}\right) = T(x^\star)$$

This point is unique. Take $y^\star \in X$ such that $T(y^\star) = y^\star \neq x^\star$, then

$$0 < d(T(x^\star), T(y^\star)) = d(x^\star, y^\star) > qd(x^\star, y^\star) \quad ↯$$

Therefore

$$\exists! x^\star \in X : T(x^\star) = x^\star$$

And $x^\star$ is the fixed point of the contractor T □

§ 2.3 Vector Spaces

Definition 2.3.1 (Vector Space)**.** A vector space $\mathcal{V}$ over a field $\mathbb{F}$ is a set, where $\mathcal{V} \neq \{\}$ and it satisfies the following properties, $\forall u, v, w \in \mathcal{V}$ and $a, b \in \mathbb{F}$

1. $u + v \in \mathcal{V}$ sum closure
2. $av \in \mathcal{V}$ scalar closure
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $\exists! 0 \in \mathcal{V} : u + 0 = 0 + u = u$
6. $\exists! v \in \mathcal{V} : u + v = 0 \implies v = -u$
7. $\exists! 1 \in \mathcal{V} : 1 \cdot u = u$
8. $(ab)u = a(bu) = b(au) = abu$
9. $(a + b)u = au + bu$
10. $a(u + v) = au + av$

Definition 2.3.2 (Norm)**.** Let $\mathcal{V}$ be a vector space over a field $\mathbb{F}$, then the *norm* is an application defined as follows

$$\|\cdot\| : \mathcal{V} \longrightarrow \mathbb{F}$$

Where it satisfies the following properties

1. $\|u\| \geq 0 \; \forall u \in \mathcal{V}$
2. $\|u\| = 0 \iff u = 0$
3. $\|cu\| = |c|\|u\| \; \forall u \in \mathcal{V} \; c \in \mathbb{F}$
4. $\|u + v\| \leq \|u\| + \|v\| \; \forall u, v \in \mathcal{V}$

Definition 2.3.3 (Normed Vector Space)**.** A *normed vector space* is defined as a couple $(\mathcal{V}, \|\cdot\|)$, where $\mathcal{V}$ is a vector space over a field $\mathbb{F}$.

Proposition 8. A normed vector space (NVS), is also a metric vector space (MVS) if we define our distance as follows

$$d(u, v) = \|u - v\| \; \forall u, v \in \mathcal{V}$$

Definition 2.3.4 (Vector Subspace)**.** Let $\mathcal{V}$ be a vector space and $\mathcal{U} \subset \mathcal{V}$. $\mathcal{U}$ is a *vector subspace* of $\mathcal{V}$ iff

1. $u, v \in \mathcal{U} \implies u + v \in \mathcal{U}$
2. $u \in \mathcal{U}, \; a \in \mathbb{F} \implies au \in \mathcal{U}$

Proposition 9. If $(\mathcal{V}, \|\cdot\|)$ is an normed vector space and $\mathcal{W} \subset \mathcal{V}$ is a subspace of $\mathcal{V}$, then $(\mathcal{W}, \|\cdot\|)$ is a normed vector space

Definition 2.3.5 (p-norm)**.** Let $(\mathcal{V}, \|\cdot\|_p)$ be a normed vector space. The norm $\|\cdot\|_p$ is said to be a *p-norm* if it's defined as follows

$$\|v\|_p := \left(\sum_{i=1}^{\dim(\mathcal{V})} (v_i)^p \right)^{\frac{1}{p}}, \; \forall v \in \mathcal{V}, \; \forall p \in \mathbb{N}^\star := \mathbb{N} \cup \{\pm\infty\} \tag{2.8}$$

Setting $p = \infty$ we have that

$$\|v\|_\infty = \max_{i \leq \dim(\mathcal{V})} |v_i| \tag{2.9}$$

Definition 2.3.6 (Dual Space)**.** Let $\mathcal{V}$ be a vector space over the field $\mathbb{F}$, we define a *linear functional* as an application $\varphi : \mathcal{V} \longrightarrow \mathbb{F}$ such that $\forall u, v \in \mathcal{V}$ and $c \in \mathbb{F}$

$$\begin{aligned} \varphi(u + v) &= \gamma(u) + \varphi(v) \\ \varphi(\lambda u) &= \lambda \varphi(u) \end{aligned} \tag{2.10}$$

Defining the sum of two linear functionals as $(\varphi_1 + \varphi_2)(v) = \varphi_1(v) + \varphi_2(v)$ we immediately see that the set of all linear functionals forms a vector space over $\mathcal{V}$, which will be called the *dual space* $\mathcal{V}^\star$.

§§ 2.3.1 Hölder and Minkowski Inequalities

Having defined p-norms, we can prove two inequalities that work with these norms, the *Minkowski inequality* and the *Hölder Inequality*

Theorem 2.5 (Hölder Inequality)**.** *Let $p_q \in \mathbb{N}^\star$, where*

$$\frac{1}{p} + \frac{1}{q} = 1$$

Then

$$\forall x, y \in \mathbb{R}^n \ \ \|x\|_p \|y\|_q \geq \sum_{k=1}^{n} |x_k y_k| \tag{2.11}$$

Proof. Taking $p = 1$, we have $q = \infty$, and the demonstration is obvious

$$\|x\|_p \|y\|_q = \|x\|_1 \|p\|_\infty = \max_{k \leq n} |y_k| \sum_{k=1}^{n} |x_k| \geq \sum_{k=1}^{n} |x_k y_k|$$

Else, if $p > 1$, we ave that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \ \forall a, b \geq 0$$

Let

$$s = \frac{x}{\|x\|_p}, \ t = \frac{y}{\|y\|_q}$$

We have

$$\sum_{k=1}^{n} \|s\|^p = \frac{1}{\|x\|_p^p} \sum_{k=1}^{n} |x_k|^p = 1 = \sum_{k=1}^{n} |t|^q = \frac{1}{\|y\|_q^q} \sum_{k=1}^{n} |y|^p$$

Therefore

$$\sum_{k=1}^{n} |s_k t_k| \leq \frac{1}{p} \sum_{k=1}^{n} |s_k|^p + \frac{1}{q} \sum_{k=1}^{n} |t_k|^q$$

Substituting again the definitions of s, t we have

$$\sum_{i=1}^{n} |y_k x_k| = \|x\|_p \|y\|_q \sum_{k=1}^{n} |s_k t_k| \leq \|x\|_p \|y\|_q$$

□

Theorem 2.6 (Minkowski Inequality)**.** *Let $p \geq 1$, therefore $\forall x, y \in \mathbb{R}^n$ we have*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p \tag{2.12}$$

Proof. We begin by writing explicitly the p-norm

$$\|x + y\|_p^p = \sum_{k=1}^{n} (|x_k| + |y_k|)^p = \sum_{k=1}^{n} (|x_k| + |y|_k) (|x_k| + |y_k|)^{p-1}$$

Letting $u_k = (|x_k| + |y_k|)^{p-1}$ we have, after imposing the condition on q of the p-norm as $q(p+1) = p$ and using that the sum is Abelian, we have

$$\begin{cases} \displaystyle\sum_{k=1}^{n} |x_k| u_k \leq \|x\|_p \|u\|_q = \|x\|_p \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \\ \displaystyle\sum_{k=1}^{n} |y_k| u_k \leq \|y\|_p \|u\|_q = \|y\|_p \left(\sum_{k=1}^{n} (|x_k| + |y_k|)^p \right)^{\frac{1}{q}} \end{cases}$$

Therefore, summing and imposing that $1 - q^{-1} = p$ we have that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_q$$

$\square$

3 Differential Analysis

§ 3.1 Digression on the Notation Used

In this chapter (and from now on, mostly), we will use a notation which is called *abstract index notation* with the *Einstein summation convention*. This is usually abbreviated in common literature as the *Einstein index notation*. We will give here a brief explanation of how this notation actually works, and why it's so useful in shortening mathematical expressions. Let $\mathcal{V}$ be a vector space and $\mathcal{V}^\star$ be the dual space associated with $\mathcal{V}$. Then we can write the elements $v \in \mathcal{V}$, $\varphi \in \mathcal{V}^\star$ with respect to some basis as follows

$$\begin{aligned} v &= (v_1, v_2, \cdots, v_n) = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \\ \varphi &= (\varphi_1, \varphi_2, \cdots, \varphi_n) = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix} \end{aligned} \tag{3.1}$$

The first notation is the *ordered tuple* notation, meanwhile the second notation is the usual column/row notation for vectors utilized in linear algebra. In Einstein notation we will have that

$$\begin{aligned} v &\longrightarrow v^i \\ \varphi &\longrightarrow \varphi_i \end{aligned} \tag{3.2}$$

Where the vector in the space will be indicated with a raised index (*index, not power!*) and the covector with a lower index, where the index will span all the values $i = 1, \cdots, \dim(\mathcal{V})$.
Let's represent the scalar product in Einstein notation. Let's say that we want to write the scalar product $\langle v, v \rangle$

$$\langle v, v \rangle = \sum_{i=1}^{\dim(\mathcal{V})} v_i v_i \longrightarrow v_i v^i \tag{3.3}$$

Note how we have omitted the sum over the repeated index. Now one might ask why it's not written as $v_i v_i$ (or $v^i v^i$, since $v \in \mathcal{V}$), and this is easily explained introducing the matrix g_{ij}, which is the matrix of the scalar product.
Applying this matrix to v^i we have $g_{ij} v^i$. Note how the low index j is free and i is being summed over, hence is a dummy index, this means that the result must have a lower index j for consistency. So we can write $v_j = g_{ij} v^i$, and due to the lower index we already know that this is a covector,

i.e. a linear functional $\mathcal{V} \longrightarrow \mathbb{F}$, hence it will "eat" a vector and "spew" a scalar (with no indices!). Feeding to this covector the vector v^j we have finally

$$\langle v, v \rangle = v_j v^j = g_{ij} v^i v^j \tag{3.4}$$

Where, algebraically we have "omitted" the definition of $\iota(\cdot) = \langle v, \cdot \rangle$, which is the canonical isomorphism between $\mathcal{V}$ and $\mathcal{V}^\star$.
With this definition we have defined what mathematically are called *musical isomorphisms*, applications which raise and lower indexes. Ironically, this operation is called *index gymnastics*, since we're raising and lowering indices. Thanks to these conventions operations with matrices (and tensors) become much much easier. Let a^i_j and b^i_j be two $n \times n$ matrices over the ordered field $\mathbb{F}$. The multiplication of these two matrices will simply be

$$c^i_j = a^i_k b^k_j \tag{3.5}$$

Note how the k index gets "eaten". This mathematical cannibalism is called *contraction of the index* k. So, the trace for a matrix a^i_j will be

$$\mathrm{tr}(a) = a^i_i \tag{3.6}$$

And now comes the tricky part. In order to write determinants we need to define a symbol, the so called *Levi-Civita symbol*, $\epsilon_{i_1 \dots i_n}$. In three dimensions it's ϵ_{ijk}, and follows the following rules

$$\epsilon_{ijk} = \begin{cases} 1 & \text{even permutation of the indices} \\ -1 & \text{uneven permutation of the indices} \\ 0 & i = j \vee j = k \vee k = i \end{cases} \tag{3.7}$$

In n dimensions, it becomes

$$\epsilon_{i_1 \dots i_k} = \begin{cases} 1 & \text{even permutation of indices} \\ -1 & \text{uneven permutation of indices} \\ 0 & i_i = i_j \text{ for some } i, j \end{cases} \tag{3.8}$$

It's obvious by definition that this weird entity is completely antysimmetrical and unitary, and therefore it's perfect for representing permutations (it's also known as permutation symbol for a reason). Therefore, remembering the definition of the determinant we can write, for an $n \times n$ matrix a^i_j

$$\det(a) = \epsilon_{i_1 \dots i_n} a^{1 i_1} a^{2 i_2} \cdots a^{n i_n} = \epsilon_{i_1 \dots i_n} g^{j i_1} a^1_j g^{k i_2} a^2_k \cdots g^{l i_n} a^n_{i_n} \tag{3.9}$$

If $\dim(\mathcal{V}) = 3$, we can therefore immediately define the cross product of two vectors as follows

$$\mathbf{c} = \mathbf{v} \times \mathbf{w} \longrightarrow c^i = g^{ij} \epsilon_{jkl} v^k w^l \tag{3.10}$$

(Note how we had to raise the index i).
From now on, we will start to use Greek letters for indices and Latin letters for labels, in order to avoid confusions, simply look again at the formula for the determinant, it's much clearer this way. In fact, letting $\mu, \nu, \cdots$ be our indices and $i, j, k, \cdots$ our labels, we can write, for a matrix A^μ_ν

$$\det(A) = A = \epsilon_{\mu_1 \dots \mu_n} g^{\nu \mu_1} g^{\sigma \mu_2} \cdots g^{\zeta \mu_n} A^1_\nu A^2_\sigma \cdots A^n_\zeta \tag{3.11}$$

See? Much clearer, at least in my opinion.
Now we might want to understand how to write norms with this notation. For the usual Euclidean norm it's quite easy. So we can easily write

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \longrightarrow \sqrt{v_\mu v^\mu} \tag{3.12}$$

In case we have a vector function $f^\mu(x^\nu)$, the following notation will be used

$$\|\mathbf{f}(\mathbf{x})\| \longrightarrow \sqrt{f_\mu f^\mu(x^\nu)} \tag{3.13}$$

Or, for the sum of two functions $g^\mu(x^\nu) \pm f^\mu(y^\nu)$

$$\|\mathbf{g}(\mathbf{x}) \pm \mathbf{f}(\mathbf{y})\| \longrightarrow \sqrt{g_\mu g^\mu(x^\nu) + f_\mu f^\mu(y^\nu) \pm 2g_\mu(x^\nu) f^\mu(y^\nu)} \tag{3.14}$$

Or

$$\|\mathbf{g}(\mathbf{x}) \pm \mathbf{f}(\mathbf{y})\| \longrightarrow= \sqrt{g_{\mu\nu}(g^\mu(x^\gamma) \pm f^\mu(y^\gamma))(g^\nu(x^\gamma) \pm f^\nu(y^\gamma))}$$

A shorthand notation can be created by directly using the norm symbol, but with the contracted index in the upper or lower position as follows

$$\|\mathbf{g}(\mathbf{x}) \pm \mathbf{f}(\mathbf{y})\| \longrightarrow \|g^\mu(x^\nu) \pm f^\mu(y^\nu)\|_\mu \tag{3.15}$$

For p-norms we have to watch out for a little detail. We have to add a square root in order to "fix" the squaring of every element. So we get

$$\|\mathbf{v}\|_p \longrightarrow \sqrt[p]{(v_\mu)^{\frac{p}{2}}(v^\mu)^{\frac{p}{2}}} = \left((v_\mu)^{\frac{p}{2}}(v^\mu)^{\frac{p}{2}}\right)^{\frac{1}{p}} \tag{3.16}$$

Theorem 3.1. $\left((v_\mu v^\mu)^{p/2}\right)^{1/p}$ *is wrong*

Proof. It's easy to see why it doesn't work by expanding the sum on μ

$$\begin{aligned} (v_\mu v^\mu)^p &= \left(v_1 v^1 + v_2 v^2 + \cdots + v_n v^n\right)^p \\ (v_\mu)^{\frac{p}{2}}(v^\mu)^{\frac{p}{2}} &= \left((v_1 v^1)^p + (v_2 v^2)^p + \cdots + (v_n v^n)^p\right) \end{aligned} \tag{3.17}$$

□

Moreover, it's time to bring down some formal rules for the usage of this notation

Theorem 3.2 (Rules for Index Calculus in Einstein Notation)**.**

1. *Free indices must be consistent in both sides of the equation. I.e.* $a^\mu_\nu b_{\mu\gamma\delta} = c_{\nu\gamma\delta}$*.* $a^\mu_\nu b_{\mu\gamma\delta} \neq c^{\gamma}_{\nu\delta}$*,* $a^\mu_\nu b_{\mu\gamma\delta} \neq c_{\nu\gamma\sigma}$

2. *An index can be repeated* only two times per factor and must be contracted diagonally*. I.e.* $a^\mu b_\mu f^\delta_\gamma = c^\delta_\gamma$ *is defined correctly,* $a_\mu b_\mu$*,* $a^\mu b^\mu$ *or* $a^\mu b_\mu f^\mu_\gamma$ *are ill defined*

3. *Dummy indices can be replaced at will, since they don't contribute to the "index equation"*

§§ 3.1.1 Differential Operators

Differential operators will be defined formally in the next sections, but for now we will simply explain how they actually work with this notation (and what are the advantages of such), alongside the usual boldface notation.
We will begin by defining the derivative along the coordinate vectors (usually indicated with x^μ).
We will use the differential operator *del* (∂).
This operator will be used as follows

1. If there is no ambiguity for the coordinate system, the derivative alongside the coordinates x^μ will be indicated as ∂_μ
2. In case of ambiguity, something will be added in order to distinguish the operators. I.e. let (x^μ, y^ν) be our coordinate system, then we will have ∂_{x^μ} or ∂_{y^ν}
3. In every single case, even the differential operator *must follow the index calculus rules*

Now let $f(x^\mu)$ be some (scalar, there are no free indices) function of the coordinates. The derivative (or gradient, it will soon be defined properly) can be written in various ways. In boldface notation it's usual to indicate this as ∇f, which can be translated as follows

$$\nabla f \longrightarrow \partial_\mu f = \frac{\partial f}{\partial x^\mu} = f_{,\mu} \tag{3.18}$$

Note how in the RHS it's obvious that this quantity must be a vector due to the free index. The last one is the *comma notation* for derivation, used for compacting (even more) the notation (Also check how in the second notation, even if the index is raised, it behaves as a lower index, we will check deeply this part in the section on differential forms).
Now comes the fun part. Higher order derivatives.
For the same function, we can define the *Hessian matrix* (the matrix of second derivatives) **Hf**, simply applying two times the ∂ operator

$$\mathbf{Hf} \longrightarrow \partial_\nu \partial_\mu f = \partial^2_{\mu\nu} f = \partial_{\mu\nu} f = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu} = f_{,\mu\nu} \tag{3.19}$$

Derivatives of order > 2 can then be defined recursively.
Now we might ask, what if we have a vector field F^μ? Nothing changes. We simply have to remember to not repeat indices in order not to represent scalar products.
We have **JF** as the *Jacobian matrix* of F^μ, basically the derivative matrix which in Einstein notation, as before, has a quite obvious nature

$$\mathbf{JF} \longrightarrow \partial_\nu F^\mu = \frac{\partial F^\mu}{\partial x^\nu} = F^\mu_{,\nu} \tag{3.20}$$

And so on, and so on...[1]
Let's now define the divergence and curl operators. Take now a vector field g^μ. We then have

$$\begin{aligned} \nabla \cdot \mathbf{g} &\longrightarrow g_{\mu\nu} g^{\mu\delta} \partial_\delta g^\nu = \partial_\mu g^\mu = \frac{\partial g^\mu}{\partial x^\mu} = g^\mu_{,\mu} \\ \nabla \times \mathbf{g} &\longrightarrow \epsilon_{\mu\nu\sigma} g^{\nu\delta} \partial_\delta g^\sigma = \epsilon_{\mu\nu\sigma} \partial^\nu g^\sigma = \epsilon_{\mu\nu\sigma} \frac{\partial g^\sigma}{\partial x_\mu} = \epsilon_{\mu\nu\sigma} g^{\sigma,\nu} \end{aligned} \tag{3.21}$$

[1] It's quite fun to dive into the dumpster of Einstein notation, isn't it?

And therefore, defining the Laplacian as $\nabla^2 = \nabla \cdot \nabla$, we will simply have, for whatever function h

$$\nabla^2 h \longrightarrow g^{\mu\nu}\partial_\nu \partial_\mu h = \partial^\mu \partial_\mu h = \frac{\partial^2 h}{\partial x^\mu \partial x_\mu} = h^{,\mu}{}_{,\mu} \tag{3.22}$$

Note how the operator ∂^μ appears. This can be seen as a derivation along the covector basis ($x_\mu = g_{\mu\nu} x^\nu$).
Now, we can go back to our mathematical rigor.

§ 3.2 Curves in $\mathbb{R}^n$

Definition 3.2.1 (Scalar Field)**.** A *scalar field* is a function $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ where A is an open set

Definition 3.2.2 (Vector Field)**.** A *vector field* is a function $f^\mu : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ where A is an open set

Definition 3.2.3 (Continuity)**.** A scalar field $f : A \longrightarrow \mathbb{R}$ is said to be continuous in a point $p^\mu \in A$ if

$$\forall \epsilon > 0 \; \exists \delta_p \; : \; \|x^\mu - p^\mu\| < \delta \implies |f(x^\mu) - f(p^\mu)| < \epsilon \tag{3.23}$$

A vector field $f^\mu : A \to \mathbb{R}^m$ is said to be continuous instead if

$$\forall \epsilon > 0 \; \exists \delta_p \; : \; \|x^\nu - p^\nu\|_\nu < \delta \implies \|f^\mu(x^\nu) - f^\mu(p^\nu)\|_\mu < \epsilon \tag{3.24}$$

If this function is continuous $\forall p^\mu \in A$, then the vector field is said to be part of the space $C(A)$, with $A \subseteq \mathbb{R}^n$

Definition 3.2.4 (Canonical Scalar Product)**.** Let $x^\mu, y^\mu \in \mathbb{R}^n$, the *canonical scalar product* is a bilinear application $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ where, if the components of the two vectors are defined as x^μ, y^μ, is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x^i y^i \longrightarrow x_\mu y^\mu \tag{3.25}$$

It's easy to see that the canonical scalar product induces the euclidean norm as follows

$$\|\mathbf{v}\| = \|\mathbf{v}\|_2 = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = \sqrt{v_\mu v^\mu} \tag{3.26}$$

Definition 3.2.5 (Curves in $\mathbb{R}^n$)**.** A *curve* is an application $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}^n$.
The function $\varphi^\mu(t) = p^\mu$, with $t \in [a,b]$ is called the *parametric representation* of the curve.
Remembering how indexes work in this notation, we already know that this application can be represented with an ordered $n-$tuple or a vector in $\mathbb{R}^n$

Definition 3.2.6 (Regular Curves)**.** A curve $\varphi^\mu(t)$ is said to be continuous if all its components are continuous. A curve is said to be regular iff

$$\begin{gathered} \varphi^\mu(t) \in C^1([a,b]) \\ \sqrt{\varphi_\mu(t)\varphi^\mu(t)} \neq 0 \quad t \in (a,b) \end{gathered} \tag{3.27}$$

A curve is said to be *piecewise regular* if it's not regular in $[a,b]$ but it's regular in a finite number of subsets $[a_n, b_n] \subset [a,b]$

Definition 3.2.7 (Homotopy of Curves)**.** Let γ^μ, η^μ be two curves from a set $[a,b]$, $[c,d]$ respectively. These two curves are said to be *homotopic* to one another, and it's indicated as $\gamma^\mu \sim \eta^\mu$ iff

$$\exists h : [c,d] \xrightarrow{\sim} [a,b], \; h \in C([c,d]), h^{-1} \in C([a,b]), \; h(s) > h(t) \text{ for } s > t \; : \; \eta^\mu = \gamma^\mu \circ h \tag{3.28}$$

Definition 3.2.8 (Tangent Vector)**.** The tangent vector of a regular curve is defined as the following vector.

$$T^\mu(t) = \frac{\dot{\gamma}^\mu}{\sqrt{(\dot{\gamma}_\mu \dot{\gamma}^\mu)}} \tag{3.29}$$

Where with $\dot{\gamma}^\mu(t)$ we indicate the derivative of γ^μ with respect to the only variable t.

Definition 3.2.9 (Tangent Line)**.** A curve $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^n$ is said to have *tangent line* at a point $t_0 \in [a,b]$ if it's regular, therefore the line will have parametric equations

$$p^\mu(t) = \gamma^\mu(t_0) + \dot{\gamma}^\mu(t_0)(t - t_0) \tag{3.30}$$

Definition 3.2.10 (Length of a Curve)**.** The *length* of a curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ is defined as follows

$$L_\gamma := \int_a^b \sqrt{\dot{\gamma}_\mu(t)\dot{\gamma}^\mu(t)}\,\mathrm{d}t \tag{3.31}$$

Remark. In $\mathbb{R}^2$, if a curve is defined in polar coordinates, it will appear as follows

$$\rho = \rho(\theta), \quad \theta \in [\theta_0, \theta_1] \tag{3.32}$$

Its length will be given from the following integral

$$L_\rho := \int_{\theta_0}^{\theta_1} \sqrt{(\rho'(\theta))^2 + (\rho(\theta))^2}\,\mathrm{d}\theta \tag{3.33}$$

The graph of a function $f : [a,b] \longrightarrow \mathbb{R},\ f \in C^1(a,b)$ can also be parametrized from a curve $\varphi^\mu(t)$, where

$$\varphi^\mu(t) \to (t, f(t)) \tag{3.34}$$

Its length will be then calculated with the following integral

$$L_\varphi := \int_a^b \sqrt{1 + (\dot{f}(x))^2}\,\mathrm{d}x \tag{3.35}$$

Definition 3.2.11 (Curviline Coordinate)**.** Let $\varphi : [a,b] \longrightarrow \mathbb{R}^n$, we can define a function $s(t)$ as follows

$$s(t) = \int_a^t \sqrt{\dot{\varphi}_\mu(\tau)\dot{\varphi}_\mu(\tau)}\,\mathrm{d}\tau \tag{3.36}$$

Then

$$\mathrm{d}s = \sqrt{\dot{\varphi}_\mu(t)\dot{\varphi}^\mu(t)}\,\mathrm{d}t \tag{3.37}$$

And the length of a curve can also be indicated as follows

$$L_\varphi = \int_\varphi \mathrm{d}s \tag{3.38}$$

Definition 3.2.12 (Curvature, Normal Vector)**.** The *curvature* of a curve is defined as follows

$$\kappa(s) = \sqrt{T_\mu(s)T^\mu(s)} = \sqrt{\ddot{\varphi}_\mu(s)\ddot{\varphi}^\mu(s)} \tag{3.39}$$

(Note that $\|\varphi'(s)\| = 1$) The *normal vector* is similarly defined as

$$N^\mu(s) = \frac{\dot{T}^\mu(s)}{\kappa(s)} \tag{3.40}$$

Definition 3.2.13 (Simple Curve, Closed Curve)**.** A *simple curve* is an injective application $\gamma : [a, b] \longrightarrow \mathbb{R}^n$. A curve is said to be closed iff $\gamma^\mu(a) = \gamma^\mu(b)$

Theorem 3.3 (Jordan Curve)**.** *Let γ^μ be a simple and closed curve in $\mathbb{R}^2$ or $\mathbb{C}$ (note that $\mathbb{C} \simeq \mathbb{R}^2$), then the set $\{\gamma\}^{\mathrm{c}}$ is defined as follows*

$$\{\gamma\}^{\mathrm{c}} = \{\gamma\}^\circ \cup \operatorname{extr}(\{\gamma\}) \tag{3.41}$$

Note that $\{\gamma\} \subset \mathbb{R}^2$ is the image of the application γ and $\operatorname{extr}(\{\gamma\})$ is the set of points that lay outside of the closed curve.

In $\mathbb{C}$ everything that was said about curves holds, however one must watch out for the definition of modulus, for a curve $\gamma^\mu \in \mathbb{C}$ we will have

$$|\dot{\gamma}(t)| = \sqrt{(\mathfrak{Re}'(\gamma))^2 + (\mathfrak{Im}'(\gamma))^2} = \sqrt{\overline{\gamma}(t)\gamma(t)} \tag{3.42}$$

§ 3.3 Differentiability in $\mathbb{R}^n$

Definition 3.3.1 (Directional Derivative)**.** Let $A \subseteq \mathbb{R}^n$ be an open set, and $f : A \longrightarrow \mathbb{R}$. The function is said to be *derivable* with respect to the direction $v^\mu \in A$ at a point $p^\mu \in A$, if the following limit is finite

$$\partial_{v^\mu} f(p^\nu) = \lim_{h\to 0} \frac{f(p^\mu + hv^\mu) - f(p^\mu)}{h} \tag{3.43}$$

If $v^\mu = x^\mu$ then this is called a *partial derivative*, and it will be indicated in the following ways

$$\frac{\partial f}{\partial x^\mu} = \partial_\mu f = \partial_{x^\mu} f \tag{3.44}$$

Definition 3.3.2 (Differentiability)**.** A scalar field $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, with A open, is said to be *differentiable* in a point $p^\mu \in A$ if and only if there exists a linear application $a_\mu(p^\nu) = a_\mu$, such that the following limit is finite

$$\lim_{\sqrt{h_\mu h^\mu}\to 0} \frac{R(p^\mu + h^\mu)}{\sqrt{h_\mu h^\mu}} = 0 \tag{3.45}$$

Where we define the function R as follows

$$R(p^\mu + h^\mu) = f(p^\mu + h^\mu) - (f(p^\mu) + a_\mu h^\mu) \tag{3.46}$$

This means that

$$f(p^\mu + h^\mu) = f(p^\mu) + a_\mu h^\mu + \mathcal{O}\left(\sqrt{h_\mu h^\mu}\right) \tag{3.47}$$

Theorem 3.4 (Consequences of Differentiability)**.** *Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable scalar field in every point of A, then*

1. *$f \in C(A)$*
2. *f is differentiable in A, and $a_\mu = \partial_\mu$, where is the vector differential operator, composed by the partial derivatives*

3. *f has directional derivatives in A and the following equation holds*

$$\partial_{v^\mu} f(p^\nu) = \partial_\mu f(p^\nu) v^\mu$$

4. *$\partial_\mu f$ indicates the maximum and minimum growth of the function f*

5. *There exist a tangent hyperplane to the graphic of the function at the point $(p^\mu, f(p^\mu)) \in \mathbb{R}^{n+1}$ and has the following equation*

$$x^{n+1} = f(p^\mu) + \partial_\mu f(p^\nu)(x^\mu - p^\mu)$$

Proof. 1. f differentiable in A implies $f \in C(A)$

$$\lim_{\sqrt{h_\mu h^\mu}\to 0} f(p^\mu + h^\mu) = \lim_{\sqrt{h_\mu h^\mu}\to 0} (f(p^\mu) + a_\mu h^\mu + \mathcal{O}\left(\sqrt{h_\mu h^\mu}\right)) = f(p^\mu)$$

2. f differentiable in A implies f derivable in A

$$\partial_i f(p^\mu) = \lim_{h\to 0} \frac{f(p^\mu + he^i) - f(p^\mu)}{h} = \lim_{h\to 0} \frac{a^i h + \mathcal{O}(h)}{h} = a^i \in \mathbb{R}$$

Then

$$R(p^\mu + h^\mu) = f(p^\mu + h^\mu) - f(p^\mu) + \partial_\mu f(p^\nu) h^\mu = \mathcal{O}\left(\sqrt{h_\mu h^\mu}\right)$$

3. $\partial_{v^\mu} f = \partial_\mu f v^\mu$

$$\partial_{v^\mu} f(p^\nu) = \lim_{h\to 0} \frac{f(p^\mu + hw^\mu) - f(p^\mu)}{h} = \lim_{h\to 0} \frac{h\partial_\mu f(p^\nu) v^\mu + \mathcal{O}(h)}{h} = \partial_\mu f(p^\nu) v^\mu$$

4. $\partial_\mu f$ indicates the direction of maximum growth.
For Cauchy-Schwartz, we have

$$\sqrt{\partial_{v^\mu} f \partial^{v_\mu} f} = \sqrt{\partial_\mu f v^\mu \partial_\nu f v^\nu} \leq \sqrt{\partial_\mu f \partial^\mu f} \sqrt{v_\nu v^\nu}$$

□

Theorem 3.5 (Continuous Differentiation)**.** *Let $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, with A open. If $f \in C^1(A)$ (i.e. the derivatives of f are continuous), then f is differentiable in A, the vice versa is also true*

Proof. We can write the following equation

$$f(p^\mu + h^\mu) = f(p^1 + h^1, \cdots, p^n) - f(p^1, \cdots, p^n) + \cdots + f(p^1, \cdots, p^n + h^n) - f(p^1, \cdots, p^n) \quad (3.48)$$

For Lagrange, we will have

$$f(p^i + h^i) = h^i \partial_i f(p^1, \cdots, q^i, \cdots, p^n) = h^i \partial_i f(c_i) \quad (3.49)$$

Therefore

$$\lim_{\sqrt{h_\mu h^\mu}\to 0} \frac{|f(p^\mu + h^\mu) - f(a^\mu) - \partial_\mu f h^\mu|}{\sqrt{h_\mu h^\mu}} \leq \lim_{h\to 0} \sum_{i=1}^{n} |\partial_i f(c_i) - \partial_i f(p^\mu)| \frac{|h^i|}{\sqrt{h_\mu h^\mu}} = 0 \quad (3.50)$$

Therefore $\partial_i f(p^\mu)$ is continuous and the function is differentiable. □

Theorem 3.6 (Differentiability of Vector Fields, Jacobian Matrix)**.** *Let $f^\mu : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a vector field and A an open set, then the function f^μ is differentiable iff exists a matrix $J^\mu_\nu \in M_{nm}(\mathbb{R})$ such that*

$$\lim_{\sqrt{h_\mu h^\mu}\to 0} \frac{\left\| f^\mu(p^\nu + h^\nu) - f^\mu(p^\nu) - J^\mu_\nu h^\nu \right\|_\mu}{\sqrt{h_\mu h^\mu}} \tag{3.51}$$

Or, equivalently

$$f_\mu(p^\nu + h^\nu) = f^\mu(p^\nu) + J^\mu_\nu h^\nu + \mathcal{O}\left(\sqrt{h_\mu h^\mu}\right) \tag{3.52}$$

The then J^μ_ν is the matrix of partial derivatives of the vector field, called the Jacobian matrix of the vector field f^μ, and can be calculated as follows

$$J^\mu_\nu(p^\sigma) = \partial_\nu f^\mu(p^\sigma) \tag{3.53}$$

Theorem 3.7 (Composite Derivation)**.** *Let $f^\mu : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^k$ and $g^\nu : B \subseteq \mathbb{R}^k \longrightarrow \mathbb{R}^p$ be two differentiable functions in $p^\sigma \in A$, $f^\mu(p^\sigma) \in B$ and A, B open sets, then $h^\nu = g^\nu \circ f^\mu$ is differentiable, and*

$$\partial_\sigma h^\nu = \partial_\mu g^\nu(f^{\tilde{\mu}})\partial_\sigma f^\mu \tag{3.54}$$

Since $\mu = 1, \cdots, k$, $\nu = 1, \cdots, p$, $\sigma = 1, \cdots, n$ it's obvious that $\partial_\sigma h^\nu \in M_{p,n}(\mathbb{R})$, $\partial_\mu g^\nu \in M_{p,k}(\mathbb{R})$, $\partial_\sigma f^\mu \in M_{k,n}(\mathbb{R})$.

Proof. We have that $(g^\nu \circ f^\mu) = g^\nu(f^\mu(p^\sigma))$. Then g^ν is differentiable at $f^\mu(p^\sigma)$ if

$$g^\nu(f^\mu(p^\sigma + s^\sigma)) = g^\nu(f^\mu(p^\sigma)) + \partial_\sigma\left(g^\nu \circ f^{\tilde{\mu}}\right)\left(f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right) + \mathcal{O}\left(\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu\right)$$

Since f^μ is differentiable, we have

$$\begin{aligned} g^\nu(f^\mu(p^\sigma + s^\sigma)) = g^\nu(f^\mu(p^\sigma)) + \partial_\sigma\left(g^\nu \circ f^{\tilde{\mu}}\right)\partial_\sigma f^\mu + \\ + \partial_\sigma\left(g^\nu \circ f^\mu\right)\mathcal{O}\left(\sqrt{s_\mu s^\mu}\right) + \mathcal{O}\left(\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu\right) \end{aligned}$$

Then, we must prove that

$$\lim_{\sqrt{s_\mu s^\mu}\to 0} \frac{\partial_\sigma\left(g^\nu \circ f^\mu\right)\mathcal{O}\left(\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu\right)}{\sqrt{s_\mu s^\mu}} = 0$$

But

$$\frac{\partial_\sigma\left(g^\nu \circ f^\mu\right)\mathcal{O}(\sqrt{s_\mu s^\mu})}{\sqrt{s_\mu s^\mu}} \to 0$$

And

$$\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu \le \sqrt{\partial^\sigma f_\mu \partial_\sigma f^\mu}\sqrt{s_\mu s^\mu} + \mathcal{O}\left(\sqrt{s_\mu s^\mu}\right) \le C\sqrt{s_\mu s^\mu}$$

Therefore

$$\frac{\mathcal{O}\left(\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu\right)}{\sqrt{s_\mu s^\mu}} = \frac{\mathcal{O}\left(\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu\right)}{\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu}\frac{\left\|f^\mu(p^\sigma + s^\sigma) - f^\mu(p^\sigma)\right\|_\mu}{\sqrt{s_\mu s^\mu}} \to 0$$

□

§ 3.4 Differentiability in $\mathbb{C}$

Definition 3.4.1 (Differentiability)**.** A function $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ with G open, is said to be *differentiable* or *derivable* at a point $a \in G$ if exists finite the following limit

$$\left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_a = f'(a) = \lim_{z\to a} \frac{f(z) - f(a)}{z - a} \tag{3.55}$$

As usual, if this holds $\forall a \in G$, the function is derivable in G

Theorem 3.8. *If $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ is derivable in $a \in G$, then f is continuous in a*

Proof.

$$\lim_{z\to a}(f(z) - f(a)) = \lim_{z\to a}(z - a) \lim_{z\to a} \frac{f(z) - f(a)}{z - a} = 0$$

□

Theorem 3.9 (Some Simple Rules)**.** *Let $f, g : G \subset \mathbb{C} \longrightarrow \mathbb{C}$*

1. $(f \pm g)'(z) = f'(z) \pm g'(z)$
2. $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$
3. $(f/g)'(z) = f'(z)/g(z) - f(z)g'(z)/g^2(z) \quad \forall z \in G \,:\, g(z) \neq 0$
4. $f(z) = c \implies f'(z) = 0$
5. $f(z) = z^n \implies f'(z) = nz^{n-1}$

Theorem 3.10 (Composite Function Derivation)**.** *Let $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ and $g : F \subset \mathbb{C} \longrightarrow \mathbb{C}$, where $f(G) \subset F$. If f is derivable in $a \in G$ and g is derivable in $f(a) \in F$, then $g \circ f$ is derivable, and its derivative is calculated as follows*

$$\left.\frac{\mathrm{d}}{\mathrm{d}z}(g \circ f)\right|_a = \left.\frac{\mathrm{d}g}{\mathrm{d}z}\right|_{f(a)} \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_a = g'(f(a))f'(a) \tag{3.56}$$

Proof. Since G is open, $\exists B_r(a) \subset G$. Therefore, taking a sequence $(z)_n \in B_r(a) \,:\, \lim_{n\to\infty}(z)_n = a$. Letting $f(z_n) \neq a$, we can directly write in the definition of derivative

$$\lim_{n\to\infty} \frac{(g \circ f)(z_n) - (g \circ f)(a)}{z_n - a} = (g \circ f)'(a) = g'(f(a))f'(a)$$

Thus, rewriting the function inside the limit

$$\frac{(g \circ f)(z_n) - (g \circ f)(a)}{z_n - a} = \frac{(g \circ f)(z_n) - (g \circ f)(a)}{f(z_n) - f(a)} \frac{f(z_n) - f(a)}{z_n - a} \to 0$$

Since f is continuous in $a \in G$ □

Theorem 3.11 (Inverse Function Derivation)**.** *Let $f : G \subset \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ be a bijective continuous map, with $f^{-1}(w) = z$. If $f(a) \neq 0$ and it's derivable at that same point, we have*

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}w}\right|_{f(a)} = \frac{1}{f'(a)} \tag{3.57}$$

Proof. Since f is bijective and continuous we can write

$$\left.\frac{\mathrm{d}f^{-1}}{\mathrm{d}w}\right|_{f(a)} = \lim_{w\to f(a)} \frac{f^{-1}(w) - f^{-1}(f(a))}{w - f(a)} = \lim_{z\to a} \frac{z-a}{f(z)-f(a)} = \frac{1}{f'(a)}$$

□

§§ 3.4.1 Holomorphic Functions

Definition 3.4.2 (Holomorphic Function)**.** A function $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *holomorphic* in its domain G if G is open, and

$$\forall z \in G\ \exists \frac{\mathrm{d}f}{\mathrm{d}z} = f'(z) \tag{3.58}$$

It is indicated as $f \in H(G)$. It's easy to demonstrate that this set is a vector space.

Theorem 3.12 (Cauchy-Riemann Equation)**.** *Let $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$, where G is open and $f \in H(G)$. Then, if we write $z = x + iy$*

$$\begin{cases} \mathfrak{Re}(f(z)) = u(x, y) \\ \mathfrak{Im}(f(z)) = v(x, y) \end{cases} \tag{3.59}$$

We have that the function is holomorphic if and only if

$$\begin{cases} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} = 0 \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} = 0 \end{cases} \tag{3.60}$$

Alternatively, it can be written as follows

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0 \tag{3.61}$$

Definition 3.4.3 (Wirtinger Derivatives)**.** Before demonstrating the previous theorem, we define the *Wirtinger derivatives* as follows.
Let $z \in \mathbb{C}$, $z = x + iy$ and $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$.

$$\begin{cases} \dfrac{\partial f}{\partial z} = \partial f(z) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) f(z) \\ \dfrac{\partial f}{\partial \overline{z}} = \overline{\partial} f(z) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) f(z) \end{cases} \tag{3.62}$$

Then, the Cauchy-Riemann equations will be equivalent to the following equation

$$\frac{\partial f}{\partial \overline{z}} = \overline{\partial} f(z) = 0 \tag{3.63}$$

Proof. Let $f(z) = u(x,y) + iv(x,y) : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a differentiable function in a point z_0, then as we defined, we have that $f \in H(B_\epsilon(z_0))$, and therefore

$$\left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z_0} = \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

And, therefore, along the imaginary axis and the real axis, we have

$$\begin{aligned}
\lim_{\mathfrak{Re}(h)\to 0} \frac{f\left(z_0 + \mathfrak{Re}(h)\right) - f(z_0)}{\mathfrak{Re}(h)} &= \left.\frac{\partial f}{\partial x}\right|_{z_0} \\
\lim_{\mathfrak{Im}(h)\to 0} \frac{f\left(z_0 + i\mathfrak{Im}(h)\right) - f(z_0)}{i\mathfrak{Im}(h)} &= \frac{1}{i}\left.\frac{\partial f}{\partial y}\right|_{z_0} = -i\left.\frac{\partial f}{\partial y}\right|_{z_0}
\end{aligned}$$

Due to the continuity of the derivative ($f \in H(B_\epsilon(z_0))$) we must have an equality between these limits

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 2\frac{\partial f}{\partial \overline{z}} = 0, \ \therefore f \in H(B_\epsilon(z_0)) \implies \frac{\partial f}{\partial \overline{z}} = 0$$

But, since $f(z) = u(x,y) + iv(x,y)$, we will have that

$$\begin{aligned}
\frac{\partial f}{\partial \overline{z}} &= \frac{\partial}{\partial \overline{z}}\left(u(x,y) + iv(x,y)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(u(x,y) + iv(x,y)\right) \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right) = 0 \\
&\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = i\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial y}
\end{aligned}$$

Rewriting the previous equation in a system, we immediately get back the Cauchy-Riemann equations

$$\begin{aligned}
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 0 \\
\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0
\end{aligned}$$

□

Definition 3.4.4 (Whole Function)**.** A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be *whole* iff $f \in H(\mathbb{C})$

Definition 3.4.5 (Singular Point)**.** Let $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ be function such that if $D = B_\epsilon(z_0) \setminus \{z_0\}$ and $f \in H(D)$, then z_0 is said to be a *singular point* of f

For functions $f : [a,b] \subset \mathbb{R} \longrightarrow \mathbb{C}$ every theorem already stated for curves in $\mathbb{R}^n$ with $n = 2$ holds, since $\mathbb{C} \simeq \mathbb{R}^2$. The only thing that should be checked thoroughly is that

$$f(t) = \begin{pmatrix} \mathfrak{Re}(f(t)) \\ \mathfrak{Im}(f(t)) \end{pmatrix} \in \mathbb{C}|_{\mathbb{R}}$$

Is written as

$$f(t) = \mathfrak{Re}(f(t)) + i\mathfrak{Im}(f(t)) \in \mathbb{C}$$

§ 3.5 Surfaces

Definition 3.5.1 (Regular Surface)**.** Let $K \subset \mathbb{R}^2$, $K = \overline{E}$ where E is an open and connected subset. A *regular surface* in $\mathbb{R}^3$ is an application

$$r^\mu : K \longrightarrow \mathbb{R}^3$$

Such that

1. $r^\mu \in C^1(K)$, i.e. $\exists \partial_\nu r^\mu \in C(K)$
2. r^μ is injective in K
3. $\operatorname{rank}(\partial_\nu r^\mu) = 2$

The image $\operatorname{Im}(r^\mu) = \Sigma \subset \mathbb{R}^2$ is then defined by the following parametric equations

$$r^\mu(u,v) = \begin{cases} x(u,v) = r^1(u,v) \\ y(u,v) = r^2(u,v) \\ z(u,v) = r^3(u,v) \end{cases} \tag{3.64}$$

The third condition can be rewritten as follows

$$\epsilon^\mu{}_{\nu\sigma}\partial_1 r^\nu \partial_2 r^\sigma = \epsilon^\mu{}_{\nu\sigma}\partial_1 r^\nu \partial_2 r^\sigma \neq 0 \quad \forall (u,v) \in K^\circ \tag{3.65}$$

Remark. A function $f \in C^1(K)$ defines automatically a surface with parametric equations $r^\mu(u,v) = (u, v, f(u,v))$. This surface is always regular since $\epsilon_{\mu\nu\sigma}\partial_1 r^\nu \partial_2 r^\sigma = (-2u, -2v, 1) \neq 0 \quad \forall (u,v) \in K$

Definition 3.5.2 (Coordinate Lines)**.** The curves obtained fixing one of the two variables are called *coordinate lines* in the surface Σ. We have therefore, for a parametric surface $r^\mu(u,v)$ and two fixed values $\tilde{u}, \tilde{v} \in I \subset \mathbb{R}$

$$\begin{aligned} x_1^\mu(t) &= r^\mu(t, \tilde{v}) \\ x_2^\mu(t) &= r^\mu(\tilde{u}, t) \end{aligned} \tag{3.66}$$

Example 3.5.1. The sphere centered in a point $p_0^\mu \in \mathbb{R}^3$, $p_0^\mu = (x_0, y_0, z_0)$ with radius $R \geq 0$ has the following parametric equations

$$\begin{aligned} x &= x_0 + R\sin(u)\cos(v) \\ y &= y_0 + R\sin(u)\sin(v) \\ z &= z_0 + R\cos(v) \end{aligned} \tag{3.67}$$

With $(u,v) \in [0,\pi] \times [0, 2\pi]$. It's a regular surface, since

$$\left\|\epsilon^\mu{}_{\nu\sigma}\partial_1 r^\nu \partial_2 r^\sigma\right\|_\mu = R^2\sin(u) > 0 \quad \forall (u,v) \in [0,\pi] \times [0,2\pi]$$

Definition 3.5.3 (Curve on a Surface)**.** Let $\gamma : [a,b] \subset \mathbb{R} \longrightarrow K \subset \mathbb{R}^3$ be a regular curve, and $\mathbf{r} : K \longrightarrow \Sigma$, with the following parametric equations

$$\gamma^\mu(t) = \begin{cases} u = u(t) \\ v = v(t) \end{cases} \tag{3.68}$$

The regular curve $p^\mu(t) = r^\mu(u(t), v(t))$ has $\operatorname{Im} p^\mu \subset \Sigma$. If it passes for a point $p_0^\mu = (u_0, v_0)$ it has tangent line

$$p^\mu(t) = p_0^\mu + \dot{r}^\mu(t)(t - t_0) = p_0^\mu + \partial_u r^\mu(u(t), v(t))\dot{u}(t) + \partial_v r^\mu(u(t), v(t))\dot{v}(t) \tag{3.69}$$

The line is contained inside the following plane

$$\det \begin{pmatrix} (x - x_0) & (y - y_0) & (z - z_0) \\ \partial_1 r^1 & \partial_1 r^2 & \partial_3 r^1 \\ \partial_2 r^1 & \partial_2 r^2 & \partial_3 r^3 \end{pmatrix} \tag{3.70}$$

For a *cartesian surface*, i.e. the surface generated from the graph of a function $f(x, y)$, the tangent plane will be

$$z = f(x_0^\mu) + \partial_\mu f(x_0^\nu)(x^\mu - x_0^\mu) \tag{3.71}$$

Definition 3.5.4 (Normal Vector)**.** The *normal vector* to a surface Σ, $n^\mu(u, v)$ is the vector

$$n^\mu(u, v) = \frac{1}{\sqrt{\epsilon^\mu{}_{\nu\sigma}\epsilon_\mu{}^{\delta\gamma}\partial_1 r^\nu \partial_2 r^\sigma \partial_1 r_\delta \partial_v r_\gamma}} \epsilon^\mu{}_{\nu\sigma}\partial_u r^\nu \partial_v r^\sigma \tag{3.72}$$

For a cartesian surface we have

$$n^\mu(x, y) = \frac{1}{\sqrt{1 + \partial_\mu f \partial^\mu f}} \begin{pmatrix} -\partial_1 f \\ -\partial_2 f \\ 1 \end{pmatrix}^\mu \tag{3.73}$$

Definition 3.5.5 (Implicit Surface)**.** Let $F : A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a function such that $F \in C^1(A)$, letting $\Sigma := \{ x^\mu \in \mathbb{R}^3 \,|\, F(x^\mu) = 0 \}$. If $x_0^\nu \in \Sigma$ and $\partial_\mu F(x_0^\nu) \neq 0$, Σ coincides locally to a cartesian surface, and the equation of the tangent plane at the point x_0^ν is the following

$$\partial_\mu F(x_0^\nu)(x^\mu - x_0^\mu) = 0 \tag{3.74}$$

Definition 3.5.6 (Metric Tensor)**.** Let $\mathrm{d}s$ be the curviline coordinate of some curve γ^μ inside a regular surface Σ. Then we have that

$$s^\mu(t) = r^\mu(u(t), v(t)) \tag{3.75}$$

And therefore

$$\mathrm{d}s^2 = \mathrm{d}r_\mu \,\mathrm{d}r^\mu = \partial_1 r^\mu \partial_1 r_\mu \left(\mathrm{d}x^1\right)^2 + 2\partial_1 r^\mu \partial_2 r_\mu \,\mathrm{d}x^1 \,\mathrm{d}x^2 + \partial_2 r^\mu \partial_2 r_\mu \left(\mathrm{d}x^2\right)^2 \tag{3.76}$$

In compact form, we can write

$$\mathrm{d}s^2 = g_{\mu\nu} \,\mathrm{d}x^\mu \,\mathrm{d}x^\nu \tag{3.77}$$

And, the *metric tensor* $g_{\mu\nu}$ can defined as follows

$$g_{\mu\nu} = \partial_\mu r^\sigma \partial_\nu r_\sigma \tag{3.78}$$

Or, in matrix notation

$$g_{\mu\nu} = \begin{pmatrix} \partial_1 r^\mu \partial_1 r_\mu & \partial_1 r^\mu \partial_2 r_\mu \\ \partial_2 r^\mu \partial_1 r_\mu & \partial_2 r^\mu \partial_2 r_\mu \end{pmatrix}_{\mu\nu} \tag{3.79}$$

In usual mathematical notation we have

$$g_{\mu\nu} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}_{\mu\nu} \tag{3.80}$$

And it's called the *first fundamental quadratic form* in the language of differential geometry. Then, we can write

$$\mathrm{d}s^2 = E\left(\mathrm{d}x^1\right)^2 + 2F\,\mathrm{d}x^1\,\mathrm{d}x^2 + G\left(\mathrm{d}x^2\right)^2 \tag{3.81}$$

§ 3.6 Optimization

§§ 3.6.1 Critical Points

Theorem 3.13 (Fermat)**.** *Let $p^\nu \in A$ be a point of local minimal or maximal for the function $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ with $f \in C^1(A)$. If f is differentiable in p^ν we have*

$$\partial_\mu f(p^\nu) = 0 \tag{3.82}$$

The point p^ν satisfying this condition is then called a stationary point or a critical point for the function f

Proof. Let $v^\mu \in A$ be a direction. The function $g(t) = f(p^\mu + tv^\mu)$ has a point of local maximal or minimal for $t = 0$. Then

$$F'(0) = \partial_{v^\mu} f(p^\nu) = \partial_\mu f(p^\nu) v^\mu = 0 \implies \partial_\mu f(p^\nu) = 0 \tag{3.83}$$

□

Definition 3.6.1 (Hessian Matrix)**.** Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, and let $f \in C^1(A)$, then we define the *Hessian matrix* as the matrix of the second partial derivatives of the function f

$$\partial_\mu \partial_\nu f(x^\gamma) = \partial_{\mu\nu} f(x^\gamma) \begin{pmatrix} \partial_{11} f & \cdots & \partial_{1n} f \\ \vdots & \ddots & \vdots \\ \partial_{n1} f & \cdots & \partial_{nn} f \end{pmatrix}_{\mu\nu} (x^\gamma) \tag{3.84}$$

Theorem 3.14 (Schwarz)**.** *Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, $f \in C^2(A)$, then*

$$\partial_{\mu\nu} f = \partial_{\nu\mu} f \tag{3.85}$$

Definition 3.6.2 (Nature of Critical Points)**.** Let p^γ be a critical point for a function $f \in C^1(A)$. Then

1. $\partial_{\mu\nu} f(p^\gamma)$ is definite positive, then p^γ is a local minimum
2. $\partial_{\mu\nu} f(p^\gamma)$ is definite negative, then p^γ is a local maximum
3. $\partial_{\mu\nu} f(p^\gamma)$ is indefinite, then p^γ is a saddle point

Theorem 3.15. *Here is a list of some rules in order to determine the definition of the matrix $\partial_{\mu\nu} f$. Let $v^\mu \in A$ be a direction, and $p^\gamma \in A$ a critial point of the function $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ then*

1. *If* $\partial_{\mu\nu} f(p^\gamma) v^\mu v^\nu > 0 \quad \forall v^\mu \in A \implies \partial_{\mu\nu} f(p^\gamma)$ *positive definite*
2. *If* $\partial_{\mu\nu} f(p^\gamma) v^\mu v^\nu < 0 \quad \forall v^\mu \in A \implies \partial_{\mu\nu} f(p^\gamma)$ *negative definite*
3. *If* $\partial_{\mu\nu} f(p^\gamma) v^\mu v^\nu \geq 0 \quad \forall v^\mu \in A \implies \partial_{\mu\nu} f(p^\gamma)$ *semi-positive definite*
4. *If* $\partial_{\mu\nu} f(p^\gamma) v^\mu v^\nu \leq 0 \quad \forall v^\mu \in A \implies \partial_{\mu\nu} f(p^\gamma)$ *semi-negative definite*
5. *If* $v^\mu \neq w^\mu$ *are two directions, and* $\partial_{\mu\nu} f(p^\gamma) v^\mu v^\nu > 0 \wedge \partial_{\mu\nu} f(p^\gamma) w^\mu w^\nu < 0 \implies \partial_{\mu\nu} f(p^\gamma)$ *indefinite*

Theorem 3.16 (Sylvester's Criteria)**.** *Let* $A^\mu_\nu \in M_{nn}(\mathbb{R})$*, and* $(A_k)^\mu_\nu$ *be the reduced matrix with order* $k \leq n$*, then*

1. $\det_{\mu\nu}\left((A_k)^\mu_\nu\right) > 0 \implies A^\mu_\nu$ *positive definite*
2. $(-1)^k \det_{\mu\nu}\left((A_k)^\mu_\nu\right) > 0 \implies A^\mu_\nu$ *negative definite*
3. *If* $\det_{\mu\nu}\left((A_{2k})^\mu_\nu\right) < 0$ *or if* $\det_{\mu\nu}\left((A_{2k+1})^\mu_\nu\right) < 0 \wedge \det_{\mu\nu}\left((A_{2n+1})^\mu_\nu\right) > 0$ *for* $k \neq n$*, then* A^μ_ν *is indefinite*

Theorem 3.17 (Compact Weierstrass)**.** *Let* $f : K \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$*,* $f \in C(K)$*, with* K *a compact set, then*

$$\exists p^\mu, q^\mu \in K \,:\, \min_K(f) = f(p^\mu) \leq f(x^\mu) \leq \max_K(f) = f(q^\mu) \ \forall x^\mu \in K \tag{3.86}$$

Proof. Being K a compact set, we have that every sequence $(p^\mu)_n$ converges inside the set, therefore, letting $(p^\mu)_n$ being a minimizing sequence for f. Then there exist a converging subsequence $(p^\mu)_{n_k}$ such that

$$f(p^\mu_{n_k}) \to f(p^\mu)$$

But, since $(p^\mu)_n$ is a minimizing sequence, we have

$$f(p^\mu) = \min_K(f)$$

By definition of minimizing sequence. Analogously, one can define a maximizing sequence and obtain the same result for the maximum of the function in K □

Theorem 3.18 (Closed Weierstrass)**.** *Let* $f : L \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$*. If* $L = \overline{L}$ *and* $f \in C(L)$ *is a coercitive function, i.e.*

$$\lim_{\sqrt{x_\mu x^\mu} \to \infty} f(x^\mu) = +\infty \tag{3.87}$$

Then

$$\exists x^\mu \in L \,:\, \min_L(f) = f(x^\mu) \tag{3.88}$$

Proof. Let $(p^\mu)_n$ be a minimizing sequence for f in L. If this sequence wasn't limited, we would have that $\sqrt{(p_\mu)_n (p^\mu)_n} \to \infty$, and therefore

$$\inf_L(f) = \lim_{n\to\infty} f(p^\mu_n) = +\infty \quad \lightning$$

Therefore $(p^\mu)_n$ must be limited, and the proof is the same as in the case of a compact set. □

Theorem 3.19 (Topology and Functions)**.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f \in C(\mathbb{R}^n)$. Then*

$$\begin{aligned} &\{ x^\mu \in \mathbb{R}^n |\, f(x^\mu) < a \in \mathbb{R}\} \\ &\{ x^\mu \in \mathbb{R}^n |\, f(x^\mu) > b \in \mathbb{R}\} \end{aligned} \tag{3.89}$$

Are open sets in $\mathbb{R}^n$ with the standard topology, and

$$\begin{aligned} &\{ x^\mu \in \mathbb{R}^n |\, f(x^\mu) \le a \in \mathbb{R}\} \\ &\{ x^\mu \in \mathbb{R}^n |\, f(x^\mu) \ge b \in \mathbb{R}\} \end{aligned} \tag{3.90}$$

Are closed sets

§§ 3.6.2 Convexity and Implicit Functions

Definition 3.6.3 (Convex Set)**.** A set $A \subset \mathbb{R}^n$ is said to be *convex* if

$$\lambda x^\mu + (1-\lambda) y^\mu \in A \quad \forall x^\mu, y^\mu \in A,\ \forall \lambda \in [0,1] \tag{3.91}$$

Analogously, a function $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be convex, if

$$f\left(\lambda x^\mu + (1-\lambda) y^\mu\right) \le \lambda f(x^\mu) + (1-\lambda) f(y^\mu) \quad \forall x^\mu, y^\mu \in A,\ \forall \lambda \in [0,1] \tag{3.92}$$

The function f is also known as a *sublinear* function
Also, the set

$$E_f = \{ (x^\mu, \lambda) \in A \times \mathbb{R} |\, f(x^\mu) \le y\} \tag{3.93}$$

Is convex

Theorem 3.20 (Convexity)**.** *Let $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$.*

1. *f convex in $A \implies f \in C(A)$*
2. *f differentiable in $A \implies f$ convex $\iff f(x^\mu) \ge f(p^\mu) + \langle \nabla f(p^\mu), x^\mu - p^\mu \rangle$*
3. *$f \in C^2(A) \implies f$ convex $\iff \partial_{\mu\nu} f(x^\gamma)$ is positive semidefinite*

Definition 3.6.4 (Matrix Infinite Norm)**.** Let $A^\mu_\nu(x^\gamma) \in \mathcal{V} \longrightarrow M_{mn}(\mathbb{F})$, where $\dim(\mathcal{V}) = n$. We can define a norm for this space as follows

$$\|A^\mu_\nu\|_\infty = \sqrt{m} \sqrt{\max_\mu \sup_{x^\gamma \in \mathcal{V}} A^{(\mu)}_\nu A^\nu_{(\mu)}(x^\gamma)} \tag{3.94}$$

Theorem 3.21 (Average Value)**.** *Let $f^\mu : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$, with $f \in C^1(A)$, A an open set and $K \subset A$ a compact convex subset, then*

$$\|f^\mu(x^\nu) - f^\mu(y^\nu)\|_\mu \le \|\partial_\nu f^\mu\|_\infty \|x^\nu - y^\nu\|_\nu \tag{3.95}$$

Proof. Let $r^\nu(t) = (1-t)y^\nu + t x^\nu$ be a smooth parametrization of a segment connecting the two points x^ν, y^ν, then

$$\begin{aligned} \|f^\mu(r^\nu(1) - f^\mu(r^\nu(0)\|_\mu^2 &\le \partial^\nu f_\mu \partial_\nu f^\mu(r^\nu(t)) \le \sup_{r^\gamma} (\partial^\nu f_\mu \partial_\nu f^\mu(r^\gamma)) \|x^\nu - y^\nu\|_\nu^2 \\ &\le m \max_\mu \sup_\gamma \left(\partial^\nu f_{(\mu)} \partial_\nu f^{(\mu)}(x^\gamma) \right) \|x^\nu - y^\nu\|_\nu^2 \end{aligned}$$

Therefore

$$\begin{aligned}&\|f^\mu(x^\nu)-f^\mu(y^\nu)\|_\mu \le \sqrt{m}\sqrt{\max_\mu \sup_\gamma \left(\partial^\nu f_{(\mu)}\partial_\nu f^{(\mu)}\right)}\|x^\nu-y^\nu\|_\nu = \\ &= \|\partial_\nu f^\mu\|_\infty \|x^\nu - y^\nu\|_\nu \quad \forall x^\nu, y^\nu \in K\end{aligned}$$

□

Theorem 3.22 (Implicit Functions, Dini)**.** *Let* $f^\mu : A \subseteq \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$*, where* $f^\mu \in C^1(A)$*. Also let* $(x_0^\nu, y_0^\gamma) \in A$ *such that*

$$f^\mu(x_0^\nu, y_0^\gamma) = 0$$
$$\det_{\mu\nu}\left(\frac{\partial f^\mu}{\partial y^\gamma}\right) \neq 0$$

Then

$$\exists B_\epsilon(x_0^\nu) = I \subset \mathbb{R}^m,\ B_\epsilon(y_0^\gamma) = J \subset \mathbb{R}^n\ :\ f^\mu(x^\nu, y^\gamma) = 0\ \forall (x^\nu, y^\gamma) \in I \times J$$

Has a unique *solution* $y^\gamma = g^\gamma(x^\nu) \in J$*, with* $g^\gamma \in C^1(I)$*, and*

$$\frac{\partial g^\gamma}{\partial x^\nu} = -\left(\frac{\partial f^\mu}{\partial y^\gamma}\right)^{-1}\frac{\partial f^\mu}{\partial x^\nu} \tag{3.96}$$

Proof. Let $B^\gamma_\mu = \left(\partial_{y_0^\gamma} f^\mu\right)^{-1}$, then we know that

$$f^\mu(x^\nu, y^\gamma) = 0 \iff B^\gamma_\mu f^\mu(x^\nu, y^\sigma) = 0 \iff G^\gamma(x^\nu, y^\sigma) = y^\gamma - B^\gamma_\mu f^\mu(x^\nu, y^\sigma) = 0$$

We have therefore

$$\begin{aligned}G^\gamma(x^\nu, g^\sigma(x^\nu)) &= g^\gamma(x^\nu) - B^\gamma_\mu f^\mu(x^\nu, g^\sigma(x^\nu)) = g^\gamma(x^\nu)\ \forall x^\nu \in \overline{B}_r(x_0^\nu) = I \\ \frac{\partial G^\gamma}{\partial y^\sigma} &= \delta^\gamma_\sigma - B^\gamma_\mu \frac{\partial f^\mu}{\partial y^\sigma} \\ \frac{\partial G^\gamma}{\partial y_0^\sigma} &= \delta^\gamma_\sigma - B^\gamma_\mu \frac{\partial f^\mu}{\partial y^\sigma} = \delta^\gamma_\sigma - \delta^\gamma_\sigma = 0\end{aligned}$$

Now take $(X, d) = (C(I, J), \|\cdot\|_\infty)$, with $J = \overline{B}_\epsilon(y_0^\gamma)$, and define an application $H : X \longrightarrow X$ such that

$$H^\gamma(w^\sigma(x^\nu)) = G^\gamma(x^\nu, w^\sigma(x^\nu))$$

We need to demonstrate that this application is a contraction, i.e. that $\exists! g^\gamma(x^\nu)\ :\ f^\mu(x^\nu, g^\gamma(x^\nu)) = 0\ \forall (x^\nu, y^\gamma) \in I \times J$

$$\begin{aligned}\|H^\gamma(w^\sigma(x^\nu)) - y_0^\gamma\|_\gamma &= \|G^\gamma(x^\nu, w^\sigma(x^\nu)) - y_0^\gamma\|_\gamma \le \\ &\le \|G^\gamma(x^\nu, w^\sigma(x^\nu)) - G^\gamma(x^\nu, y_0^\sigma)\|_\gamma + \|G^\gamma(x^\nu, y_0^\sigma) - G^\gamma(x_0^\nu, y_0^\nu)\|_\gamma \le \\ &\le \left\|\frac{\partial G^\gamma}{\partial y^\sigma}\right\|_\infty \|w^\sigma(x^\nu) + y_0^\sigma\|_\sigma + \|G^\gamma(x^\nu, y_0^\sigma) - G^\gamma(x_0^\nu, y_0^\sigma)\|_\gamma \le \epsilon\end{aligned}$$

Since $\|G^\gamma(x^\nu, y_0^\gamma) - G^\gamma(x_0^\nu, y_0^\gamma)\|_\gamma \le \epsilon/2$ and $\|w^\sigma(x^\nu) - y_0^\sigma\|_\sigma \le \epsilon$, $\forall (x^\nu, y^\gamma) \in I \times J$, we have

$$\left\|\frac{\partial G^\gamma}{\partial y^\sigma}\right\|_\infty \le \frac{1}{2}$$

Therefore

$$\|H^\gamma(w^\gamma(x^\nu)) - H^\gamma(v^\sigma(x^\nu))\|_\gamma \leq \frac{1}{2}\|w^\gamma(x^\nu) - v^\gamma(x^\nu)\|_\gamma$$

I.e. H is a contraction in $C(I,J)$.
Due to the differentiability of f^μ we can write

$$\begin{aligned}\forall\epsilon > 0\ \exists\eta_\epsilon\ :\ \|h^\nu\|_\nu, \|k^\gamma\|_\gamma \leq \eta_\epsilon \implies \|f^\mu(x^\nu, g^\gamma(x^\nu) + k^\gamma) - f^\mu(x^\nu, g^\gamma(x^\nu)) - \partial_{x^\nu} f^\mu h^\nu - \partial_{y^\gamma} f^\mu k^\gamma\|_\mu \leq \\ \leq \epsilon(\|h^\nu\|_\nu + \|k^\gamma\|_\gamma)\end{aligned}$$

Letting $k^\gamma = g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)$ we have by definition

$$f^\mu(x^\nu + h^\nu, g^\gamma(x^\nu) + k^\gamma) - f^\mu(x^\nu, g^\gamma(x^\nu)) = 0$$

And therefore, putting $\partial_{y^\gamma} f^\mu = \delta^\mu_\gamma$

$$\|g^\gamma(x^\nu + h^\nu) + \partial_{x^\nu} f^\mu h^\nu\|_\mu \leq \epsilon\left(\|h^\nu\|_\nu + \|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)\|_\gamma\right)$$

Letting $\epsilon = 1/2$ we have $\|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)\| \leq \eta_{1/2}$, and we have

$$\begin{aligned}\|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)\|_\gamma &\leq \|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu) - \partial_{x^\nu} f^\gamma h^\nu\|_\gamma \|\partial_{x^\nu} f^\mu h^\nu\|_\gamma \\ &\leq \frac{1}{2}\left(\|h^\nu\|_\nu + \|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)\|_\gamma\right) + \|\partial_{x^\nu} f^\mu\|_\infty \|h^\nu\|\end{aligned}$$

Which implies

$$\|g^\gamma(x^\nu + h^\nu) - g^\gamma(x^\nu)\|_\gamma \leq \|h^\nu\|_\nu \left(1 + 2\|\partial_{x^\nu} f^\mu\|_\infty\right)$$

Which implies that $g^\gamma(x^\nu)$ is continuously differentiable in I. Whenever $\partial_{y^\gamma} f^\mu \neq \delta^\mu_\gamma$ we can find a transformed function $\tilde{f}^\mu$ such that $\partial_{y^\gamma}\tilde{f}^\mu = \delta^\mu_\gamma$ □

§§ 3.6.3 Lagrange Multipliers

Definition 3.6.5 (Vinculated Critical Points)**.** Let $f : K \subset A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ with A open and $f \in C^1(A)$. Let $\partial K = \bigcup_{k \leq n} \gamma_k$ with $\gamma_k : [a_k, b_k] \longrightarrow \mathbb{R}^2$. The critical points of $f|_{\partial K}$ are found in $P_{ik} = \gamma_k(t_i) \in \partial K$, for which

$$\frac{\mathrm{d}}{\mathrm{d}t} f(P_{ik}) = 0$$

Definition 3.6.6 (Argmax, Argmin)**.** Let $f^\mu : A \longrightarrow \mathbb{R}^n$ a function which reach its maximum in $x^\nu_i \in A$ $i = 1, \cdots, m$ and its minimum at $y^\nu_j \in A$ $j = 1, \cdots, k$ Then we can define

$$\begin{aligned}\operatorname*{Arg\,max}_A(f) &:= \{x^\nu_1, \cdots, x^\nu_m\} \\ \operatorname*{Arg\,min}_A(f) &:= \{y^\nu_1, \cdots, y^\nu_k\}\end{aligned} \tag{3.97}$$

Theorem 3.23 (Lagrange Multipliers)**.** *Let $f, g : A \longrightarrow \mathbb{R}$, $f, g \in C^1(A)$, $A \subseteq \mathbb{R}^n$ open, and $\mathcal{M} = \{\, x^\mu \in A |\ g(x^\mu) = 0\}$ and let $x^\mu_0 \in \mathcal{M} : \partial_m u g(x^\mu_0) \neq 0$, then $x^\mu_0 \in \text{Arg}\max_{\mathcal{M}} f \vee \text{Arg}\min_{\mathcal{M}} f$ if it's a free critical point of the Lagrangian*

$$\mathcal{L}(x^\mu, \lambda) = f(x^\mu) - \lambda g(x^\mu) \quad (x^\mu, \lambda) \in A \times \mathbb{R} \tag{3.98}$$

I.e. $\exists \lambda_0 \in \mathbb{R} \, : \, (x_0^\mu, \lambda_0)$ *solves*

$$\begin{cases} \partial_\mu f(x^\nu) = \lambda \partial_\mu g(x^\nu) \\ \quad g(x^\mu) = 0 \end{cases} \tag{3.99}$$

Or, that

$$\operatorname{rank} \begin{pmatrix} \partial_\mu f(x_0^\nu) \\ \partial_\mu g(x_0^\nu) \end{pmatrix} = 1$$

Proof. Let $\partial_n g \neq 0$, then we can see $\mathcal{M}$ as a graph of a regular implicit function of g, $h : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, where

$$g(x^\mu, h(x^\mu)) = 0 \quad \forall x^\mu \in B_r(\tilde{x^\mu}_0) \subset \mathbb{R}^{n-1}$$

Letting $\varphi : (-\epsilon, \epsilon) \longrightarrow B_r(x_0^\mu)$ a smooth curve, such that $\varphi^\mu(0) = x_0^\mu$, we have that $\psi^\nu(t) = (\varphi^\mu(t), h(t)) \in \mathcal{M}$ is the parameterization of a smooth curve that passes through $x_0^\mu \in \mathcal{M}$. We have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} f(\psi^\nu(0)) = \partial_\mu f \dot{\phi}^\mu(0) + \partial_n f \dot{h}(\phi^\mu(0)) = \partial_\nu f(x_0^\mu) s^\nu \\ &\frac{\mathrm{d}}{\mathrm{d}t} g(\psi^\nu(0)) = \partial_\nu g(x_0^\mu) s^\nu \end{aligned}$$

With $s^\nu = \dot{\psi}^\nu(0)$, therefore $\partial_\nu f \|_{\psi^\nu(0)} \partial_\nu g$ □

Theorem 3.24 (Generalized Lagrange Multiplier Method)**.** *Let* $f, g_i : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$*,* $0 < i < n$*,* $f, g_i \in C^1(A)$ *with* A *an open set, let* $\mathcal{M} := \{ x^\nu \in A | \, g(x^\nu) = 0 \}$*. Take* $x_0^\nu \in \mathcal{M}$ *such that*

$$\operatorname{rank} \partial_\nu g^\mu(x_0^\gamma) = k$$

Then x_0^ν *is a critical point for* $f|_{\mathcal{M}}$*, and it's a free critical point for the Lagrangian* $\mathcal{L}$

$$\mathcal{L}(x^\gamma, \lambda^\mu) = f(x^\gamma) - \lambda_\nu g^\nu(x^\gamma)$$

I.e. $\exists (x_0^\gamma, \lambda_0^\nu) \in A \times \mathbb{R}$ *solution of the system*

$$\begin{cases} \partial_\nu f(x^\gamma) = \partial_\nu g^\mu(x^\gamma) \lambda^\nu \\ \quad g^\nu(x^\gamma) = 0 \end{cases}$$

Alternatively, one can check that

$$\operatorname{rank}(A) = \begin{pmatrix} \partial_\mu f(x_0^\gamma) \\ \partial_\nu g^\mu(x^\gamma) \end{pmatrix} = k$$

4 Tensors and Differential Forms

§ 4.1 Tensors and k-forms

§§ 4.1.1 Basic Definitions, Tensor Product and Wedge Product

Definition 4.1.1 (Multilinear Functions, Tensors)**.** Let $\mathcal{V}$ be a real vector space, and take $\mathcal{V}^k = \mathcal{V}\times\cdots\times\mathcal{V}$ $k-$times. A function $T : \mathcal{V}^k \longrightarrow \mathbb{R}$ is called *multilinear* if $\forall i = 1, \cdots, k$, $\forall a \in \mathbb{R}$, $\forall v, w \in \mathcal{V}$

$$T(v_1, \cdots, av_i + w_i, \cdots, v_k) = aT(v_1, \cdots, v_i, \cdots, v_k) + T(v_1, \cdots, w_i, \cdots, v_k) \tag{4.1}$$

A multilinear function of this kind is called *k-tensor* on $\mathcal{V}$. The set of all $k-$tensors is denoted as $\mathcal{T}^k(\mathcal{V})$ and is a real vector space.
The tensor T is usually denoted as follows

$$T_{\mu_1\dots\mu_k} \tag{4.2}$$

Where each index indicates a slot of the multilinear application $T(-, \cdots, -)$

Definition 4.1.2 (Tensor Product)**.** Let $S \in \mathcal{T}^k(V), T \in \mathcal{T}^l(\mathcal{V})$, we define the *tensor product* $S \otimes T \in \mathcal{T}^{k+l}(\mathcal{V})$ as follows

$$(S \otimes T)(v_1, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = S(v_1, \cdots, v_k)T(v_{k+1}, \cdots, v_{k+l}) \tag{4.3}$$

This product has the following properties

$$\begin{aligned} (S_1 + S_2) \otimes T &= S_1 \otimes T + S_2 \otimes T \\ S \otimes (T_1 + T_2) &= S \otimes T_1 + S \otimes T_2 \\ (aS) \otimes T &= S \otimes (aT) = a(S \otimes T) \\ (S \otimes T) \otimes U &= S \otimes (T \otimes U) = S \otimes T \otimes U \end{aligned} \tag{4.4}$$

If $S = S_{\mu_1\dots\mu_k}$ and $T = T_{\mu_{k+1}\dots\mu_{k+l}}$ we have

$$(S \otimes T)_{\mu_1\dots\mu_k\mu_{k+1}\dots\mu_{k+l}} = S_{\mu_1\dots\mu_k}T_{\mu_{k+1}\dots\nu_{k+l}} \tag{4.5}$$

Definition 4.1.3 (Dual Space)**.** We define the *dual space* of a real vector space $\mathcal{V}$ as the space of all *linear functionals* from the space to the field over it's defined, and it's indicated with $\mathcal{V}^\star$. I.e. let $\varphi^\mu \in \mathcal{V}^\star$, then $\varphi^\mu : \mathcal{V} \longrightarrow \mathbb{R}$.
It's easy to see how $\mathcal{V}^\star = \mathcal{T}^1(\mathcal{V})$.

Theorem 4.1. *Let $\mathcal{B} = \{v_{\mu_1}, \cdots, v_{\mu_n}\}$ be a basis for the space $\mathcal{V}$, and let $\mathcal{B}^\star := \{\varphi^{\mu_1}, \cdots, \varphi^{\mu_n}\}$ be the basis of the dual space, i.e. $\varphi^\mu v_\nu = \delta^\mu_\nu \ \forall \varphi^\mu \in \mathcal{B}^\star, \ v_\mu \in \mathcal{B}$, then the set of all k-fold tensor products has basis $\mathcal{B}_\mathcal{T}$, where*

$$\mathcal{B}_\mathcal{T} := \{\varphi^{\mu_1} \otimes \cdots \otimes \varphi^{\mu_k}, \ \forall i = 1, \cdots, n\} \tag{4.6}$$

Theorem 4.2 (Linear Transformations on Tensor Spaces)**.** *If $f^\mu_\nu : \mathcal{V} \longrightarrow \mathcal{W}$ is a linear transformation, $f^\nu_\mu \in \mathcal{L}(\mathcal{V}, \mathcal{W})$, one can define a linear transformation $f^\star : \mathcal{T}^k(W) \longrightarrow \mathcal{T}^k(V)$ as follows*

$$f^\star T(v_{\mu_1}, \cdots, v_{\mu_k}) = T(f^\mu_\nu v_{\mu_1}, \cdots, f^\mu_\nu v_{\mu_k})$$

Theorem 4.3. *If g is an inner product on $\mathcal{V}$ (i.e. $g : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$, with the properties of an inner product), there is a basis $v_{\mu_1}, \cdots, v_{\mu_n}$ of $\mathcal{V}$ such that $g(v_\mu, v_\nu) = g_{\mu\nu} = g_{\nu\mu} = g(v_\nu, v_\mu) = \delta_{\mu\nu}$. This basis is called orthonormal with respect to T. Consequently there exists an isomorphism $f^\mu_\nu : \mathbb{R}^n \xrightarrow{\sim} \mathcal{V}$ such that*

$$g(f^\mu_\nu x^\nu, f^\mu_\nu y^\nu) = x_\mu y^\mu = g_{\mu\nu} x^\mu y^\nu \tag{4.7}$$

I.e.

$$f^\star g(\cdot, \cdot) = g_{\mu\nu} \tag{4.8}$$

Definition 4.1.4 (Alternating Tensor)**.** Let $\mathcal{V}$ be a real vector space, and $\omega \in \mathcal{T}^k(\mathcal{V})$. ω is said to be *alternating* if

$$\begin{aligned} &\omega(v_{\mu_1}, \cdots, v_{\mu_i}, \cdots, v_{\mu_j}, \cdots, v_{\mu_k}) = -\omega(v_{\mu_1}, \cdots, v_{\mu_j}, \cdots, v_{\mu_i}, \cdots, v_{\mu_k}) \\ &\omega(v_{\mu_1}, \cdots, v_{\mu_i}, \cdots, v_{\mu_i}, \cdots, v_{\mu_k}) = 0 \end{aligned} \tag{4.9}$$

Or, compactly

$$\begin{aligned} \omega_{\mu \ldots \nu \ldots \gamma \ldots \sigma} &= -\omega_{\mu \ldots \gamma \ldots \nu \ldots \sigma} \\ \omega_{\mu \ldots \nu \ldots \nu \ldots \gamma} &= 0 \end{aligned} \tag{4.10}$$

The space of all alternating $k-$tensors on $\mathcal{V}$ is indicated as $\Lambda^k(\mathcal{V})$, and we obviously have that $\Lambda^k(\mathcal{V}) \subset \mathcal{T}^k(\mathcal{V})$.
We can define an application $\mathrm{Alt} : \mathcal{T}^k(\mathcal{V}) \longrightarrow \Lambda^k(\mathcal{V})$ as follows

$$\mathrm{Alt}(T)(v^\mu_1, \cdots, v^\mu_k) = \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \mathrm{sgn}(\sigma) T(v^\mu_{\sigma(1)}, \cdots, v^\mu_{\sigma(k)}) \tag{4.11}$$

With $\sigma = (i, j)$ a permutation and Σ_k the set of all permutations of natural numbers $1, \cdots, k$ Compactly, we define an operation on the indices, indicated in square brackets, called the *antisymmetrization* of the indices inside the brackets.
This definition is much more general, since it lets us define a partially antisymmetric tensor, i.e. antisymmetric on only some indices.

$$\mathrm{Alt}(T_{\mu_1 \ldots \mu_k}) = \frac{1}{k!} T_{[\mu_1 \ldots \mu_k]} \tag{4.12}$$

As an example, for a $2-$tensor $a_{\mu\nu}$ we can write

$$a_{[\mu\nu]} = \frac{1}{2}\left(a_{\mu\nu} - a_{\nu\mu}\right) = \tilde{a}_{\mu\nu} \in \Lambda^2(\mathcal{V}) \tag{4.13}$$

This is valid for general tensors. If we define a $k-$tensor over the product repeated k times for $\mathcal{V}$ and k for its dual space $\mathcal{V} \times \cdots \mathcal{V} \times \mathcal{V}^\star \times \cdots \times \mathcal{V}^\star$, we can define the space $\mathcal{T}^k(\mathcal{V} \times \mathcal{V}^\star) = \mathcal{W}$. Let the basis for this space be the following

$$\mathcal{B}_{\mathcal{W}} := \{v_{\mu_1} \otimes \cdots \otimes v_{\mu_k} \otimes \varphi^{\nu_1} \otimes \cdots \otimes \varphi^{\nu_k}\}$$

Then an element $\mathcal{Y}$ of the space $\mathcal{W}$ can be written as follows

$$\mathcal{Y}(v_{\mu_1}, \cdots, v_{\mu_k}, \varphi^{\nu_1}, \cdots, \varphi^{\nu_k}) = \mathcal{Y}^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_k}$$

We can define a new element $Y \in \Lambda^k(\mathcal{V} \times \mathcal{V}^\star)$ using the antisymmetrization brackets

$$Y^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_k} = \mathcal{Y}^{[\nu_1 \dots \nu_k]}_{[\mu_1 \dots \mu_k]}$$

We can define also partially antisymmetric parts as follows

$$R^{\nu_1 \dots \nu_k}_{\mu_1 \dots \mu_k} = \mathcal{Y}^{\nu_1 \dots [\nu_i \nu_{i+1}] \dots \nu_k}_{\mu_1 \dots [\mu_l \mu_{l+1}] \dots \mu_k} = \frac{1}{4!}\left(\mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} - \mathcal{Y}^{\nu_1 \dots \nu_{i+1} \nu_i \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} + \mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_l \mu_{l+1} \dots \mu_k} - \mathcal{Y}^{\nu_1 \dots \nu_i \nu_{i+1} \dots \nu_k}_{\mu_1 \dots \mu_{l+1} \mu_l \dots \mu_k}\right)$$

Note how the indexes in the expressions with the label i and l simply got switched, and in the new definition, the tensor R is antisymmetric in both the *covariant* (lower) indexes μ_l, μ_{l+1} and in the *contravariant* (upper) indexes ν_i, ν_{i+1}, where obviously $i, l \leq k$

Theorem 4.4. *Let $T \in \mathcal{T}^k(\mathcal{V})$ and $\omega \in \Lambda^k(\mathcal{V})$. Then*

$$\begin{aligned} T_{[\mu_1 \dots \mu_k]} &\in \Lambda^k(\mathcal{V}) \\ \omega_{[\mu_1 \dots \mu_k]} &= \omega_{\mu_1 \dots \mu_k} \\ T_{[[\mu_1 \dots \mu_k]]} &= T_{[\mu_1 \dots \mu_k]} \end{aligned} \tag{4.14}$$

Definition 4.1.5 (Wedge Product)**.** Let $\omega \in \Lambda^k(\mathcal{V}),\ \eta \in \Lambda^l(\mathcal{V})$. In general $\omega \otimes \eta \notin \Lambda^{k+l}(\mathcal{V})$, hence we define a new product, called the *wedge product*, such that $\omega \wedge \eta \in \Lambda^{k+l}(\mathcal{V})$

$$\omega_{\mu_1 \dots \mu_k} \wedge \eta_{\nu_1 \dots \nu_k} = \frac{(k+l)!}{k!l!} \omega_{[\mu_1 \dots \mu_k} \eta_{\nu_1 \dots \nu_l]} \tag{4.15}$$

With the following properties
$\forall \omega, \omega_1, \omega_2 \in \Lambda^k(\mathcal{V}),\ \forall \eta, \eta_1, \eta_2 \in \Lambda^l(\mathcal{V}),\ \forall a \in \mathbb{R}, \forall f^\star \in \mathcal{L} : \mathcal{T}^k(\mathcal{V}) \longrightarrow \mathcal{T}^l(\mathcal{V})\ \forall \theta \in \Lambda^m(\mathcal{V})$

$$\begin{aligned} (\omega_1 + \omega_2) \wedge \eta &= \omega_1 \wedge \eta + \omega_2 \wedge \eta \\ \omega \wedge (\eta_1 + \eta_2) &= \omega \wedge \eta_1 + \omega \wedge \eta_2 \\ (\omega \wedge \eta) \wedge \theta &= \omega \wedge (\eta \wedge \theta) \\ a\omega \wedge \eta &= \omega \wedge a\eta = a(\omega \wedge \eta) \\ \omega \wedge \eta &= (-1)^{kl} \eta \wedge \omega \\ f^\star(\omega \wedge \eta) &= f^\star(\omega) \wedge f^\star(\eta) \end{aligned} \tag{4.16}$$

Theorem 4.5. *The set*

$$\{\varphi^{\mu_1} \wedge \cdots \wedge \varphi^{\mu_k},\ k < n\} \subset \Lambda^k(\mathcal{V}) \tag{4.17}$$

Is a basis for the space $\Lambda^k(\mathcal{V})$, and therefore

$$\dim(\Lambda^k(\mathcal{V})) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Where $\dim(\mathcal{V}) = n$.
Therefore, $\dim(\Lambda^n(\mathcal{V})) = 1$

Theorem 4.6. *Let $v_{\mu_1}, \cdots, v_{\mu_n}$ be a basis for $\mathcal{V}$, and take $\omega \in \Lambda^n(\mathcal{V})$, then, if $w_\mu = a^\nu_\mu v_\nu$*

$$\omega(w_{\mu_1} \cdots w_{\mu_n}) = \det_{\mu\nu}(a^\mu_\nu)\omega(v_{\mu_1}, \ldots, v_{\mu_n}) \tag{4.18}$$

Or using the basis representation of a vector $t^\mu = t^\mu w_\mu = t^\mu a^\nu_\mu v_\nu$ we have

$$\omega_{\mu_1\ldots\mu_n} t^{\mu_1} \cdots t^{\mu_n} = \det_{\mu\nu}(a^\mu_\nu)\omega_{\nu_1\ldots\nu_n} t^{\nu_1} \cdots t^{\nu_n} \tag{4.19}$$

Proof. Define $\eta_{\mu_1\ldots\mu_n} \in \mathcal{T}^n(\mathbb{R}^n)$ as

$$\eta_{\mu_1\ldots\mu_n} a^{\mu_1}_{\nu_1} a^{\mu_2}_{\nu_2} \cdots a^{\mu_n}_{v_n} = \omega_{\mu_1\ldots\mu_n} a^{\mu_1}_{\nu_1} \cdots a^{\mu_n}_{\nu_n}$$

Hence $\eta \in \Lambda^n(\mathbb{R}^n)$ so $\eta = \lambda \det(\cdot)$ for some λ, and

$$\lambda = \eta_{\mu_1\ldots\mu_n} e^{\mu_1} \cdots e^{\mu_n} = \omega_{\mu_1\ldots\mu_n} v^{\mu_1} \cdots v^{\mu_n}$$

□

§§ 4.1.2 Volume Elements and Orientation

Definition 4.1.6 (Orientation)**.** The previous theorem shows that a $\omega \in \Lambda^n(\mathcal{V}),\ \omega \neq 0$ splits the bases of $\mathcal{V}$ in two disjoint sets.
Bases for which $\omega(\mathcal{B}_v) > 0$ and for which $\omega(\mathcal{B}_w) < 0$. Defining $w^\mu = a^\mu_\nu v^\nu$ we have that the two bases belong to the same group iff $\det_{\mu\nu}(a^\mu_\nu) > 0$. We call this the *orientation* of the basis of the space. The *usual orientation* of $\mathbb{R}^n$ is

$$[e_\mu]$$

Given another two basis of $\mathbb{R}^n$ we can define (taking the first two examples)

$$\begin{gathered} [v_\mu] \\ -[w_\mu] \end{gathered}$$

Definition 4.1.7 (Volume Element)**.** Take a vector space $\mathcal{V}$ such that $\dim(\mathcal{V}) = n$ and it's equipped with an inner product g, such that there are two bases $(v^{\mu_1}, \cdots, v^{\mu_n}),\ (w^{\mu_1}, \cdots, w^{\mu_n})$ that satisfy the *orthonormality condition* with respect to this scalar product

$$g_{\mu\nu} v^{\mu_i} v^{\nu_j} = g_{\sigma\gamma} w^{\sigma_i} w^{\gamma_j} = \delta_{ij} \tag{4.20}$$

Then

$$\omega_{\mu_1\ldots\mu_n} v^{\mu_1} \cdots v^{\mu_n} = \omega_{\mu_1\ldots\mu_n} w^{\mu_1} \cdots w^{\mu_n} = \det_{\mu\nu}(a^\mu_\nu) = \pm 1$$

Where

$$w^\mu = a^\mu_\nu v^\nu$$

Therefore

$$\exists! \omega \in \Lambda^n(\mathcal{V}) \,:\, \exists! [w^{\mu_1}, \cdots, w^{\mu_n}] = O$$

Where O is the *orientation* of the vector space.

Definition 4.1.8 (Cross Product)**.** Let $v^\mu_1, \cdots, v^\mu_n \in \mathbb{R}^{n+1}$ and define $\varphi_\nu w^\nu$ as follows

$$\varphi_\nu w^\nu = \det \begin{pmatrix} v^{\mu_1} \\ \vdots \\ v^{\mu_n} \\ w^\nu \end{pmatrix}$$

Then $\varphi \in \Lambda^1(\mathbb{R}^{n+1})$, and

$$\exists! z^\mu \in \mathbb{R}^{n+1} \,:\, z^\mu w_\mu = \varphi_\nu w^\nu$$

z^μ is called the *cross product*, and it's indicated as

$$z^\mu = v^{\nu_1} \times \cdots \times v^{\nu_n} = \epsilon^\mu_{\ \nu_1 \dots \nu_n} v^{\nu_1} \cdots v^{\nu_n}$$

§ 4.2 Tangent Space and Differential Forms

Definition 4.2.1 (Tangent Space)**.** Let $p \in \mathbb{R}^n$, then the set of all pairs $\{(p, v^\mu) |\, v^\mu \in \mathbb{R}^n\}$ is denoted as $T_p\mathbb{R}^n$ and it's called the *tangent space* of $\mathbb{R}^n$ (at the point. This is a vector space defining the following operations

$$(p, av^\mu) + (p, aw^\mu) = (p, a(v^\mu + w^\mu)) = a(p, v^\mu + w^\mu) \quad \forall v^\mu, w^\mu \in \mathbb{R}^n,\ a \in \mathbb{R}$$

Remark. If a vector $v^\mu \in \mathbb{R}^n$ can be seen as an arrow from 0 to the point v, a vector $(p, v^\mu) \in T_p\mathbb{R}^n$ can be seen as an arrow from the point p to the point $p + v$. In concordance with the usual notation for vectors in physics, we will write $(p, v^\mu) = v^\mu$ directly, or v^μ_p when necessary to specify that we're referring to the vector $v^\mu \in T_p\mathbb{R}^n$. The point $p + v$ is called the *end point* of the vector v^μ_p.

Definition 4.2.2 (Inner Product in $T_p\mathbb{R}^n$)**.** The *usual inner product* of two vectors $v^\mu_p, w^\mu_p \in T_p\mathbb{R}^n$ is defined as follows

$$\begin{aligned} \langle \cdot, \cdot \rangle_p :& T_p\mathbb{R}^n \times T_p\mathbb{R}^n \longrightarrow \mathbb{R} \\ v^\mu_p w^p_\mu =& v^\mu w_\mu = k \end{aligned} \tag{4.21}$$

Analogously, one can define the usual orientation of $T_p\mathbb{R}^n$ as follows

$$[(e^{\mu_1})_p, \cdots, (e^{\mu_n})_p]$$

Definition 4.2.3 (Vector Fields, Again)**.** Although we already stated a definition for a vector field, we're gonna now state the actual precise definition of vector field
Let $p \in \mathbb{R}^n$ be a point, then a function $f^\mu(p) : \mathbb{R}^n \longrightarrow T_p\mathbb{R}^n$ is called a vector field, if $\forall p \in A \subseteq \mathbb{R}^n$ we can define

$$f^\mu(p) = f^\mu(p)(e_\mu)_p \tag{4.22}$$

Where $(e_\mu)_p$ is the canonical basis of $T_p\mathbb{R}^n$
All the previous *(and already stated)* considerations on vector fields hold with this definition.

Definition 4.2.4 (Differential Form)**.** Analogously to vector fields, one can define $k-$forms on the tangent space. These are called *differential (k-)forms* and "live" on the space $\Lambda^k(T_p\mathbb{R}^n)$.
Let $\varphi_p^{\mu_1}, \cdots, \varphi_p^{\mu_k} \in (T_p\mathbb{R}^n)^\star$ be a basis on such space, then the differential form $\omega \in \Lambda^k\left(T_p\mathbb{R}^n\right)$ is defined as follows

$$\omega_{\mu_1\ldots\mu_k}(p) = \omega_{\mu_1\ldots\mu_k}\varphi_p^{[\mu_1}\cdots\varphi_p^{\mu_k]} \to \sum_{i_1<\ldots<i_k}\omega_{i_1\ldots i_k}(p)\varphi_{i_1}(p)\wedge\cdots\wedge\varphi_{i_k}(p) \tag{4.23}$$

A function $f : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ is defined as $f \in \Lambda^0(T_p\mathbb{R}^n)$, or a $0-$form. In general, so, we can write without incurring in errors

$$f(p)\omega = f(p)\wedge\omega = f(p)\omega_{\mu_1\ldots\mu_k} \tag{4.24}$$

§§ 4.2.1 External Differentiation, Closed and Exact Forms

Definition 4.2.5 (Differential)**.** Now we will omit that we're working on a point $p \in \mathbb{R}^n$ and we'll use the usual notation.
Let $f : T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth (i.e. continuously differentiable) function, where $f \in C^\infty$, then, using operatorial notation we have that $\partial_\mu f(v) \in \Lambda^1(\mathbb{R}^n)$, therefore, with a small modification, we can define

$$\mathrm{d}f(v_p^\nu) = \partial_\mu f(v^\nu) \tag{4.25}$$

It's obvious how $\mathrm{d}x^\mu(v_p^\nu) = \partial_\nu x^\mu(v^\nu) = v^\mu$, therefore $\mathrm{d}x^\mu$ is a basis for $\Lambda^1(T_p\mathbb{R}^n)$, which we will indicate as $\mathrm{d}x^\mu$, therefore $\forall\omega \in \Lambda^k(T_p\mathbb{R}^n)$

$$\omega_{\mu_1\ldots\mu_k} = \omega_{\mu_1\ldots\mu_k}\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_k]} \to \sum_{i_1<\ldots<i_k}\omega_{i_1\ldots i_k}(p)\,\mathrm{d}x^{i_1}\wedge\cdots\wedge\mathrm{d}x^{i_k} \tag{4.26}$$

Basically, the vectors $\mathrm{d}x^\mu$ are the *dual basis* with respect to the canonical basis $(e_\mu)_p$

Theorem 4.7. *Since* $\mathrm{d}f(v_p^\nu) = \partial_\nu f(v^\nu)$ *we have, expressing the differential of a function with the basis vectors,*

$$\mathrm{d}f = \frac{\partial f}{\partial x^\mu}\,\mathrm{d}x^\mu = \partial_\mu f\,\mathrm{d}x^\mu \tag{4.27}$$

Definition 4.2.6. Having defined a smooth linear transformation $f_\nu^\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, it induces another linear transformation $\partial_\gamma f_\nu^\mu : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, which with some modifications becomes the application $(f_\star)_\nu^\mu : T_p\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}^m$ defined such that

$$(f_\star)_\nu^\mu(v^\nu) = \left(\mathrm{d}f|_{f(p)}\right)_\nu^\mu(v^\nu) \tag{4.28}$$

Which, in turn, also induces a linear transformation $f^\star : \Lambda^k(T_{f(p)}\mathbb{R}^m) \longrightarrow \Lambda^k(T_p\mathbb{R}^n)$, defined as follows. Let $\omega_p \in \Lambda^k(\mathbb{R}^m)$, then we can define $f^\star\omega \in \Lambda^k(T_{f(p)}\mathbb{R}^n)$ as follows

$$(f^\star\omega_p)(v_{\mu_1},\ldots,v_{\mu_k}) = \omega_{f(p)}\left((f_\star)_{\nu_1}^{\mu_1}v_{\mu_1},\cdots,(f_\star)_{\nu_k}^{\mu_k}v_{\mu_k}\right) \tag{4.29}$$

(Just remember that in this way we are writing explicitly the chosen base, watch out for the indexes!)

Theorem 4.8. *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ *be a smooth function, then*

1. $(f^\star)_\nu^\mu(\mathrm{d}x^\nu) = \mathrm{d}f = \partial_\nu f^\mu\,\mathrm{d}x^\nu$

2. $f^\star(\omega_1 + \omega_2) = f^\star\omega_1 + f^\star\omega_2$
3. $f^\star(g\omega) = (g \circ f) f^\star\omega$
4. $f^\star(\omega \wedge \eta) = f^\star\omega \wedge f^\star\eta$
5. $f^\star\left(h\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_n]}\right) = h \circ f \det_{\mu\nu}(\partial_\nu f^\mu)\,\mathrm{d}x^{[\mu_1}\cdots\mathrm{d}x^{\mu_n]}$

Definition 4.2.7 (Exterior Derivative)**.** We define the operator d as an operator $\Lambda^k(T_p\mathcal{V}) \xrightarrow{\mathrm{d}} \Lambda^{k+1}(T_p\mathcal{V})$ for some vector space $\mathcal{V}$. For a differential form ω it's defined as follows

$$(\mathrm{d}\omega)_{\nu\mu_1\dots\mu_k} = \partial_{[\nu}\omega_{\mu_1\dots\mu_k]} \tag{4.30}$$

This, using the classical mathematical notation can be written as follows

$$\begin{aligned} \mathrm{d}\omega &= \sum_{i_1<\dots<i_k} \mathrm{d}\omega_{i_1,\dots,i_k} \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \\ \mathrm{d}\omega &= \sum_{i_1<\dots<i_k}\sum_{j=1}^{n} \frac{\partial}{\partial x^j}\omega_{i_1,\dots,i_k}\,\mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \end{aligned} \tag{4.31}$$

Theorem 4.9 (Properties of d)**.** *1.* $\mathrm{d}(\omega + \eta) = \mathrm{d}\omega + \mathrm{d}\eta$

2. $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\omega \wedge \mathrm{d}\eta$ *for* $\omega \in \Lambda^k(\mathcal{V}),\ \eta \in \Lambda^l(\mathcal{V})$
3. $\mathrm{d}\mathrm{d}\omega = \mathrm{d}^2\,\omega = 0$
4. $f^\star(\mathrm{d}\omega) = \mathrm{d}(f^\star\omega)$

Definition 4.2.8 (Closed and Exact Forms)**.** A form ω is called *closed* iff

$$\mathrm{d}\omega = 0 \tag{4.32}$$

It's called exact iff

$$\omega = \mathrm{d}\eta \tag{4.33}$$

Theorem 4.10. *Let ω be an exact differential form. Then it's closed*

Proof. The proof is quite straightforward. Since ω is exact we can write $\omega = \mathrm{d}\rho$ for some differential form ρ, therefore

$$\mathrm{d}\omega = \mathrm{d}\mathrm{d}\rho = \mathrm{d}^2\,\rho = 0$$

Hence $\mathrm{d}\omega = 0$ and ω is closed. □

Example 4.2.1. Take $\omega \in \Lambda^1(\mathbb{R}^2)$, where it's defined as follows

$$\omega_\mu = p\,\mathrm{d}x + q\,\mathrm{d}y \tag{4.34}$$

The external derivative will be of easy calculus by remembering the mnemonic rule $\mathrm{d} \to \partial_\mu \wedge \mathrm{d}x^\mu$, or also as $\partial_{[\nu}$ then we have

$$\mathrm{d}\omega_{\mu\nu} = \partial_{[\nu}\omega_{\mu]}$$

But

$$\partial_\nu \omega_\mu = \begin{pmatrix} \partial_1\omega_1 & \partial_1\omega_2 \\ \partial_2\omega_1 & \partial_2\omega_2 \end{pmatrix}_{\mu\nu}$$

And

$$\partial_{[\nu}\omega_{\mu]} = \frac{1}{2}(\partial_\nu\omega_\mu - \partial_\mu\omega_\nu) = \frac{1}{2}(\partial\omega - \partial\omega^{\mathrm{T}})$$

Therefore

$$\mathrm{d}\omega_{\mu\nu} = \frac{1}{2}\begin{pmatrix} 0 & \partial_x q - \partial_y p \\ \partial_y p - \partial_x q & 0 \end{pmatrix}_{\mu\nu}$$

Which, expressed in terms of the basis vectors of $\Lambda^2(\mathbb{R}^2)$, $\mathrm{d}x \wedge \mathrm{d}y$, we get

$$\mathrm{d}\omega = \frac{1}{2}(\partial_x q - \partial_y p)\,\mathrm{d}x \wedge \mathrm{d}y + \frac{1}{2}(\partial_y p - \partial_x q)\,\mathrm{d}y \wedge \mathrm{d}x = (\partial_x q - \partial_y p)\,\mathrm{d}x \wedge \mathrm{d}y \tag{4.35}$$

Therefore

$$\mathrm{d}\omega = 0 \iff \partial_x q - \partial_y p = 0 \tag{4.36}$$

Definition 4.2.9 (Star Shaped Set)**.** A set A is said to be *star shaped with respect to a point a* iff $\forall x \in A$ the line segment $[a, x] \subset A$

Lemma 4.2.1 (Poincaré's)**.** Let $A \subset \mathbb{R}^n$ be an open star shaped set, with respect to 0. Then every closed form on A is exact

§ 4.3 Chain Complexes and Manifolds

§§ 4.3.1 Singular $n-$cubes and Chains

Definition 4.3.1 (Singular $n-$cube)**.** A *singular n-cube* is an application $c : [0,1]^n \longrightarrow A \subset \mathbb{R}^n$. In general. A singular 0-cube is a function $f : \{0\} \longrightarrow A$ and a singular $1-$cube is a curve.

Definition 4.3.2 (Standard $n-$cube)**.** We define a *standard n-cube* as a function $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ such that $I^n(x^\mu) = x^\mu$.

Definition 4.3.3 (Face)**.** Given a standard $n-$cube I^n we define the $(i,\alpha)-$face of the cube as

$$I^n_{(i,\alpha)} = (x^1, \cdots, x^{i-1}, \alpha, x^i, \cdots, x^{n-1}) \quad \alpha = 0, 1 \tag{4.37}$$

Definition 4.3.4 (Chain)**.** Given n $k-$cubes c_i, we define a *n-chain* s as follows

$$s = \sum_{i=1}^{n} a_i c_i \quad a_i \in \mathbb{R} \tag{4.38}$$

Definition 4.3.5 (Boundary)**.** Given an $n-$cube c_i we define the *boundary* as ∂c_i. For a standard $n-$cube we have

$$\partial I^n = \sum_{i=1}^{n}\sum_{\alpha=0,1}(-1)^{i+\alpha} I^n_{(i,\alpha)} \tag{4.39}$$

For a $k-$chain s we define

$$\partial s = \partial(\sum_i a_i c_i) = \sum_i a_i \partial c_i \tag{4.40}$$

Where ∂s is a $(k-1)$-chain

Theorem 4.11. *For a chain c, we have that $\partial\partial c = \partial^2 c = 0$*

§§ 4.3.2 Manifolds

Definition 4.3.6 (Manifold)**.** Given a set $M \subset \mathbb{R}^n$, it is said to be a *k-dimensional manifold* if $\forall x^\mu \in M$ we have that

1. $\exists U \subset \mathbb{R}^k$ open set $x^\mu \in U$ and $V \subset \mathbb{R}^n$ and φ a diffeomorphism such that $U \simeq V$ and $\varphi(U \cap M) = V \cap \left(\mathbb{R}^k \times \{0\}\right)$, i.e. $U \cap M \simeq \mathbb{R}^k \cap \{0\}$
2. $\exists U \subset \mathbb{R}^k$ open and $W \subset \mathbb{R}^k$ open, $x^\mu \in U$ and $f : W \longrightarrow \mathbb{R}^n$ a diffeomorphism
   (a) $f(W) = M \cap U$
   (b) $\operatorname{rank}(f) = k \ \forall x^\mu \in W$
   (c) $f^{-1} \in C(f(W))$

The function f is said to be a *coordinate system in M*

Definition 4.3.7 (Half Space)**.** We define the *k-dimensional half space* $\mathbb{H}^k \subset \mathbb{R}^k$ as

$$\mathbb{H}^k := \left\{ x^\mu \in \mathbb{R}^k \middle| \, x^i \geq 0 \right\} \tag{4.41}$$

Definition 4.3.8 (Manifold with Boundary)**.** A *manifold with boundary* (MWB) is a manifold M such that, given a diffeomorphism h, an open set $U \supset M$ and an open set $V \subset \mathbb{R}^n$

$$h(U \cap V) = V \cap \left(\mathbb{H}^k \cap \{0\}\right) \tag{4.42}$$

The set of all points that satisfy this forms the set ∂M called the *boundary of M*

Definition 4.3.9 (Tangent Space)**.** Given a manifold M and a coordinate set f around $x^\mu \in M$, we define the *tangent space of M at* $x^\mu \in M$ as follows

$$f : W \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n \implies f_\star\left(T_x\mathbb{R}^k\right) = T_x M \tag{4.43}$$

Definition 4.3.10 (Vector Field on a Manifold)**.** Given a vector field f^μ we identify it as a vector field on a manifold M if $f^\mu(x^\nu) \in T_x M$. Analogously we define a $k-$differential form

5 Integral Analysis

§ 5.1 Measure Theory

Definition 5.1.1 (Lower and Upper Sums)**.** We define the *upper* and *lower Riemann sums* as follows.
Let $f(x)$ be a function, then

$$\begin{cases} \mathcal{U}(f,x) := \sum_{i=1}^{n} \sup_{t\in[x_k,x_{k+1}]} (f(t)) \\ \mathcal{L}(f,x) := \sum_{i=1}^{n} \inf_{t\in[x_k,x_{k+1}]} (f(t)) \end{cases} \tag{5.1}$$

A function is said to be Riemann integrable if $\lim_{n\to\infty}(\mathcal{L}(f,x) - \mathcal{U}(f,x)) = 0$

Definition 5.1.2 (Set Function)**.** Let A be a set. We define the following function $\mathbb{1}_A(x)$ as follows

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{5.2}$$

Theorem 5.1. *The function $\mathbb{1}_{\mathbb{Q}}$ is not integrable over the set $[0,1]$ with the usual definition of the integral (Riemann sums)*

Proof. Indicating the integral I as usual

$$I = \int_0^1 \mathbb{1}_{\mathbb{Q}}(x)\,\mathrm{d}x$$

We see immediately that

$$\mathcal{U}(\mathbb{1}_{\mathbb{Q}}, x) = 1$$
$$\mathcal{L}(\mathbb{1}_{\mathbb{Q}}, x) = 0$$

Therefore $\mathbb{1}_{\mathbb{Q}}(x)$ is not integrable in $[0,1]$ (with the Riemann integral) □

Definition 5.1.3 (Measure)**.** Let $A \subset X$ be a subset of a metric space. We define the measure of the set A, $\mu(A)$ as follows

$$\mu(A) = \int_X \mathbb{1}_A(x)\,\mathrm{d}x \tag{5.3}$$

Basically, what we did before, was demonstrating that the set $\mathbb{Q} \cap [0,1]$ is not measurable in the Riemann integration theory. This is commonly indicated with saying that the set $\mathbb{Q} \cap [0,1]$ *is not Jordan measurable.*
For clarity, let K be some measure theory. We will say that a set is K-*measurable* if the following calculation exists

$$\mu_K(A) = \int_X \mathbb{1}_A(x)\,\mathrm{d}x \tag{5.4}$$

Definition 5.1.4 (Algebra)**.** Let $X \neq \{\}$ be a set. An *algebra* $\mathcal{A}$ over X is a collection of subsets of X such that

1. $\{\} \in \mathcal{A}$
2. $X \in \mathcal{A}$
3. $A \in \mathcal{A} \implies A^{\mathrm{c}} \in \mathcal{A}$
4. $A_1, \cdots, A_n \in \mathcal{A} \implies \bigcup_{i=1}^n A_i, \bigcap_{i_1}^n A_i \in \mathcal{A}$

Example 5.1.1 (Simple Set Algebra). Let $X = \mathbb{R}^2$ and call R the set of all rectangles $I_i \subset \mathbb{R}^\star \times \mathbb{R}^\star$, where $\mathbb{R}^\star = \mathbb{R} \cup \{\pm\infty\}$. It's easy to see that this is not an algebra, since, by taking $[0,1] \in R$, we have that $[0,1]^{\mathrm{c}} \notin R$, hence it cannot be an algebra.
But, taken $\mathcal{S}$ as follows

$$\mathcal{S} := \left\{ A \subset \mathbb{R}^2 \middle| \, A = \bigcup_{i=1}^n I_i \quad I_i \in R \right\}$$

We can see easily, using De Morgan law, that $\mathcal{S}$ is an algebra.

§§ 5.1.1 Jordan Measure

Definition 5.1.5 (Disjoint Union)**.** Taken two sets A, B, we define their *disjoint union* the binary operation $A \sqcup B$ as follows

$$A \sqcup B := A \cup B \setminus A \cap B \tag{5.5}$$

Definition 5.1.6 (Simple Set)**.** A set A is a *simple set* iff, for some $R_i \in \mathcal{S}$, we have

$$A = \bigsqcup_{i=1}^n R_i$$

Definition 5.1.7 (Measure of a Simple Set)**.** Let A be a simple set, the *Jordan measure* of a simple set is given by the sum of the measure of the rectangles, i.e. the "area" of A is given by the sum of the area of each rectangle R_i

$$\mu_J(A) = \sum_{i=1}^n \mu_J(R_i) \tag{5.6}$$

Definition 5.1.8 (External and Internal Measure)**.** We define the external measure $\overline{\mu}_J$ and the internal measure $\underline{\mu}_J$ as follows.
Taken a limited set B and a simple set A we have

$$\begin{aligned} \overline{\mu}_J(B) &= \inf\{\, \mu_J(A) |\, B \subset A\} \\ \underline{\mu}_J(B) &= \sup\{\, \mu_J(A) |\, A \subset B\} \end{aligned} \tag{5.7}$$

A set is said to be *Jordan measurable* iff

$$\overline{\mu}_J(B) = \underline{\mu}_J(B) = \mu_J(B)$$

Remark (A Non Measurable Set). A good example for showing that the Jordan measure is the set we were trying to measure, the set $\mathbb{Q} \cap [0,1]$. We can easily see that

$$\overline{\mu}_J(\mathbb{Q} \cap [0,1]) = 1$$
$$\underline{\mu}_J(\mathbb{Q} \cap [0,1]) = 0$$

Therefore it's not Jordan measurable.
From this we can jump to a new definition of measure, which is the *Lebesgue measure* where instead of covering $\mathbb{Q} \cap [0,1]$ with a *finite* number of simple sets, we use sets which are formed from the union of *countable infinite* simple sets.
We can define

$$\mathbb{Q} \cap [0,1] := \{q_1, q_2, \cdots\}$$

We then take $\epsilon > 0$ and define the following set

$$A = \bigcup_{n=1}^{\infty} \left[q_n - \frac{\epsilon}{2^n}, q_n + \frac{\epsilon}{2^n}\right]$$

We have that

$$\mu(A) \leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n-1}} = 2\epsilon$$

But $\overline{\mu}(\mathbb{Q} \cap [0,1]) \leq \mu(A) \leq 2\epsilon \to 0$, therefore $\mathbb{Q} \cap [0,1]$ is measurable with $\mu(\mathbb{Q} \cap [0,1]) = 0$

§§ 5.1.2 Lebesgue Measure

Definition 5.1.9 ($\sigma-$Algebras and Borel Spaces)**.** Given a non empty set X a $\sigma-$*algebra* on X is a collection of subsets $\mathcal{F}$ such that

1. $\forall A \in \mathcal{F},\ A \subset X$
2. Let $A_i \in \mathcal{F},\ i \in \mathcal{I} : |\mathcal{I}| = \aleph_0$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

The couple $(X, \mathcal{F})$ is called a *Borel space* or also a *measurable space*

Definition 5.1.10 (Measure)**.** Given a Borel space $(X, \mathcal{F})$, we can define an application

$$\mu : \mathcal{F} \longrightarrow [0, \infty] = \mathbb{R}^{\star}_{+} \tag{5.8}$$

Which satisfies the following properties

1. σ-additivity, given $A_i \in \mathcal{F}$ with $i \in I \subset \mathbb{N},\ |I| \leq \aleph_0$, such that $A_n \cap A_k = \{\}$ for $n \neq k$

$$\mu\left(\bigsqcup_{i\in I} A_i\right) = \sum_{i \in I} \mu(A_i)$$

2. If $Y_j \subset X$, with $j \in J \subseteq \mathbb{N}$, $\mu(Y_j) < \infty$ then $X = \bigcup_{j=1}^{\infty} Y_j$

Definition 5.1.11 (Measure Space)**.** A *measure space* is a triplet $(X, \mathcal{F}, \mu)$ with $\mathcal{F}$ a $\sigma-$algebra and μ a measure.

Remark. The empty set has null measure.

Proof. Due to $\sigma-$additivity we have that

$$\mu(\{\}) = \mu(\{\} \cup \{\}) = \mu(\{\}) + \mu(\{\})$$

Therefore, $\mu(\{\}) = 0$ necessarily. □

Definition 5.1.12 (Lebesgue Measure)**.** Consider again $X = \mathbb{R}^2$ and $\mathcal{S}$ the algebra of simple sets. The *external Lebesgue measure* of a set $B \subset \mathbb{R}^2$ is then defined as follows

$$\overline{\mu}_L(B) := \inf \left\{ \sum_{i=1}^{\infty} \text{Area}(R_i) \middle| R_i \in \mathcal{S},\ B \subset \bigcup_{i=1}^{\infty} R_i \right\} \tag{5.9}$$

The set B is said to be *Lebesgue measurable* iff, $\forall C \subset \mathbb{R}^2$

$$\overline{\mu}_L(C) = \overline{\mu}_L(C \cap B) + \overline{\mu}_L(C \setminus B) \tag{5.10}$$

If it's measurable, then, $\overline{\mu}_L(B) = \mu_L(B)$ and it's called the *Lebesgue measure* of the set.
In other words $\exists \epsilon > 0 : \exists A, C \subset \mathbb{R}^2$, with $A = A^\circ$, $C = \overline{C}$ such that

$$C \subset B \subset A \ \vee \ \overline{\mu}_L(A \setminus C) < \epsilon \tag{5.11}$$

Definition 5.1.13 (Borel Algebra)**.** Let R be the set of all rectangles. The smallest $\sigma-$algebra containing R is called the *Borel algebra* and it's indicated as $\mathcal{B}$

Definition 5.1.14 (Lebesgue Algebra)**.** The set of (Lebesgue) measurable sets is a $\sigma-$algebra, which we will indicate as $\mathcal{L}$. In particular, we have that, if I is a rectangle, $I \in \mathcal{L}$.
If we add the fact that in $\mathcal{B}$ there are null measure sets which have subsets which aren't part of $\mathcal{B}$, we end up with the conclusion that $\mathcal{B} \subset \mathcal{L}$

Definition 5.1.15 (Null Measure Sets)**.** A set with null measure is a set $X \subset \mathcal{F}$ such that

$$\mu(X) = 0 \tag{5.12}$$

Where μ is a measure function.
It's obvious that sets formed by a single point have null measure.
I.e take a set $A = \{a\}$, then it can be seen as a rectangle with 0 area, and therefore

$$\mu(\{a\}) = 0 \tag{5.13}$$

Theorem 5.2. *Every set such that* $|A| = \aleph_0$ *has null measure*

Corollary 5.1.1. Every line in $\mathbb{R}^2$ has null measure

Proof. Take the set $A = \{a_1, a_2, a_3, \cdots\}$. Then, due to $\sigma-$additivity, we have

$$\mu(\{a_1, a_2, a_3, \cdots\}) = \mu\left(\bigsqcup_{k=1}^{\infty} \{a_k\}\right) = \sum_{k=1}^{\infty} \mu(\{a_k\}) = 0 \tag{5.14}$$

For the corollary, it's obvious if the line is thought as a rectangle in $\mathbb{R}^2$ with null area □

§ 5.2 Integration

Definition 5.2.1 (Measurable Function)**.** Given a Borel space $(X, \mathcal{F})$ a *measurable function* is a function $f : X \longrightarrow \mathbb{F}$ such that, $\forall k \in \mathbb{F}$ the following set is measurable

$$I_f := \{ k \in \mathbb{F} |\, f(x) < k \} \tag{5.15}$$

Or, in other words $I_f \in \mathcal{F}$, with $\mathcal{F}$ the given $\sigma-$algebra of the Borel space.
The space of all measurable functions on X will be identified as $\mathcal{M}(X)$

Theorem 5.3. *Given a set $A \in \mathcal{F}$ with $\mathcal{F}$ a $\sigma-$algebra, the function $\mathbb{1}_A(x)$ is measurable*

Proof. We have that

$$I_{\mathbb{1}_A} = \begin{cases} A & k > 1 \\ \{\} & t \leq 1 \end{cases}$$

Therefore $I_{\mathbb{1}_A} \in \mathcal{F}$ and $\mathbb{1}_A(x)$ is measurable ☐

Definition 5.2.2 (Simple Measurable Function)**.** Given a Borel space $(X, \mathcal{F})$, a *simple measurable function* is a function $f : X \longrightarrow \mathbb{F}$ which can be written as follows

$$f(x) = \sum_{k=1}^{n} c_k \mathbb{1}_{A_k}(x) \tag{5.16}$$

Where $A_k \in \mathcal{F}, \ c_k \in \mathbb{F} \quad 0 \leq k \leq n$

Definition 5.2.3 (Integral)**.** Given a measure space $(X, \mathcal{F}, \mu)$ and a simple function $f(x)$, we can define the *integral* of the function f with respect to the measure μ over the set X as follows

$$\int_X f(x)\mu\,(\mathrm{d}x) = \sum_{k=1}^{n} c_k \mu(A_k) \tag{5.17}$$

For non negative functions we define the integral as follows

$$\int_X f(x)\mu\,(\mathrm{d}x) = \sup \left\{ \int_X g(x)\mu\,(\mathrm{d}x) \right\} \tag{5.18}$$

Where $g(x)$ is a simple measurable function such that $0 \leq g \leq f$.
If f assumes both negative and positive values we can write

$$f = f^+ - f^- \tag{5.19}$$

Where

$$\begin{cases} f^+ = \max f, 0 \\ f^- = \max -f, 0 \end{cases} \tag{5.20}$$

The integral, due to linearity, then will be

$$\int_X f(x)\mu\,(\mathrm{d}x) = \int_X f^+(x)\mu\,(\mathrm{d}x) - \int_X f^-(x)\mu\,(\mathrm{d}x) \tag{5.21}$$

With the only constraint that the function $f(x)$ must be misurable in the σ-algebra $\mathcal{F}$

§§ 5.2.1 Lebesgue Spaces

Definition 5.2.4 ($\mathcal{L}^p$ spaces)**.** With the previous definitions, we can define an *infinite dimensional function space* with the following properties
Given a measure space $(X, \mathcal{F}, \mu)$ we have the following definition

$$\mathcal{L}^p\left(X, \mathcal{F}, \mu\right) = \mathcal{L}^p(\mu) := \left\{ f : X \longrightarrow \mathbb{F} |\, I_f \in \mathcal{F} \wedge \int_X |f|^p \mu\left(\mathrm{d}x\right) < \infty \right\} \tag{5.22}$$

Defining the integral as an *operator* $\hat{K}_\mu[f]$ we can see easily that this is a vector spaces due to the properties of $\hat{K}_\mu$.
It's easy to note that if the chosen $\sigma-$algebra and measure are the Lebesgue ones, then this integral is simply an extension of the usual Riemann integral.
It's important to note that a norm in $\mathcal{L}^p(\mu)$ can't be defined as an usual integral $p-$norm, since there are nonzero functions which have actually measure zero.

Definition 5.2.5 (Almost Everywhere Equality)**.** Taken two functions $f, g \in \mathcal{L}^p(\mu)$ we say that these two function are *almost everywhere equal* if, given a set $A := \{ x \in X |\, f(x) \neq g(x) \}$ has null measure. Therefore

$$f \sim g \iff \mu(A) = 0 \tag{5.23}$$

This equivalence relation creates equivalence classes of functions compatible with the vector space properties of $\mathcal{L}^p(\mu)$.

Definition 5.2.6 (L^p-Spaces)**.** With the definition of the almost everywhere equality we can then define a quotient space as follows

$$L^p(\mu) = \mathcal{L}^p(\mu) \backslash \sim \tag{5.24}$$

This is a vector space, obviously, where the elements are the equivalence classes of functions $f \in \mathcal{L}^p(\mu)$, indicated as $[f]$.
If we define our $\sigma-$algebra and measure as the Lebesgue ones, this space is called the *Lebesgue space* $L^p(X)$, where an integral $p-$norm can be defined.

§§ 5.2.2 Lebesgue Integration

Note:
In this section the differential $\mathrm{d}x$ will actually indicate the Lebesgue measure $\mu\left(\mathrm{d}x\right)$ used previously, unless stated otherwise.

Theorem 5.4. *Let $f : E \longrightarrow \mathbb{F}$ be a measurable function over E.*
Given

$$F_{+\infty} = x \in E |\, f(x) = +\infty \;\wedge\; F_{-\infty} = x \in E |\, f(x) = -\infty$$

Assuming $E \subset X$, with $(X, \mathcal{L}, \mu)$ a Lebesgue measure space, we have that

$$\mu\left(F_{+\infty}\right) = \mu\left(F_{-\infty}\right) = 0$$

Proof. We can immediately say that

$$F_{+\infty} = \bigcap_{k \geq 0} F_k \in \mathcal{L}$$

Letting $r > 0$ we will indicate with $\mathbb{1}_r(x)$ the set function of the set $F_{+\infty} \cap B_r(0)$, therefore we have that

$$f^+(x) \geq k\mathbb{1}_r(x) \quad \forall k \in \mathbb{N}$$

Therefore

$$\mu\left(F_{+\infty} \cap B_r(0)\right) = \int \mathbb{1}_r(x)\,\mathrm{d}x \leq \frac{1}{k}\int_E f^+(x)\,\mathrm{d}x \longrightarrow 0$$

The proof is analogous for $F_{-\infty}$ □

Theorem 5.5. *Let $(X, \mathcal{L}, \mu)$ be a measure space, where $\mathcal{L}$ is the Lebesgue $\sigma-$algebra and μ is the Lebesgue measure. Given a function $f \in L^1(X)$ we have that*

$$\int_X f(x)\,\mathrm{d}x = 0 \iff f \sim 0 \tag{5.25}$$

Proof. Let $F_0 = x \in X|\, f(x) > 0 = \bigcap_{k\geq 0} F_{1/k}$.
Since $f(x) > 1/k,\ \forall x \in F_{1/k}$, we have that, $\forall k \in \mathbb{N}$

$$\mu(F_{1/k}) \leq \int_X f(x)\,\mathrm{d}x = 0$$

Through induction, we obtain that $\mu(F_0) = 0$ □

Theorem 5.6 (Monotone Convergence (B. Levi))**.** *Let $(f)_k$ be a sequence of measurable functions over a Borel space E, such that*

$$0 \leq f_1(x) \leq \cdots \leq f_k(x) \leq \cdots \quad \forall x \in F \subset E,\ \mu(F) = 0$$

If $f_k(x) \to f(x)$, we have that

$$\int_E f(x)\,\mathrm{d}x = \lim_{k\to\infty}\int_E f_k(x)\,\mathrm{d}x \tag{5.26}$$

Or, in another notation

$$\int_E f_k(x)\,\mathrm{d}x \to \int_E f(x)\,\mathrm{d}x \tag{5.27}$$

Proof. Let $F_{0k} = 0 < y < f_k(x)$ and $F_0 = 0 < y < f(x)$ be two sets defined as seen. They are all measurable since $f_k(x), f(x)$ are measurable, and due to the monotony of $f_k(x)$ we have that

$$F_{01} \subset F_{02} \subset \cdots \subset F_{0k} \subset \cdots \ \wedge\ F_0 = \bigsqcup_{k=1}^{\infty} F_{0k}$$

Due to $\sigma-$additivity of the measure function, we have that F_0 is measurable, and that

$$\mu\left(F_0\right) = \sum_{k=1}^{\infty} \mu\left(F_{0k}\right) \quad \therefore \mu\left(F_0\right) = \lim_{k\longrightarrow\infty} \mu\left(F_{0k}\right)$$

□

Notation (For Almost All)**.** We now introduce a new (unconventional) symbol in order to avoid writing too much, which would complicate the already difficult to understand theorems.
In order to indicate that we're picking *almost all* elements of a set we will use a new quantifier, which means that we're picking all elements of a null measure subset of the set in question. The quantifier in question will be the following

$$\forall^\dagger \tag{5.28}$$

Corollary 5.2.1. Let $f_k(x)$ be a sequence of non-negative measurable functions over a measurable set E, then $\forall^\dagger\, x \in E$

$$\int_E \sum_{k\geq 0} f_k(x)\,\mathrm{d}x = \sum_{k\geq 0}\int_E f_k(x)\,\mathrm{d}x \tag{5.29}$$

Theorem 5.7 (Fatou)**.** *Let $f_k(x)$ be a sequence of measurable functions over a measurable set E, such that $\forall^\dagger\, x \in E\ \exists\Phi(x)$ measurable: $f_k(x) > \Phi(x)$, then*

$$\int_E \liminf_{k\to\infty} f_k(x)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_E f_k(x)\,\mathrm{d}x$$

Analogously happens with the $\limsup$ *of the sequence*

Proof. Let $h_k(x) = f_k(x) - \Phi(x) \geq 0\ \ \forall^\dagger\, x \in E$ and $g_j(x) = \inf_{k\geq k} h_k(x)$, then $\forall k \geq j$ we have

$$\int_E g_j(x)\,\mathrm{d}x \leq \int_E h_k(x)\,\mathrm{d}x$$

It's also (obviously) true taking the $\limsup$ of the RHS, and for the theorem on the monotone convergence, we have that

$$\int_E \lim_{j\to\infty} g_j(x)\,\mathrm{d}x = \lim_{j\to\infty}\int_E g_j(x)\,\mathrm{d}x \leq \int_E h_k(x)\,\mathrm{d}x$$
$$\therefore \lim_{j\to\infty} g_j(x) = \sup_j g_j(x) = \sup_j \inf_{k\geq j} h_k(x) = \liminf_{k\to\infty} h_k(x)$$

□

Theorem 5.8 (Dominated Convergence (Lebesgue))**.** *Let $h(x) \geq 0$ be a measurable function on the measurable set E such that for a sequence of measurable functions $f_k(x)$ we have that*

$$|f_k(x)| \leq h(x) \quad \forall^\dagger\, x \in E$$

And

$$f(x) = \lim_{k\to\infty} f_k(x) \quad \forall^\dagger\, x \in E$$

Then

$$\int_E f(x)\,\mathrm{d}x = \lim_{k\to\infty}\int_E f_k(x)\,\mathrm{d}x$$

Proof. By definition we have that $-h(x) \leq f_k(x) \leq h(x)\ \ \forall^\dagger\, x \in E$, and we can apply Fatou's theorem

$$\int_E f(x)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_E f_k(x)\,\mathrm{d}x \leq \limsup_{k\to\infty}\int_E f_k(x)\,\mathrm{d}x \leq \int_E f(x)\,\mathrm{d}x$$

□

Corollary 5.2.2. Let E be a measurable set such that $\mu(E) < \infty$ and let $f_k(x)$ be a sequence of functions in E such that $|f_k(x)| \le M \ \forall^\dagger x \in E$ and $f_k(x) \to f(x), \ \forall^\dagger x \in E$. Then the theorem (5.8) is valid.

Example 5.2.1. Take the sequence of functions $f_k(x) = kxe^{-kx}$ over $E = [0,1]$. We already know that $f_k(x) \longrightarrow f(x) = 0$ for $x \in E$, but $f_k(x) \not\rightrightarrows f(x)$ in E.
We have that

$$\sup_E f_k(x) = e^{-1} = h(x) \neq f(x)$$

We have that $h(x)$ is measurable in E and we can apply the theorem (5.8)

Definition 5.2.7 (Carathéodory Function)**.** Let $(X, \mathcal{L}, \mu)$ be a measure space and $A \subset \mathbb{R}^n$. $f : X \times A \longrightarrow \mathbb{R}$ is a *Carathéodory function* iff $f(x^\mu, a^\nu) \in C(A) \ \forall a^\nu \in A$ and $f(x^\mu, a^\nu) \in \mathcal{M}(X) \ \forall^\dagger x^\mu \in X$

Definition 5.2.8 (Locally Uniformly Integrably Bounded)**.** Let $f : X \times A \longrightarrow \mathbb{R}$ be a Carathéodory function. It's said to be *locally uniformly integrably bounded* if $\forall a^\nu \in A \ \exists h_{a^\nu} : X \longrightarrow \mathbb{R}$ measurable, and $\exists B_\epsilon(a^\nu) \subset A$, such that

$$\forall y^\nu \in B_\epsilon(x^\mu) \ |f(x^\mu, y^\nu)| \le h_{a^\nu}(x^\mu)$$

Note that if μ is a finite measure, then f bounded $\implies$ f locally uniformly integrably bounded or LUIB.

Theorem 5.9 (Leibniz's Derivation Rule)**.** *Let $(X, \mathcal{F}, \mu)$ be a measure space and $A \subset \mathbb{R}^n$ an open set. If $f : X \times A \longrightarrow \mathbb{R}$ is a LUIB Carathéodory function we can define*

$$g(a^\mu) = \int_X f(x^\nu, a^\mu) \, \mathrm{d}\mu\,(x^\sigma) \in C(A)$$

Then

$$\partial_{x^\mu} f(x^\nu, a^\sigma) \in C(A)$$

Is LUIB, and therefore

$$g(a^\mu) \in C^1(A)$$

And

$$\partial_\mu g = \int_X \partial_{a^\mu} f(a^\nu, x^\sigma) \, \mathrm{d}\mu\,(x^\gamma)$$

In other terms

$$\partial_{a^\mu} \int_X f(a^\nu, x^\sigma) \, \mathrm{d}\mu(x^\gamma) = \int_X \partial_{a^\mu} f(a^\nu, x^\sigma) \, \mathrm{d}\mu(x^\gamma) \tag{5.30}$$

Proof. Since f is a LUIB Carathéodory function we have that $\exists h_{a^\mu}(x^\nu) : X \longrightarrow \mathbb{R}$ and $B_\epsilon(a^\mu) \subset A : \forall y^\mu \in B_\epsilon(a^\nu)$

$$|f(y^\mu, x^\nu)| \le h_{a^\mu}(x^\nu)$$

Therefore

$$|g(a^\mu)| \le \int_X h_{a^\mu}(x^\nu) \, \mathrm{d}\mu(x^\sigma) < \infty$$

Now take a sequence $(a^\mu)_n : (a^\mu)_n \to a^\mu$, then $f \in C(A) \implies f(a^\mu_n, x^\nu) \to f(a^\mu, x^\nu) \ \forall^\dagger x^\mu \in X, \forall a^\mu_n \in B_\epsilon(a^\mu)$

$$\therefore \exists N \in \mathbb{N} : \forall n \ge N \ |f(a^\mu_n, x^\nu)| \le h_{a^\mu}(x^\nu)$$

Then

$$g(a_n^\mu) = \int_X f(a_n^\mu, x^\nu)\,\mathrm{d}\mu(x^\sigma) \to \int_X f(a^\mu, x^\nu)\,\mathrm{d}\mu(x^\sigma) = g(a^\mu)$$

Since f is differentiable and its derivative is measurable, we have for the mean value theorem

$$f(a^\mu + te^\mu, x^\nu) - f(a^\mu, x^\nu) = t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)$$

If $\xi^\mu(t, x^\nu) \in B_\epsilon(a^\mu)$ we have that

$$|t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)| \le h_{a^\mu}(x^\nu)$$

And therefore

$$\frac{g(a^\mu + te^\mu) - g(a^\mu)}{t} = \frac{1}{t}\int_X t\partial_\mu f(\xi^\nu(t, x^\sigma), x^\gamma)\,\mathrm{d}\mu(x^\delta)$$

For $t \to 0$ $\partial_\mu f(\xi^\nu, x^\sigma) \to \partial_\mu f(a^\nu, x^\sigma)$, and the LHS is simply the gradient of g. Therefore for theorem (5.8)

$$\partial_\mu g(a^\nu) = \frac{\partial}{\partial a^\mu}\int_X f(a^\nu, x^\sigma)\,\mathrm{d}\mu(x^\gamma) = \int_X \partial_\mu f(a^\nu, x^\sigma)\,\mathrm{d}\mu(x^\gamma)$$

□

§ 5.3 Calculus of Integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$

§§ 5.3.1 Double Integration

Theorem 5.10. *Let $E \subset \mathbb{R}^2$ and $F \subset \mathbb{R}^3$. Define $E_x := \{y \in \mathbb{R} |\, (x, y) \in E\}$ the sections of E parallel to the y axis, then*

$$\mu(E) = \int_{\mathbb{R}} \mu_1(E_x)\,\mathrm{d}y \tag{5.31}$$

Where with μ_i we indicate the $i-$dimensional measure on $\mathbb{R}^n$.
Analogously, we define $F_z := \left\{ (x, y) \in \mathbb{R}^2 \middle|\, (x, y, z) \in F \right\}$ then

$$\mu(F) = \int_{\mathbb{R}} \mu_2(F_z)\,\mathrm{d}z \tag{5.32}$$

If we define $F_{xy} := \{ z \in \mathbb{R} |\, (x, y, z) \in F\}$ we have

$$\mu(F) = \iint_{\mathbb{R}^2} \mu_1(F_{xy})\,\mathrm{d}x\,\mathrm{d}y \tag{5.33}$$

Proof. Let $A \subset \mathbb{R}^2$ open, and let $Y_k \subset \mathbb{R}^2$ be rectangles such that

$$Y_1 \subset Y_2 \subset Y_3 \subset \cdots$$
$$A = \bigsqcup_{k=1}^{\infty} Y_k$$

Then, due to $\sigma-$additivity, we have

$$\mu_2(A) = \lim_{k\to\infty} \mu_2(Y_k) = \lim_{k\to\infty} \int_{\mathbb{R}} \mu_1(Y_{kx})\,\mathrm{d}x$$

But

$$Y_{1x} \subset Y_{2x} \subset \cdots$$
$$A_x = \bigsqcup_{k=1}^{\infty} Y_{kx}$$

Due to $\sigma-$additivity and the Beppo-Levi theorem we have that

$$\int_{\mathbb{R}} \mu_1(A_x)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}} \mu_1(Y_{kx})\,\mathrm{d}x$$

Let $E \subset \mathbb{R}^2$ be a measurable set. Define a sequence of compact sets K_i and a sequence of open sets A_j such that

$$K_1 \subset \cdots \subset K_j \subset E \subset A_j \subset \cdots \subset A_1$$

We have that $\lim_{j\to\infty} \mu_2(A_j) = \lim_{j\to\infty} \mu_2(K_j) = \mu_2(E)$ and that $K_{jx} \subset E \subset A_{jx}$.
From the previous derivation we can write that

$$\lim_{j\to\infty} \int_{\mathbb{R}} (\mu_1(A_{jx}) - \mu_1(K_{jx}))\,\mathrm{d}x = 0$$

Building a sequence of non-negative functions $f_j(x) = \mu_1(A_{jx}) - \mu_1(K_{jx})$ we have that $f_j(x) \leq f_{j-1}(x)$ and due to Beppo-Levi we have that

$$\lim_{j\to\infty} \int_{\mathbb{R}} f_j(x)\,\mathrm{d}x = \int_{\mathbb{R}} \lim_{j\to\infty} f_j(x)\,\mathrm{d}x$$

And therefore $\mu_1(K_{jx}) = \mu_1(A_{jx})$, and

$$\forall^\dagger\, x \in \mathbb{R} \quad \mu_2(K_j) = \int_{\mathbb{R}} \mu_1(K_{jx})\,\mathrm{d}x \leq \int_{\mathbb{R}} \overline{\mu_1}(E_x)\,\mathrm{d}x \leq \int_{\mathbb{R}} \mu_1(A_{jx}) = \mu_2(A_j)$$

□

Theorem 5.11 (Fubini). *Let $f(x, y)$ be a measurable function in $\mathbb{R}^2$, then*

1. $\forall^\dagger\, x \in \mathbb{R} \quad y \mapsto f(x, y)$ *is measurable in* $\mathbb{R}$
2. $g(x) = \int_{\mathbb{R}} f(x, y)\,\mathrm{d}y$ *is measurable in* $\mathbb{R}$
3. $\iint_{\mathbb{R}^2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$

Proof. Let $f(x, y) \geq 0$. Defining $F_0 := \{\, (x, y) \in E \times \mathbb{R} |\, 0 < z < f(x, y)\} \subset \mathbb{R}^3$, we have that F_0 is measurable, and

$$\mu_3(F_0) = \iint_{\mathbb{R}^2} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

But F_{0x} is also measurable $\forall^\dagger\, x \in \mathbb{R}$ and therefore

$$\mu_3(F_0) = \int_{\mathbb{R}} \mu_2(F_{0x})\,\mathrm{d}x = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$$

□

Theorem 5.12 (Tonelli)**.** *Let $f(x,y)$ be a measurable function and $E \subset \mathbb{R}^2$ be a measurable set. If one of these integrals exists, the others also exist and have the same value*

$$\iint_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y \quad \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x$$

Theorem 5.13 (Integration Over Rectangles)**.** *Let $R = [a,b] \times [c,d] \subset \mathbb{R}^2$ be a rectangle, and $f(x,y)$ a measurable function over R. Then*

1. *If $\forall^\dagger\, x \in [a,b]\ \exists G(x) = \int_c^d f(x,y)\,\mathrm{d}y$, the function $G(x)$ is measurable in $[a,b]$ and*
$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b G(x)\,\mathrm{d}x = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$
2. *If $\forall^\dagger\, y \in [c,d]\ \exists F(y) = \int_a^b f(x,y)\,\mathrm{d}x$, the function $F(y)$ is measurable in $[c,d]$ and*
$$\int_{\mathbb{R}^2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d F(y)\,\mathrm{d}y = \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

If both are true, then

$$\int_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_c^d f(x,y)\,\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x,y)\,\mathrm{d}x \tag{5.34}$$

Definition 5.3.1 (Normal Set)**.** A set $E \subset \mathbb{R}^2$ is said to be *normal* with respect to the x axis if

$$E = \left\{ (x,y) \in \mathbb{R}^2 \middle|\, a \le x \le b\ \alpha(x) \le y \le \beta(x) \right\}$$

The definition is analogous for the other axes.

Theorem 5.14 (Integration over Normal Sets)**.** *Let $E \subset \mathbb{R}^2$ be a normal set with respect to the x axis, and $f(x,y)$ is a measurable function over E. Then*

$$\int_E f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_{\alpha(x)}^{\beta(x)} f(x,y)\,\mathrm{d}y \tag{5.35}$$

Theorem 5.15 (Dirichlet Inversion Formula)**.** *Take the triangle $T := \left\{ (x,y) \in \mathbb{R}^2 \middle|\, a \le y \le x \le b \right\}$. It can be considered normal with respect to both axes, and we can use the inversion formula*

$$\iint_T f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \mathrm{d}x \int_a^x f(x,y)\,\mathrm{d}y = \int_a^b \mathrm{d}y \int_y^b f(x,y)\,\mathrm{d}x \tag{5.36}$$

§§ 5.3.2 Triple Integration

Theorem 5.16 (Wire Integration)**.** *Let $E \subset \mathbb{R}^3$ be a normal set with respect to the z axis. If $f(x,y,z)$ is measurable in E we have*

$$\iiint_E f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_D \mathrm{d}x\,\mathrm{d}y \int_{h(x,y)}^{g(x,y)} f(x,y,z)\,\mathrm{d}z \tag{5.37}$$

This is called the wire integration formula

Theorem 5.17 (Section Integration). *Let $F \subset \mathbb{R}^3$ be a measurable set bounded by the planes $z = a$ and $z = b$ with $a < b$. Taken $z \in [a, b]$ we can define F_z and we have*

$$\iiint_F f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \int_a^b \mathrm{d}z \iint_{F_z} f(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \tag{5.38}$$

This is called the section integration formula

Theorem 5.18 (Center of Mass). *Take a plane $E \subseteq \mathbb{R}^2$ with surface density $\rho(x, y) > 0$. We define the total mass M as follows*

$$M = \iint_E \rho(x, y) \,\mathrm{d}x \,\mathrm{d}y \tag{5.39}$$

The coordinates of the center of mass will be the following

$$\begin{aligned} x_G &= \frac{1}{M} \iint_E \rho(x, y) x \,\mathrm{d}x \,\mathrm{d}y \\ y_G &= \frac{1}{M} \iint_E \rho(x, y) y \,\mathrm{d}x \,\mathrm{d}y \end{aligned} \tag{5.40}$$

Theorem 5.19 (Moment of Inertia). *Taken the same plane E, we define the moment of inertia with respect to a line r as the following integral*

$$I_r = \iint_E \rho(x, y) \left(d(p^\mu, r)\right)^2 \mathrm{d}x \,\mathrm{d}y \tag{5.41}$$

Where $d(p^\mu, r)$ is the distance function between the point (x, y) and the rotation axis r.
Both formulas are easily generalizable in $\mathbb{R}^3$

§§ 5.3.3 Change of Variables

Definition 5.3.2 (Diffeomorphism). Let $M, N \subset X$ be two subsets of a metric space X. The two sets are said to be *diffeomorphic* if $\exists f : M \xrightarrow{\sim} N$ an isomorphism such that $f \in C^1(M)$ and $f^{-1} \in C^1(N)$. The application f is called a *diffeomorphism.*
Two diffeomorphic sets are indicated as follows

$$M \simeq N$$

Theorem 5.20. *Let $A, B \subset \mathbb{R}^n$ be two open sets and $\varphi^\mu : A \xrightarrow{\sim} B$ a diffeomorphism, such that*

$$\varphi^\mu(E) = F$$

If $f : E \subset B \longrightarrow \mathbb{R}$ is measurable, we have that

$$\int_E f(y^\mu) \,\mathrm{d}y^\mu = \int_{\varphi^{-1}(E)} f(\varphi^\mu(x^\nu)) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu = \int_F f(\varphi^\mu) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu$$

Theorem 5.21 (Change of Variables). *Let $\varphi^\mu : \mathbb{R}^n \xrightarrow{\sim} \mathbb{R}^n$ be a diffeomorphism such that*

$$\varphi^\mu(x^\nu) = x^\mu \quad \forall \|x^\mu\|_\mu > 1$$

And $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ a function such that $\operatorname{supp} f = K \subset \mathbb{R}^n$ *is a compact set. If f is measurable, we have that*

$$\int_{\mathbb{R}^n} f(y^\mu) \,\mathrm{d}y^\mu = \int_{\mathbb{R}^n} f(\varphi^\mu(x^\nu)) \left| \det_{\mu\nu} \partial_\mu \varphi^\nu \right| \mathrm{d}x^\mu \tag{5.42}$$

Proof. Take $n=2$ without loss of generality. We can immediately write that

$$g(y^1,y^2)=\int_{-\infty}^{y^1} f(\eta,y^2)\,\mathrm{d}\eta$$

Then, for the fundamental theorem of integral calculus

$$\partial_1 g(y^1,y^2)=f(y^1,y^2)$$

Taken $c\in\mathbb{R},\ c>1\ :\ K\subset Q=[-c,c]\times[-c,c]$, we have that $\varphi^\mu(x^\nu)=\delta^\mu_\nu\ \forall\|x^\mu\|_\mu>1\ \wedge\ f(x^\mu)=0\ \forall x^\mu\notin Q$.
Therefore $f(\varphi^\mu)=0$ also and we have

$$\int_{\mathbb{R}^n} f(\varphi^\mu)\det_{\mu\nu}\partial_\mu\varphi^\nu\,\mathrm{d}x^\gamma=\int_Q f(\varphi^\mu)\det_{\mu\nu}\partial_\mu\varphi^\nu\,\mathrm{d}x^\gamma=\int_Q \partial_1 g(\varphi^\mu)\det_{\mu\nu}\partial_\mu\varphi^\nu\,\mathrm{d}x^\gamma$$

But we have that

$$g(y^\mu)=0\quad\forall\left|y^1\right|\geq c\ \vee\ \left|y^1\right|<-c$$

Define the following matrix $H_{\mu\nu}$

$$H_{\mu\nu}=\begin{pmatrix}\partial_\mu g(\phi^\gamma)\\ \partial_\mu\varphi^2\end{pmatrix}$$

Then we have that

$$\det_{\mu\nu}H_{\mu\nu}=\partial_1 g(\varphi^\mu)\det_{\mu\nu}\partial_\mu\varphi^\nu$$

Writing $g(\varphi^\mu)=G(x^\mu)$ we have

$$\det_{\mu\nu}H_{\mu\nu}=\partial_1 G\partial_2\varphi^2-\partial_2 G\partial_1\varphi^2$$

Thanks to the integration formula (5.34) we can then write

$$\int_Q\det_{\mu\nu}H_{\mu\nu}\,\mathrm{d}x^\gamma=\int_{-c}^{c}\mathrm{d}x^2\int_{-c}^{c}\partial_1 G\partial_2\varphi^2\,\mathrm{d}x^\nu$$

Integrating by parts we get

$$\int_Q\det_{\mu\nu}H_{\mu\nu}\,\mathrm{d}x^\gamma=\left.G\partial_2\varphi^2\right|_{-c}^{c}-\int_{-c}^{c}G\partial_{21}^2\varphi^2\,\mathrm{d}x^1-\left.G\partial_1\varphi^2\right|_{-c}^{c}-\int_{-c}^{c}G\partial_{12}^2\varphi^2\,\mathrm{d}x^2$$

But $\forall x^\mu\in\partial Q\quad\varphi^\mu(x^\nu)=x^\mu\implies G(-c,x^2)=g(-c,x^2)=0\ \wedge\ G(c,x^2)=g(c,x^2)$

$$\therefore\int_Q\det_{\mu\nu}H_{\mu\nu}\,\mathrm{d}x^\gamma=\int_Q f(x^\mu)\,\mathrm{d}x^\gamma$$

□

Theorem 5.22 (Common Coordinate Transformation in $\mathbb{R}^2$ and $\mathbb{R}^3$). *1. Polar Coordinates*

$$\varphi^\mu(x^\nu)=\begin{cases}x(\rho,\theta)=\rho\cos\theta & \rho\in\mathbb{R}^+\\ y(\rho,\theta)=\rho\sin\theta & \theta\in[0,2\pi)\end{cases}\tag{5.43a}$$

$$\begin{aligned} \partial_\mu \varphi^\nu &= \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \\ \det_{\mu\nu} \partial_\mu\varphi^\nu &= \rho \end{aligned} \tag{5.43b}$$

2. Spherical Coordinates

$$\varphi^\mu(x^\nu) = \begin{cases} x(\rho,\theta,\phi) = \rho\sin\phi\cos\theta & \rho \in \mathbb{R}^+ \\ y(\rho,\theta,\phi) = \rho\sin\phi\sin\theta & \theta \in [0, 2\pi) \\ z(\rho,\theta,\phi) = \rho\cos\phi & \phi \in [0,\pi] \end{cases} \tag{5.44a}$$

$$\begin{aligned} \partial_\mu\varphi^\nu &= \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} \\ \det_{\mu\nu}\partial_\mu\varphi^\nu &= \rho^2\sin\phi \end{aligned} \tag{5.44b}$$

3. Cylindrical Coordinates

$$\varphi^\mu(x^\nu) = \begin{cases} x(\rho,\theta,z) = \rho\cos\theta & \rho\in\mathbb{R}^+ \\ y(\rho,\theta,z) = \rho\sin\theta & \theta\in[0,2\pi) \\ z(\rho,\theta,z) = z & z\in\mathbb{R} \end{cases} \tag{5.45a}$$

$$\det_{\mu\nu}\partial_\mu\varphi^\nu = \rho \tag{5.45b}$$

Definition 5.3.3 (Rotation Solids)**.** Let $D \subset \mathbb{R}^2$ be a bounded measurable set contained in the half-plane $y = 0, x > 0$. Suppose we let D "pop up" into $\mathbb{R}^3$ through a rotation by an angle θ_0 around the z axis. What has been obtained is a *rotation solid* $E \subset \mathbb{R}^3$. We have that

$$\mu(E) = \iiint_E \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iint_D \int_0^{\theta_0} \rho\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z = \theta_0 \iint_D \rho\,\mathrm{d}\rho\,\mathrm{d}z = \theta_0 \iint_D x\,\mathrm{d}x\,\mathrm{d}y \tag{5.46}$$

Or

$$\mu(E) = \theta_0 x_G \mu_2(D)$$

Theorem 5.23 (Guldino)**.** *The measure of a rotation solid is given by the measure of the rotated figure times the circumference described by the center of mass of the solid.*
This is exactly the previous formula.

§§ 5.3.4 Line Integrals

Definition 5.3.4 (Line Integral of the First Kind)**.** Given a scalar field $f : A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ and a smooth curve $\{\gamma\} \subset \mathbb{R}^3$, we define the *line integral of the first kind* as follows

$$\int_\gamma f\,\mathrm{d}s = \int_a^b f(\gamma^\mu) \left\| \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t} \right\|_\mu \mathrm{d}t \tag{5.47}$$

Theorem 5.24 (Center of Mass of a Curve)**.** *Given a curve $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ with linear mass density $m : \{\gamma\} \longrightarrow \mathbb{R}$, we define the total mass of γ as follows*

$$M = \int_\gamma m \,\mathrm{d}s = \int_a^b m(\gamma^\mu) \left\| \frac{\mathrm{d}\gamma^\mu}{\mathrm{d}t} \right\|_\mu \mathrm{d}t \tag{5.48}$$

The center of mass is then defined as follows

$$x^\mu_G = \frac{1}{M} \int_\gamma x^\mu m(x^\nu) \,\mathrm{d}s \tag{5.49}$$

Definition 5.3.5 (Line Integral of the Second Kind)**.** Given a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ and a smooth curve $\gamma^\mu : [a,b] \longrightarrow A \subset \mathbb{R}^3$ we define the *line integral of the second kind* as follows

$$\int_\gamma f^\mu T_\mu \,\mathrm{d}s = \int_a^b f^\mu(\gamma^\nu) \frac{\mathrm{d}\gamma_\mu}{\mathrm{d}t} \,\mathrm{d}t \tag{5.50}$$

Defining a differential form $\omega = f^\mu \,\mathrm{d}x_\mu$ we can also see this integral as follows

$$\int_\gamma \omega = \int_\gamma f^\mu T_\mu \,\mathrm{d}s \tag{5.51}$$

Where T^μ is the tangent vector of the curve

Definition 5.3.6 (Conservative Field)**.** Let $f^\mu : A \longrightarrow \mathbb{R}^3$ be a vector field such that $f^\mu \in C^1(A)$ and A is open and connected. This field is said to be *conservative*, if $\forall x^\mu \in A$

$$\exists U(x^\mu) \in C^2(A) \,:\, f^\mu = -\partial^\mu U \tag{5.52}$$

The function $U(x^\mu)$ is called the *potential* of the field.

Theorem 5.25 (Line Integral of a Conservative Field)**.** *Given a conservative field $f^\mu : A \longrightarrow \mathbb{R}^3$ and a smooth curve $\{\gamma\} \subset A,\ \gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ with A open and connected, we have that*

$$\int_\gamma f^\mu T_\mu \,\mathrm{d}s = U(\gamma(a)) - U(\gamma(b)) \tag{5.53}$$

Where $U(x^\mu)$ is the potential of the vector field.

Definition 5.3.7 (Rotor)**.** Given a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ with $f^\mu \in C^1(A)$, we define the *rotor* of the vector field as follows

$$\mathrm{rot}(f^\mu) = \epsilon^\mu_{\nu\gamma} \partial^\nu f^\gamma \tag{5.54}$$

Theorem 5.26. *Given f^μ a conservative vector field on an open connected set A, we have that*

$$\epsilon^\mu_{\nu\gamma} \partial^\nu f^\gamma = 0 \tag{5.55}$$

Alternatively, if $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$ is the parameterization of a smooth closed curve, we have that

$$\oint_\gamma f^\mu T_\mu \,\mathrm{d}s = 0 \tag{5.56}$$

§§ 5.3.5 Surface Integrals

Definition 5.3.8 (Area of a Surface)**.** Given $r^\mu : K \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$ a smooth surface, we have that given its metric tensor $g_{\mu\nu}(u,v)$ we have that

$$\mu(\Sigma) = \int_\Sigma \mathrm{d}\sigma = \iint_K \sqrt{\det_{\mu\nu} g_{\mu\nu}}\, \mathrm{d}u\, \mathrm{d}v = \iint_K \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v \tag{5.57}$$

For a cartesian surface S we have that

$$\mu(S) = \int_S \mathrm{d}s = \iint_K \sqrt{1 + \left(\|\partial_\mu f\|_\mu\right)^2}\, \mathrm{d}x\, \mathrm{d}y \tag{5.58}$$

Definition 5.3.9 (Rotation Surface)**.** Given a smooth curve $\gamma^\mu : [a,b] \longrightarrow \mathbb{R}^3$, the rotation of this curve around the $z-$axis generates a smooth surface Σ with the following parameterization

$$r^\mu(t,\theta) = \begin{cases} \gamma^1(t)\cos\theta \\ \gamma^2(t)\sin\theta \qquad (t,\theta) \in [a,b] \times [0,\theta_0] \\ \gamma^3(t) \end{cases} \tag{5.59}$$

The area of a rotation surface is calculated as follows

$$\mu(\Sigma) = \theta_0 \int_a^b \gamma^1(t) \sqrt{\left(\frac{\mathrm{d}\gamma^1}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}\gamma^2}{\mathrm{d}t}\right)^2}\, \mathrm{d}t \tag{5.60}$$

Theorem 5.27 (Guldino II)**.** *Given Σ a smooth rotation surface defined as before, we have that its area will be*

$$\mu(\Sigma) = \theta_0 \int_\gamma x^1\, \mathrm{d}s = \theta_0 x_G^1 L_\gamma \tag{5.61}$$

Where x_G^1 is the first coordinate of the center of mass of the curve, calculated as follows

$$x_G^1 = \frac{1}{L_\gamma} \int_\gamma x^1\, \mathrm{d}s$$

Definition 5.3.10 (Surface Integral)**.** Given a smooth surface $\Sigma \subset \mathbb{R}^3$ with parameterization $r^\mu : K \longrightarrow \Sigma$ and a scalar field $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$, we define the *surface integral* of h as follows

$$\int_\Sigma h(x^\mu)\, \mathrm{d}\sigma = \iint_K h(r^\mu) \sqrt{\det_{\mu\nu} g_{\mu\nu}}\, \mathrm{d}u\, \mathrm{d}v \tag{5.62}$$

If Σ is a cartesian surface, we have

$$\int_\Sigma h(x^\mu)\, \mathrm{d}\sigma = \iint_K h(x^1, x^2, f) \sqrt{1 + (\|\partial_\mu f\|^\mu)^2}\, \mathrm{d}x\, \mathrm{d}y \tag{5.63}$$

Definition 5.3.11 (Center of Mass of a Surface)**.** Given a smooth surface Σ with parameterization $r^\mu(u,v)$ and mass density δ, we define its total mass as follows

$$M = \int_\Sigma \delta\, \mathrm{d}\sigma \tag{5.64}$$

Its center of mass x_G^μ will be calculated as follows

$$x_G^\mu = \frac{1}{M} \int_\Sigma x^\mu \delta(x^\nu)\, \mathrm{d}\sigma \tag{5.65}$$

Definition 5.3.12 (Moment of Inertia of a Surface). Given a smooth surface Σ with parameterization $r^\mu(u, v)$ and mass density δ we define its moment of inertia around an axis r, I, as the following integral

$$I = \int_\Sigma \delta(x^\mu)\,(d(p^\mu, r))^2\,\mathrm{d}\sigma \quad p^\mu \in \Sigma \tag{5.66}$$

Definition 5.3.13 (Orientable Surface). A smooth surface with parameterization $r^\mu : K \subset \mathbb{R}^2 \longrightarrow \Sigma \subset \mathbb{R}^3$ is said to be *orientable* if $\forall \gamma : [a, b] \longrightarrow \Sigma$ smooth closed curve, we have, given n^μ the normal vector of the surface

$$n^\mu(\gamma^\nu(a)) = n^\mu(\gamma^\nu(b)) \tag{5.67}$$

Another way of formulating it is

$$n^\mu(x^\nu) \in C(K) \tag{5.68}$$

Definition 5.3.14 (Boundary of a Surface). Given a smooth surface as before, we define the *boundary* $\partial\Sigma$ as follows

$$\partial\Sigma = \overline{\Sigma} \setminus \Sigma \tag{5.69}$$

Note how, given the parameterization r^μ, we have $r^\mu(\partial K) = \partial\Sigma$

Definition 5.3.15 (Closed Surface). A surface $\Sigma \subset \mathbb{R}^3$ is said to be *closed* iff $\partial\Sigma = \{\}$

Definition 5.3.16 (Flux). Given a vector field $f^\mu : A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ and a smooth orientable surface $\Sigma \subset A$, we define the *flux* of the vector field f^μ on the surface as follows

$$\Phi_\Sigma(f^\mu) = \int_\Sigma f^\mu n_\mu\,\mathrm{d}\sigma = \iint_K f^\mu(r^\nu)\epsilon_{\mu\gamma\sigma}\partial_1 r^\gamma \partial_2 r^\sigma\,\mathrm{d}u\,\mathrm{d}v \tag{5.70}$$

§ 5.4 Integration in $\mathbb{C}$

Definition 5.4.1 (Piecewise Continuous Function). Let $\gamma : [a, b] \longrightarrow \mathbb{C}$ be a piecewise continuous curve such that $\{\gamma\} \subset D \subset \mathbb{C}$, and $f : D \longrightarrow \mathbb{C}, \quad f \in C(D)$. Then the function $(f \circ \gamma)\,\gamma'(t) : [a, b] \longrightarrow \mathbb{C}$ is a *piecewise continuous function*

Definition 5.4.2 (Line Integral in $\mathbb{C}$). Let $\gamma : [a, b] \longrightarrow D \subset \mathbb{C}$ be a piecewise continuous curve and $f : D \longrightarrow \mathbb{C}$ a measurable function $f \in C(D)$.
We define the *line integral over* γ the result of the application of the integral operator $\hat{K}_\gamma[f]$, where

$$\hat{K}_\gamma[f] = \int_\gamma f(z)\,\mathrm{d}z = \int_a^b (f \circ \gamma)\,\gamma'(t)\,\mathrm{d}t \tag{5.71}$$

Where $\forall^\dagger\, z \in \{\gamma\}\ f(z)$ is defined

Theorem 5.28 (Properties of the Line Integral). *Let $z, w, t \in \mathbb{C}$, $f, g \in \mathcal{M}(\mathbb{C})$ and $\{\gamma\}, \{\eta\}, \{\kappa\}$ three smooth curves, then*

1. $\hat{K}_\gamma[zf + wg] = z\hat{K}_\gamma[f] + w\hat{K}_\gamma[g]$
2. $\gamma \sim \eta \implies \hat{K}_\gamma[f] = \hat{K}_\eta[f]$

3. $\gamma = \eta + \kappa \implies \hat{K}_\gamma[f] = \hat{K}_{\eta+\kappa}[f] = \hat{K}_\eta[f] + \hat{K}_\kappa[f]$

4. $\hat{K}_{\gamma+w}[f(z)] = \hat{K}_\gamma[f(z+w)]$

Notation. If a measurable function $f(z)$ has the same value of the integral for different curves between two points $z_1, z_2 \in \mathbb{C}$, we will write directly

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$$

Theorem 5.29 (Darboux Inequality)**.** *Let $f : D \longrightarrow \mathbb{C}$ be a measurable function and $\gamma : [a,b] \longrightarrow \{\gamma\} \subset D \subseteq \mathbb{C}$ piecewise smooth. Then*

$$\left\| \int_\gamma f(z)\,\mathrm{d}z \right\| \le L_\gamma \sup_{z\in\{\gamma\}} \|f(z)\|$$

Proof. The proof is quite straightforward using the definition given for the line integral

$$\begin{aligned}\left\| \int_\gamma f(z)\,\mathrm{d}z \right\| = \left\| \int_a^b (f\circ\gamma)\,\gamma'(t)\,\mathrm{d}t \right\| \le \int_a^b \|(f\circ\gamma)\,\gamma'(t)\|\,\mathrm{d}t \le \\ \le \sup_{z\in\{\gamma\}} \|f(z)\| \int_a^b \|\gamma'(t)\|\,\mathrm{d}t = L_\gamma \sup_{z\in\{\gamma\}} \|f(z)\|\end{aligned}$$

□

§§ 5.4.1 Integration of Holomorphic Functions

Definition 5.4.3 (Primitive)**.** Let $f : D \longrightarrow \mathbb{C}$ and $F : D \longrightarrow \mathbb{C}$ be two functions and $D \subset \mathbb{C}$ an open and connected set. $F(z)$ is said to be the *primitive function* or *antiderivative* of f in D if

$$\frac{\mathrm{d}F}{\mathrm{d}z} = f(z) \quad \forall z \in D \tag{5.72}$$

Notation. Given a closed curve γ and a measurable function $f(z)$ we define the following notation

$$\int_\gamma f(z)\,\mathrm{d}z = \oint_\gamma f(z)\,\mathrm{d}z$$

Theorem 5.30 (Existence of the Primitive Function)**.** *Let $f : D \longrightarrow \mathbb{C} \quad f \in C(D)$ with $D \subset \mathbb{C}$ open and connected. Then these statements are equivalent*

1. $\exists F : D \longrightarrow \mathbb{C} \,:\, F'(z) = f(z)$

2. $\forall z_1, z_2 \in D,\ \forall \{\gamma\} \subset D$ *piecewise smooth* $\int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$

3. $\forall \gamma : [a,b] \longrightarrow \{\gamma\} \subset D$ *closed piecewise smooth* $\oint_\gamma f(z)\,\mathrm{d}z = 0$

Proof. $1 \implies 2$. As with the hypothesis we have that $\exists F : D \longrightarrow \mathbb{C} \, : \, F'(z) = f(z) \; \forall z \in D$. Given two points $z_1, z_2 \in D$ and taken a smooth curve $\gamma : [a, b] \longrightarrow D \, : \, \gamma(a) = z_1 \, \wedge \, \gamma(b) = z_2$. Therefore

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b (f \circ \gamma)\,\gamma'(t)\,\mathrm{d}t = \int_a^b (F' \circ \gamma)\,\gamma'(t)\,\mathrm{d}t$$

The result of the integral is obviously $F(z_2) - F(z_1)$, therefore we can immediately write that, if

$$\exists F : D \longrightarrow \mathbb{C} \, : \, F'(z) = f(z) \implies \int_\gamma f(z)\,\mathrm{d}z = \int_{z_1}^{z_2} f(z)\,\mathrm{d}z$$

$2 \implies 1$ Taken a point $z_0 \in D$, any point $z \in D$ can be connected with a polygonal to z_0 since D is connected. The integral of f over this polygonal is obviously path-independent, hence we can define the following function

$$F(z) = \int_{z_0}^{z} f(w)\,\mathrm{d}w$$

Since D is open we can define $\delta_z \in \mathbb{R}, \; \delta_z > 0 \, \wedge \, \exists B_{\delta_1}(z) \subset D$. Taken $\Delta z \in \mathbb{C} \, : \, \|\Delta z\| < \delta_1$ we have that

$$F(z + \Delta z) - F(z) = \int_z^{z+\Delta z} f(w)\,\mathrm{d}w$$

Dividing by Δz and taking the limit as $\Delta z \to 0$ we have that using the Darboux inequality we get that

$$\left\| \frac{F(z + \Delta z) - F(z)}{\Delta z} - f(z) \right\| = \frac{1}{\|\Delta z\|} \left\| \int_z^{z+\Delta z} f(w)\,\mathrm{d}w \right\| \leq \epsilon$$

$2 \implies 3$. Taken an arbitrary piecewise smooth curve γ and $z_1 \neq z_2 \in \{\gamma\}$. We can now find two curves such that $\gamma(t) = \gamma_1(t) - \gamma_2(t)$. Since the integral of f is path independent, we get

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z - \int_{\gamma_2} f(z)\,\mathrm{d}z = 0$$

$3 \implies 2$ is exactly as before but with the opposite reasoning. □

Example 5.4.1. Let's calculate the integral of functions $f_n(x) = z^{-n} \quad n \in \mathbb{N}$ for a closed simple piecewise smooth curve γ such that $0 \notin \{\gamma\}$.
For $n > 1$ we have that $f \in C(D)$ where $D = \mathbb{C} \setminus \{0\}$, and we have that

$$\int \frac{1}{z^n}\,\mathrm{d}z = -\frac{z^{-(n-1)}}{n-1} + w \quad w \in \mathbb{C}$$

Therefore, for every closed simple piecewise smooth curve $\gamma \, : \, 0 \notin \{\gamma\}$ we have

$$\oint_\gamma \frac{1}{z^n}\,\mathrm{d}z = 0$$

For $n = 1$ we still have that $f \in C(D)$ but $\nexists F(z) : D \longrightarrow \mathbb{C}$ primitive of $f_1(z)$, but there exists one in the domain G of holomorphy of the logarithm.

Although we have that $G \subset D$, and we can take a curve $\gamma : 0 \in \operatorname{extr}\gamma$, and therefore $\{\gamma\} \subset G$ and we have that

$$\oint_\gamma \frac{1}{z}\,\mathrm{d}z = 0$$

If we otherwise have $0 \in \gamma^\circ$ the integral is non-zero.
Take a branch of the logarithm σ and a curve η has only one point of intersection with such branch at $z_i = u_0 e^{i\alpha}$. Taken $\eta(a) = \eta(b) = u_0 e^{i\alpha}$, we define $\eta_\epsilon : [a+\epsilon, b+\epsilon] \longrightarrow \mathbb{C}$ with $\epsilon > 0 : \eta_\epsilon(t) = \eta(t)\ \forall t \in [a+\epsilon, b+\epsilon]$, then

$$\oint_\eta \frac{1}{z}\,\mathrm{d}z = \lim_{\epsilon\to 0}\oint_{\eta_\epsilon}\frac{1}{z}\,\mathrm{d}z$$

Therefore, $\forall z \in \mathbb{C}\setminus\{\sigma\}$ we have that

$$\frac{\mathrm{d}\log z}{\mathrm{d}z} = \frac{1}{z}$$

And therefore

$$\oint_{\eta_\epsilon}\frac{1}{z}\,\mathrm{d}z = \log\left(\eta(b-\epsilon)\right) - \log\left(\eta\left(a+\epsilon\right)\right)$$

For $\epsilon \to 0$ we have

$$\int_\eta \frac{1}{z}\,\mathrm{d}z = (\log(u_0) + i(\alpha+2\pi)) - (\log(u_0) + i\alpha) = 2\pi i$$

Example 5.4.2. Let's calculate the integral of $f(z) = \sqrt{z}$ along a closed simple piecewise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C} : 0 \in \gamma^\circ$ and it intersects the line $\sigma_\alpha = u_0 e^{i\alpha}$, where

$$\sqrt{z} = \sqrt{r}e^{i\frac{\theta}{2}} \quad r \in \mathbb{R}^+,\ \theta \in (\alpha, \alpha+2\pi],\ \alpha \in \mathbb{R}$$

Taken a parametrization $\gamma(t) : \gamma(a) = \gamma(b) = u_0 e^{i\alpha}$ we have that $f(z) \in H(D)$ where $D = \mathbb{C}\setminus\{\sigma_\alpha\}$. Proceding as before, we have

$$\oint_\gamma \sqrt{z}\,\mathrm{d}z = \lim_{\epsilon\to 0}\oint_{\gamma_\epsilon}\sqrt{z}\,\mathrm{d}z$$

Since it has a primitive in D we can write

$$\lim_{\epsilon\to 0}\oint_{\gamma_\epsilon}\sqrt{z}\,\mathrm{d}z = \frac{2}{3}\lim_{\epsilon\to 0} z\sqrt{z}\Big|_{\gamma_\epsilon(a+\epsilon)}^{\gamma_\epsilon(b-\epsilon)} = \frac{2}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i(\alpha+2\pi)} - \frac{2}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i\alpha} = -\frac{4}{3}u_0\sqrt{u_0}e^{\frac{3}{2}i\alpha}$$

Lemma 5.4.1. Taken a closed simple pointwise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ and taken $D = \{\gamma\}^\circ \cup \gamma = \overline{\{\gamma\}^\circ}$ and a function $f \in H(D)$, for a finite cover of D, $\mathcal{Q}$ composed by squares $Q_j \in \mathcal{Q}\ \forall j \in [1,N] \subset \mathbb{N}$, we have that

$$\exists z_j \in Q_j \cap \overline{\{\gamma\}^\circ} : \left\|\frac{f(z)-f(z_j)}{z-z_j} - \frac{\mathrm{d}f}{\mathrm{d}z}\bigg|_{z_j}\right\| < \epsilon\ \forall z \in Q_j \cap \overline{\{\gamma\}^\circ}\setminus\{z_j\}$$

Proof. Going by contradiction, let's say that

$$\exists \epsilon > 0 : \nexists z_j \in Q_j \cap \overline{\{\gamma\}^\circ}$$

Taken a finite subcover $\mathcal{Q}_n$ where $\operatorname{diam}(Q_j^n) = \frac{d}{2^n}$ $\forall Q_j \in \mathcal{Q}$ we can define for some $k \in K \subset \mathbb{N}$

$$A_n = \bigcup_{k \in K} Q_k^n \cap \overline{\{\gamma\}^\circ} \quad \forall n \in \mathbb{N}$$

We have that $A_{n+1} \subset A_n$, and taking a sequence $(w)_n \in \overline{\{\gamma\}^\circ}$ we have due to the compactness of $\overline{\{\gamma\}^\circ}$ that $\exists (w)_{n_j} \to w \in \overline{\{\gamma\}^\circ}$. Since $f \in H(\overline{\{\gamma\}^\circ})$ we have that f is holomorphic in w, therefore

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0\ :\ \left\| \frac{f(z) - f(w)}{z - w} - \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_w \right\| < \epsilon\ \forall z \in B_{\delta_\epsilon}(w) \setminus \{w\}$$

Taken an $\tilde{n}$ such that $\operatorname{diam}(Q_j^{\tilde{n}}) = \frac{\sqrt{2}}{2^{\tilde{n}}} d < \delta$ we have that still $w \in A_n$ $\forall n \in \mathbb{N}$, and due to its closedness we can also say

$$\exists N_j \in \mathbb{N}\ :\ \forall n_j > N_j\ (w)_{n_j} \in A_n$$

Therefore

$$\exists k_0 \in \mathbb{N}\ :\ w \in Q_{k_0}^{\tilde{n}} \cap \overline{\{\gamma\}^\circ} \subset A_{\tilde{n}}\ \lightning$$

$\square$

Theorem 5.31 (Cauchy-Goursat)**.** *Taken $\gamma : [a, b] \longrightarrow \mathbb{C}$ a closed simple piecewise smooth curve and $D = \{\gamma\} \cup \{\gamma\}^\circ$ and a function $f \in H(D)$, we have*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \tag{5.73}$$

Proof. Using the previous lemma we can say that for a finite cover $\{\gamma\}, Q_j \in \mathcal{Q}$ $\exists z_j \in Q_j \cap \overline{\{\gamma\}^\circ}$ and a function

$$\delta_j(z) = \begin{cases} \dfrac{f(z) - f(z_j)}{z - z_j} - f'(z_j) & z \neq z_j \\ 0 & z = z_j \end{cases}$$

Which is countinuous and $\delta_j(z) < \epsilon$ $\forall z \in Q_j \cap \overline{\{\gamma\}^\circ}$.
Taken a curve $\{\eta_j\} = \partial \left(Q_j \cap \overline{\{\gamma\}^\circ}\right)$, and the expansion of $f(z)$ in the region, we have that

$$\begin{gathered} f(z) = f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j) \\ \oint_{\eta_j} f(z)\,\mathrm{d}z = (f(z_j) - z_j f'(z_j)) \oint_{\eta_j} \mathrm{d}z + f'(z_j) \oint_{\eta_j} z\,\mathrm{d}z + \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z \end{gathered}$$

The first two integrals on the second line are null, and we have therefore

$$\oint_{\eta_j} f(z)\,\mathrm{d}z = \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z$$

By definition $\{\gamma\} = \bigcup_{j=1}^N \{\eta_j\}$ and therefore

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^N \oint_{\eta_j} \delta_j(z)(z - z_j)\,\mathrm{d}z$$

Using the Darboux inequality we have immediately that

$$\left\| \oint_\gamma f(z)\,\mathrm{d}z \right\| \le \sum_{j=1}^N \left\| \oint_{\eta_j} \delta_j(z)(z-z_j)\,\mathrm{d}z \right\| \le \sum_{j=1}^N \epsilon\sqrt{2}d(4d+L_j)$$

Using the theorem on the Jordan curve, we have that $\exists Q_n \in \mathcal{Q}$ such that $\{\gamma\} \subset Q_n$. Taken $\operatorname{diam}(Q_n) = D$

$$\left\| \oint_\gamma f(z)\,\mathrm{d}z \right\| \le \sum_{j=1}^N \epsilon\sqrt{2}D(4D+L) \to 0$$

□

Definition 5.4.4 (Simple Connected Set)**.** An open set $G \subset X$ with X some metric space, is said to be *simply connected* iff $\forall\{\gamma_j\} \subset G$ simple curves we have that $\gamma_j \sim 0$.
$\gamma \sim 0$ implies that the curve is homotopic to a point

Theorem 5.32 (Cauchy-Goursat II)**.** *Let $G \subset \mathbb{C}$ open and simply connected. Then, $\forall f \in H(G), \{\gamma\} \subset G$ with γ simple closed and smooth*

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

Proof.

1. The curve γ doesn't intersect itself.

$$\oint_\gamma f(z)\,\mathrm{d}z = \oint_0 f(z)\,\mathrm{d}z = 0$$

2. The curve γ intersects itself $n-1$ times.
Then $\{\gamma\} = \bigcup_{k=1}^n \{\gamma_k\}$ with γ_k simple smooth non intersecting curves. Since $\{\gamma_k\} \subset G$ $\forall k = 1, \cdots, n,\ \{\gamma_k\} \sim 0$, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=1}^n \oint_{\gamma_k} f(z)\,\mathrm{d}z = 0$$

□

Theorem 5.33. *Let $G \subset \mathbb{C}$ be a simply connected open set. If $f \in H(G)$, then there exists a primitive for $f(z)$*

§§ 5.4.2 Integral Representation of Holomorphic Functions

Definition 5.4.5 (Positively Oriented Curve)**.** The parametrization of a curve in $\mathbb{C}$ is said to be *positively oriented* if its parametrization is taken such the path taken results counterclockwise.

Notation. The integral over a closed positively oriented parametrization of a curve γ is indicated as follows

$$\oint_\gamma$$

Theorem 5.34 (Cauchy Integral Representation). *Taken a positively oriented closed simple piecewise smooth curve $\gamma : [a,b] \longrightarrow \mathbb{C}$ and a function $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ such that if $D = \{\gamma\} \cup \{\gamma\}^\circ \subset G,\ f \in H(D)$, we have that*

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z} \, \mathrm{d}w \quad \forall w \in \{\gamma\}^\circ \tag{5.74}$$

Proof. Taken $\gamma_\rho(\theta) = z + \rho e^{i\theta}$ such that $\gamma_\rho \sim \gamma,\ \{\gamma_\rho\} \subset \{\gamma\}^\circ$ is a simple curve, we have

$$\oint_\gamma \frac{f(w)}{w-z} \, \mathrm{d}w = \oint_{\gamma_\rho} \frac{f(w)}{w-z} \, \mathrm{d}w$$

Then, using that

$$\oint_\gamma \frac{1}{w-z} \, \mathrm{d}w = 2\pi i$$

We get

$$\oint_\gamma \frac{f(z)}{w-z} \, \mathrm{d}w - 2\pi i f(z) = \oint_{\gamma_\rho} \frac{f(w) - f(z)}{w-z} \, \mathrm{d}w$$

Since $f \in H(\{\gamma\}^\circ)$ we have that

$$\forall \epsilon > 0\ \exists \delta_\epsilon > 0 \ :\ \|z - w\| < \delta_\epsilon \implies \|f(z) - f(w)\| < \epsilon$$

Taken $\rho < \delta_\epsilon$ we get, using the Darboux inequality

$$\left\| \oint_{\gamma_\rho} \frac{f(w) - f(z)}{w-z} \, \mathrm{d}w \right\| \le 2\pi\epsilon \implies \oint_\gamma \frac{f(w) - f(z)}{w-z} \, \mathrm{d}w = 0$$

□

Theorem 5.35 (Derivatives of a Holomorphic Function). *Let $D \subset \mathbb{C}$ be an open set and $f : D \longrightarrow \mathbb{C}$ a function $f \in H(D)$, then $f \in C^\infty(D)$ and*

$$\frac{\mathrm{d}^n f}{\mathrm{d}z^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(w)}{(w-z)^{n+1}} \, \mathrm{d}w \tag{5.75}$$

Where γ is a closed simple piecewise smooth curve such that $z \in \{\gamma\}^\circ$ and $\overline{\{\gamma\}} \subset D$

Corollary 5.4.1. Let $f \in H(D)$, then

$$\forall n \in \mathbb{N} \quad \frac{\mathrm{d}^n f}{\mathrm{d}z^n} \in H(D)$$

Theorem 5.36 (Morera). *Let $D \subset \mathbb{C}$ be an open and connected set. Take $f : D \longrightarrow \mathbb{C} \ :\ f \in C(D)$. Then, if $\forall \{\gamma\} \subset D$ closed piecewise smooth*

$$\oint_\gamma f(z) \, \mathrm{d}z = 0 \implies f \in H(D) \tag{5.76}$$

Proof. Since $f \in C(D)$ $\exists F(z) \in C^1(D)$: $f(z) = F'(z)$. Since $C^1(\mathbb{C}) \simeq H(\mathbb{C})$ we have that, due to the previous corollary

$$\frac{\mathrm{d}F}{\mathrm{d}z} = f(z) \in H(D)$$

□

Theorem 5.37 (Cauchy Inequality). *Let $f \in H(B_R(z_0))$ with $z_0 \in \mathbb{C}$. If $\|f(z)\| \leq M$ $\forall z \in B_R(z_0)$*

$$\left\| \left. \frac{\mathrm{d}f}{\mathrm{d}z} \right|_{z_0} \right\| \leq \frac{n!M}{R^n} \tag{5.77}$$

Proof. Take $\gamma_r(\theta) = z_0 + re^{i\theta}$ with $\theta \in [0, 2\pi]$, $r > R$, then the derivative $\left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0}$ can be written using the Cauchy integral representation, since $f \in H(B_r(z_0))$

$$\left. \frac{\mathrm{d}^n f}{\mathrm{d}z^n} \right|_{z_0} = \frac{n!}{2\pi i} \oint_{\gamma_r} \frac{f(w)}{(w - z_0)^{n+1}} \,\mathrm{d}w$$

Using the Darboux inequality we have then

$$\left\| \left. \frac{\mathrm{d}^n f}{\mathrm{d}z^n} \right|_{z_0} \right\| \leq \frac{n!}{r^n} \sup_{z \in \{\gamma_r\}} \|f(z)\| \leq \frac{n!M}{r^n}$$

Since $r < R$ we therefore have

$$\left\| \left. \frac{\mathrm{d}^n f}{\mathrm{d}z^n} \right|_{z_0} \right\| \leq \frac{n!M}{R^n}$$

□

Theorem 5.38 (Liouville). *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ a function such that $f \in H(\mathbb{C})$, i.e. whole. If $\exists M > 0$: $\|f(z)\| \leq M$ $\forall z \in \mathbb{C}$ the function $f(z)$ is constant*

Proof. $f \in H(\mathbb{C})$, $\|f(z)\| \leq M$ and we can write, taken $\gamma_R(\theta) = z + Re^{i\theta}$ with $\theta \in [0, 2\pi]$

$$f'(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(w)}{(w - z)^2} \,\mathrm{d}z$$

For Darboux

$$\|f'(z)\| \leq \frac{1}{2\pi} \left\| \oint_{\gamma_R} \frac{f(w)}{(w - z)^2} \right\| \mathrm{d}z \leq \frac{\sup_{z \in \{\gamma_R\}} \|f(z)\|}{R} \leq \frac{M}{R}$$

Since $R > 0$ is arbitrary, we can say directly that $\|f'(z)\| = 0$ and therefore $f(z)$ is constant $\forall z \in \mathbb{C}$. □

Theorem 5.39 (Fundamental Theorem of Algebra). *Take a polynomial $P_n(z) \in \mathbb{C}_n[z]$, where $\mathbb{C}_n[z]$ is the space of complex polynomials with variable z and degree n. If we have*

$$P_n(z) = \sum_{k=0}^{n} a_k z^k, \quad z, a_k \in \mathbb{C},\ a_n \neq 0$$

We can say that $\exists z_0 \in \mathbb{C}$: $P_n(z_0) = 0$

Proof. As an absurd, say that $\forall z \in \mathbb{C},\ P_n(z) \neq 0$. Then $f(z) = 1/P_n(z) \in H(\mathbb{C})$.
Since $\lim_{z\to\infty} P_n(z) = \infty$, we have that $\|f(z)\| \leq M\ \forall z \in \mathbb{C}$, and $\lim_{z\to\infty} f(z) = 0$.
Therefore $\exists R > 0\ :\ \forall \|z\| > R,\ \|f(z)\| < 1$. Since $f \in H(\mathbb{C})$, we have that $f \in C(\overline{B}_R(z))$. Due to the Liouville theorem we have that $f(z)$ is constant $\lightning$ $\square$

§ 5.5 Integral Theorems in $\mathbb{R}^2$ and $\mathbb{R}^3$

Theorem 5.40 (Gauss-Green). *Given $D \subset \mathbb{R}^2$ a set with a piecewise smooth parameterization of ∂D and two functions $\alpha, \beta : A \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\overline{D} \subset A$*

$$\iint_D \partial_x \beta \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} \beta(x,y)\,\mathrm{d}y, \quad \iint_D \partial_y \alpha \,\mathrm{d}x\,\mathrm{d}y = -\int_{\partial^D} \alpha(x,y)\,\mathrm{d}x\,\mathrm{d}y \tag{5.78}$$

Theorem 5.41 (Stokes). *Given $D \subset \mathbb{R}^2$ an open set with ∂D piecewise smooth and a vector field $f^\mu : A \longrightarrow \mathbb{R}^2$ with $D \subset A$*

$$\int_D \epsilon_{3\mu\nu} \partial^\mu f^\nu \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} f^\mu t_\mu \,\mathrm{d}s \tag{5.79}$$

Where t^μ is the vector tangent to $\partial^+ D$

Theorem 5.42 (Gauss 1). *Given $D \subset \mathbb{R}^n$ open set with ∂D piecewise smooth and a vector field $f^\mu : A \longrightarrow \mathbb{R}^n$ with $D \subset A$*

$$\iiint_D \partial_\mu f^\mu \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial^+ D} f^\mu n_\mu \,\mathrm{d}s \tag{5.80}$$

Where n^μ is the normal vector to $\partial^+ D$

Theorem 5.43 (Stokes for Surfaces). *Given a smooth surface $\Sigma \subset \mathbb{R}^3$ with parameterization r^μ and a vector field $f^\mu : A \longrightarrow \mathbb{R}^3$ with $\Sigma \subseteq A$*

$$\int_\Sigma n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}\sigma = \int_{\partial^+ \Sigma} f^\mu t_\mu \,\mathrm{d}s \tag{5.81}$$

Where t^μ is the tangent vector to the border of the surface

Theorem 5.44 (Useful Identities). *Given $u, v \in C^2(\Omega)$ and a vector field $f^\mu \in C^2(\Omega, \mathbb{R}^3)$*

$$\begin{aligned}
\int_\Omega \partial_\mu \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\partial\Omega} n^\mu \partial_\mu v \,\mathrm{d}\sigma \\
\int_\Omega u \partial_\mu f^\mu \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= -\int_\Omega f^\mu \partial_\mu w \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\partial\Omega} u f^\mu n_\mu \,\mathrm{d}\sigma \\
\int_\Omega u \partial_\mu \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= -\int_\Omega \partial_\mu u \partial^\mu v \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{\partial\Omega} u n^\mu \partial_\mu v \,\mathrm{d}\sigma \\
\int_\Omega (u \partial_\mu \partial^\mu v - w \partial_\mu \partial^\mu u) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_{\partial\Omega} (u n^\mu \partial_\mu v - w n^\mu \partial_\mu u) \,\mathrm{d}\sigma
\end{aligned} \tag{5.82}$$

We can analogously write these theorems in the language of differential forms and manifolds, after giving a couple of definitions

Definition 5.5.1 (Volume Element)**.** Given a $k-$dimensional compact oriented manifold M with boundary and $\omega \in \Lambda^k(M)$ a $k-$differential form on M, we define the *volume* of M as follows

$$V(M) = \int_M \mathrm{d}V = \int_M \omega \tag{5.83}$$

Where $\mathrm{d}V$ is the *volume element* of the manifold, given by the unique $\omega \in \Lambda^k(M)$, defined as follows

$$\omega = f\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} \tag{5.84}$$

With f an unique function.
For $M \subset \mathbb{R}^3$ with n^μ as outer normal and $\omega \in \Lambda^2(M)$ we can write immediately, by definition

$$\omega_{\mu\nu} v^\mu w^\nu = n^\mu \epsilon_{\mu\nu\gamma} v^\nu w^\gamma = \mathrm{d}A$$

Therefore

$$\mathrm{d}A = \|\epsilon_{\mu\nu\gamma} v^\nu w^\gamma\|^\mu \tag{5.85}$$

Which is the already known formula.
For a 2−manifold we can write immediately the following formulas

$$\mathrm{d}A = n^1\,\mathrm{d}y \wedge \mathrm{d}z + n^2\,\mathrm{d}z \wedge \mathrm{d}x + n^3\,\mathrm{d}x \wedge \mathrm{d}y \tag{5.86}$$

And, on M

$$\begin{cases} n^1\,\mathrm{d}A = \mathrm{d}y \wedge \mathrm{d}z \\ n^2\,\mathrm{d}A = \mathrm{d}z \wedge \mathrm{d}x \\ n^3\,\mathrm{d}A = \mathrm{d}x \wedge \mathrm{d}y \end{cases} \tag{5.87}$$

Theorem 5.45 (Gauss-Green-Stokes-Ostogradskij)**.** *Given M a smooth manifold with boundary, c a $p-$cube in M and $\omega \in \Lambda(M)$ we have*

$$\int_c \mathrm{d}\omega = \int_{[0,1]^p} c^\star\,\mathrm{d}\omega = \int_{\partial c} \omega \tag{5.88}$$

In general, we can write

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega \tag{5.89}$$

Definition 5.5.2 (Gauss-Green, Differential Forms)**.** Given $M \subset \mathbb{R}^2$ a compact 2−manifold with boundary and two functions $\alpha, \beta : M \longrightarrow \mathbb{R}$ with $\alpha, \beta \in C^1(M)$ defining

$$\omega = \alpha\,\mathrm{d}x + \beta\,\mathrm{d}y \tag{5.90}$$

We have

$$\int_{\partial M} \alpha\,\mathrm{d}x + \beta\,\mathrm{d}y = \int_{\partial M} \omega = \int_M \mathrm{d}\omega = \iint_M \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \tag{5.91}$$

Proof. Take $\omega = \alpha\,\mathrm{d}x + \beta\,\mathrm{d}y$, then

$$\mathrm{d}\omega = \mathrm{d}(\alpha\,\mathrm{d}x + \beta\,\mathrm{d}y) = \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y$$

□

Theorem 5.46 (Gauss, Differential Forms)**.** *Given M a $3-$manifold smooth with boundary and compact with outer normal n^μ and a vector field $f^\mu \in C^1(M)$, we have*

$$\int_M \partial_\mu f^\mu \,\mathrm{d}V = \int_{\partial M} f^\mu n_\mu \,\mathrm{d}A \tag{5.92}$$

Proof. Taken the following differential form

$$\omega = f^1 \,\mathrm{d}y \wedge \mathrm{d}z + f^2 \,\mathrm{d}z \wedge \mathrm{d}x + f^3 \,\mathrm{d}x \wedge \mathrm{d}y$$

We have, using the formulas (5.87)

$$\omega = f^\mu n_\mu \,\mathrm{d}A$$

And

$$\mathrm{d}\omega = \partial_\mu f^\mu \,\mathrm{d}V$$

Therefore

$$\int_M \partial_\mu f^\mu \,\mathrm{d}V = \int_M \mathrm{d}\omega = \int_{\partial M} \omega = \int_{\partial M} f^\mu n_\mu \,\mathrm{d}A$$

□

Theorem 5.47 (Stokes, Differential Forms)**.** *Given $M \subset \mathbb{R}^3$ a compact oriented smooth $2-$manifold with boundary with n^μ as outer normal and t^μ as tangent vector in ∂M, given a vector field $f^\mu \in C^1(A)$ where $M \subset A$, we have*

$$\int_M n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A = \int_{\partial M} f^\mu t_\mu \,\mathrm{d}s \tag{5.93}$$

Proof. Taking the following differential form

$$\omega = f^\mu \,\mathrm{d}x_\mu$$

We have that

$$\mathrm{d}\omega = (\partial_2 f^3 - \partial_3 f^2) \,\mathrm{d}y \wedge \mathrm{d}z + (\partial_3 f^1 - \partial_1 f^3) \,\mathrm{d}z \wedge \mathrm{d}x + (\partial_1 f^2 - \partial_2 f^1) \,\mathrm{d}x \wedge \mathrm{d}y$$

Using the formulas (5.87) we have

$$\mathrm{d}\omega = n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A$$

Since in $\mathbb{R}^2$ we have $t^\mu \,\mathrm{d}s = \mathrm{d}x^\mu$ we therefore have

$$f^\mu t_\mu \,\mathrm{d}s = f^\mu \,\mathrm{d}x_\mu = \omega$$

And therefore

$$\int_M n^\mu \epsilon_{\mu\nu\gamma} \partial^\nu f^\gamma \,\mathrm{d}A = \int_M \mathrm{d}\omega = \int_{\partial M} \omega = \int_{\partial M} f^\mu t_\mu \,\mathrm{d}s$$

□

These last formulas are a good example on how they can be generalized through the use of differential forms, bringing an easy way of calculus in $\mathbb{R}^n$ of the various integral theorems, all condensed in one formula, the *Gauss-Green-Stokes-Ostogradskij theorem*

6 Sequences, Series and Residues

§ 6.1 Sequences of Functions

Definition 6.1.1 (Sequence of Functions)**.** Let S be a set and (X, d) a metric space, a *sequence of functions* is defined as follows

$$\begin{aligned} f_n :& S \longrightarrow (X, d) \\ & s \to f_n(s) \end{aligned} \tag{6.1}$$

Where, $\forall n \in \mathbb{N}$ a function $f_{(n)} : S \longrightarrow (X, d)$ is defined

Definition 6.1.2 (Pointwise Convergence)**.** A sequence of functions $(f_n)_{n\geq 0}$ is said to converge pointwise to a function $f : S \longrightarrow (X, d)$, and it's indicated as $f_n \to f$, if

$$\forall \epsilon > 0,\ \forall x \in S\ \exists N_\epsilon(x) \in \mathbb{N} : d(f_n(x), f(x)) < \epsilon\ \forall n \geq N_\epsilon(x) \tag{6.2}$$

It can be indicated also as follows

$$\lim_{n\to\infty} (f_n(x)) = f(x) \tag{6.3}$$

Definition 6.1.3 (Uniform Convergence)**.** Defining an $\|\cdot\|_\infty = \sup_{i\leq n} |\cdot|$ we have that the convergence of a sequence of functions is uniform, and it's indicated as $f_n \rightrightarrows f$, iff

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} : d(f_n(x), f(x)) < \epsilon\ \forall n \geq N_\epsilon\ \forall x \in S \tag{6.4}$$

Or, using the norm $\|\cdot\|_\infty$

$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} : \|f_n - f\|_\infty < \epsilon \tag{6.5}$$

Theorem 6.1 (Continuity of Uniformly Convergent Sequences)**.** *Let $(f_n)_{n\geq 0} : (S, d_S) \longrightarrow (X, d)$ be a sequence of continuous functions. Then if $f_n \rightrightarrows f$, we have that $f \in C(S)$, where $C(S)$ is the space of continuous functions*

Proof.

$$\begin{aligned} & \forall x \in S,\ \exists \epsilon > 0 : f_n \rightrightarrows f,\ \therefore \forall n \geq N_\epsilon \in \mathbb{N} : d(f_n(x), f(x)) < \frac{\epsilon}{3} \\ & f_n \in C(S) \implies \exists \delta_\epsilon > 0 : d(f_n(x), f_n(y)) < \frac{\epsilon}{3},\ \forall x, y \in S : d_S(x, y) < \delta \\ & \therefore d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) < \epsilon \iff d_S(x, y) < \delta_\epsilon \end{aligned} \tag{6.6}$$

□

Theorem 6.2 (Integration of Sequences of Functions)**.** *Let $(f_n)_{n\geq 0}$ be a sequence of functions such that $f_n \rightrightarrows f$ Then we can define the following equality*

$$\lim_{n\to\infty}\int_a^b f_n(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \tag{6.7}$$

Proof. We already know that in the closed set $[a,b]$ we can say, since $f_n \rightrightarrows f$, that

$$\forall\epsilon > 0\ \exists N_\epsilon \in \mathbb{N} : \forall n \geq N_\epsilon\ \|f_n - f\|_\infty < \frac{\epsilon}{b-a} \tag{6.8}$$

Then, we have that

$$\forall n \geq N_\epsilon\ \left|\int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x\right| \leq \|f_n - f\|_\infty (b-a) < \epsilon \tag{6.9}$$

□

Theorem 6.3 (Differentiation of a Sequence of Functions)**.** *Define a sequence of functions as $f_n : I \longrightarrow \mathbb{R}$, with $f_n(x) \in C^1(I)$. If*

1. $\exists x_0 \in I : f_n(x_0) \to l$
2. $f_n' \rightrightarrows g\ \forall x \in I$

Then

$$f_n(x) \rightrightarrows f \implies \forall x \in I,\ f'(x) = \lim_{n\to\infty} f_n'(x) = g(x) \tag{6.10}$$

Proof. For the fundamental theorem of integral calculus, we can write, using the regularity of the $f_n(x)$ that

$$f_n(x) = f_n(x_0) + \int_{x_0}^x f_n(t)\,\mathrm{d}t$$

Taking the limit we have

$$\lim_{n\to\infty} f_n(x) = l + \int_{x_0}^x g(t)\,\mathrm{d}t = f(x)$$
$$\therefore f'(t) = g(t)$$

But, we also have that

$$\forall\epsilon > 0\ \|f_n' - f'\|_\infty \leq |f_n(x_0) - l| + \|f_n' - g\|_\infty (b-a) < \epsilon$$
$$\therefore f_n \rightrightarrows f,\ f_n' \rightrightarrows f'$$

□

§ 6.2 Series of Functions

Let now, for the rest of the section, $(X, d) = \mathbb{C}$.

Definition 6.2.1 (Series of Functions)**.** Let $(f_n)_{n\geq 0} \in \mathbb{C}$ be a sequence of functions, such that $f_n : S \to \mathbb{C}$. We can define the *series of functions* as follows

$$s_n(x) = \sum_{k=1}^{n} f_k(x) \tag{6.11}$$

Definition 6.2.2 (Convergent Series)**.** A series of functions $s_n(x) : S \to \mathbb{C}$ is said to be *convergent* or *pointwise convergent* if

$$s_n(x) = \sum_{k=0}^{n} f_k(x) \longrightarrow s(x) \tag{6.12}$$

Where $s(x) : S \to \mathbb{C}$ is the *sum* of the series.
This means that

$$\forall x \in S, \ \lim_{k\to\infty} s_k(x) = \sum_{k=0}^{\infty} f_k(x) = s(x) \tag{6.13}$$

Theorem 6.4. *Necessary Condition for the convergence of a series of functions:*
Let $(f_n) \in \mathbb{C}$ be a succession, then the series $s_n(x)$ defined as follows, converges to the function $s(x)$

$$s_n(x) = \sum_{k=0}^{n} f_k(x) = s(x) = \sum_{k=0}^{\infty} f_k(x)$$

Proof.

$$\forall x \in S \ \lim_{k\to\infty} f_k(x) = \lim_{n\to\infty} (s_n(x) - s_{n+1}(x)) = 0$$

□

Definition 6.2.3 (Uniform Convergence)**.** A series of functions is said to be *uniformly convergent* if and only if

$$\sum_{k=0}^{\infty} f_k(x) \rightrightarrows s(x) \iff s_n(x) = \sum_{k=0}^{n} f_k(x) \rightrightarrows s(x) \tag{6.14}$$

Definition 6.2.4 (Absolute Convergence)**.** A series of functions is said to be *absolutely convergent* if and only if

$$\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x) \implies \sum_{k=0}^{\infty} |f_k(x)| \to s(x) \tag{6.15}$$

Theorem 6.5. *Let $\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x)$, then*

$$\sum_{k=0}^{\infty} f_k(x) \xrightarrow{\text{A}} s(x) \implies \sum_{k=0}^{\infty} f_k(x) \to s(x) \tag{6.16}$$

Proof. Let

$$
\begin{aligned}
s_n(x) = \sum_{k=0}^{n} f_k(x) \ \therefore \exists g(x) : (S,d) \longrightarrow \mathbb{C},\ \exists N_\epsilon(x) \in \mathbb{N} : \left| g(x) - \sum_{k=0}^{\infty} f_k(x) \right| = \\
&= \sum_{k=n+1}^{\infty} |f_k(x)| < \epsilon \ \forall n \geq N_\epsilon(x) \\
&\therefore \forall n, m \in \mathbb{N}, m > n \\
&|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| < \epsilon \ \forall x \in S \\
&\therefore (s_n(x)) \text{ is a Cauchy series in } \mathbb{C} \implies s_k(x) \to s(x)
\end{aligned}
$$

□

Definition 6.2.5 (Total Convergence)**.** A series of functions $s_k(x)$ is said to be *totally convergent* if

1. $\exists M_k : \sup_S |f_k(x)| \leq M_k \ \forall k \geq 1$
2. $\sum_{k=0}^{\infty} M_k \to M$

The total convergence is then indicated as $s_k(x) \xrightarrow{\mathrm{T}} s(x)$

Proposition 10. Let

$$s_n(x) = \sum_{k=0}^{n} f_n(x)$$

Then

1. $f_n(x) \in C(S) \wedge s_k(x) \rightrightarrows s(x) \implies s(x) \in C(S)$
2. $f_n(x) \in C(S),\ s_k(x) \rightrightarrows s(x) \implies \int s(x)\, \mathrm{d}x = \lim_{k\to\infty} \int s_k(x)\, \mathrm{d}x$
3. $s_k(x) \xrightarrow{\mathrm{A}} s(x) \implies s_k(x) \to s(x)$
4. $s_k(x) \rightrightarrows s(x) \implies s_k(x) \xrightarrow{\mathrm{A}} s(x)$
5. $s_k(x) \xrightarrow{\mathrm{T}} s(x) \implies s_k(x) \rightrightarrows s(x)$

§§ 6.2.1 Power Series and Convergence Tests

Theorem 6.6 (Weierstrass Test)**.** *Let $(f_n) : (S,d) \to \mathbb{C}$ a sequence of functions. If we have that*

$$
\begin{gathered}
\forall n > N_\epsilon \in \mathbb{N} \ \exists M_n > 0 : |f_n(x)| \leq M_n \\
\therefore \forall x \in S \ \sum_{k=0}^{n} f_k(x) \leq \sum_{k=1}^{\infty} M_k \to M \therefore \sum_{k=0} f_k(x)^n \rightrightarrows s(x)
\end{gathered}
$$

Definition 6.2.6 (Power Series)**.** Let $z, z_0, (a_n) \in \mathbb{C}$. A *power series centered in* z_0 is defined as follows

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \tag{6.17}$$

Example 6.2.1. Take the *geometric series*. This is the best example of a power series centered in $z_0 = 0$, and it has the following form

$$\sum_{k=0}^{\infty} z^k \tag{6.18}$$

We can expand it as follows

$$\sum_{k=0}^{m} z^k = (1-z)\left(1 + z + z^2 + \cdots + z^m\right) = 1 - z^{n+1} = \frac{1 + z^{n+1}}{1 - z} \;\forall |z| \neq 1 \tag{6.19}$$

Taking the limit, we have, therefore

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z} \;\forall |z| < 1 \tag{6.20}$$

Theorem 6.7 (Cauchy-Hadamard Criteria)**.** *Let* $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ *be a power series, with* $a_n, z, z_0 \in \mathbb{C}$*. We define the Radius of convergence* $R \in \mathbb{R}^\star = \mathbb{R} \cup \{\pm\infty\}$*, with the Cauchy-Hadamard criteria*

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{\frac{1}{n}} = \begin{cases} +\infty & \frac{1}{R} = 0 \\ l & 0 < \frac{1}{R} = l < \infty \\ 0 & \frac{1}{R} = +\infty \end{cases} \tag{6.21}$$

Then $s_k(z) \rightrightarrows s(z) \;\forall |z| \in (-R, R)$

Theorem 6.8 (D'Alambert Criteria)**.** *From the power series we have defined before, we can write the D'Alambert criteria for convergence as follows*

$$\frac{1}{R} = \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| \implies R = \lim_{k \to \infty} \left| \frac{a_k}{a_{k+1}} \right| \tag{6.22}$$

Where R *is the previously defined radius of convergence*

Theorem 6.9 (Abel)**.** *Let* $R > 0$*, then if a power series converges for* $|z| = R$*, it converges uniformly* $\forall |z| \in [r, R] \subset (-R, R]$*. It is valid analogously for* $x = -R$

Remark (Power Series Integration). If the series has $R > 0$ and it converges in $|z| = R$, calling $s(x)$ the sum of the series, with $x = |z|$ we can say that

$$\int_0^R s(x)\,\mathrm{d}x = \sum_{k=0}^{\infty} \int_0^R a_k x^k \,\mathrm{d}x = \int_0^R \sum_{k=1}^{\infty} a_k x^k \,\mathrm{d}z = \sum_{k=0}^{\infty} a_k \frac{R^{k+1}}{k+1} \tag{6.23}$$

Remark (Power Series Derivation). If Abel's theorem holds, we have also that, if we have $s(x)$ our power series sum, we can define the $n-$th derivative of this series as follows

$$\frac{\mathrm{d}^n s}{\mathrm{d}x^n} = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1) a_k x^{k-n} \tag{6.24}$$

§ 6.3 Series Representation of Functions

§§ 6.3.1 Taylor Series

Theorem 6.10 (Taylor Series Expansion)**.** *Let $f : D \longrightarrow \mathbb{C}$ be a function such that $f \in H(B_R(z_0))$, with $B_r(z_0) \subseteq D$. Then*

$$f(z) = \sum_{n=0}^{n} \frac{1}{n!} \left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} (z - z_0)^n \quad \|z - z_0\| < r \tag{6.25}$$

Proof. Taken $z \in B_r(z_0)$ and $\gamma(t) = z_0 + re^{it}$ $t \in [0, 2\pi]$ and $\|z - z_0\| < r < R$ we can write, using the integral representation of f

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z)} \,\mathrm{d}w = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0) - (z - z_0)} \,\mathrm{d}w$$

From basic calculus we know already that if $z \neq w$

$$\frac{1}{w - z} = \frac{1}{w} \left(\frac{1 - (z/w)^n}{1 - z/w} + \frac{1}{1 - z/w} \left(\frac{z}{w}\right)^n \right) =$$

$$= \frac{1}{w - z} \left(\frac{z}{w}\right)^n + \sum_{k=0}^{n-1} \frac{1}{w} \left(\frac{z}{w}\right)^n$$

Therefore, inserting it back into the integral representation, we have

$$f(z) = \sum_{k=0} \frac{(z - z_0)^k}{2\pi i} \oint_\gamma \frac{f(w)}{(w - z_0)^{k+1}} \,\mathrm{d}w + \frac{(z - z_0)^n}{2\pi i} \oint_\gamma \frac{f(w)}{[(w - z_0) - (z - z_0)]\,(w - z_0)^n} \,\mathrm{d}w$$

On the RHS as first term we have the $k-$th derivative of f and on the right there is the so called *remainder* $R_n(z)$. Therefore

$$f(z) = \sum_{k=0}^{n} \frac{1}{k!} \left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} (z - z_0)^k + R_n(z)$$

It's easy to demonstrate that $R_n(z) \overset{n\to\infty}{\longrightarrow} 0$, and therefore

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} (z - z_0)^k$$

□

Definition 6.3.1 (Taylor Series for Scalar Fields)**.** Given a function $f : A \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ $f \in C^m(A)$, given a multi-index α one can define the Taylor series of the scalar field as follows

$$f(x) = \sum_{|\alpha| \leq m} \frac{1}{\alpha!} \partial^\alpha f(x_0)(x - x_0)^\alpha + R_m(x)$$

Where, the remainder is defined in integral form as follows

$$R_m(x) = (m + 1) \sum_{|\alpha| = m+1} \frac{(x - x_0)^\alpha}{\alpha!} \int_0^1 (1 - t)^m \partial^\alpha f(x_0 + tx - tx_0) \,\mathrm{d}t$$

Definition 6.3.2 (MacLaurin Series)**.** Taken a Taylor series, such that $z_0 = 0$, we obtain a MacLaurin series.

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \left. \frac{\mathrm{d}^k f}{\mathrm{d}z^k} \right|_{z=0} z^k \tag{6.26}$$

Definition 6.3.3 (Remainders)**.** We can have two kinds of remainder functions while calculating series:

1. Peano Remainders, $R_n(z) = \mathcal{O}(\|z - z_0\|^n)$
2. Lagrange Remainders, $R_n(x) = (n+1)!^{-1} f^{(n+1)}(\xi)(x - x_0)^{n+1},\ x, x_0 \in \mathbb{R}\ \xi \in (x, x_0)$

What we saw before as $R_n(z)$ is the remainder function for functions $f : D \subset \mathbb{C} \longrightarrow \mathbb{C}$.
A particularity of remainder function is that $R_n(z) \to 0$ always, if f is holomorphic

Theorem 6.11 (Integration of Power Series II)**.** *Let $f, g : B_R(z_0) \longrightarrow \mathbb{C}$ and $\{\gamma\} \subset B_R(z_0)$ a piecewise smooth path. Taken*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad g \in C(\{\gamma\})$$

We have that

$$\sum_{n=0}^{\infty} a_n \int_\gamma g(z)(z - z_0)^n \,\mathrm{d}z = \int_\gamma g(z) f(z) \,\mathrm{d}z \tag{6.27}$$

Proof. Since $f, g \in C(\{\gamma\})$ by definition, and $f \in H(\overline{B_r}(z_0))$ with $r < R$, we have that $\exists \hat{K}_\gamma[fg]$.
Firstly we can write that $\forall z \in B_R(z_0)$

$$g(z)f(z) = \sum_{k=0}^{n-1} a_k g(z)(z - z_0)^k + g(z)R_n(z) = \sum_{k=0}^{n-1} a_k g(z)(z - z_0)^k + g(z) \sum_{k=n}^{\infty} a_k (z - z_0)^k$$

Then we can write

$$\int_\gamma g(z)f(z) \,\mathrm{d}z = \sum_{k=0}^{n-1} a_k \oint_\gamma g(z)(z - z_0)^k \,\mathrm{d}z + \int_\gamma g(z)R_n(z) \,\mathrm{d}z$$

Letting $M = \sup_{z \in \{\gamma\}} \|g(z)\|$, and noting that $\|R_n(z)\| < \epsilon$ for $\forall \epsilon > 0$ and for some $n \geq N_\epsilon \in \mathbb{N},\ z \in \{\gamma\}$ we have, using the Darboux inequality

$$\left\| \int_\gamma g(z)R_n(z) \,\mathrm{d}z \right\| \leq M L_\gamma \epsilon \to 0$$

$\square$

Theorem 6.12 (Holomorphy of Power Series)**.** *If a function $f(z)$ is expressable as a power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k,\ \|z - z_0\| < R$ we have that $f \in H(B_R(z_0))$*

Proof. Take the previous theorem on the integration of power series, and choose $g(z) = 1$. Since $g(z) \in H(\mathbb{C})$ we also have that it'll be continuous on all paths $\{\gamma\} \subset \mathbb{C}$ piecewise smooth.
Take now a closed piecewise smooth path $\{\gamma\}$, then we can write

$$\oint_\gamma f(z)g(z)\,\mathrm{d}z = \oint_\gamma \sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k \oint_\gamma (z-z_0)^k\,\mathrm{d}z$$

Since the function $h(z) = (z-z_0)^k \in H(\mathbb{C})\ \forall k \neq 1$, we have, thanks to the Morera and Cauchy theorems

$$\oint_\gamma f(z)\,\mathrm{d}z = 0 \implies f(z) \in H(B_R(\overline{\{\gamma\}}))$$

□

Corollary 6.3.1 (Derivative of a Power Series II)**.** Take $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k\ \|z-z_0\| < R$. Then, $\forall z \in B_R(z_0)$ we have that

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \sum_{k=1}^{\infty} a_k k(z-z_0)^{k-1} \tag{6.28}$$

Proof. Taken $z \in B_R(z_0)$ and a continuous function $g(w) \in C(\{\gamma\})$, with $\{\gamma\} \subset B_R(z_0)$ a closed simple piecewise smooth path. If $z \in \{\gamma\}^\circ$ and

$$g(w) = \frac{1}{2\pi i}\left(\frac{1}{(w-z)^2}\right)$$

We have, using the integral representation for holomorphic functions

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z)^2}\,\mathrm{d}w = \sum_{k=0}^{\infty} \frac{a_k}{2\pi i}\oint_\gamma \frac{(w-z_0)^k}{(w-z_0)^2}\,\mathrm{d}w$$

Since $h(w) = (w-z_0)^k \in H(\mathbb{C})\ \forall k \neq 1$ we have, using again the integral representation for holomorphic functions

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z)^2}\,\mathrm{d}w = \frac{\mathrm{d}f}{\mathrm{d}z}$$
$$\frac{1}{2\pi i}\oint_\gamma \frac{(w-z_0)^k}{(w-z)^2}\,\mathrm{d}w = k(z-z_0)^{k-1}$$

Therefore

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{(w-z)^2}\,\mathrm{d}w = \sum_{k=0}^{\infty} a_k k(z-z_0)^k = \frac{\mathrm{d}f}{\mathrm{d}z}$$

□

Corollary 6.3.2 (Uniqueness of the Taylor Series)**.** Taken an holomorphic function $f \in H(D)$ with $D \subset \mathbb{C}$ some connected open set, we have that

$$f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k \quad a_k = \frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0} \quad \forall \|z-z_0\| < R$$

Proof. Taken $g(z)$ a continuous function over a closed piecewise simple path $\{\gamma\} \subset \mathbb{C}$, where

$$g(z) = \frac{1}{2\pi i}\left(\frac{1}{(z-z_0)^{k+1}}\right)$$

We have that

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \sum_{k=1}^{\infty}\frac{a_k}{2\pi i}\oint_\gamma (z-z_0)^{k-n-1}\,\mathrm{d}z$$

The integral on the RHS evaluates to δ_n^k, and thanks to the integral representation of $f(z)$ we can write

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z = \frac{1}{n!}\left.\frac{\mathrm{d}^n f}{\mathrm{d}z^n}\right|_{z_0} = n!a_n$$

□

§§ 6.3.2 Laurent Series

Definition 6.3.4 (Annulus Domain)**.** Let $0 \le r < R \le \infty$ and $z_0 \in \mathbb{C}$, we define the *annulus set* as follows

$$A_{rR}(z_0) := \{\, z \in \mathbb{C} |\, r < \|z - z_0\| < R\} \tag{6.29}$$

Special cases of this are the ones where $r = 0,\ R = \infty$ and $r = 0,\ R = \infty$

$$\begin{aligned} A_{0,R}(z_0) &= B_R(z_0) \setminus \{z_0\} \\ A_{r,\infty}(z_0) &= \mathbb{C} \setminus \overline{B}_r(z_0) \\ A_{0,\infty}(z_0) &= \mathbb{C} \setminus \{z_0\} \end{aligned}$$

Theorem 6.13 (Laurent Series Expansion)**.** *Let $f : A_{R_1R_2}(z_0) \longrightarrow \mathbb{C}$ be a function such that $f \in H\left(A_{R_1R_2}(z_0)\right)$, and $\{\gamma\} \subset A_{R_1R_2}(z_0)$ a closed simple piecewise smooth curve.*
Then f is expandable in a generalized power series or a Laurent series as follows

$$f(z) = \sum_{n=0}^{\infty} c_n^+ (z-z_0)^n + \sum_{n=1}^{\infty}\frac{c_n^-}{(z-z_0)^n} = \sum_{k=-\infty}^{\infty} c_k (z-z_0)^k \tag{6.30}$$

Where the coefficients are the following

$$\begin{aligned} c_n^- &= \frac{1}{2\pi i}\oint_\gamma f(z)(z-z_0)^{n-1}\,\mathrm{d}z \quad n \ge 0 \\ c_n^+ &= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,\mathrm{d}z \quad n > 0 \\ c_k &= \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,\mathrm{d}z \quad k \in \mathbb{Z} \end{aligned} \tag{6.31}$$

Proof. Taken a random point $z \in A_{R_1R_2}(z_0)$, a closed simple piecewise smooth curve $\{\gamma\} \subset A_{R_1R_2}(z_0)$ and two circular smooth paths $\{\gamma_2\}, \{\gamma_3\} : \{\gamma_2\} \cup \{\gamma_3\} = \partial A_{r_1r_2}(z_0) \subset A_{R_1R_2}(z_0) \ \wedge$

$\{\gamma\} \subset A_{r_1 r_2}(z_0)$ and a third circular path $\{\gamma_3\} \subset A_{r_1 r_2}(z_0)$, we can write immediately, using the omotopy between all the paths

$$\oint_{\gamma_2} \frac{f(w)}{w-z}\,\mathrm{d}w = \oint_{\gamma_1} \frac{f(w)}{w-z}\,\mathrm{d}w + \oint_{\gamma_3} \frac{f(w)}{w-z}\,\mathrm{d}w$$

Using the Cauchy integral representation we have that the integral on γ_3 yields immediately $2\pi i f(z)$, hence we can write

$$f(z) = \frac{1}{2\pi i}\oint_{\gamma_2} \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w + \frac{1}{2\pi i}\oint_{\gamma_1} \frac{f(w)}{(z_0-z)-(w-z_0)}\,\mathrm{d}w$$

Using the two following identities for $z \neq w$

$$\begin{aligned}
\frac{1}{w-z} &= \frac{1}{w-z}\left(\frac{z}{w}\right)^n + \sum_{k=0}^{n-1} \frac{1}{w}\left(\frac{z}{w}\right)^k \\
\frac{1}{z-w} &= \frac{1}{z-w}\left(\frac{w}{z}\right)^n + \sum_{k=1}^{n} \frac{1}{w}\left(\frac{w}{z}\right)^k
\end{aligned}$$

We obtain that

$$f(z) = \sum_{k=0}^{n-1} \frac{(z-z_0)^k}{2\pi i}\oint_{\gamma_2} \frac{f(w)}{(w-z_0)^{k+1}}\,\mathrm{d}w + \rho_n(z) + \sum_{k=1}^{n} \frac{1}{2\pi i (z-z_0)^k}\oint_{\gamma_1} f(w)(w-z_0)^{k-1}\,\mathrm{d}w + \sigma_n(z)$$

Where, after choosing appropriate substitutions with some coefficients c_k^+, c_k^- we have

$$f(z) = \sum_{k=0}^{n-1} c_k^+ (z-z_0)^k + \rho_n(z) + \sum_{k=1}^{n} \frac{c_k^-}{(z-z_0)^k} + \sigma_n(z)$$

Where ρ_n, σ_n are the two remainders of the series expansion, and are

$$\begin{aligned}
\rho_n(z) &= \frac{(z-z_0)^n}{2\pi i}\oint_{\gamma_2} \frac{f(w)}{\left[(w-z_0)-(z-z_0)\right](w-z_0)^n}\,\mathrm{d}w \\
\sigma_n(z) &= \frac{1}{2\pi i (z-z_0)^n}\oint_{\gamma_1} \frac{f(w)}{(w-z_0)-(z-z_0)}\,\mathrm{d}w
\end{aligned}$$

In order to prove the theorem we now need to demonstrate that $\rho_n, \sigma_n \overset{n\to\infty}{\longrightarrow} 0$. Taken $M_1 = \sup_{z\in\{\gamma_1\}} \|f(z)\|$, $M_2 = \sup_{z\in\{\gamma_2\}} \|f(z)\|$, we have, using the fact that both γ_1, γ_2 are circular

$$\begin{aligned}
\|\rho_n(z)\| &\le \frac{M_2}{1-\frac{1}{r_2}\|z-z_0\|}\left(\frac{\|z-z_0\|}{r_2}\right)^n \overset{n\to\infty}{\longrightarrow} 0 \quad \|z-z_0\| < r_2 \\
\|\sigma_n(z)\| &\le \frac{M_1}{\frac{1}{r_1}\|z-z_0\|-1}\left(\frac{r_1}{\|z-z_0\|}\right)^n \overset{n\to\infty}{\longrightarrow} 0 \quad r_1 < \|z-z_0\|
\end{aligned}$$

And the theorem is proved. $\square$

Theorem 6.14 (Convergence of a Laurent Series). *Being defined on an annulus set, the Laurent series of a function must have* two *radii of convergence. Given a function f holomorphic on a set $A_{R_1R_2}(z_0)$ we have*

$$\begin{aligned} \frac{1}{R_2} &= \limsup_{n\to\infty} \sqrt[n]{\|c_n\|} \\ R_1 &= \limsup_{n\to\infty} \sqrt[n]{\|c_{-n}\|} \end{aligned} \tag{6.32}$$

It's equivalent of showing the convergence of the two series

$$f(z) = \sum_{k=0}^{\infty} c_k^+ (z-z_0)^k + \sum_{k=1}^{\infty} \frac{c_k^-}{(z-z_0)^k}$$

Theorem 6.15 (Integral of a Laurent Series). *Let $f(z) \in H\left(A_{R_1R_2}(z_0)\right)$ and take $\{\gamma\} \subset A_{R_1R_2}(z_0)$ a piecewise smooth curve, and $g \in C(\{\gamma\})$, then we have*

$$\sum_{n=-\infty}^{\infty} c_n \oint_\gamma g(z)(z-z_0)^n \,\mathrm{d}z = \oint_\gamma g(z) f(z) \,\mathrm{d}z$$

Proof. We begin by separating the sum in two parts, ending up with the following

$$\begin{aligned} \sum_{n=0}^{\infty} c_n^+ \oint_\gamma g(z)(z-z_0)^n \,\mathrm{d}z &= \oint_\gamma g(z) f_+(z) \,\mathrm{d}z \\ \sum_{n=1}^{\infty} c_n^- \oint_\gamma \frac{g(z)}{(z-z_0)^n} \,\mathrm{d}z &= \oint_\gamma g(z) f_-(z) \,\mathrm{d}z \end{aligned}$$

Which is analogous to the integration of Taylor series. The same could be obtained keeping the bounds of the sum in all $\mathbb{Z}$ □

As for Taylor series, in a completely analogous fashion, a Laurent series is holomorphic and unique.
The derivative of a Laurent series, is then obviously the following

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \sum_{n=-\infty}^{\infty} c_n n (z-z_0)^{n-1}$$

§§ 6.3.3 Multiplication and Division of Power Series

Theorem 6.16 (Product of Power Series). *Take $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,\ z \in B_{R_1}(z_0)$ and $g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n,\ z \in B_{R_2}(z_0)$. Then*

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n, \quad c_n = \sum_{k=0}^{n} a_k b_{n-k} \quad \|z-z_0\| < \min(R_1, R_2) = R$$

Proof. Due to the holomorphy of both f and g, we have that the function fg has a Taylor series expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n (z-z_0) \quad \|z-z_0\| < R$$

We have then, using Leibniz's derivation rule

$$
\begin{aligned}
c_n = \frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}f(z)g(z) &= \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0}\left.\frac{\mathrm{d}^{n-k} g}{\mathrm{d}z^{n-k}}\right|_{z_0} = \\
&= \sum_{k=0}^{n}\frac{1}{k!}\left.\frac{\mathrm{d}^k f}{\mathrm{d}z^k}\right|_{z_0}\frac{1}{(n-k)!}\left.\frac{\mathrm{d}^{n-k} g}{\mathrm{d}z^{n-k}}\right|_{z_0} = \\
&= \sum_{k=0}^{n} a_k b_{n-k}
\end{aligned}
$$

□

Theorem 6.17 (Division of Power Series)**.** *Taken the two functions as before, with the added necessity that $g(z) \neq 0$, we have that*

$$
\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z-z_0)^n \quad d_n = \frac{1}{b_0}\left(a_n - \sum_{k=0}^{n-1} d_k b_{n-k}\right)
$$

Proof. Everything hold as in the previous proof. Remembering that $(f/g)g = f$ and using the previous theorem, we obtain

$$
a_n = \sum_{k=0}^{n} d_k b_{k-n}
$$

And therefore, inverting

$$
d_n = \frac{a_n}{b_0} - \frac{1}{b_0}\sum_{k=0}^{n-1} d_k b_{n-k}
$$

□

§§ 6.3.4 Useful Expansions

$$
e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!} \quad \|z\| < \infty \tag{6.33}
$$

$$
\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \quad \|z\| < \infty \tag{6.34}
$$

$$
\cos(z) = \frac{\mathrm{d}}{\mathrm{d}z}\sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n} \quad \|z\| < \infty \tag{6.35}
$$

$$
\cosh(z) = \cos(iz) = \sum_{n=0}^{\infty}\frac{z^{2n}}{(2n)!} \quad \|z\| < \infty \tag{6.36}
$$

$$
\sinh(z) = \frac{\mathrm{d}}{\mathrm{d}z}\cosh(z) = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!} \quad \|z\| < \infty \tag{6.37}
$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \|z\| < 1 \tag{6.38}$$

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad \|z\| < 1 \tag{6.39}$$

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad \|z-1\| < 1 \tag{6.40}$$

$$(1+z)^s = \sum_{n=0}^{\infty} \binom{s}{n} z^n \quad s \in \mathbb{C},\ \|z\| < 1 \tag{6.41}$$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} \quad 0 < \|z\| < \infty \tag{6.42}$$

§ 6.4 Residues

§§ 6.4.1 Singularities and Residues

Definition 6.4.1 (Singularity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ we define a *singularity* a point $z_0 \in G$ such that

$$\forall \epsilon > 0\ \exists z \in B_\epsilon(z_0)\ :\ f(z) \text{ is holomorphic} \tag{6.43}$$

Definition 6.4.2 (Isolated Singularity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ we define an *isolated singularity* a point $z_0 \in G$ such that

$$\exists r > 0\ :\ f \in H\left(A_{0r}(z_0)\right) \tag{6.44}$$

Definition 6.4.3 (Residue)**.** Let $z_0 \in G$ be an isolated singularity of $f : G \longrightarrow \mathbb{C}$, then $\exists r > 0 :$ $\forall z \in A_{0r}(z_0)$ the following Laurent series expansion holds

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

The *residue* of the function f in z_0 is defined as follows

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = c_{-1} \tag{6.45}$$

A second definitiion is given by the following contour integral

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i} \oint_\gamma f(z)\,\mathrm{d}z$$

Where γ is a simple closed path around z_0

Definition 6.4.4 (Winding Number)**.** Given a closed curve $\{\gamma\}$ we define the *winding number* or *index* of the curve around a point z_0 the following integral

$$n(\gamma, z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{\mathrm{d}z}{z-z_0} \tag{6.46}$$

Theorem 6.18 (Residue Theorem)**.** *Given a function $f : G \longrightarrow \mathbb{C}$ such that $f \in H(D)$ where $D = G \setminus \{z_1, \cdots, z_n\}$ and z_k are isolated singularities, we have, taken a closed piecewise smooth curve $\{\gamma\}$, such that $\{z_1, \cdots, z_n\} \subset \{\gamma\}^\circ$*

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=0}^{\infty} n(\gamma, z_k) \operatorname*{Res}_{z=z_k} f(z) \tag{6.47}$$

Proof. Firstly we can say that $\gamma \sim \sum_k \gamma_k$ where γ_k are simple curves around each z_k, then since the function is holomorphic in the regions $A_{0r}(z_k)$ with $k = 1, \cdots, n$ we can write

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_k)^n$$

Therefore, we have

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \oint_{\gamma_k} f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \sum_{j=-\infty}^{\infty} c_j \oint_{\gamma_k} (z - z_k)^j\,\mathrm{d}z$$

We can then use the linearity of the integral operator and write

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} \sum_{j=-\infty}^{-2} c_j \oint_{\gamma_k} (z - z_k)^j\,\mathrm{d}z + c_{-1} \oint_{\gamma_k} \frac{\mathrm{d}z}{z - z_k} + \sum_{j=0}^{\infty} c_j \oint_{\gamma_k} (z - z_k)^j\,\mathrm{d}z$$

Thanks to the Cauchy theorem we already know that the first and last integrals on the RHS must be null, therefore

$$\oint_\gamma f(z)\,\mathrm{d}z = \sum_{k=0}^{n} c_{-1} \oint_{\gamma_k} \frac{\mathrm{d}z}{z - z_k}$$

Recognizing the definition of residue and the winding number of the curve, we have the assert

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=0} n(\gamma, z_k) \operatorname*{Res}_{z=z_k} f(z)$$

□

Definition 6.4.5 (Residue at Infinity)**.** Given a function $f : G \longrightarrow \mathbb{C}$ and a piecewise smooth closed curve γ. If $f \in H(\{\gamma\} \cup \operatorname{extr}\{\gamma\})$ we have

$$\oint_\gamma f(z)\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{z=\infty} f(z) = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right) \tag{6.48}$$

Theorem 6.19. *Given a function $f : G \longrightarrow \mathbb{C}$ as before, if the function has z_k singularities with $k = 1, \cdots, n$*

$$\operatorname*{Res}_{z=\infty} f(z) = \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \tag{6.49}$$

§§ 6.4.2 Classification of Singularities, Zeros and Poles

Definition 6.4.6 (Pole)**.** Given a function $f(z)$ with an isolated singular point in $z_0 \in \mathbb{C}$, we have that in $A_{0r}(z_0)$ the function can be expanded with a Laurent series

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k + \sum_{k=1}^{\infty} \frac{b_k}{(z-z_0)^k}$$

The point z_0 is called a *pole of order* m if $b_k = 0\ \forall k > m$

Definition 6.4.7 (Removable Singularity)**.** Given $f(z), z_0$ as before, we have that z_0 is a *removable singularity* if $b_k = 0\ \forall k \geq 1$

Definition 6.4.8 (Essential Singularity)**.** Given $f(z), z_0$ as before, we have that z_0 is an *essential singularity* if $b_k \neq 0$ for infinite values of k

Definition 6.4.9 (Meromorphic Function)**.** Let $f : G \subset \mathbb{C} \longrightarrow \mathbb{C}$ be a function. f is said to be *meromorphic* if $f \in H(\tilde{G})$ where $\tilde{G} = G \setminus \{z_1, \cdots, z_n\}$ where $z_k \in G$ are poles of the function

Theorem 6.20. *Let z_0 be an isolated singularity of a function $f(z)$. z_0 is a pole of order m if and only if*

$$f(z) = \frac{1}{(m-1)!} \left.\frac{\mathrm{d}^{m-1} g}{\mathrm{d}z^{m-1}}\right|_{z_0} \qquad g \in H(B_\epsilon(z_0))\ \epsilon > 0 \tag{6.50}$$

Proof. Let $f : G \longrightarrow \mathbb{C}$ be a meromorphic function and $g : G \longrightarrow \mathbb{C}$, $g \in H(G)$ where $f(z)$ has a pole in $z_0 \in G$ and $g(z_0) \neq 0$

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

Since $g(z)$ is holomorphic in z_0 we have that, for some r

$$g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}z_0^k} (z-z_0)^k \quad z \in B_r(z_0)$$

And therefore, $\forall z \in A_{0r}(z_0)$

$$f(z) = \frac{1}{(z-z_0)^m} g(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d}z_0^k} (z-z_0)^{k-m}$$

Since $g(z_0) \neq 0$ we have the assert.
Alternatively we start by hypothesizing that z_0 is already a pole of order m for f, and therefore we can write the following Laurent expansion for some $r > 0$

$$f(z) = \sum_{k=-m}^{\infty} c_k (z-z_0)^k \quad \forall z \in A_{0r}(z_0)$$

Where $c_{-m} \neq 0$. Therefore, we write

$$g(z) = \begin{cases} (z-z_0)^m f(z) & z \in A_{0r}(z_0) \\ c_{-m} & z = z_0 \end{cases}$$

And, expanding $g(z)$ for $z \in B_r(z_0)$ we obtain

$$g(z) = c_{-m} + c_{-m+1}(z - z_0) + \cdots + c_{-1}(z - z_0)^{m-1} + \sum_{k=0}^{\infty} c_k (z - z_0)^{k+m}$$

$g(z)$ is holomorphic in the previous domain of expansion, and therefore we have, since the Taylor expansion is unique

$$c_{-1} = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1} g}{\mathrm{d} z_0^{m-1}} = \operatorname*{Res}_{z=z_0} f(z)$$

□

Definition 6.4.10 (Zero)**.** Let $f : G \longrightarrow \mathbb{C}$ be a holomorphic function. Taken $z_0 \in G$, it's said to be a *zero of order* m if

$$\begin{cases} \dfrac{\mathrm{d}^k f}{\mathrm{d} z_0^k} = 0 \quad k = 1, \cdots, m-1 \\ \dfrac{\mathrm{d}^m f}{\mathrm{d} z_0^m} \neq 0 \end{cases}$$

Theorem 6.21. *The point $z_0 \in G$ is a zero of order m for f if and only if*

$$f(z) = \frac{g(z)}{(z - z_0)^m} \quad g(z_0) \neq 0,\ g \in H(G)$$

Proof. Taken $f(z) = (z - z_0)^m g(z)$ such that $g(z_0) \neq 0$ we can expand $g(z)$ with Taylor and at the end obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\mathrm{d}^k g}{\mathrm{d} z_0^k} (z - z_0)^{k+m}$$

Since this is a Taylor expansion also for $f(z)$ we have that, for $j = 1, \cdots, m-1$

$$\frac{\mathrm{d}^j f}{\mathrm{d} z_0^j} = 0 \qquad \frac{\mathrm{d}^m f}{\mathrm{d} z_0^m} = m! g(z_0) \neq 0$$

The same is obtainable with the vice versa demonstrating the theorem □

Notation. Let f be a meromorphic function. We will define the following sets of points accordingly

1. Z_f^m as the set of zeros of order m
2. S_f as the set of isolated singularities of f
3. P_f^m as the set of poles of order m

We immediately see some special cases

1. P_f^∞ is the set of essential singularities of f
2. P_f^1 is the set of removable singularities of f

Theorem 6.22. *Let $f : D \longrightarrow \mathbb{C}$ be a function such that $f \in H(D)$, with D an open set, then*

1. *$f(z) = 0\ \forall z \in D$*
2. *$\exists z_0 \, : \, f^{(k)}(z_0) = 0\ \forall k \geq 0$*
3. *$Z_f \subset D$ has a limit point*

Proof. 3) $\implies$ 2)
Take $z_0 \in D$ as the limit point of Z_f. Since $f \in C(D)$ we have that $z_0 \in Z_f^m$. therefore

$$f(z) = (z - z_0)^m g(z) \quad g(z_0) \neq 0,\ g \in H(D) \implies \exists \delta > 0 \, : \, g(z) \neq 0 \quad \forall z \in B_\delta(z_0)$$

Therefore

$$f(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \lightning$$

2) $\implies$ 1)
Suppose that $Z_{f^{(k)}} := \left\{ z \in D |\ f^{(k)}(z) = 0 \right\} \neq \{\}$. We have to demonstrate that this set is clopen in D.
Take $z \in \overline{Z_{f^{(k)}}}$ and a sequence $(z)_k \in Z_{f^{(k)}}$ such that $z_k \to z$. We have then

$$f^{(k)}(z) = \lim_{k \to \infty} f^{(k)}(z_k) = 0$$

Therefore $Z_{f^{(k)}} = \overline{Z_{f^{(k)}}}$ and the set is closed.
Take then $z \in Z_{f^{(k)}} \subset D$, since D is open we have that $\exists r > 0 \, : \, B_r(z) \subset D$, therefore

$$\forall w \in B_r(z),\ z \neq w \quad f(w) = \sum_{k=0}^{\infty} a_k (w - z)^k = 0 \implies \begin{cases} z = w \\ a_k = 0 \quad \forall k \geq 0 \end{cases}$$

Since $w \neq z$ we have that $B_r(z) \subset Z_{f^{(k)}}$ and the set is open. Taking both results we have that the set is clopen and $D = Z_{f^{(k)}}$ □

Corollary 6.4.1. Let $f, g : D \longrightarrow \mathbb{C}$ and $f, g \in H(D)$. We have that $f = g$ iff the set $\{f(z) = g(z)\}$ has a limit point in D

Corollary 6.4.2 (Zeros of Holomorphic Functions)**.** Let $f : D \longrightarrow \mathbb{C}$ be a non-constant function $f \in H(D)$ with D an open connected set. Then

$$\forall z \in Z_f^m \quad m < \infty$$

Proof. Take $z_0 \in Z_f$, then since f is non-constant we have that Z_f has no limit points in D, therefore

$$\exists \delta > 0 \, : \, f(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \ \wedge \ \exists m \geq 1 \, : \, \frac{\mathrm{d}^k f}{\mathrm{d} z_0^k} = 0 \ k \in [0, m),\ \frac{\mathrm{d}^m f}{\mathrm{d} z_0^m} \neq 0$$

Therefore $z_0 \in Z_f^m$ □

Theorem 6.23. *Let $f : D \longrightarrow \mathbb{C}$ be a meromorphic function, such that*

$$f(z) = \frac{p(z)}{q(z)} \quad p, q \in H(D)$$

If $z_0 \in Z_q^m$ such that $p(z_0) \neq 0$, then $z_0 \in P_f^m$

Proof. $z_0 \in Z_q^m$ is an isolated singularity of q, therefore

$$\exists \delta > 0 \,:\, q(z) \neq 0 \quad \forall z \in A_{0\delta}(z_0) \;\therefore z_0 \in S_{p/q}$$

We therefore can take $q(z) = (z - z_0)^m g(z)$ and we have

$$f(z) = \frac{p(z)}{g(z)(z - z_0)^m} = \frac{h(z)}{(z - z_0)^m}$$

Where $h(z)$ is a holomorphic function such that $h(z_0) \neq 0$. By definition of pole we have $z_0 \in P_f^m$ □

Theorem 6.24 (Quick Calculus of Residues for Rational Functions)**.** *If $f(z) = p(z)/q(z)$ as before, there is a quick rule of thumb for calculating the residue in z_0. We can write*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1} h}{\mathrm{d} z_0^{m-1}}$$

If the pole is a removable singularity, we have $z_0 \in P_f^1$ and

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

Theorem 6.25. *Let f be a meromorphic function. If $z_0 \in P_f^m$ we have*

$$\lim_{z \to z_0} f(z) = \infty$$

Proof.

$$z_0 \in P_f^m \implies f(z) = \frac{g(z)}{(z - z_0)^m},\; z_0 \notin Z_g$$

Then

$$\lim_{z \to z_0} \frac{1}{f(z)} = \lim_{z \to z_0} \frac{(z - z_0)}{g(z)} = 0$$

□

Theorem 6.26. *If $z_0 \in P_f^1$, $\exists \epsilon > 0$ such that $f \in A_{0\epsilon}(z_0)$ and $\|f(z)\| \leq M$, $\forall z \in A_{0\epsilon}(z_0)$*

Proof. By definition we have that

$$\exists r > 0 \,:\, f \in H(A_{0\epsilon}(z_0))$$

And therefore the function is Laurent representable in this set as follows

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k \quad 0 < \|z - z_0\| < \epsilon$$

Taken the following holomorphic function

$$g(z) = \begin{cases} f(z) & z \in A_{0\epsilon}(z_0) \\ \displaystyle\sum_{z=0}^{\infty} c_k (z - z_0) & z = z_0 \end{cases}$$

We have that $g \in H\left(\overline{B_\epsilon}(z_0)\right)$ and therefore $\|f(z)\| \leq M \quad \forall z \in A_{0\epsilon}(z_0)$ □

Lemma 6.4.1 (Riemann)**.** Take a function $f \in H\left(A_{0\epsilon}(z_0)\right)$ for some $\epsilon > 0$, then if $\|f(z)\| \leq M\ \forall z \in A_{0\epsilon}(z_0)$
The point z_0 is a removable singularity for f

Proof. In the set of holomorphy the function is representable with Laurent, therefore

$$f(z) = \sum_{k=0}^{\infty} c_k^+ (z - z_0)^k + \sum_{k=1}^{\infty} \frac{c_k^-}{(z - z_0)^k}$$

We have that the coefficients c_k^- are the following, where we integrate over a curve $\{\gamma\} := \{ z \in \mathbb{C} |\ \|z - z_0\| = \rho < \epsilon \}$

$$c_k^- = \frac{1}{2\pi i} \oint_\gamma f(z)(z - z_0)^{k-1}\,\mathrm{d}z$$

The function is limited, and therefore for Darboux

$$c_k^- \leq \rho^k M \to 0 \quad \forall k \geq 1$$

Therefore $z_0 \in P_f^1$ □

Theorem 6.27 (Quick Calculus Methods for Residues)**.** *Let f be a meromorphic function, then*

1. *$z_0 \in P_f^n$ then*

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} (z - z_0)^n f(z) \tag{6.51}$$

2. *$z_0 \in P_f^m$ and $f(z) = p(z)/(z - z_0)^m$, where $p \in \mathbb{C}_k[z]$ with $k \leq m - 2$ and $p(z_0) \neq 0$, then*

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{(z - z_0)^m} = 0$$

§ 6.5 Applications of Residue Calculus

§§ 6.5.1 Improper Integrals

Definition 6.5.1 (Improper Integral)**.** An *improper integral* is defined as the integral of a function in a domain where such function has a divergence, or where the interval is infinite. Some examples of such integrals, given a function $f(x)$ with divergences at $a, b \in \mathbb{R}$ are the following

$$\begin{aligned}
\int_c^\infty f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \int_c^R f(x)\,\mathrm{d}x \\
\int_{-\infty}^d f(x)\,\mathrm{d}x &= \lim_{R\to\infty} \int_{-R}^d f(x)\,\mathrm{d}x \\
\int_{-\infty}^\infty f(x)\,\mathrm{d}x &= \int_{\mathbb{R}} f(x)\,\mathrm{d}x = \lim_{R\to\infty} \int_{-R}^R f(x)\,\mathrm{d}x \\
\int_a^b f(x)\,\mathrm{d}x &= \lim_{\epsilon\to 0^+} \int_{a+\epsilon}^{b-\epsilon} f(x)\,\mathrm{d}x \\
\int_e^h f(x)\,\mathrm{d}x &= \lim_{\epsilon\to 0^+} \left(\int_e^{a-\epsilon} f(x)\,\mathrm{d}x + \int_{a+\epsilon}^h f(x)\,\mathrm{d}x \right) \quad a \in (e, h)
\end{aligned}$$

Definition 6.5.2 (Cauchy Principal Value)**.** The previous definitions give rise to the following definition, the *Cauchy principal value.* Given an improper integral we define the Cauchy principal value as follows
Let $f(x)$ be a function with a singularity $c \in (a, b)$, and $g(x)$ another function then

$$\begin{aligned} \mathrm{PV}\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x &= \mathrm{PV}\int_{\mathbb{R}} g(x)\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} g(x)\,\mathrm{d}x \\ \mathrm{PV}\int_{a}^{b} f(x)\,\mathrm{d}x &= \lim_{\epsilon\to 0^+}\left(\int_{a}^{c-\epsilon} f(x)\,\mathrm{d}x + \int_{c+\epsilon}^{b} f(x)\,\mathrm{d}x\right) \end{aligned}$$

In the first case. PV is usually omitted.
For a complex integral, if $\gamma_R(t) = Re^{it}$ is a circumference, we have

$$\mathrm{PV}\int_{\gamma_R} f(z)\,\mathrm{d}z = \lim_{R\to\infty}\int_{\gamma_R} f(z)\,\mathrm{d}z$$

Notation (Circumferences and Parts of Circumference)**.** For a quick writing of the integrals in this section, we will use this notation for the following circumferences

$$\begin{aligned} C_R(t) &= Re^{it} \quad t \in [0, 2\pi] \\ C_{R\alpha\beta} &= Re^{it} \quad t \in [\alpha, \beta] \\ C_R^+(t) &= Re^{it} \quad t \in [0, \pi] \\ C_R^-(t) &= Re^{-it} \quad t \in [0, \pi] \\ \tilde{C^{\pm}}_R &= C_R^{\pm} \times [-R, R] \end{aligned}$$

Hypothesis 1. Let $R_0 > 0$ and $f \in C(D)$, where $D := \{\, z \in \mathbb{C} |\ \|z\| \geq R_0\} \cup \mathbb{R}$ and

$$\lim_{z\to\infty} zf(z) = 0$$

Hypothesis 2. Let $R_0 > 0$ and $f \in C(D)$, where $D := \{\, z \in \mathbb{C} |\ \|z\| \geq R_0\} \cup \mathbb{R}$ and

$$\lim_{z\to\infty} f(z) = 0$$

Theorem 6.28. *If* (1) *holds true, then*

$$\mathrm{PV}\int_{\gamma_R} f(z)\,\mathrm{d}z = 0 \quad \gamma_R = C_R, C_R^+, C_R^- \tag{6.52}$$

Also, if $f(x)$ is a real function

$$\int_{\mathbb{R}} f(x)\,\mathrm{d}x = \mathrm{PV}\int_{\tilde{C}_R^+} f(z)\,\mathrm{d}z = \mathrm{PV}\int_{\tilde{C}_R^-} f(z)\,\mathrm{d}z \tag{6.53}$$

Theorem 6.29. *Let $f(z)$ be an even function, if (1) holds we have*

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{2}\,\mathrm{PV}\int_{\tilde{C}_R^+} f(z)\,\mathrm{d}z = \frac{1}{2}\,\mathrm{PV}\int_{\tilde{C}_R^-} f(z)\,\mathrm{d}z \tag{6.54}$$

Theorem 6.30. *Let $f(z) = g(z^k)$, $k \geq 2$. If (1) holds*

$$\int_0^\infty f(x)\,\mathrm{d}x = \frac{1}{1-e^{\frac{2i\pi}{k}}}\,\mathrm{PV}\int_{\tilde{C}_{R0,2\pi/k}} f(z)\,\mathrm{d}z \tag{6.55}$$

Theorem 6.31. *If (2) holds*

$$\begin{aligned} \int_{\mathbb{R}} f(x)e^{i\lambda x}\,\mathrm{d}x &= \mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z \quad \lambda > 0 \\ \int_{\mathbb{R}} f(x)e^{i\lambda x}\,\mathrm{d}x &= \mathrm{PV}\int_{\tilde{C}_R^-} f(z)e^{i\lambda z}\,\mathrm{d}z \quad \lambda > 0 \end{aligned} \tag{6.56}$$

From this, we can write then, for $\lambda > 0$

$$\begin{aligned} \int_{\mathbb{R}} f(x)\cos(i\lambda x)\,\mathrm{d}x &= \mathfrak{Re}\left(\mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z\right) \quad \lambda > 0 \\ \int_{\mathbb{R}} f(x)\sin(i\lambda x)\,\mathrm{d}x &= \mathfrak{Im}\left(\mathrm{PV}\int_{\tilde{C}_R^+} f(z)e^{i\lambda z}\,\mathrm{d}z\right) \quad \lambda > 0 \end{aligned} \tag{6.57}$$

Hypothesis 3. Let $f(z) = g(z)h(z)$ with $g(z)$ a meromorphic function such that $S_g \not\subset \mathbb{R}^+$ and

1. $h \in H(\mathbb{C} \setminus \mathbb{R}^+)$
2. $\lim_{z\to\infty} zf(z) = 0$
3. $\lim_{z\to 0} zf(z) = 0$

Definition 6.5.3 (Pacman Path)**.** Let $\Gamma_{Rr\epsilon}$ be what we will call as the *pacman path*, this path is formed by 4 different paths

$$\begin{aligned} \gamma_1(t) &= re^{it} \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_2 &= [-R, R] \\ \gamma_3(t) &= Re^{it} \quad t \in [\epsilon, 2\pi - \epsilon] \\ \gamma_4 &= [-R, R] \end{aligned} \tag{6.58}$$

We will abbreviate this as Γ

Theorem 6.32. *Given $f(x)$ a function such that (3) holds, we have that*

$$\int_0^\infty g(x)\Delta h(x)\,\mathrm{d}x = \mathrm{PV}\int_\Gamma g(z)h(z)\,\mathrm{d}z \tag{6.59}$$

Where

$$\Delta h(x) = \lim_{\epsilon \to 0^+} \left(h(x + i\epsilon) - h(x - i\epsilon) \right) \tag{6.60}$$

In general, we have the following conversion table

$h(z)$	$\Delta h(x)$
$-\frac{1}{2\pi i} \log_+(z)$	1
$\log_+(z)$	$-2\pi i$
$\log_+^2(z)$	$-2\pi i \log(x) + 4\pi^2$
$\log_+(z) - 2\pi i \log_+(z)$	$-4\pi i \log(x)$
$\frac{i}{4\pi} \log_+^2(z) + \frac{1}{2} \log_+(z)$	$\log(x)$
$[z^\alpha]^+$	$x^\alpha \left(1 - e^{2\pi i \alpha}\right)$

(6.61)

All the previous integrals are solved through a direct application of the residue theorem.

§§ 6.5.2 General Rules

Theorem 6.33 (Integrals of Trigonometric Functions)**.** *Let $f(\cos\theta, \sin\theta)$ be some rational function of cosines and sines. Then we have that*

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, \mathrm{d}\theta = \int_{\|z\|=1} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right) \frac{\mathrm{d}z}{iz} \tag{6.62}$$

Theorem 6.34 (Integrals of Rational Functions)**.** *Let $f(x) = p_n(x)/q_m(x)$ with $m \geq n + 2$ and $q_m(x) \neq 0 \quad \forall x \in \mathbb{R}$, then*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)}\, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} \tag{6.63}$$

Lemma 6.5.1 (Jordan's Lemma)**.** Let $f(z)$ be a holomorphic function in $A := \{\, z \in \mathbb{C} |\ \|z\| > R_0,\ \mathfrak{Im}(z) \geq 0\}$. Taken $\gamma(t) = Re^{it}$ $0 \leq t \leq \pi$ with $R > R_0$.
If $\exists M_R > 0 : \|f(z)\| \leq M_R\ \forall z \in \{\gamma\}$ and $M_R \to 0$, we have that

$$\mathrm{PV} \int_\gamma f(z) e^{iaz}\, \mathrm{d}z = 0 \quad a > 0 \tag{6.64}$$

Theorem 6.35. *Let $f(x) = p_n(x)/q_m(x)$ and $m \geq n + 1$ with $q_m(x) \neq 0\ \forall x \in \mathbb{R}$, then $\forall a > 0$ we have that*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)} e^{iax}\, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} e^{iaz} \tag{6.65}$$

Lemma 6.5.2. Let $f(z)$ be a meromorphic function such that $z_0 \in P^1_f$ and $\gamma_r^{\pm}$ are semi circumferences parametrized as follows

$$\gamma_r^{\pm}(t) = z_0 + re^{\pm i\theta} \quad \theta \in [-\pi, 0]$$

Then

$$\mathrm{PV} \int_{\gamma_r^{\pm}} f(z) \, \mathrm{d}z = \pm \pi i \operatorname*{Res}_{z=z_0} f(z) \tag{6.66}$$

Theorem 6.36. *Let $f(x) = p_n(x)/q_m(x)$ with $m \geq n+2$ and $q_m(x)$ has $x_j \in Z^1_g\big|_{\mathbb{R}}$ then*

$$\int_{\mathbb{R}} \frac{p_n(x)}{q_m(x)} \, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{p_n(z)}{q_m(z)} + \pi i \sum_j \operatorname*{Res}_{z=x_j} \frac{p_n(z)}{q_m(z)} \tag{6.67}$$

If $g(x) = r_\alpha(x)/s_\beta(x)e^{iax}$ and $\beta \geq \alpha + 1$ with $x_j \in Z^1_g\big|_{\mathbb{R}}$, then $\forall a > 0$

$$\int_{\mathbb{R}} \frac{r_\alpha(x)}{s_\beta(x)} e^{iax} \, \mathrm{d}x = 2\pi i \sum_k \operatorname*{Res}_{z=z_k} \frac{r_\alpha(z)}{s_\beta(z)} e^{iaz} + \pi i \sum_j \operatorname*{Res}_{z=x_j} \frac{r_\alpha(z)}{s_\beta(z)} e^{iaz} \tag{6.68}$$

z_k are all the zeros of q, s contained in the plane $\{\mathfrak{Im}(z) > 0\}$

7 Hilbert and Banach Spaces

§ 7.1 Banach Spaces

§§ 7.1.1 Sequence Spaces

Definition 7.1.1 (Banach Space)**.** Given a space and a norm $(X, \|\cdot\|)$, the space is said to be a *Banach space* if it's complete with respect to the norm $\|\cdot\|$.
I.e. remembering the definition of completeness, we have that $\forall (x)_k \in X$ Cauchy sequence, $x_k \to x \in X$

Notation (The Field $\mathbb{F}$)**.** Here in this section, the field $\mathbb{F}$ should be intended as either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$

Definition 7.1.2 (Sequence Space)**.** As a n-tuple in the field $\mathbb{F}^n$ can be seen as a sequence, as follows

$$x \in \mathbb{F}^n,\ x = (x_1, x_2, \cdots, x_n) = (x_k)_{k=1}^n$$

We can imagine a sequence as a point in a space. We will call this space $\mathbb{F}^{\mathbb{N}}$, and an element of this space will be indicated as follows

$$x \in \mathbb{F}^{\mathbb{N}},\ x = (x)_n = (x_1, x_2, \cdots, x_n, \cdots) = (x_k)_{k=1}^\infty$$

Therefore, every point in $\mathbb{F}^{\mathbb{N}}$ is a sequence. Note that the infinite sequence of 0s and 1s will be indicated as $0 = (0)_n,\ 1 = (1)_n$

Definition 7.1.3 (Sequence of Sequences)**.** We can see a sequence of sequences as a mapping from $\mathbb{N}$ to the space $\mathbb{F}^{\mathbb{N}}$, as follows

$$\begin{aligned} x :& \mathbb{N} \longrightarrow \mathbb{F}^{\mathbb{N}} \\ & n \to ((x)_k)_n \end{aligned}$$

It's important to note how there are two indexes, since every element of the sequence is a sequence in itself (i.e. $((x)_k)_n \in \mathbb{F}^{\mathbb{N}}$ for any fixed $n \in \mathbb{N}$)

Definition 7.1.4 (Convergence of a Sequence of Sequences)**.** A sequence of sequences is said to converge to a sequence in $\mathbb{F}^{\mathbb{N}}$ if and only if

$$\lim_{n \to \infty} \|(x)_k - ((x)_k)_n\| = 0 \tag{7.1}$$

For some norm $\|\cdot\|$

Definition 7.1.5 (Pointwise Convergence)**.** A sequence of sequence is said to converge *pointwise* to a sequence in $\mathbb{F}^{\mathbb{N}}$ if and only if

$$\forall k \in \mathbb{N} \ \lim_{n\to\infty} ((x)_k)_n = (x)_k \tag{7.2}$$

And it's indicated as $((x)_k)_n \to (x)_k$

Example 7.1.1. Take the following sequence of sequences in $\mathbb{F}^{\mathbb{N}}$

$$\left((x)_k\right)_n = \frac{k}{n}$$

This sequence converges pointwise to the null sequence, since

$$\lim_{n\to\infty} \left((x)_k\right)_n = \lim_{n\to\infty} \frac{k}{n} = (0)_k$$

Space of Bounded Sequences

Definition 7.1.6 (Limited Sequence Space)**.** Let $(x)_k \in \mathbb{F}^{\mathbb{N}}$. Calling the space of bounded sequences as $\ell^\infty(\mathbb{F})$, we have that $(x)_k \in \ell^\infty(\mathbb{F})$ if and only if

$$\sup_{n\in\mathbb{N}} |(x)_n| = M \in \mathbb{F} \tag{7.3}$$

Therefore, this space is defined as follows

$$\ell^\infty(\mathbb{F}) := \{\, (x)_n \in \mathbb{F}^{\mathbb{N}} \big| \sup_{n\in\mathbb{N}} |(x)_n| < M,\ M \in \mathbb{F}\} \tag{7.4}$$

Theorem 7.1. *The application $\|\cdot\|_\infty = \sup_{n\in\mathbb{N}} |\cdot|$ is a norm in $\ell^\infty(\mathbb{F})$*

Proof. 1) $\|(x)_n\| \geq 0 \ \forall (x)_n \in \mathbb{F}^{\mathbb{N}}$, $\|(x)_n\|_\infty = 0 \iff (x)_n = (0)_n$, by definition of sup the first statement is obvious, meanwhile for the second

$$0 \leq |(x)_n| \leq \sup_{n\in\mathbb{N}} |(x)_n| = 0 \implies |(x)_n| = 0 \ \therefore (x)_n = (0)_n$$

2) $\|c(x)_n\|_\infty = |c|\|(x)_n\|_\infty$

$$\|c(x)_n\|_\infty = \sup_{n\in\mathbb{N}} |c(x)_n| = \sup_{n\in\mathbb{N}} |c||(x)_n| = |c| \sup_{n\in\mathbb{N}} |(x)_n| = |c|\|(x)_n\|_\infty$$

3) $\|(x)_n + (y)_n\|_\infty \leq \|(x)_n\|_\infty + \|(y)_n\|_\infty$

$$\sup_{n\in\mathbb{N}} |(x)_n + (y)_n| \leq \sup_{n\in\mathbb{N}} (|(x)_n| + |(y)_n|) = \sup_{n\in\mathbb{N}} |(x)_n| + \sup_{n\in\mathbb{N}} |(y)_n| = \|(x)_n\|_\infty + \|(y)_n\|_\infty$$

Since $\ell^\infty(\mathbb{F})$ is a vector space, the couple $(\ell^\infty(\mathbb{F}), \|\cdot\|_\infty)$ is a normed vector space ☐

Remark. Let $\mathcal{V}$ be a vector space over some field $\mathbb{F}$. If $\dim(\mathcal{V}) = \infty$, a closed and bounded subset $\mathcal{W} \subset \mathcal{V}$ isn't necessarily compact, whereas, a compact subset $\mathcal{Z} \subset \mathcal{V}$ is necessarily closed and bounded.

Example 7.1.2. Take $\mathcal{V} = \ell^\infty(\mathbb{F})$ and $\mathcal{W} = \overline{B_1((0)_n)}$, where

$$\overline{B_1((0)_n)} := \{ (x)_n \in \mathbb{F}^\infty |\ \|(x)_n\|_\infty \leq 1\}$$

We have that $\operatorname{diam}(\overline{B_1}) = 2$, therefore this set is bounded and closed by definition.
Take the *canonical sequence of sequences* $\left((e)_k\right)_n$, defined as follows:

$$\left((e)_k\right)_n = ((0)_k, (0)_k, \cdots, (0)_k, (1)_k, (0)_k, \cdots),\ \text{for some } k \in \mathbb{N}$$

Therefore, $\forall n \neq m$

$$\left\|\left((e)_k\right)_n - \left((e)_k\right)_m\right\|_\infty = \|(1)_k\|_\infty = 1$$

Therefore there aren't converging subsequences, and therefore $\overline{B_1}$ can't be compact.

Space of Sequences Converging to 0

Definition 7.1.7 (Space of Sequences Converging to 0)**.** The space of sequences converging to 0 is indicated as $\ell_0(\mathbb{F})$ and is defined as follows

$$\ell_0(\mathbb{F}) := \{ (x)_n \in \mathbb{F}^\mathbb{N} \big|\ (x)_n \to 0\} \tag{7.5}$$

Proposition 11. $\ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})$, and the couple $(\ell_0(\mathbb{F}), \|\cdot\|_\infty)$ is a normed vector space, where the norm $\|\cdot\|_\infty$ gets induced from the space $\ell^\infty(\mathbb{F})$

Proof.

$$\begin{aligned}
&\lim_{k\to\infty} (x)_k = 0 \implies \forall\epsilon > 0\ \exists N \in \mathbb{N} : |(x)_n| < \epsilon\ \forall n \geq N \\
&\therefore \sup_{n\in\mathbb{N}} |(x)_n| = \epsilon \leq M \in \mathbb{F} \implies (x)_n \in \ell^\infty(\mathbb{F}),\ \therefore \ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})
\end{aligned}$$

□

$\ell^p(\mathbb{F})$ **Spaces**

Definition 7.1.8. The sequence space $\ell^p(\mathbb{F})$ is defined as follows

$$\ell^p(\mathbb{F}) := \{ (x)_n \in \mathbb{F}^\mathbb{N} \big|\ \|(x)_n\|_p^p = M \in \mathbb{F}\} \tag{7.6}$$

Where $\|\cdot\|_p$ is the usual $p-$norm

Proposition 12. The application $\|\cdot\|_p : \ell^p(\mathbb{F}) \longrightarrow \mathbb{F}$ is a norm in $\ell^p(\mathbb{F})$, and the couple $(\ell^p(\mathbb{F}), \|\cdot\|_p)$ is a normed vector space

Proof. We begin by proving that $\ell^p(\mathbb{F})$ is actually a vector space, therefore 1) $\forall (x)_n, (y)_n \in \ell^p(\mathbb{F}),\ (x)_n + (y)_n = (x+y)_n \in \ell^p(\mathbb{F})$

$$\begin{aligned}
&(x+y)_n \in \ell^p(\mathbb{F}) \implies \sum_{n=0}^{\infty} |(x)_n + (y)_n|^p = \|(x)_n + (y)_n\|_p^p < M \in \mathbb{F} \\
&\|(x)_n + (y)_n\|_p^p \leq \|(x)_n\|_p^p + \|(y)_n\|_p^p < M \in \mathbb{F}
\end{aligned}$$

2) $\forall (x)_n \in \ell^p(\mathbb{F}),\ c \in \mathbb{F},\ c(x)_n \in \ell^p(\mathbb{F})$

$$c(x)_n \in \ell^p(\mathbb{F}) \implies \|c(x)_n\|_p^p < M \in \mathbb{F}$$
$$\|c(x)_n\|_p^p = \sum_{n=0}^{\infty} |c(x)_n|^p = |c|^p \sum_{n=0}^{\infty} |(x)_n|^p = |c|^p \|(x)_n\|_p^p < M \in \mathbb{F}$$

□

Remark. $(x)_n \in \ell^p(\mathbb{F}) \implies (x)_n \in \ell_0(\mathbb{F})$.

Proof. The proof is simple, taking $(y)_n = |(x)_n|^p$, we can see that $(y)_n \to 0$, therefore $(x)_n \to 0$ and $(x)_n \in \ell_0(\mathbb{F})$ □

Space of Finite Sequences

Definition 7.1.9 (Space of Finite Sequences)**.** The space of finite sequences is indicated as $\ell_f(\mathbb{F})$ and it's defined as follows

$$\ell_f(\mathbb{F}) := \left\{ (x)_n \in \mathbb{F}^{\mathbb{N}} \middle| (x)_n = 0 \ \forall n > N \in \mathbb{N} \right\} \tag{7.7}$$

It's already obvious that $\ell_f(\mathbb{F}) \subset \ell^p(\mathbb{F}) \subset \ell^q(\mathbb{F}) \subset \ell_0(\mathbb{F}) \subset \ell^\infty(\mathbb{F})$, where $p < q \in \mathbb{R}^+ \setminus \{0\}$ where $p < q \in \mathbb{R}^+ \setminus \{0\}$

§§ 7.1.2 Function Spaces

Notation. In this case, when there will be written the field $\mathbb{F}$, we might either mean $\mathbb{R}$ only, i.e. functions $\mathbb{R} \longrightarrow \mathbb{R}$, or $\mathbb{R}; \mathbb{C}$, i.e. functions $\mathbb{R} \longrightarrow \mathbb{C}$.

Definition 7.1.10 (Some Function Spaces)**.** We are already familiar from the basic courses in one dimensional real analysis, about the space of continuous functions $C(A)$, where $A \subset \mathbb{R}$. We can define three other spaces directly, adding some restrictions.

1. $C_b(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \sup_{x \in \mathbb{F}}(f(x)) \leq M \in \mathbb{F}\}$
2. $C_0(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \lim_{x \to \infty}(f(x)) = 0\}$
3. $C_c(\mathbb{F}) := \{ f \in C(\mathbb{F}) | f(x) = 0 \quad \forall x \in A^c \subset \mathbb{F}\}$ i.e. $C_c(\mathbb{F}) := \{ f \in C(\mathbb{F}) | \operatorname{supp}(f) \text{ is compact } \}$, where with supp we indicate the following set $\operatorname{supp}_{\mathbb{F}}(f) := \overline{\{ x \in \mathbb{F} | f(x) \neq 0\}}$

Due to the properties of continuous functions, these spaces are obviously vector spaces.

Proposition 13. We have $C_c(\mathbb{F}) \subset C_0(\mathbb{F}) \subset C_b(\mathbb{F}) \subset C(\mathbb{F})$, the application

$$\|f\|_u = \|f\|_\infty = \sup_{x \in A} |f(x)| \tag{7.8}$$

Is a norm in $C(A)$, whereas

$$\|f\|_u = \|f\|_\infty = \sup_{x \in \mathbb{F}} |f(x)| \tag{7.9}$$

Is a norm in the other three spaces

Proof. The inclusion of these spaces is obvious, due to the definition of these. For the proof that the application $\|\cdot\|_u$ is a norm, it's immediately given from the proof that the application $\|\cdot\|_\infty$ is a norm in $\ell^\infty(\mathbb{F})$, and that $\|\cdot\|_u = \|\cdot\|_\infty$ □

Remark. Take $f_n \in C_b(\mathbb{F})$ a sequence of functions. The uniform convergence of this sequence means that $f_n \to f$ in the norm $\|\cdot\|_u = \|\cdot\|_\infty$

Proposition 14. If $f \in C_0(\mathbb{F})$, then f is uniformly continuous

Proof. Let $f \in C_0(\mathbb{F})$, then

$$\forall \epsilon > 0\ \exists l\ :\ |x| \geq l \implies |f(x)| < \frac{\epsilon}{2}$$

Since every continuous function is uniformly continuous in a closed set, then

$$\forall \epsilon > 0\ \exists \delta\ :\ \forall x, y \in [-L-1, L+1] \wedge |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Hence we can have two cases. We either have $|x-y| < \delta$ or $x, y \in [-L-1, L+1]$. Hence we have, in the first case

$$|f(x) - f(y)| \leq |f(x)| + |f(y)| < \epsilon$$

Or, in the second case

$$\forall \epsilon > 0\ \exists \delta > 0\ :\ |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Demonstrating our assumption □

$C_p(\mathbb{F})$ spaces

Definition 7.1.11. We can define a set of function spaces analogous to the $\ell^p(\mathbb{F})$ spaces. These spaces are the $C_p(\mathbb{F})$ spaces. We define analogously the $p-$norm for functions as follows

$$\|f\|_p := \sqrt[p]{\int_{\mathbb{F}} |f(x)|^p \,\mathrm{d}x} \tag{7.10}$$

Thanks to what said about $\ell^p(\mathbb{F})$ spaces and $p-$norms, it's already obvious that these spaces are normed vector spaces

Remark. Watch out! $C_p(\mathbb{F}) \not\subset C_0(\mathbb{F})$, and $C_p(\mathbb{F}) \not\subset C_q(\mathbb{F})$ for $1 \leq p \leq q$. It's easy to find counterexamples

Proposition 15. If $1 \leq p \leq q$, then

$$C_p(\mathbb{F}) \cap C_b(\mathbb{F}) \subset C_q(\mathbb{F})$$

Proof. Let $f \in C_p(\mathbb{F}) \cap C_b(\mathbb{F})$. Therefore $\sup_{x \in \mathbb{F}} |f(x)| < M \in \mathbb{F}$, then

$$\int_{\mathbb{F}} |f(x)|^q \,\mathrm{d}x = \int_{\mathbb{F}} |f(x)|^p |f(x)|^{q-p} \,\mathrm{d}x \leq M^{q-p} \int_{\mathbb{F}} |f(x)|^p \,\mathrm{d}x \leq \infty$$

Therefore $f \in C_p(\mathbb{F}) \cap C_b(\mathbb{F}) \implies f \in C_q(\mathbb{F})$ □

§§ 7.1.3 Function Spaces in $\mathbb{R}^n$

Definition 7.1.12 (Seminorm)**.** A *seminorm* is an application $\|\cdot\|_{\alpha,\beta} : A \longrightarrow \mathbb{F}$ with A a function space and α, β multiindices, where

$$\|f\|_{\alpha,\beta} := \left\|x^\alpha \partial^\beta f\right\|_\infty = \sup_{x\in\mathbb{F}} \left|x^\alpha \partial^\beta f(x)\right| \tag{7.11}$$

Definition 7.1.13 (Schwartz Space)**.** The space $\mathcal{S}(\mathbb{R}^n)$ is called the *Schwartz space*, and it's defined as follows

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) |\ \|f\|_{\alpha,\beta} < \infty, \alpha, \beta \text{ multiindices}\right\} \tag{7.12}$$

Example 7.1.3. Taken $p(x) \in \mathbb{R}[x]$ a polynomial, a common example of functions $f(x) \in \mathcal{S}(\mathbb{R})$ is the following.

$$f(x) = p(x)e^{-a|x|^{2n}} \tag{7.13}$$

With $a > 0,\ n \in \mathbb{N}$

Theorem 7.2. *A function $f \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{S}(\mathbb{R}^n)$ if $\forall\beta$ multiindex, $\forall a > 0$ $\exists C_{\alpha,\beta}$ such that*

$$\left\|\partial^\beta f(x)\right\| \leq \frac{C_{\alpha,\beta}}{\left(1 + \|x\|^2\right)^{\frac{a}{2}}} \quad \forall x \in \mathbb{R}^n \tag{7.14}$$

Proof. Taken $n = 1$ and $f \in C^\infty(\mathbb{R})$, then

$$\left|x^j \partial^k f(x)\right| = |x|^j \left|\partial^k f(x)\right| \leq \frac{C_{j,k}|x|^j}{(1 + x^2)^{\frac{j}{2}}} \leq C_{j,k} \quad \forall x \in \mathbb{R}$$

Therefore

$$\|f\|_{j,k} \leq C_{j,k} < \infty \implies f \in \mathcal{S}(\mathbb{R})$$

Taken $f \in S(\mathbb{R})$ we have that, if $|x| \geq 1$

$$(1 + x^2)^{\frac{a}{2}} \leq 2^{\frac{a}{2}} |x|^a$$

Taken $j = \lceil a \rceil$

$$\left|\partial^k f(x)\right| = \frac{\left|x^j \partial^k f(x)\right|}{|x|^j} \leq \frac{\|f\|_{j,k}}{|x|^a} \leq \frac{2^{\frac{a}{2}} \|f\|_{j,k}}{(1 + x^2)^{\frac{a}{2}}} \leq \frac{2^{\frac{a}{2}} \|f\|_{0,k}}{(1 + x^2)^{\frac{a}{2}}} \quad |x| < 1$$

Taken $C_{j,k} = 2^{\frac{a}{2}} \max \|f\|_{\lceil a \rceil,k}, \|f\|_{0,k}$ the assertion is demonstrated ☐

Definition 7.1.14 (Space of Compact Support Function)**.** Given a function with compact support f, we define the space of compact functions $C_c^\infty(\mathbb{R}^n)$ as the space of all such functions.
We have obviously that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$

Theorem 7.3. *Both $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ are dense in $(C_p(\mathbb{R}^n), \|\cdot\|_p)$*

Theorem 7.4 (Other Function Spaces)**.**

1. *$C(\mathbb{R})$ Space of continuous functions*
2. *$\mathbb{R}[x]$ Space of real polynomials*

3. *$C^k(\mathbb{R})$ Space of continuous $k-$derivable functions*

4. *$C^k_c(\mathbb{R})$ Space of functions $f \in C^k(\mathbb{R})$ with compact support*

5. *$C^\infty(\mathbb{R})$ Space of infinitely differentiable (smooth) functions*

6. *$C_0(\mathbb{R})$ Space of smooth functions with $\lim_{x\to\pm\infty} f(x) = 0$*

7. *$C^\infty_c(\mathbb{R})$ Space of smooth functions with compact support*
   We have the obvious inclusions

$$\begin{aligned}
\ell_f \subset \ell^p \subset \cdots \subset \ell_0 \subset \mathbb{R}^{\mathbb{N}} \\
C_c(\mathbb{R}) \subset C_0(\mathbb{R}) \subset \cdots \subset C_p(\mathbb{R}) \subset C(\mathbb{R}) \\
\mathbb{R}[x] \subset C^\infty(\mathbb{R}) \subset \cdots \subset C(\mathbb{R}) \\
C^\infty_c(\mathbb{R}) \subset C^\infty_0(\mathbb{R}) \subset C^\infty(\mathbb{R})
\end{aligned}$$

§ 7.2 Hilbert Spaces

Definition 7.2.1 (Hermitian Product)**.** Given $\mathcal{V}$ a complex vector space, and an application $\langle\cdot,\cdot\rangle : \mathcal{V} \longrightarrow \mathbb{C}$ such that $\forall u, v, z \in \mathcal{V},\ c, d \in \mathbb{C}$

1. $\langle v, v\rangle \geq 0$

2. $\langle v, v\rangle = 0 \iff v = 0$

3. $\langle u, v\rangle = \overline{\langle v, u\rangle}$

4. $\langle u + v, z\rangle = \langle u, z\rangle + \langle v, z\rangle$

5. $\langle cu, dv\rangle = c\overline{d}\langle u, v\rangle$

The application $\langle\cdot,\cdot\rangle$ is called an *Hermitian product* in $\mathcal{V}$, and the couple $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ is called an *Euclidean space*

Remark. It's usual in physics that for a Hermitian product, we have that

$$\langle cu, v\rangle = \overline{c}\langle u, v\rangle \tag{7.15}$$

Definition 7.2.2 (Hilbert Space)**.** Given $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ an euclidean space. It's said to be a *Hilbert space* if it's complete

Theorem 7.5 (Cauchy-Schwartz Inequality)**.** *Given $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ a complex euclidean space, then $\forall u, v \in \mathcal{V}$*

$$\|\langle u, v\rangle\|^2 \leq \langle u, u\rangle\langle v, v\rangle \tag{7.16}$$

Proof. Taken $t \in \mathbb{C}$, we define $p(t) = \langle tu + v, tu + v\rangle$. Then by definition of the Hermitian product, we have

$$p(t) = \|t\|^2\langle u, u\rangle + t\langle u, v\rangle + \overline{t}\langle v, u\rangle + \langle v, v\rangle$$

Writing $\langle u, v\rangle = \rho e^{i\theta},\ t = se^{-i\theta}$ we have

$$p(se^{-i\theta}) = s^2\langle u, u\rangle + 2s\rho + \langle v, v\rangle \geq 0 \quad \forall s \in \mathbb{R}$$

Then, by definition, we have

$$\rho^2 = \|\langle u, v\rangle\|^2 \leq \langle u, u\rangle\langle v, v\rangle$$

□

Theorem 7.6 (Induced Norm)**.** *Given a Hermitian product $\langle\cdot,\cdot\rangle$ we can define an induced norm $\|\cdot\|$ by the definition*

$$\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle} \tag{7.17}$$

Theorem 7.7. *Addition, multiplication by a scalar and the scalar product are continuous in an euclidean space $\mathcal{V}$, then, given two sequences $u_n \longrightarrow u \in \mathcal{V}$, $v_n \to v \in \mathcal{V}$ and*

$$\begin{aligned} u_n + v_n &\longrightarrow u + v \\ c \in \mathbb{C} \implies cu_n &\longrightarrow u \\ \langle u_n, v_n\rangle &\longrightarrow \langle u, v\rangle \end{aligned}$$

Proof. Thanks to Cauchy-Schwartz we have

$$\begin{aligned} |\langle u, v\rangle - \langle u_n, v_n\rangle| &= |\langle u, v\rangle - \langle u, v_n\rangle + \langle u, v_n\rangle - \langle u_n, v_n\rangle| \leq |\langle u, v\rangle - \langle u, v_n\rangle| + |\langle u, v_n\rangle - \langle u_n, v_n\rangle| = \\ &= \|u\|\|v - v_n\| + \|v_n\|\|u - u_n\| \end{aligned}$$

Since the successions are convergent, we have that $\exists M > 0 \,:\, \|v_n\| \leq M \;\forall n \in \mathbb{N}$, therefore

$$|\langle u, v\rangle - \langle u_n, v_n\rangle| \leq \max\{\|u\|, M\}\left(\|v - v_n\| + \|u - u_n\|\right) \to 0$$

□

Example 7.2.1 (Some Euclidean Spaces)*.* 1) $\ell^2(\mathbb{C})$
Given $x, y \in \ell^2(\mathbb{C})$ we define the scalar product as

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i\overline{y_i}$$

2) $\ell^2(\mu)$, a weighted sequence space, where

$$\ell^2(\mu) := \left\{ x \in \mathbb{C}^{\mathbb{N}} \middle| \sum_{i=1}^{\infty} \mu_i|x_i|^2 < \infty,\; \mu_i \in \mathbb{R},\; \mu_i > 0 \;\forall i \right\}$$

Given $x, y \in \ell^2(\mu)$ we define

$$\langle x, y\rangle = \sum_{i=1}^{\infty} \mu_i x_i\overline{y_i}$$

3) $C_2(\mathbb{C})$ Given $f, g \in C_2(\mathbb{C})$ we define

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}\,\mathrm{d}x$$

4) $C_2(\mathbb{C}, p(x)\,\mathrm{d}x)$, weighted function spaces, where

$$C_2(\mathbb{C}, p(x)\,\mathrm{d}x) := \left\{ f \in C(\mathbb{C}) \middle| \int_{\mathbb{R}} f(x)\overline{f(x)}p(x)\,\mathrm{d}x < \infty,\; p(x) \in C(\mathbb{R}; \mathbb{R}^+) \right\}$$

Given $f,g \in C_2(\mathbb{C}, p(x)\,\mathrm{d}x)$ we define

$$\langle f,g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}p(x)\,\mathrm{d}x$$

The spaces C_2 *aren't complete* therefore they're not Hilbert spaces. The spaces $L^2(\mathbb{C})$ and the weighted alternative are the completion of such spaces and are therefore Hilbert spaces

Theorem 7.8 (Polarization Identity)**.** *Given a complex euclidean space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ we have, $\forall u,v \in \mathcal{V}$*

$$\langle u,v\rangle = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2 + i\left(\|u+iv\|^2\|u-iv\|^2\right)\right) \tag{7.18}$$

Theorem 7.9 (Parallelogram Rule)**.** *Let $(\mathcal{V}, \|\cdot\|)$ be a normed vector space. A necessary and sufficient condition that the norm is induced by a scalar product is that*

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right) \quad \forall u,v \in \mathcal{V} \tag{7.19}$$

§ 7.3 Projections

§§ 7.3.1 Orthogonality

Definition 7.3.1 (Angle)**.** Given a real euclidean space $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ we define the angle $\theta = u\angle v$ as follows

$$\theta = \arccos\left(\frac{\langle u,v\rangle}{\|u\|\|v\|}\right) \tag{7.20}$$

Definition 7.3.2 (Orthogonal Complement)**.** Given an euclidean vector space $\mathcal{V}$ and two vectors u,v, we say that the two vectors are orthogonal $u \perp v$ if

$$\langle u,v\rangle = 0 \tag{7.21}$$

If $X \subset \mathcal{V}$ and $\forall x \in X$ we have that

$$\langle u,x\rangle = 0$$

We say that $u \in X^\perp$ where this space is called the *Orthogonal Complement* of X, i.e.

$$X^\perp := \{\, v \in \mathcal{V} |\ \langle v,w\rangle = 0\ \forall w \in X\} \tag{7.22}$$

Theorem 7.10. *Given $X \subset \mathcal{V}$ with $\mathcal{V}$ an euclidean space, the set $X^\perp$ is a closed subspace of $\mathcal{V}$*

Proof. $X^\perp$ is a subspace, hence $\forall v_1, v_2 \in X^\perp$ and $c_1, c_2 \in \mathbb{C}$ we have

$$\langle c_1v_1 + c_2v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle \quad \forall w \in X$$

Hence $c_1v_1 + c_2v_2 \in X^\perp$.
Given a sequence $(v)_n \in X^\perp : (v)_n \longrightarrow v \in \mathcal{V}$ we have, given $w \in X$

$$\langle v_n, w\rangle = 0 \quad \forall n \in \mathbb{N}$$

Due to the continuity of the scalar product we have that

$$\lim_{n\to\infty} \langle v_n, w\rangle = 0 = \langle v,w\rangle$$

Therefore $v \in X^\perp$ and the subspace $X^\perp$ is closed in $\mathcal{V}$ □

Theorem 7.11. *Given $X, Y \subset \mathcal{V}$ with $\mathcal{V}$ an euclidean space, we have*

$$X \subset Y \implies Y^\perp \subset X^\perp$$
$$X^\perp = (\overline{X})^\perp = \left(\overline{\text{span}(X)}\right)^\perp$$

Proof. Taken $v \in Y^\perp$ we have by definition

$$\langle v, y\rangle = 0 \quad \forall y \in Y$$

Since $X \subset Y$ we have then

$$\langle v, x\rangle = 0 \quad \forall x \in X$$

Therefore $Y^\perp \subset X^\perp$.
By definition we have that $X \subset \overline{X} \subset \overline{\text{span}(X)}$, and thanks to the previous proof

$$X^\perp \supset (\overline{X})^\perp \supset (\overline{\text{span}(X)})^\perp$$

Taken $w \in \text{span}(X)$ we have

$$w = \sum_i c_i w_i \quad w_i \in X$$

And given $v \in X^\perp$, we get

$$\langle v, w\rangle = \sum_i c_i \langle v, w_i\rangle = 0$$

Now take $w \in \overline{\text{span}(X)}$. Take a sequence $(w)_n \in \overline{\text{span}(X)}$ such that $(w)_n \longrightarrow w$. Thanks to the continuity of the scalar product we have

$$\langle v, w\rangle = \langle v, \lim_{n\to\infty} w_n\rangle = \lim_{n\to\infty} \langle v, w_n\rangle = 0$$

Demonstrating that $X^\perp = (\overline{\text{span}(X)})^\perp$ □

Lemma 7.3.1. Let $\mathcal{V}$ be a Hilbert space. Given $\mathcal{W} \subset \mathcal{V}$ a closed subspace. Given $v \in \mathcal{V}$

$$\exists! w_0 \in \mathcal{W} : \forall w \in \mathcal{W} \; d = \|v - w_0\| \leq \|v - w\|$$

Proof. Take $d = \inf_{w\in\mathcal{W}} \|v - w\|$. By definition of infimum we have that $\exists (w)_n \in \mathcal{W}$ such that

$$\lim_{n\to\infty} \|v - w_n\| = d$$

Using the parallelogram rule, we have that

$$\|w_n - w_k\|^2 = \|(w_n - v) + (v - w_k)\|^2 = 2\|v - w_n\|^2 + 2\|v - w_k\|^2 - 4\left\|\frac{1}{2}(w_n + w_k) - v\right\|^2$$

Since $1/2(w_n + w_k) \in \mathcal{W}$ we have by definition of d

$$\left\|\frac{1}{2}(w_n + w_k) - v\right\| \geq d$$

Therefore, we can rewrite

$$\|w_n - w_k\|^2 \leq 2\|v - w_n\|^2 + 2\|v - w_k\|^2 - 4d^2$$

Therefore, we have

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} : \forall n, k \geq N\ \|w_n - w_k\|^2 \leq 4(d+\epsilon)^2 - 4d^2$$

Hence $(w)_n$ is a Cauchy sequence. Since by definition $\mathcal{V}$ is complete and $\mathcal{W} \subset \mathcal{V}$ is closed, we have that $\mathcal{W}$ is also complete, therefore $(w)_n \to w_0 \in \mathcal{W}$ and we have

$$\|v - w_0\| = d$$

Now suppose that $\exists w_1, w_2 \in \mathcal{W}$ such that the previous is true, i.e.

$$\|v - w_1\| = \|v - w_2\| \leq \|v - w\| \quad \forall w \in \mathcal{W}$$

Taken $w_3 = 1/2(w_1 + w_2)$ we have

$$\|v - w_3\|^2 = \|v - w_1\|^2 - \frac{1}{4}\|w_2 - w_1\|^2$$

Taken $d = \|v - w_1\| = \|v - w_2\|$, $z_1 = v - w_3$ and $z_2 = 1/2(w_1 - w_2)$ we get

$$\|z_1 + z_2\|^2 + \|z_1 - z_2\|^2 = 2\left(\|z_1\|^2 + \|z_2\|^2\right)$$

And therefore

$$d^2 = \frac{1}{2}\left(\|v - w_1\|^2 + \|v - w_2\|^2\right) = \|v - w_3\|^2 + \frac{1}{4}\|w_1 - w_2\|^2$$

I.e. if $w_1 \neq w_2$, w_3 is the infimum between $v \in \mathcal{V}$ and $\mathcal{W}$ $\lightning$ $\square$

§§ 7.3.2 Projections and Orthogonal Projections

Theorem 7.12 (Projection)**.** *Given $\mathcal{W} \subset \mathcal{V}$ closed subspace of a Hilbert space, we have*

$$v = w + z \quad \forall v \in \mathcal{V},\ w \in \mathcal{W},\ z \in \mathcal{W}^\perp$$

Proof. Given $v \in \mathcal{V}$, due to the previous lemma we have that $\exists! w \in \mathcal{W}$ such that

$$d = \|v - w\| \leq \|v - w'\| \quad \forall w' \in \mathcal{W}$$

Taken $z = v - w$, and an element $x \in \mathcal{W}$, define the vector $w + tx$ with $t \in \mathbb{C}$. Since $\mathcal{W}$ is a subspace $w + tx \in \mathcal{W}$ and $\forall t \in \mathbb{C}$ we have

$$d^2 \leq \|v - (w + tx)\|^2 = \|v - w\|^2 - \bar{t}\langle v - w, x, -\rangle t\langle x, v - w\rangle + \|t\|^2\|x\|^2$$

Writing $\langle x, v - w\rangle = \|\langle x, v - w\rangle\| e^{i\theta}$ and $t = se^{-i\theta}$ with $s \in \mathbb{R}$ we have

$$-2s\|\langle v - w, x\rangle\| + s^2\|x\|^2 \geq 0 \quad \forall s \in \mathbb{R}$$

Which implies

$$\langle v-w, x\rangle = 0 \implies z = v - w \in \mathcal{W}^{\perp}$$

Therefore there exists a representation $v = w + z$ with $w \in \mathcal{W},\ z \in \mathcal{W}^{\perp}$ Now, we suppose that $v = w' + z'$, then

$$0 = (w - w') + (z - z')$$

Therefore

$$0 = \|(w-w') + (z-z')\|^2 = \|w - w'\| + \|z - z'\|^2$$

Therefore the representation is unique. □

Theorem 7.13. *If $\mathcal{W} \subset \mathcal{V}$ with $\mathcal{V}$ a Hilbert space, we have*

$$\left(\mathcal{W}^{\perp}\right)^{\perp} = \overline{\mathcal{W}}$$

If $\mathcal{W}$ is closed

$$\left(\mathcal{W}^{\perp}\right)^{\perp} = \mathcal{W}$$

Proof. Taken $w \in \mathcal{W}$ we have that $w \perp v$ with $v \in \mathcal{W}^{\perp}$, therefore $w \in (\mathcal{W}^{\perp})^{\perp}$. Therefore

$$\mathcal{W} \subset \left(\mathcal{W}^{\perp}\right)^{\perp}$$

Since the space on the right is closed, we have

$$\overline{\mathcal{W}} = \overline{(\mathcal{W}^{\perp})^{\perp}} = \left(\mathcal{W}^{\perp}\right)^{\perp}$$

Now taken $w \in \left(\mathcal{W}^{\perp}\right)^{\perp}$, since $\overline{\mathcal{W}}$ is a closed subspace by definition, we can write

$$w = v + z \quad v \in \overline{\mathcal{W}},\ z \in \overline{\mathcal{W}}^{\perp} = \mathcal{W}^{\perp}$$

We have $w \perp z$, and therefore

$$\|z\|^2 = \langle z, w - v\rangle = 0 \implies w = v \in \overline{\mathcal{W}}$$

□

Definition 7.3.3 (Orthogonal Projection)**.** Given a closed subspace $\mathcal{W} \subset \mathcal{V}$ we can define an operator $\hat{\pi}_{\mathcal{W}} : \mathcal{V} \longrightarrow \mathcal{W}$ such that

$$\hat{\pi}_{\mathcal{W}} v = w \iff w \in \mathcal{W},\ v - w \in \mathcal{W}^{\perp} \tag{7.23}$$

$\hat{\pi}_{\mathcal{W}}$ is linear and called a *orthogonal projection*

Theorem 7.14. *Given $\mathcal{W} \subset \mathcal{V}$ a closed subspace of the Hilbert space $\mathcal{V}$, then given an orthogonal projection $\hat{\pi}_{\mathcal{W}} : \mathcal{V} \longrightarrow \mathcal{W}$ we have, $\forall v, z \in \mathcal{V}$ and another closed subspace $\mathcal{Z} \subset \mathcal{V}$*

1. $\hat{\pi}_{\mathcal{W}}^2 = \hat{\pi}_{\mathcal{W}}$
2. $\langle \hat{\pi}_{\mathcal{W}} v, z\rangle = \langle v, \hat{\pi}_{\mathcal{W}} z\rangle \geq 0$
3. *If $\mathcal{Z} \subseteq \mathcal{W}^{\perp}$* $\quad \hat{\pi}_{\mathcal{W}} \circ \hat{\pi}_{\mathcal{Z}} = \hat{\pi}_{\mathcal{Z}} \circ \hat{\pi}_{\mathcal{W}} = 0$

4. *If* $\mathcal{Z} \subset \mathcal{W} \quad \hat{\pi}_{\mathcal{W}} \circ \hat{\pi}_{\mathcal{Z}} = \hat{\pi}_{\mathcal{Z}} \circ \hat{\pi}_{\mathcal{W}} = \hat{\pi}_{\mathcal{W}}$

Definition 7.3.4 (Direct Sum)**.** An euclidean space $\mathcal{V}$ is called the *direct sum* of closed subspaces $\mathcal{V}_i \subset \mathcal{V}$ and it's indicated as follows

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots = \bigoplus_{k=1}^{\infty} \mathcal{V}_k \tag{7.24}$$

If

1. The spaces $\mathcal{V}_k$ are orthogonal in couples
2. $\forall v \in \mathcal{V}\ v = \sum_{k=1}^{\infty} v_k$ with $v_k \in \mathcal{V}_k$

Corollary 7.3.1. Given a Hilbert space $\mathcal{V}$ and a closed subspace $\mathcal{W}$, then

$$\mathcal{V} = \mathcal{W} \oplus \mathcal{W}^{\perp} \tag{7.25}$$

§§ 7.3.3 Orthogonal Systems and Bases

Definition 7.3.5 (Orthogonal System)**.** A set of vectors $X \subset \mathcal{V} \quad X \neq \{\}$ is said to be an *orthogonal system* if $\forall u, v \in X,\ u \neq v \quad u \perp v$.

Definition 7.3.6 (Orthonormal System)**.** Given an orthogonal system $X \subset \mathcal{V}$ such that $\forall u \in X$ we have $\|u\| = 1$, the system X is called an *orthonormal system*

Theorem 7.15. *Given an orthogonal system $X \subset \mathcal{V}$ with $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ an euclidean space, then we have that X is a system of linearly independent vectors*

Definition 7.3.7 (Basis)**.** Given an orthogonal and complete set of vectors $(v)_\alpha$ in an euclidean space $\mathcal{V}$ it's said to be an *orthogonal basis* of $\mathcal{V}$. If it's an orthonormal and complete set of vectors it's said to be an *orthonormal basis* of $\mathcal{V}$

Lemma 7.3.2. Given an orthogonal system $(v)_{k=1}^n \in \mathcal{V}$ and let $u \in \mathcal{V}$ an arbitrary vector. Then given $z \in \mathcal{V}$ as follows

$$z := u - \sum_{k=1}^{n} \frac{\langle u, v_k\rangle}{\|v_k\|^2} v_k$$

We have that $z \perp v_i\ \forall 1 \leq i \leq n$ and therefore $z \perp \operatorname{span}\{v_1, \cdots, v_n\}$

Proof. $\forall i = 1, \cdots, n$ it's obvious that $\langle z, v_i\rangle = 0$, therefore

$$z \in \{v_1, \cdots, v_n\}^{\perp}$$

Therefore

$$z \in \{v_1, \cdots, v_n\}^{\perp} = \overline{\operatorname{span}\{v_1, \cdots, v_n\}}^{\perp} = \operatorname{span}\{v_1, \cdots, v_n\}^{\perp}$$

$\square$

Theorem 7.16 (Gram-Schmidt Orthonormalization)**.** *Given $\mathcal{V}$ an euclidean space and $(v)_{n\in\mathbb{N}} \in \mathcal{V}$ a set of linearly independent vectors. Then $\exists (u)_{n\in\mathbb{N}} \in \mathcal{V}$ orthonormal system such that*

1. *u_n is a linear combination of v_i $\forall 0 \leq i \leq n$, i.e.*

$$u_n = \sum_{k=1}^{n} a_{nk} v_k \quad a_{nn} \neq 0$$

2. *v_n can be written as follows*

$$v_n = \sum_{k=1}^{n} b_{nk} u_k \quad b_{nn} \neq 0$$

Therefore, $\forall n \in \mathbb{N}$ we have that

$$\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{u_1, \cdots, u_n\}$$

Proof. Defining

$$w_n = v_n - \sum_{k=1}^{n-1} \frac{\langle v_n, v_k \rangle}{\|w_k\|^2} w_k \quad u_n = \frac{w_n}{\|w_n\|}$$

We can say immediately that $\forall n \geq 1$ $w_n \in \{w_1, \cdots, w_{n-1}\}^{\perp}$. By induction we can say that it holds $\forall (w)_{n\in\mathbb{N}}$, therefore $(w)_n$ is an orthogonal system and $(u)_{n\in\mathbb{N}}$ is an orthonormal system
We can also say that

$$v_n = w_n + \sum_{j=1}^{k-1} \beta_{nj} w_j$$

I.e. $\forall n \geq 1$ w_n is a linear combination of $\{v_1, \cdots, v_n\}$, therefore, by definition

$$\operatorname{span}\{v_1, \cdots, v_n\} = \operatorname{span}\{w_1, \cdots, w_n\} = \operatorname{span}\{u_1, \cdots, u_n\}$$

□

Example 7.3.1. 1) Legendre Polynomials
Using the Gram-Schmidt orthonormalization procedure, we can find an orthonormal system $\{p_0, \cdots, p_n\} \subset C_2[-1, 1]$ starting from the following system

$$(v)_n := \left\{1, x, x^2, x^3, x^4, \cdots, x^n\right\}$$

The final result will be called the *Legendre Polynomials* We begin by taking the canonical scalar product in $C_2[-1, 1]$ as follows

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) \, \mathrm{d}x$$

And using that

$$\int_{-1}^{1} x^n \, \mathrm{d}x = \begin{cases} \dfrac{2}{n+1} & n = 2k \in \mathbb{N} \\ 0 & n = 2k \in \mathbb{N} \end{cases}$$

Therefore, we have that

$$
\begin{aligned}
&w_0 = 1 \quad \|w_0\|^2 = \int_{-1}^{1} \mathrm{d}x = 2 \\
&w_1 = x - \frac{1}{2}\int_{-1}^{1} x \,\mathrm{d}x = x \quad \|w_1\|^2 = \frac{-1}{1} x^2 \,\mathrm{d}x = \frac{2}{3} \\
&w_2 = x^2 - \frac{1}{3} \quad \|w_2\|^2 = \int_{-1}^{1} \left(x^2 \frac{1}{3}\right)^2 \mathrm{d}x = \frac{8}{45} \\
&\vdots
\end{aligned}
$$

Normalizing, we find

$$
\begin{aligned}
p_0 &= \frac{1}{\sqrt{2}} \\
p_1(x) &= \sqrt{\frac{3}{2}} x \\
p_2(x) &= \sqrt{\frac{45}{8}} \left(x^2 - \frac{1}{3}\right) = \frac{1}{2}\sqrt{\frac{5}{2}}\left(3x^2 - 1\right) \\
&\vdots
\end{aligned}
$$

And so on. The set $(p)_n$ is called the set of *Legendre polynomials* 2) Hermite Polynomials
Using the same procedure, we can find the *Hermite polynomials* $H_n(x)$ in the space $C_2(\mathbb{R}, e^{-x^2}\,\mathrm{d}x)$. The first 5 are the following

$$
\begin{aligned}
H_1 &= 1 \\
H_2(x) &= x \\
H_3(x) &= x^2 - \frac{1}{2} \\
H_4(x) &= x^3 - \frac{3}{2} x \\
H_5(x) &= x^4 - 3x^2 + \frac{3}{4}
\end{aligned}
$$

Theorem 7.17 (Existence of an Orthonormal Basis)**.** *Given a separable or complete euclidean space $\mathcal{V}$, there always exists an orthonormal basis*

Proof. Taken $\mathcal{V}$ a separable euclidean space and $(v)_{n\in\mathbb{N}}$ a dense subset of $\mathcal{V}$.
Removing the linearly independent elements of this subset, we can call the new linearly independent set $(w)_{n\in\mathbb{N}}$. We have obviously

$$
\begin{aligned}
\operatorname{span}\{(w)_{n=1}^{\infty}\} &= \operatorname{span}\{(v)_{n=1}^{\infty}\} \\
\overline{\operatorname{span}\{(w)_{n=1}^{\infty}\}} &= \overline{\operatorname{span}\{(v)_{n=1}^{\infty}\}} = \mathcal{V}
\end{aligned}
$$

Orthonormalizing the system $(w)_{n\in\mathbb{N}}$ with the Gram-Schmidt procedure I obtain then a new set $(u)_{n\in\mathbb{N}}$ such that

$$
\overline{\operatorname{span}\{(u)_{n\in\mathbb{N}}\}} = \overline{\operatorname{span}\{(w)_{n\in\mathbb{N}}\}} = \mathcal{V}
$$

$(u)_{n\in\mathbb{N}}$ is complete and therefore a basis. ∎

Remark. If $\mathcal{V}$ is a complete euclidean space but not separable, we can find thanks to Zorn's lemma a maximal orthonormal basis, but

1. There isn't a standard procedure for finding this basis
2. Taken the basis $(u)_{\alpha\in I}$ it can't be numerable. If it was then $\mathcal{V} = \overline{\operatorname{span}\{(u)_\alpha\}}$. Taken $X = \operatorname{span}_{\mathbb{Q}}\{(u)_k\}$ as the set of finite linear combinations with rational coefficients, we have that $\overline{X} = \operatorname{span}\{(u)_k\}$ and therefore $\overline{X} = \mathcal{V}$ contradicting the fact that $\mathcal{V}$ is not separable

8 Distributions

§ 8.1 Linear Functionals

§§ 8.1.1 Dual Spaces and Functionals

Definition 8.1.1 (Dual Space II)**.** Given $(\mathcal{V}, \|\cdot\|)$ a normed vector space over a field $\mathbb{F}$, we define the following set

$$\mathcal{V}^\star := \{ f : \mathcal{V} \longrightarrow \mathbb{F} |\, v \mapsto f(v) \}$$

An element $f \in \mathcal{V}^\star$ is called a *continuous linear functional*, with the following properties, $\forall f, g \in \mathcal{V}^\star,\ u, v \in \mathcal{V},\ \lambda, \lambda_1, \lambda \in \mathbb{F}$

$$\begin{aligned} f(\lambda_1 u + \lambda_2 v) &= \lambda_1 f(u) + \lambda_2 f(v) \\ (f+g)(u) &= f(u) + g(v) \\ (\lambda f)(u) &= f(\lambda u) = \lambda f(u) \end{aligned} \tag{8.1}$$

With these properties the set $\mathcal{V}^\star$ has a vector space structure, and it's called the *dual space*

Definition 8.1.2 (Bounded Linear Functional)**.** Given $f \in \mathcal{V}^\star$, we call f a *bounded linear functional* if, $\forall x \in \mathcal{V}$

$$\sup_{\|x\| \leq 1} |f(x)| < \infty \tag{8.2}$$

Definition 8.1.3 (Dual Norm)**.** Given a normed vector space $(\mathcal{V}, \|\cdot\|)$ we can give a structure of normed vector space to its dual with the couple $(\mathcal{V}^\star, \|\cdot\|_\star)$, where the application $\|\cdot\|_\star$ is called the *dual norm*

$$\|\cdot\|_\star : \mathcal{V}^\star \longrightarrow \mathbb{F}$$

The dual norm is defined in two ways. $\forall f \in \mathcal{V}^\star,\ v \in \mathcal{V}$

$$\|f\|_{\star_1} = \sup_{v \neq 0} \frac{|f(v)|}{\|v\|} \tag{8.3a}$$

$$\|f\|_{\star_2} = \sup_{\|v\| \leq 1} |f(x)| \tag{8.3b}$$

Where $f \in \mathcal{V}^\star$ is a bounded linear functional.

Theorem 8.1. *For a bounded linear functional $f \in \mathcal{V}^\star$ the two definitions of the dual norm coincide, i.e.*

$$\|f\|_{\star_1} = \|f\|_{\star_2} = \|f\|_\star$$

And we can define the following inequality

$$|f(v)| \leq \|f\|_\star \|v\| \quad \forall v \in \mathcal{V} \tag{8.4}$$

Proof. Since $f \in \mathcal{V}^\star$ is bounded we have that both norms exist and must be finite. We can therefore write, thanks to the homogeneity of f, taken $v \in \mathcal{V}$

$$\|f\|_{\star_1} = \sup_{v \neq 0} \left| f\left(\frac{v}{\|v\|}\right) \right| = \sup_{\|v\|=1} |f(v)| \leq \sup_{\|v\| \leq 1} |f(v)| = \|f\|_{\star_2}$$

I.e. $\|f\|_{\star_1} \leq \|f\|_{\star_2}$, analogously

$$\|f\|_{\star_2} = \sup_{\|v\| \leq 1} |f(v)| \leq \sup_{0 < \|v\| \leq 1} \frac{|f(v)|}{\|v\|} \leq \sup_{v \neq 0} \frac{|f(v)|}{\|v\|} = \|f\|_{\star_2}$$

I.e. $\|f\|_{\star_1} \geq \|f\|_{\star_2}$, therefore we have

$$\|f\|_{\star_1} = \|f\|_{\star_2} = \|f\|_\star$$

The inequality is obvious taken the definition of supremum □

Theorem 8.2. *Given $f \in \mathcal{V}^\star$ with $\mathcal{V}$ normed vector space, we have that the following assumptions are equivalent*

1. *f is continuous*
2. *f is continuous at the origin*
3. *f is bounded*

Proof. 1) $\implies$ 2)
Since $f \in \mathcal{V}^\star$ is linear by definition, we have that it's also continuous and injective, therefore

$$\lim_{v \to 0} f(v) = f(0) = 0$$

2) $\implies$ 3)
Since f is continuous at the origin, we have by definition of continuity and limit

$$\forall \epsilon > 0 \; \exists \delta > 0 \; : \; \|x\| < \delta \implies |f(x)| < \epsilon \quad x \in \mathcal{V}$$

Taken $u = \delta x \in \mathcal{V}$, we have $\|u\| = |\delta| \|x\|$ and

$$|f(x)| = \left| f\left(\frac{u}{\delta}\right) \right| = \frac{1}{\delta} |f(u)| \leq \epsilon$$

Therefore, if $\|x\| \leq 1$ we have that $|f(u)| \leq \delta\epsilon$, therefore

$$|f(u)| \leq \|f\|_\star \|u\| \implies \|f\|_\star \|x\| \leq \|f\|_\star \leq \epsilon$$

3) $\implies$ 1) By definition of continuity and boundedness we have

$$\forall v, w \in \mathcal{V}, \forall \epsilon > 0 \; \exists \delta > 0 : \|u - w\| < \delta \implies |f(u) - f(w)| < \epsilon$$

Through the linearity of f we have

$$|f(u) - f(w)| = |f(u - w)| \leq \|f\|_\star \|u - w\| \leq \delta \|f\|_\star$$

Taken $\delta = \epsilon \|f\|_\star^{-1}$, we have

$$|f(u) - f(v)| \leq \delta \|f\|_\star = \epsilon$$

□

Corollary 8.1.1. Given $f \in \mathcal{V}^\star$, we have that, given $C \in \mathbb{F}$ a constant

$$\|f\|_\star = C \iff \begin{cases} \forall x \in \mathcal{V} \quad |f(x)| \leq C\|x\| \\ \forall \epsilon > 0 \; \exists x \in \mathcal{V} : |f(x)| \geq (C - \epsilon)\|x\| \end{cases}$$

Proof. We have, by definition of $\|f\|_\star$

$$\|f\|_\star = \sup_{\|x\| \leq 1} |f(x)| \leq \sup_{\|x\| \leq 1} |c\|x\|| = C$$

In the second case, supposing $\|f\|_\star < C$ and taken $\epsilon = 1/2(C - \|f\|_\star)$

$$\exists x \in \mathcal{V} : |f(x)| \geq (C - \|f\|_\star)\|x\| \quad \lightning$$

□

Definition 8.1.4 (Kernel)**.** Given a linear functional $f \in \mathcal{V}^\star$ we define the *kernel* of f as the set of zeros of f, i.e.

$$\ker f = \{ v \in \mathcal{V} | \, f(v) = 0\} \tag{8.5}$$

Theorem 8.3 (Riesz Representation Theorem)**.** *Given $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ a Hilbert space, we can uniquely define its dual $\mathcal{V}^\star$ through the isomorphism $\phi_v : \mathcal{V} \xrightarrow{\sim} \mathbb{F}$ defined as follows*

$$\forall u, v \in \mathcal{V} \quad \phi_v(u) = \langle u, v \rangle$$

I.e. $v \mapsto \langle \cdot, v \rangle$.
This isomorphism has the following properties

$$\begin{aligned} &1) \; \forall v \in \mathcal{V} \; \phi_v \in \mathcal{V}^\star, \quad \|\phi_v\|_\star = \|v\| \\ &2) \; \forall c_1, c_2 \in \mathbb{F}, \; v_1, v_2 \in \mathcal{V} \quad \phi_{c_1 v_1 + c_2 v_2} = \overline{c_1}\phi_{v_1} + \overline{c_2}\phi_{v_2} \\ &3) \; f \in \mathcal{V}^\star \implies \exists v \in \mathcal{V} : f = \phi_v \end{aligned} \tag{8.6}$$

Proof. Taken $w, v, z \in \mathcal{V}$ and $c, s \in \mathbb{F}$ we have by definition

$$\begin{aligned} \phi_v(w + z) &= \langle w + z, v \rangle = \langle w, v \rangle + \langle z, v \rangle = \phi_v(w) + \phi_v(z) \\ \phi_v(cw) &= \langle cw, v \rangle = c\langle w, v \rangle = c\phi_v(w) \\ \phi_{cw+sz}(v) &= \langle v, cw + sz \rangle = \overline{c}\langle v, w \rangle + \overline{s}\langle v, z \rangle = \overline{c}\phi_w(v) + \overline{s}\phi_z(v) \end{aligned}$$

We also have, thanks to Cauchy-Schwartz

$$|\phi_v(z)| = |\langle z, v\rangle| \leq \|z\|\|v\|$$
$$|\phi_v(v)| = |\langle v, v\rangle| \leq \|v\|^2 \quad \therefore \|\phi_v\|_\star = \|v\|$$

For the last one we take $f \in \mathcal{V}^\star$. If $\ker f = \mathcal{V}$ we have by definition that

$$f(v) = 0 \quad \forall v \in \mathcal{V} \quad \therefore f = 0$$

Taken $v = 0$ we then have

$$\phi_0(v) = \langle 0, v\rangle = 0 = f(v) \quad \forall v \in \mathcal{V}$$

Supposing now $\ker f \neq \mathcal{V}$ there must be $w \in \ker f^\perp \, : \, w \neq 0$. Taken $u \in \mathcal{V}$ we can write

$$u = u_1 + \frac{f(u)}{f(w)} w = u - \frac{f(u)}{f(w)} w + \frac{f(u)}{f(w)} w$$

Therefore, through the linearity of f

$$f(u_1) = f\left(u - \frac{f(u)}{f(w)} w\right) = f(u) - f(w) = f(u - v) = 0 \quad \therefore u_1 \in \ker f$$

Since $w \in \ker f^\perp$, we have that

$$\langle u_1, w\rangle = 0$$

By definition of ϕ_v, then

$$\phi_v(u) = \langle v, u_1\rangle + \langle v, \frac{f(u)}{f(w)} w\rangle = \langle \frac{f(w)}{\|w\|^2} w, u_1\rangle + \langle \frac{f(w)}{\|w\|^2} w, \frac{f(u)}{f(w)} w\rangle$$

Then

$$\phi_v(u) = \frac{f(w)}{\|w\|^2}\langle w, u_1\rangle + \frac{f(w)}{\|w\|^2}\frac{f(u)}{f(w)}\langle w, w\rangle = f(u)$$

Therefore

$$\phi_v(u) = f(u) \quad \forall f \in \mathcal{V}^\star, \ \forall u \in \mathcal{V}$$

□

§ 8.2 Distributions

§§ 8.2.1 Local Integrability

Definition 8.2.1 (Weak Convergence)**.** Given a normed vector space $(\mathcal{V}, \|\cdot\|)$, a sequence $(v)_{k\in\mathbb{N}} \in \mathcal{V}$ is said to be *weakly convergent* to $v \in \mathcal{V}$ and it's indicated as $v_k \rightharpoonup_{\mathrm{w}} v$ if

$$\forall f \in \mathcal{V}^\star \quad \lim_{k\to\infty} f(v_k) = f(v)$$

Theorem 8.4. *Given a normed vector space $\mathcal{V}$ and a sequence $(v)_{k\in\mathbb{N}} \in \mathcal{V}$ such that $v_k \to v \in \mathcal{V}$ (i.e. converges strongly), we have that*

$$v_k \to v \implies v_k \rightharpoonup_{\mathrm{w}} v$$

Proof. Given $v_k \to v$, we have by definition that

$$v_k \to v \iff \lim_{k\to\infty} \|v_k - v\| = 0$$

Now, writing the definition of weak convergence and applying the linearity of the limit

$$v_k \rightharpoonup_{\mathrm{w}} v \iff \forall f \in \mathcal{V}^\star \quad \lim_{k\to\infty} (f(v_k) - f(v)) = 0$$

Therefore

$$\forall f \in \mathcal{V}^\star \quad \lim_{k\to\infty} (f(v_k - v)) \le \|f\|_\star \lim_{k\to\infty} \|v_k - v\|$$

Therefore we have that, since f is bounded

$$\exists C \in \mathbb{F} : \lim_{k\to\infty} (f(v_k - v)) \le C \lim_{k\to\infty} \|v_k - v\| \quad \forall f \in \mathcal{V}^\star$$

And therefore

$$v_k \to v \implies v_k \rightharpoonup_{\mathrm{w}} v$$

□

Remark. The opposite isn't necessarily true.

Proof. Take $(\mathcal{V}, \|\cdot\|) = (\ell^2, \|\cdot\|_2)$ and take the standard sequence $(e_i)_j$ where

$$(e_i)_j = (0, 0, \cdots, 0, \underbrace{(1)_j}_{i-\mathrm{th}}, 0, \cdots)$$

And $(1)_j$ is the identity sequence.
We have that $(e_i)_j \not\to (a)_i$ since

$$\|(e_i)_j - (e_i)_k\|_2 = \sqrt{2} \quad \forall j \ne k$$

Now, considering that $(\ell^2, \|\cdot\|_2)$ is a Hilbert space, we have that, due to Riesz representation theorem

$$\forall f \in \ell^{2\star} \quad f = \langle \cdot, a \rangle = \phi_a \quad a \in \ell^2$$

Therefore

$$f(e_{ij}) = \langle e_{ij}, a_i \rangle = \sum_{i=1}^{\infty} a_i e_{ij} = a_j \to 0$$

Therefore

$$\forall (a)_j \in \ell^2 \; \forall f \in \ell^{2\star} \quad \lim_{j\to\infty} f(e_{ij}) = 0$$

Hence, by definition $e_{ij} \rightharpoonup_{\mathrm{w}} 0$ □

Definition 8.2.2 (Isolated Singularities II)**.** Given a function $f : \mathbb{R} \longrightarrow \mathbb{C}$, it's said to have *isolated singularities* if

$$S_f := \left\{ x \in \mathbb{R} | \lim_{y\to x} f(y) \ne f(x) \right\}$$

Doesn't have accumulation points, i.e. if $|S_f| < \infty$

Definition 8.2.3 (Piecewise Continuity)**.** A function $f : [a, b] \subset \mathbb{R} \longrightarrow \mathbb{C}$ is said to be *piecewise continuous* if

1. $S_f \neq \{\}$ and $|S_f| < \infty$
2. $\forall u \in S_f$
$$\begin{aligned} &\exists \lim_{x \to u^+} f(x) = f(u^+) \\ &\exists \lim_{x \to u^-} f(x) = f(u^-) \end{aligned} \tag{8.7}$$
3.
$$\begin{aligned} &\exists \lim_{x \to a^+} f(x) = f(a^+) \\ &\exists \lim_{x \to b^-} f(x) = f(b^-) \end{aligned} \tag{8.8}$$

Definition 8.2.4 (Jump)**.** Given a piecewise continuous function f we define the *jump* of the function at a discontinuity $x \in S_f$ as follows

$$\Delta f(x) = f(x^+) - f(x^-)$$

Definition 8.2.5 (Piecewise Differentiability)**.** A piecewise continuous function $f : [a, b] \longrightarrow \mathbb{C}$ is said to be *piecewise differentiable* if and only if

1. f is piecewise continuous in $[a, b]$
2. f' is piecewise continuous in $[a, b]$

Definition 8.2.6 (Local Integrability)**.** A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be *locally integrable* if

1. $S_f \neq \{\}$ and $|S_f| < \infty$
2. $\forall a, b \in \mathbb{R},\ a < b$ we have that
$$\int_a^b |f(x)| \, \mathrm{d}x < \infty$$

The set of locally integrable functions forms a subspace of $L^1(\mathbb{R})$ and it's indicated as $L^1_{loc}(\mathbb{R})$

Theorem 8.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a piecewise continuous function, then $f \in L^1_{loc}(\mathbb{R})$*

Proof. Take $f(x)$ a piecewise continuous function, then $\forall a, b \in \mathbb{R}\ \exists (u)_{0 \leq k \leq n} \in \mathbb{R} : u_0 = a,\ u_n = b$ for which we have $f \in C((u_{k-1}, u_k))\ \forall 0 \leq k \leq n$. I can therefore define $\tilde{f}_k : [u_{k-1}, u_k] \longrightarrow \mathbb{C}$, where

$$\tilde{f}_k(x) = \begin{cases} f(x) & x \in (u_{k-1}, u_k) \\ f(u_{k-1}^+) & x = u_{k-1} \\ f(u_k^-) & x = u_k \end{cases}$$

We have that

$$f(x) = \sum_{k=0}^{\infty} \tilde{f}_k(x)$$

And therefore

$$\int_a^b |f(x)| \, \mathrm{d}x = \sum_{k=0}^{n} \int_{u_{k-1}}^{u_k} |f(x)| \, \mathrm{d}x = \sum_{k=0}^{n} \int_{u_{k-1}}^{u_k} \left|\tilde{f}_k(x)\right| \mathrm{d}x$$

Therefore $f(x) \in L^1_{loc}(\mathbb{R})$ and the theorem is proved □

Theorem 8.6 (Integration by Parts in $L^1_{loc}(\mathbb{R})$)**.** *Let $f, g \in L^1_{loc}([a, b])$, then indicating the evaluation of a function at two points as $[f(x)]_z^w$ we have*

$$\begin{aligned} [fg]_{a^+}^{b^-} = & \int_a^b (f'(x)g(x) + f(x)g'(x)) \, \mathrm{d}x + \\ & + \sum_{x \in S_f \cup S_g} \big(f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x) \Delta g(x)\big) \end{aligned} \tag{8.9}$$

Proof. Take $H : [a, b] \longrightarrow \mathbb{C}$, then for the fundamental theorem of calculus

$$H(b^-) - H(a^+) = \int_a^b \frac{\mathrm{d}H}{\mathrm{d}x} \, \mathrm{d}x + \sum_{x \in S_H} \Delta H(x)$$

Taken $H(x) = f(x)g(x)$ we have

$$\Delta(fg)(x) = f(x^+)g(x^+) - f(x^-)g(x^-)$$

With some manipulation we have

$$\begin{aligned} \Delta(fg)(x) = f(x^+)g(x^+) - f(x^+)g(x^-) + f(x^+)g(x^-) - f(x^-)g(x^-) = f(x^+)\Delta g(x) + g(x^-)\Delta f(x) = \\ = \big(f(x^-) + \Delta f(x)\big) \Delta g(x) + g(x^-)\Delta f(x) = f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x) \Delta g(x) \end{aligned}$$

Therefore

$$(fg)(b^-) - (fg)(a^+) = \int_a^b (fg)'(x) \, \mathrm{d}x + \sum_{x \in S_f \cup S_g} \big(f(x^-)\Delta g(x) + g(x^-)\Delta f(x) + \Delta f(x)\Delta g(x)\big)$$

□

§§ 8.2.2 Regular and Singular Distributions

Definition 8.2.7 (Test Functions)**.** A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be a *test function* if $\overline{\operatorname{supp}\{f\}}$ is compact and $f \in C^\infty(\mathbb{R})$, i.e. $f \in C_c^\infty(\mathbb{R})$
This space is usually denoted as follows $C_c^\infty(\mathbb{R}) = \mathcal{K}$

Definition 8.2.8 ($\mathcal{K}$-convergence)**.** Given a sequence $(f)_{n \in \mathbb{N}} \in \mathcal{K}$, it's said to be $\mathcal{K}-$convergent if

1. $\exists I \subset \mathbb{R} : \forall x \in I^c \; f_n(x) = 0$
2. $\forall k \in \mathbb{N} \; f_n^{(k)}(x) \rightrightarrows f^{(n)}(x)$

Then, it's indicated as $f_n \rightharpoonup_{\mathcal{K}} f$

Definition 8.2.9 (Distribution)**.** A *distribution* is a continuous linear functional $\varphi : \mathcal{K} \longrightarrow \mathbb{C}$, i.e.

$$\forall (f)_n \in \mathcal{K},\ f_n \rightharpoonup_{\mathcal{K}} f \implies \varphi(f_n) = \varphi(f)$$

By definition of dual space, we have that $\varphi \in \mathcal{K}^\star$

Theorem 8.7. *Given $g \in L^1_{loc}(\mathbb{R})$ and $\varphi_g : \mathcal{K} \longrightarrow \mathbb{C}$, if we have*

$$\varphi_g(f) = \int_{\mathbb{R}} g(x) f(x)\,\mathrm{d}x \quad \forall f \in \mathcal{K}$$

Then $\varphi_g \in \mathcal{K}^\star$

Proof. Using the fact that $f \in \mathcal{K}$ we can immediately say that $|f(x)| \le M\ \forall x \in [-a, a] \subset \mathbb{R}$, therefore

$$\int_{\mathbb{R}} |g(x)f(x|)\,\mathrm{d}x = \int_{-a}^{a} |g(x)f(x)|\,\mathrm{d}x \le M \int_{-a}^{a} |g(x)|\,\mathrm{d}x < \infty$$

Alternatively, using the defintion of integral we can say that

$$\varphi_g(\alpha f + \beta h) = \alpha \varphi_g(f) + \beta \varphi_g(h) \quad \forall \alpha, \beta \in \mathbb{C},\ \forall f, h \in \mathcal{K}$$

We only need to show that this application is $\mathcal{K}-$continuous. I take $f_n \rightharpoonup_{\mathcal{K}} f$, with $f_n \in \mathcal{K}$.
It's obvious that $f_n \rightrightarrows f$, and using the linearity and that $g \in L^1_{loc}(\mathbb{R})$, calling $A = \|g\|$, we have

$$|\varphi_g(f) - \varphi_g(f_n)| \le A \|f - f_n\|_u \to 0$$

Therefore $\varphi_g(f_n) \to \varphi_g(f)\ \forall f_n \in \mathcal{K},\ f_n \rightharpoonup_{\mathcal{K}} f$ $\square$

Definition 8.2.10 (Regular Distribution)**.** A distribution $f \in \mathcal{K}^\star$ is said to be *regular* if $\exists g \in L^1_{loc}(\mathbb{R})$ such that

$$f = \langle \cdot, g \rangle = \varphi_g$$

I.e.

$$f(h) = \varphi_g(h) = \langle h, g \rangle = \int_{\mathbb{R}} h(x) g(x)\,\mathrm{d}x$$

Definition 8.2.11 (Ceiling and Floor Functions)**.** We define the *ceiling function* $\lceil \cdot \rceil : \mathbb{R} \longrightarrow \mathbb{Z}$ and the *floor functions* $\lfloor \cdot \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$ as follows

$$\begin{aligned} \lfloor x \rfloor &= \max \{ m \in \mathbb{Z} |\ m \le x \} \\ \lceil x \rceil &= \min \{ n \in \mathbb{Z} |\ n \ge x \} \end{aligned} \tag{8.10}$$

Definition 8.2.12 (Heaviside Function)**.** We define the *Heaviside function* as a function $H : \mathbb{R} \longrightarrow \mathbb{R}$ as follows

$$H(x) = \begin{cases} 1 & x \ge 0 \\ 0 & x < 0 \end{cases}$$

It's obviously piecewise continuous and therefore $H \in L^1_{loc}(\mathbb{R})$
A secondary definition is the one that follows $\tilde{H} : \mathbb{R} \longrightarrow \mathbb{R}$

$$\tilde{H}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

Example 8.2.1 (Floor and Ceiling Distributions). Since both the floor and ceiling distributions are locally integrable, we can build a regular distribution $\varphi \in \mathcal{K}^\star$ as follows

$$\begin{aligned}\varphi_{\lfloor x \rfloor}(f) &= \int_{\mathbb{R}} f(x)\lfloor x \rfloor \,\mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_k^{k+1} kf(x)\,\mathrm{d}x \\ \varphi_{\lceil x \rceil}(f) &= \int_{\mathbb{R}} f(x)\lceil x \rceil \,\mathrm{d}x = \sum_{k\in\mathbb{Z}} \int_k^{k+1} (k+1)f(x)\,\mathrm{d}x\end{aligned}$$

Example 8.2.2 (Theta Distribution). Given $H \in L^1_{loc}(\mathbb{R})$ we can define the associated *theta distribution* $\varphi_H = \vartheta \in \mathcal{K}$, as follows

$$\vartheta(f) = \int_{\mathbb{R}} f(x)H(x)\,\mathrm{d}x = \int_{\mathbb{R}^+} f(x)\,\mathrm{d}x$$

It's already obvious that $\varphi_{\tilde{H}} = \varphi_H$, but it's better to formalize it in the following theorem

Theorem 8.8. *Let $f, g \in L^1_{loc}(\mathbb{R})$ such that $\{x \in \mathbb{R} |\, f(x) \neq g(x)\} = \{u_1, \cdots, u_n\}$. Then we have $\varphi_g = \varphi_f \in \mathcal{K}^\star$*

Proof. We have by Riesz theorem that

$$\varphi_g(f) = \int_{\mathbb{R}} g(x)f(x)\,\mathrm{d}x = \int_{-\infty}^{u_1} f(x)g(x)\,\mathrm{d}x + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)g(x) + \int_{u_n}^{\infty} f(x)\,\mathrm{d}x$$

But, since $f(x) = g(x), \quad \forall x \in (u_k, u_{k+1})$ we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x &= \int_{-\infty}^{u_1} f(x)g(x)\,\mathrm{d}x + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)g(x)\,\mathrm{d}x + \int_{u_n}^{\infty} f(x)g(x)\,\mathrm{d}x = \\ &= \int_{-\infty}^{u_1} f(x)h(x)\,\mathrm{d}x + \sum_{k=2}^{n-1} \int_{u_k}^{u_{k+1}} f(x)h(x)\,\mathrm{d}x + \int_{u_n}^{\infty} f(x)h(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)h(x)\,\mathrm{d}x\end{aligned}$$

Therefore

$$\varphi_g = \varphi_h$$

□

Definition 8.2.13 (Singular Distribution)**.** A *singular distribution* is a distribution $f \in \mathcal{K}^\star$ for which, given $g \in L^1_{loc}(\mathbb{R})$, $f \neq \varphi_g$, where

$$\varphi_g(h) = \int_{\mathbb{R}} h(x)g(x)\,\mathrm{d}x$$

Definition 8.2.14 (Dirac Delta Distribution)**.** An example of singular distribution is the *Dirac delta distribution* $\delta_a \in \mathcal{K}^\star$. This distribution is defined as follows

$$\delta_a(f) = f(a) \quad \forall f \in \mathcal{K},\ a \in \mathbb{R}$$

Theorem 8.9. *Given the Dirac delta distribution δ_a, $\nexists \delta(x) \in L^1_{loc}(\mathbb{R})$ such that (taken $a = 0$ without loss of generality)*

$$\delta_0(f) = \int_{\mathbb{R}} f(x)\delta(x)\,\mathrm{d}x$$

Proof. Let's say that $\exists \delta(x) \in L^1_{loc}(\mathbb{R})$, therefore, we could define $\delta_0 = \langle \cdot, \delta(x) \rangle$. This function therefore must have these properties

$$\delta(x) \neq 0 \iff x = 0,\ x \in S_\delta$$

But, since for $b \in \mathbb{R}$, $b \neq 0$, we have that $\delta(x) \to \delta(b)$ continuously, i.e. $\delta(b) = A > 0$, therefore

$$\exists \epsilon > 0 \,:\, \delta(x) \geq \frac{A}{2} \quad \forall x \in [b - \epsilon, b + \epsilon]$$

But, then $\exists f \in \mathcal{K}$ such that $f(x) \to f(b)$ continuously, $f > 0$, $f(b) = 1$. Taken $\epsilon < b$ we can say

$$\exists \tilde{\epsilon} \in (0, \epsilon) \,:\, f(x) \geq \frac{1}{2} \quad \forall x \in [b - \epsilon', b + \epsilon']$$

Therefore, we have that

$$\int_{\mathbb{R}} f(x)\delta(x)\,\mathrm{d}x = \int_{-\epsilon}^{\epsilon} f(x)\delta(x)\,\mathrm{d}x \geq \int_{-\epsilon}^{\epsilon} \frac{A}{2}\frac{1}{2}\,\mathrm{d}x = \frac{\epsilon' A}{2}$$

But, by definition $f(0) = 0$, therefore

$$\int_{\mathbb{R}} f(x)\delta(x)\,\mathrm{d}x = 0 \implies \varphi_\delta(f) = \varphi_0(f) \quad \lightning$$

Therefore, the distribution $\delta_0 \neq \langle \cdot, \delta(x) \rangle$ □

Notation (Common Abuse of Notation)**.** A common abuse of notation in the usage of the Dirac delta distribution, is supposing that it's writable like a regular distribution, i.e. supposing that $\exists \delta(x) \in L^1_{loc}(\mathbb{R})$, the "Dirac delta function" such that

$$\delta_a(f) = \int_{\mathbb{R}} f(x)\delta(x - a)\,\mathrm{d}x = f(a) \quad \forall f \in \mathcal{K}$$

This is obtainable only if the "delta function" is defined as follows

$$\delta(x) = \begin{cases} +\infty & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{8.11}$$

Note that

$$\delta_0(1) = \int_{\mathbb{R}} \delta(x)\,\mathrm{d}x = 1$$

Theorem 8.10. *Given $g_n \in L^1_{loc}(\mathbb{R})$ such that $g_n \in C[-a_n, a_n]$ with $a_n \to 0$, $a_n > 0 \quad \forall n \in \mathbb{N}$, we have that, if*

$$\int_{\mathbb{R}} g_n(x)\,\mathrm{d}x = 1$$

Then

$$\lim_{n\to\infty} \varphi_{g_n}(f) = \delta_0(f)$$

Proof. We use the definition of limit, therefore we have

$$\left|\int_{\mathbb{R}} f(x)g_n(x)\,\mathrm{d}x - \delta_0(f)\right| \leq \int_{-a_n}^{a_n} |f(x) - f(0)|g_n(x)\,\mathrm{d}x$$

Using that the integral over the real axis of g_n is unitary

$$\left|\int_{\mathbb{R}} f(x)g_n(x)\,\mathrm{d}x - \delta_0(f)\right| \leq \sup_{|x|\leq a_n} |f(x) - f(0)| \to 0$$

Therefore

$$\lim_{n\to\infty} \varphi_{g_n}(f) = \delta_0(f)$$

□

Corollary 8.2.1. Given $g \in C_1(\mathbb{R})$ a non-negative function $g \geq 0$ such that

$$\int_{\mathbb{R}} g(x)\,\mathrm{d}x = 1$$

Then, if we put $g_n(x) = ng(nx)$ we have that

$$\lim_{n\to\infty} \varphi_{g_n}(f) = \delta_0(f)$$

Proof. As before, we use the definition, and therefore we have

$$\left|\int_{\mathbb{R}} f(x)g_n(x)\,\mathrm{d}x - \delta_0(f)\right| \leq \int_{\mathbb{R}} |f(x) - f(0)|g_n(x)\,\mathrm{d}x = n\int_{\mathbb{R}} |f(x) - f(0)|g(nx)\,\mathrm{d}x$$

Using the substitution $u = nx$ we therefore get

$$\int_{\mathbb{R}} |f(x) - f(0)|g(nx)n\,\mathrm{d}x = \int_{\mathbb{R}} \left|f\left(\frac{u}{n}\right) - f(0)\right|g(u)\,\mathrm{d}u$$

But, by definition of $g(x)$, we have that $\exists L > 0 : \exists \epsilon > 0$ for which

$$1 - \epsilon \leq \int_{-L}^{L} g(u)\,\mathrm{d}u \leq 1 \quad \wedge \quad \int_{|u|\geq L} g(u)\,\mathrm{d}u \leq \epsilon$$

And therefore

$$\int_{\mathbb{R}} \left|f\left(\frac{u}{n}\right) - f(0)\right|g(u)\,\mathrm{d}u = \int_{-L}^{L} \left|f\left(\frac{u}{n}\right) - f(0)\right|g(u)\,\mathrm{d}u + \int_{|u|\geq L} \left|f\left(\frac{u}{n}\right) - f(0)\right|g(u)\,\mathrm{d}u$$

By using the properties of the integral, we know that

$$\int_{\mathbb{R}} \left|f\left(\frac{u}{n}\right) - f(0)\right|g(u)\,\mathrm{d}u \leq \sup_{x\leq \frac{L}{n}} |f(x) - f(0)| + \epsilon\left(\sup_{u\in\mathbb{R}}\left|f\left(\frac{u}{n}\right)\right| - |f(0)|\right)$$

Since $f(x) \to f(0)$ continuously, due to the fact that $f \in \mathcal{K}$, we have that

$$\int_{\mathbb{R}} |f(x) - f(0)|g(nx)n\,\mathrm{d}x \leq 2\epsilon\|f\|_u$$

Therefore

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}} f(x)g_n(x)\,\mathrm{d}x - \delta_0(f)\right| = 0$$

$\square$

Definition 8.2.15 (PV (x^{-n}) −Distribution)**.** Taken $n > 0$, $n \in \mathbb{N}$, we have that $x^{-n} \notin L^1_{loc}(x)$ and therefore there is no associated distribution $\varphi_{x^{-n}} \in \mathcal{K}^\star$.
A useful thing we could do is utilizing the definition of the Cauchy principal value of the function. Therefore, we define the following singular distribution

$$\mathrm{PV}\left(\frac{1}{x^n}\right)[f] = \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{x^n}\left(f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k\right)\mathrm{d}x \tag{8.12}$$

Theorem 8.11. *The application* $\mathrm{PV}(x^{-n}) : \mathcal{K} \longrightarrow \mathbb{R}$ *defined as before is a distribution, hence* $\mathrm{PV}(x^{-n}) \in \mathcal{K}^\star$

Proof. We already know that this distribution is linear, since $\forall f, g \in \mathcal{K}$, $\forall c, d \in \mathbb{R}$

$$\mathrm{PV}\left(\frac{1}{x^n}\right)[cf+dg] = \lim_{a\to\infty}\int_{-a}^{a}\frac{1}{x^n}\left(cf(x)+dg(x)+\sum_{k=0}^{n-1}\frac{1}{k!}\left(cf^{(k)}(0)+dg^{(k)}(0)\right)x^k\right)\mathrm{d}x =$$
$$= c\,\mathrm{PV}\left(\frac{1}{x^n}\right)[f] + d\,\mathrm{PV}\left(\frac{1}{x^n}\right)[g]$$

Secondly we must show that the integral is well defined.
Let $f \in \mathcal{K}$, then

$$I_R^n(f) = \int_{-R}^{R}\frac{1}{x^n}\left(f(x) - \sum_{k=0}^{n-1}\frac{1}{k!}f^{(k)}(0)x^k\right)\mathrm{d}x, \quad n \in \mathbb{N},\ R > 0$$

Using the Lagrange formulation for the remainder of the McLaurin expansion, we have

$$R_{n-1}(x) = f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}x^k = x^n r_{n-1}(x)$$

And therefore we have

$$I_R^n(x) = \int_{-R}^{R} r_{n-1}(x)\,\mathrm{d}x$$

Since $r_{n-1}(x) \in C^\infty(\mathbb{R})$ (it's a polynomial) and $\operatorname{supp} f \subset [-b, b]$ for some $b \in \mathbb{R},\ b > 0$, we have that for $R > b$

$$I_R^n(f) = I_b^n(f) - \sum_{k=0}^{n-1}\frac{f^{(k)}(0)}{k!}\int_{b\le|x|\le R}\frac{1}{x^{n-k}}\,\mathrm{d}x$$

The last integral evaluates to

$$\int_{b\le|x|\le R}\frac{1}{x^{n-k}}\,\mathrm{d}x = \begin{cases}\dfrac{2}{n-k-1}\left[\dfrac{1}{b^{n-k-1}} - \dfrac{1}{R^{n-k-1}}\right] & n-k \mod 2 = 0,\ n-k > 2 \\ 0 & \text{else}\end{cases}$$

Therefore, introducing the following dummy index $j = n - k$ where $j \mod 2 = 0$, we have

$$\mathrm{PV}\left(\frac{1}{x^n}\right)(f) = I_b^n(f) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} \frac{2}{(j-1)b^{j-1}}$$

Therefore the integral defining the distribution is well defined.
Lastly we need to prove that $\mathrm{PV}(x^{-n})(f_n) \rightharpoonup_{\mathcal{K}} \mathrm{PV}(x^{-n})(f)$, $\forall (f)_n \in \mathcal{K} : f_n \to f$, i.e. that it's a continuous functional.
Going back to the previous definition we have that

$$r_{n-1}(x) = \frac{R_{n-1}(x)}{x^n} = \frac{1}{x^n} \frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}}(\xi(x)) \quad \xi(x) \in (0, x)$$

Therefore

$$|I_R^n(f)| \leq 2 \sup_{x \in [-R,R]} |r_{n-1}(x)| \leq \frac{2R}{n!} \|f\|_u$$

Which means

$$\left|\mathrm{PV}\left(\frac{1}{x^n}\right)(f)\right| \leq \frac{2b}{n!} \left\|f^{(n)}\right\|_u + \sum_{j=0}^{n-1} \frac{\left|f^{(j)}(0)\right|}{j!} \frac{2}{(j-1)b^{j-1}}$$

Taken $b_1 = \max\{b, 1\}$ we have that $\operatorname{supp} f \subset [-b_1, b_1]$ and whenever $j - 1 > 1,\ b_1 \geq 1$

$$\frac{2}{(j-1)b_1^{j-1}} \leq 2$$

Therefore, finally

$$\left|\mathrm{PV}\left(\frac{1}{x^n}\right)(f)\right| \leq \frac{2}{n!} \max\{b, 1\} \left\|f^{(n)}\right\|_u + 2 \sum_{j=0}^{n-1} \frac{\left|f^{(j)}(0)\right|}{j!}$$

Now, defined $f_j \in \mathcal{K}$ and $h_j(x) = f(x) - f_j(x)$ with $f_j \to f$ we have that $\operatorname{supp} h_j \subset [-b, b],\ \forall j \in \mathbb{N}$.
By definition we have that $h_j \rightrightarrows 0$ and $h_j^{(n)} \rightrightarrows 0\ \forall n \in \mathbb{N}$

$$\mathrm{PV}\left(\frac{1}{x^n}\right)(h_j) \leq \frac{2}{n!} \max\{b, 1\} \left\|h_j^{(n)}\right\|_u + 2 \sum_{j=0}^{n-1} \frac{\left|h_j^{(j)}(0)\right|}{j!}$$

This means

$$\forall k \in \mathbb{N}, \forall \epsilon > 0,\ \exists N_R \in \mathbb{N} : \forall k \geq N_j \ \left|h_j^{(k)}(x)\right| \leq \frac{\epsilon}{2(b+1+\epsilon)}$$

Chosen $N = \max_{0 \leq k \leq n-1} N_k$ we have that $\forall j \geq N$

$$\left|\mathrm{PV}\left(\frac{1}{x^n}\right)(h_j)\right| \leq \frac{2\epsilon}{2(b+1+\epsilon)} \left[\max\{b, 1\} + \sum_{j=0}^{n-1} \frac{1}{j!}\right] \leq \epsilon \to 0$$

Therefore, $\mathrm{PV}(x^{-n}) \in \mathcal{K}^\star$ $\square$

§§ 8.2.3 Operations with Distributions

Definition 8.2.16 (Weak Derivative)**.** Given $u \in L^1([a,b])$, we define the weak derivative $v \in L^1([a,b])$ if, $\forall h \in C_c^\infty([a,b])$ we have

$$\int_a^b u(x)h'(x)\,\mathrm{d}x = -\int_a^b v(x)h(x)\,\mathrm{d}x$$

The function $v(x)$ will then be identified as follows

$$v(x) = \mathrm{D}\,u(x)$$

Theorem 8.12 (Operations with Distributions)**.** *Given $\varphi, \gamma \in \mathcal{K}^\star$, $f \in \mathcal{K}$, $h \in C^\infty(\mathbb{R})$ and $c \in \mathbb{C}$ we define the following operations in $\mathcal{K}^\star$*

$$\begin{aligned} + &: \mathcal{K}^\star \times \mathcal{K}^\star \longrightarrow \mathcal{K}^\star \\ \cdot &: \mathbb{C} \times \mathcal{K}^\star \longrightarrow \mathcal{K}^\star \\ \circ &: C^\infty \times \mathcal{K}^\star \longrightarrow \mathcal{K}^\star \\ \mathrm{D} &: \mathcal{K}^\star \longrightarrow \mathcal{K}^\star \end{aligned} \tag{8.13}$$

Where, they act as follows

$$\begin{aligned} +(\varphi(f), \gamma(f)) &= (\varphi + \gamma)(f) = \varphi(f) + \gamma(f) \\ \cdot(c, \varphi(f)) &= (c\varphi)(f) = c\varphi(f) \\ \circ(h, \varphi(f)) &= (h\varphi)(f) = \varphi(hf) \\ \mathrm{D}\,\varphi(f) &= -\varphi(f') \end{aligned}$$

The last operation is the distributional derivative.

Theorem 8.13. *Given $g(x) \in L^1_{loc}(\mathbb{R})$ and $\varphi_g \in \mathcal{K}^\star$ its associated distribution. Then, $\forall h \in C^\infty$, $f \in \mathcal{K}$*

$$(h\varphi_g)(f) = \varphi_g(hf) = \varphi_{hg}(f) \tag{8.14}$$

Proof. The proof is quite straightforward

$$(h\varphi_g)(f) = \varphi_g(hf) = \int_{\mathbb{R}} g(x)\,(h(x)f(x))\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,(g(x)h(x))\,\mathrm{d}x = \varphi_{hg}(f)$$

□

Theorem 8.14. *Taken $g \in L^1_{loc}(\mathbb{R})$ and $g \mapsto \varphi_g \in \mathcal{K}^\star$ its associated distribution, we have that*

$$\mathrm{D}\,\varphi_g = \varphi_{\mathrm{D}\,g}$$

Where $\mathrm{D}\,g$ is the weak derivative of g

Proof. $\forall f \in \mathcal{K}$ we have that

$$\mathrm{D}\,\varphi_g(f) = -\varphi_g(f') = -\int_{\mathbb{R}} f'(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathrm{D}\,g(x)\,\mathrm{d}x = \varphi_{\mathrm{D}\,g}(f)$$

□

Theorem 8.15. *$\forall g, f \in L^1_{loc}(\mathbb{R})$ and given $\varphi_g, \gamma_f \in \mathcal{K}^\star,\ c \in \mathbb{C}$ their associated distributions, we have that*

$$\varphi_g + \varphi_f = \varphi_{g+f}$$
$$\varphi_{cf} = c\varphi_f$$

Proof. The proof is obvious using the linearity of the integral operator □

Theorem 8.16 (Useful Identities)**.** *Here is a list of some useful identities*

$$\begin{aligned} x\delta_0 &= 0 \\ x\mathrm{PV}\left(\frac{1}{x}\right) &= \varphi_1 \\ x\,\mathrm{D}\,\delta_0 &= -\delta_0 \end{aligned} \tag{8.15}$$

Proof. 1) $x\delta_0 = 0$.
Taken $f \in \mathcal{K}$

$$x\delta_0(f) = \delta_0(xf) = 0f(0) = 0$$

2) $x\mathrm{PV}\left(\frac{1}{x^n}\right) = \varphi_1$
Again, taken $g \in \mathcal{K}$

$$x\,\mathrm{PV}\left(\frac{1}{x}\right)(g) = \mathrm{PV}\left(\frac{1}{x}\right)(xg) = \lim_{\epsilon\to 0}\int_{|x|\geq 0} f(x)\,\mathrm{d}x = \varphi_1$$

For the last one, taken $h \in \mathcal{K}$

$$x\,\mathrm{D}\,\delta_0(f) = \mathrm{D}\,\delta_0(xf) = -\delta_0\left(f + xf'\right) = -f(0) - 0f(0) = -f(0) = -\delta_0(f)$$

□

Notation (Abuse of Notation)**.** Given $g \in L^1_{loc}(\mathbb{R})$ and its associated distribution $g \mapsto \varphi_g$, is quite common to use the original function to indicate actually the distribution. Together with this, the distributional derivative D gets indicated as an usual derivative, therefore

$$\varphi_g \to g(x)$$
$$\mathrm{D}\,\varphi_g \to g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x}$$

Therefore it isn't uncommon to see identities like this

$$x\mathrm{PV}\left(\frac{1}{x}\right) = 1$$

Where actually $1 \to \varphi_1$ is the identity distribution.
Or

$$\delta_0' = -\delta_0$$

This makes an easy notation for calculating distributional derivatives and have some calculations, but one should watch out to this common abuse of notation

Definition 8.2.17 ($\mathcal{K}^\star$ Convergence)**.** Given $(\sigma)_{n\in\mathbb{N}} \in \mathcal{K}^\star$ a sequence of distributions, it's say to $\mathcal{K}^\star$*−converge* if

$$(\sigma)_n(f) \to \sigma(f) \in \mathcal{K}^\star$$

It's indicated as follows

$$(\sigma)_n \rightharpoonup_\star \sigma$$

Theorem 8.17. *Given $\kappa \in \mathcal{K}^\star$ a distribution, we have that $f \in C^\infty(\mathcal{K})$ in the sense of distributional derivatives, where the $n-th$ derivative is defined as follows*

$$\mathrm{D}^n \kappa(f) = (-1)^n \kappa(f^{(n)}) \quad \forall f \in \mathcal{K}$$

Proof. Taken $f \in \mathcal{K}$ we have that $f \in C_c^\infty(\mathbb{R}) \subset C^\infty(\mathbb{R})$ hence f is smooth, therefore we have, for $\kappa \in \mathcal{K}^\star$

$$\mathrm{D}\,\kappa(f) = -\kappa(f')$$

Iterating, we get for $n = 2$

$$\mathrm{D}^2 \kappa(f) = -\mathrm{D}\,\kappa(f') = \kappa(f'')$$

Iterating till n we get finally

$$\mathrm{D}^n \kappa(f) = (-1)^n \kappa(f^{(n)})$$

Hence the derivability depends only on the test function f, and since it's smooth, the distributional derivatives can be defined $\forall n \in \mathbb{N}$ □

Example 8.2.3. Take for example the delta distribution $\delta_0 \in \mathcal{K}^\star$, given $f \in \mathcal{K}$ we want to calculate the following value

$$\mathrm{D}^n \delta_0(f)$$

Using the previous identity, we have that

$$\mathrm{D}^n \delta_0(f) = (-1)^n \delta_0[f^{(n)}] = (-1)^n f^{(n)}(0)$$

Theorem 8.18 (Chain Rule)**.** *Given $h \in C^\infty(\mathbb{R})$ and $\eta \in \mathcal{K}^\star$, we have that*

$$\mathrm{D}(h\eta) = h'\eta + h\,\mathrm{D}\,\eta$$

Where the distributional derivative is identical to the usual derivative for h and it's the usual distributional derivative for η

Proof. Taken $h \in C^\infty$, $\eta \in \mathcal{K}^\star$ and $f \in \mathcal{K}$ we have

$$(h\eta)(f) = \eta(fh)$$

Therefore, using the identity for derivating a distribution we have that

$$\begin{aligned}\mathrm{D}(h\eta)[f] &= -(h\eta)[f'] = \\ &= -\eta[hf'] = -\eta[(hf)' - h'f] = -\eta[(hf)'] + \eta[h'f] = \\ &= (h\,\mathrm{D}\,\eta)[f] + (h'\eta)[f]\end{aligned}$$

□

Theorem 8.19. *Given $g_n(x) : \mathbb{R} \longrightarrow \mathbb{C}$ a continuous sequence of functions such that $g_n \rightrightarrows g \in C(\mathbb{R})$. Then, taken $g_n \mapsto \varphi_{g_n} \in \mathcal{K}^\star$ the associated distribution, we have that*

$$\varphi_{g_n} \rightharpoonup_\star \varphi_g \in \mathcal{K}^\star$$

Proof. By definition of convergence, we have that, given $f \in \mathcal{K}$, where $\operatorname{supp} f \subset [-a, a]$

$$|\varphi_{g_n}(f) - \varphi_g(f)| \leq \varphi_{|g_n - g|}(|f|) \leq 2a\|g_n - g\|_u \|f\|_u \to 0$$

Therefore, $\varphi_{g_n} \rightharpoonup_\star \varphi_g$ □

Theorem 8.20. *Given $(\eta)_n \in \mathcal{K}^\star$ a sequence of distribution, then if $(\eta)_n \rightharpoonup_\star \eta$ we have that*

$$\mathrm{D}\,\eta_n \rightharpoonup_\star \mathrm{D}\,\eta$$

Proof. By definition, we have

$$\lim_{n\to\infty} \mathrm{D}\,\eta_n(f) = -\lim_{n\to\infty} \eta_n(f') = -\eta(f') = \mathrm{D}\,\eta(f)$$

□

Example 8.2.4 (The Absolute Value Distribution). Taken $g(x) = |x|$ we have that (obviously) $|x| \in L^1_{loc}(\mathbb{R})$, therefore there exists a distribution $\varphi_{|x|} \in \mathcal{K}^\star$ defined as follows

$$\varphi_{|x|}(f) = \int_{\mathbb{R}} |x| f(x)\,\mathrm{d}x$$

We have that the distributional derivative it's actually the function $\operatorname{sgn}(x)$, (in this case, since it's a locally integrable function it coincides with its weak derivative), therefore it's not unusual to see expressions like this

$$\frac{\mathrm{d}}{\mathrm{d}x}|x| = \operatorname{sgn}(x) \quad \text{weakly/distributionally}$$

The proof of this is quite easy. Taken $f \in \mathcal{K}$

$$\begin{aligned}
\mathrm{D}\,\varphi_{|x|}(f) = -\varphi_{|x|}(f') = -\int_{\mathbb{R}} |x| f'(x)\,\mathrm{d}x = \\
&= [xf(x)]_{\mathbb{R}^-} - [xf(x)]_{\mathbb{R}^+} + \int_{\mathbb{R}^+} f(x)\,\mathrm{d}x - \int_{\mathbb{R}^-} f(x)\,\mathrm{d}x = \\
&= \int_{\mathbb{R}} \operatorname{sgn}(x) f(x)\,\mathrm{d}x = \varphi_{\operatorname{sgn}(x)}(f)
\end{aligned}$$

Since we have that

$$\mathrm{D}\,\varphi_g = \varphi_{\mathrm{D}\,g}$$

Where $\mathrm{D}\,g$ is intended as the weak derivative of g, we have that

$$\mathrm{D}\,|x| = \operatorname{sgn}(x)$$

Where in order to emphasize tha this is a weak derivative, one could use the notation

$$\mathrm{D}_{\mathrm{w}}\,|x| = \operatorname{sgn}(x)$$

Note that in literature it's common to use the abuse of notation written beforehand in general and in this particular case.
Also note that if the function we had used would have been an ordinarily derivable function, the weak derivative would have coincided with the ordinary derivative.

Theorem 8.21. *In general, given $g \in C^1(\mathbb{R})$, we can say that $g(|x|)$ has the following weak derivative*

$$\mathrm{D}\, g(|x|) = \mathrm{D}\, g(|x|) \,\mathrm{sgn}(x)$$

Proof. Since $g(|x|) \in L^1_{loc}(\mathbb{R})$ we can say that exists $g \mapsto \varphi_g \in \mathcal{K}^\star$ such that

$$\mathrm{D}\, \varphi_{g(|x|)} = \varphi_{\mathrm{D}\, g(|x|)}$$

Where $\mathrm{D}\, g$ is the weak derivative of $g(|x|)$. Since it's a derivative (and it's also demonstrable) we can use the rules for composite derivation, and knowing that $\mathrm{D}\,|x| = \mathrm{sgn}(x)$ we have

$$\mathrm{D}\, \varphi_{g(|x|)} = \varphi_{\mathrm{D}\, g(|x|)} = \varphi_{\mathrm{D}\, g\, \mathrm{D}\,|x|} = \varphi_{\mathrm{sgn}\, \mathrm{D}\, g}$$

□

Theorem 8.22 (Derivative of the ϑ Distribution)**.** *Given $\vartheta \in \mathcal{K}^\star$ the theta distribution, we define*

$$\vartheta(f) = \int_{\mathbb{R}} f(x) H(x) \,\mathrm{d}x = \int_{\mathbb{R}^+} f(x) \,\mathrm{d}x$$

The derivative of this distribution is

$$\mathrm{D}\, \vartheta = \delta_0$$

Proof. We have $\forall f \in \mathcal{K}$

$$\mathrm{D}\, \vartheta(f) = -\vartheta(f') = -\int_{\mathbb{R}} f'(x) H(x) \,\mathrm{d}x = -\int_{\mathbb{R}^+} f'(x) \,\mathrm{d}x = f(0) = \delta_0(f)$$

Therefore

$$\mathrm{D}\, \vartheta = \delta_0$$

Or, using a common abuse of notation

$$\vartheta = \vartheta(x) = H(x), \qquad \mathrm{D}\, H(x) = \delta(x)$$

□

Notation (Piecewise Derivative)**.** Given $g : \mathbb{R} \longrightarrow \mathbb{C}$ a piecewise differentiable function, we define the differential operator $\mathrm{D}_\circ$ as follows

$$\mathrm{D}_\circ\, g(x) = \begin{cases} g'(x) & \exists g'(x) \\ 0 & \nexists g'(x) \end{cases} \tag{8.16}$$

Theorem 8.23. *Given $f : \mathbb{R} \longrightarrow \mathbb{C}$ a piecewise differentiable function and $S_f := \{u_1, \cdots, u_k\}$ isolated singularities, then*

$$\mathrm{D}\, f(x) = \mathrm{D}_\circ\, f(x) + \sum_{i=1}^{\infty} \Delta f(u_i) \delta(u_i)$$

Proof. Taken $S_f := \{u\}$ without loss of generality, we have that

$$\begin{aligned}\lim_{x\to u^+} f(x) &= f(u^+)\\ \lim_{x\to u^-} f(x) &= f(u^-)\\ \Delta f(u) &= f(u^+) - f(u^-)\end{aligned}$$

And, $\forall g \in \mathcal{K}$, taken $f \mapsto \varphi_f \in \mathcal{K}^\star$

$$\mathrm{D}\,\varphi_f(g) = -\varphi_f(g') = -\int_{-\infty}^{u} g'(x)f(x)\,\mathrm{d}x - \int_{u}^{\infty} g(x)f(x)\,\mathrm{d}x$$

Integrating by parts and rebuilding the definition of $\Delta f(u)$ we have

$$\mathrm{D}\,\varphi_f(g) = g(u)\Delta f(u) + \int_{\mathbb{R}} g(x)\,\mathrm{D}_\circ f(x)\,\mathrm{d}x = \varphi_{\mathrm{D}_\circ g}(f) + \Delta f(u)\delta_u(f)$$

And therefore

$$\mathrm{D}\,f(x) = \mathrm{D}_\circ f + \Delta f(u)\delta(u)$$

□

Example 8.2.5 (Deriving the Sign Function). Take $a \in \mathbb{R}$ and the function $\operatorname{sgn}(x-a)$, using the previous formula we have that $S_{\operatorname{sgn}(x-a)} = \{a\}$ and $\Delta\operatorname{sgn}(a) = 2$, therefore

$$\mathrm{D}\operatorname{sgn}(x-a) = \mathrm{D}_\circ \operatorname{sgn}(x-a) + 2\delta(x-a)$$

Since $\mathrm{D}_\circ \operatorname{sgn}(x-a) = 0$ we have finally

$$\mathrm{D}\operatorname{sgn}(x-a) = 2\delta(x-a)$$

And equivalently

$$\mathrm{D}\operatorname{sgn}(a-x) = -2\delta(x-a)$$

Example 8.2.6 (A General Piecewise Differentiable Function). Take the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined as follows

$$g(x) = \begin{cases} 1 & x < 0 \\ x-2 & x \geq 0 \end{cases}$$

We have that

$$\mathrm{D}_\circ g(x) = H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Since the discontinuity is in the origin and we have $\Delta g(0) = -3$ we have

$$\mathrm{D}\,g(x) = H(x) - 3\delta(x)$$

Example 8.2.7 (Derivative of the Floor Function). Given the floor function $\lfloor x \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$ we have that $S_{\lfloor x \rfloor} = \mathbb{Z}$, $\Delta\lfloor x \rfloor = 1$ and $\mathrm{D}_\circ \lfloor x \rfloor = 0$ therefore

$$\mathrm{D}\lfloor x \rfloor = \sum_{k\in\mathbb{Z}} \delta(x-k) = \sum_{k=-\infty}^{\infty} \delta(x-k)$$

Given instead a different function, $\lfloor x^2 \rfloor$ we have that $\mathrm{D}_\circ \lfloor x^2 \rfloor = 0$ and $S_{\lfloor x^2 \rfloor} = \left\{ k \in \mathbb{Z} \middle| \pm \sqrt{k} \right\} \subset \mathbb{R}$ with $\Delta \lfloor \pm\sqrt{k} \rfloor = \pm 1$, therefore

$$\mathrm{D}\lfloor x^2 \rfloor = \sum_{k \geq 1} \delta(x - \sqrt{k}) - \delta(x - \sqrt{-k}) = \sum_{k=1}^{\infty} \delta(x - \sqrt{k}) - \delta(x - \sqrt{-k})$$

Theorem 8.24. *Given $h \in C^\infty(\mathbb{R})$ a smooth function and $a \in \mathbb{R}$, we have that*

$$h(x)\,\mathrm{D}^n\,\delta(x-a) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} h^{(k)}(a)\,\mathrm{D}^{n-k}\,\delta(x-a)$$

Proof. Taken $f \in \mathcal{K}$ we have that

$$(h(x)\,\mathrm{D}^n\,\delta_a)(f) = \mathrm{D}^n\,\delta_a(hf) = \delta_a\left((hf)^{(n)}\right)$$

Using now Leibnitz's composite derivation rule, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} h(x)f(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}^k h}{\mathrm{d}x^k} \frac{\mathrm{d}^{n-k} f}{\mathrm{d}x^{n-k}}$$

We have that

$$\begin{aligned}(h(x)\,\mathrm{D}^n\,\delta_a)(f) &= \delta_a\left[(-1)^n \sum_{k=0}^{n} \binom{n}{k} h^{(k)}(x) f^{(n-k)}(x)\right] = \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k h^{(k)}(x)\,\mathrm{D}^{n-k}\,\delta_a(f)\end{aligned}$$

□

Example 8.2.8. Take as an example the following distributional derivative

$$\mathrm{D}^7\left(xe^x\,\mathrm{D}^2\,\delta_0\right)$$

We have from that formula that

$$xe^x\,\mathrm{D}^2\,\delta_0 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}(xe^x)\delta_0 = \frac{\mathrm{d}}{\mathrm{d}x}(e^x + xe^x)\delta_0 = (2+x)e^x\delta_0$$

And therefore $h'(0) = 1$, $h''(0) = 2$
Which means that

$$xe^x\,\mathrm{D}^2\,\delta_0 = \binom{2}{2} h(0)\,\mathrm{D}^2\,\delta_0 - \binom{2}{1} h'(0)\,\mathrm{D}\,\delta_0 + \binom{2}{0} h''(0)\delta_0$$

Inserting the values, we get

$$xe^x\,\mathrm{D}^2\,\delta_0 = 2\,\mathrm{D}\,\delta_0 - 2\delta_0$$

The derivative is now trivial, and we get

$$\mathrm{D}^7\left(xe^x\,\mathrm{D}^2\,\delta_0\right) = 2\,\mathrm{D}^7\left(\delta_0 - \mathrm{D}\,\delta_0\right) = 2\left(\mathrm{D}^7\,\delta_0 - \mathrm{D}^8\,\delta_0\right)$$

Theorem 8.25. *Given $k, m \in \mathbb{N}$ and a function $f \in C^k(\mathbb{R})$ we have that*

$$\mathrm{D}^k\left(x^m f\right)(0) = \begin{cases} 0 & k < m \\ \binom{k}{m} m! f^(k-m)(0) & k \geq m \end{cases}$$

And

$$\mathrm{D}^i(x^m)(0) = m!\delta^i_m$$

Proof. We apply immediately the Leibnitz chain rule and we have

$$\mathrm{D}^k\left(x^m f\right)(0) = \sum_{j=0}^{m} \binom{k}{j} f^{(k-j)}(0) m! \delta_{jm} = m! \binom{k}{m} f^{(k-m)}(0) \quad \forall k \geq m$$

□

Theorem 8.26 (Properties of the $\mathrm{PV}(x^{-n})$ Distribution)**.** *Here there will be listed some properties of the* $\mathrm{PV}(x^{-n})$ *distribution*

1. $\mathrm{D}\log|x| = \mathrm{PV}(x^{-n})$
2. *Given $n, m \in \mathbb{N}$, then*
$$x^m \mathrm{PV}\left(\frac{1}{x^n}\right) = \begin{cases} x^{m-n} & m \geq n \\ \mathrm{PV}\left(\frac{1}{x^{n-m}}\right) & m < n \end{cases}$$
3. $\mathrm{D}^m\,\mathrm{PV}(x^{-n}) = (-m)!\mathrm{PV}(x^{-n-m})$

Proof. 1) Taken $\log|x| \in L^1_{loc}(\mathbb{R})$ we know that $\exists \varphi_{\log|x|} \in \mathcal{K}^\star$ such that $\log|x| \mapsto \varphi_{\log|x|}$ and therefore we can write as follows the previous derivative, that $\forall f \in \mathcal{K}$

$$\mathrm{D}\,\varphi_{\log|x|}(f) = -\varphi_{\log|x|}(f') = -\lim_{\epsilon\to 0^-}\int_{-\infty}^{\epsilon} f'(x)\log(-x)\,\mathrm{d}x - \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\infty} f(x)\log(x)\,\mathrm{d}x$$

Integrating by parts we get

$$\mathrm{D}\,\varphi_{\log|x|}(f) = \lim_{\epsilon\to 0^+}\log\epsilon\left(f(\epsilon) - f(-\epsilon)\right) + \lim_{\epsilon\to 0^+}\int_{|x|\geq\epsilon}\frac{f(x)}{x}\,\mathrm{d}x = \mathrm{PV}\left(\frac{1}{x^n}\right)(f)$$

Or

$$\mathrm{D}\log|x| = \mathrm{PV}\left(\frac{1}{x^n}\right)$$

2) Taken $f \in \mathcal{K}$

$$\left(x^m \mathrm{PV}\left(\frac{1}{x^n}\right)\right)(f) = \mathrm{PV}\left(\frac{1}{x^n}\right)(x^m f) = \lim_{R\to\infty}\int_{-R}^{R}\frac{1}{x^n}\left(x^m f(x) - \sum_{k=0}^{n-1}\frac{\mathrm{D}^k(x^m f)(0)}{k!}x^k\right)\mathrm{d}x$$

Taken $m \geq n$ we have that the sum is null $\forall k \leq m$, therefore since $k \leq n-1 < m$ it sums to 0
Therefore

$$\lim_{R\to\infty} I_R^n(x^m f) = \mathrm{PV}\left(\frac{1}{x^n}\right)(x^m f) = \int_{\mathbb{R}} x^{m-n} f(x)\,\mathrm{d}x = \varphi_{x^{m-n}}(f)$$

Taken $m < n$ we have that

$$\sum_{k=0}^{n-1} \frac{x^k}{k!}\,\mathrm{D}^k(x^m f)(0) = \sum_{k=m}^{n-1} \binom{k}{m} \frac{m!}{k!} f^{(k-m)}(0) = \sum_{k=0}^{n-m-1} \frac{f^{(k)}(0)}{k!} x^k$$

Therefore

$$\lim_{R\to\infty} I_R^n(x^m f) = \lim_{R\to\infty}\lim_{\epsilon\to 0} \int_{-R}^{R} \frac{1}{x^{n-m}} \left(f(x) - \sum_{k=0}^{n-m-1} \frac{f^k(0)}{k!} x^k \right) \mathrm{d}x = \mathrm{PV}\left(\frac{1}{x^{n-m}}\right)(f)$$

3) Taken $n \in \mathbb{N}$, $n > 0$, $m = 1$ we have

$$\mathrm{D}\,\mathrm{PV}\left(\frac{1}{x^n}\right)(f) = -\mathrm{PV}\left(\frac{1}{x^n}\right)(f') = -\lim_{R\to\infty} I_R^n(f')$$

Integrating by parts we have

$$\begin{aligned} I_R^n(f') &= \left[\frac{1}{x^n} \left(f(x) - f(0) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{(k+1)!} x^{k+1} \right) \right]_{-R}^{R} + \\ &+ \int_{-R}^{R} \frac{n}{x^{n+1}} \left(f(x) - f(0) - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(0)}{(k+1)!} x^k \right) \mathrm{d}x = \\ &= \frac{1}{R^n} \left(f(R) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} R^k \right) - \frac{1}{(-R)^n} \left(f(-R) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} (-R)^k \right) + n I_R^{n+1}(f) \end{aligned}$$

Since $\operatorname{supp} f \subset [-b, b]$ we have two cases, $R > b$ and $R < b$.
Supposing $R > b$ we have

$$I_R^n(f') = n I_R^{n+1}(f) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} \left(\frac{1}{(-R)^{n-k}} - \frac{1}{R^{n-k}} \right)$$

Taken a new index $j = n - k$ such that $j \mod 2 = 0$ we have

$$I_R^n(f') = n I_R^{n+1} - 2 \sum_{j=0}^{n} \frac{f^{(j)}(0)}{j!} \frac{1}{R^{n-k}} \longrightarrow -n\mathrm{PV}\left(\frac{1}{x^{n+1}}\right)(f)$$

In case that $m > 1$, by iteration we get

$$\mathrm{D}^m\,\mathrm{PV}\left(\frac{1}{x^n}\right) = (-m)!\mathrm{PV}\left(\frac{1}{x^{n+m}}\right)$$

$\square$

§§ 8.2.4 A Physicist's Trick, The Dirac δ "Function"

Definition 8.2.18 (Composite Delta)**.** Supposing that a Dirac δ exists also as a function, we can imagine describing it as a composite function $(\delta \circ f)(x) = \delta(f(x))$.
Watch out, since $\delta(f(x)) \neq \delta_0(f)$

Definition 8.2.19 (Giving it Some Meaning)**.** Taken $g_n \in L^1_{loc}(\mathbb{R})$ a sequence of functions such that $g_n \in C[-a_n, a_n]$ with $a_n > 0$, $a_n \to 0$, we have that $\exists! g_n \mapsto \varphi_{g_n}$ such that $\varphi_{g_n} \rightharpoonup_\star \delta_0$
A good choice for this sequence would be the Gaussian function γ_n defined as follows

$$\gamma_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \tag{8.17}$$

Then, we have that $\varphi_{\gamma_n} \rightharpoonup_\star \delta_0$, or written in a clear abuse of notation with non-existing functions

$$\lim_{n\to\infty} \gamma_n(x) = \delta(x)$$

Which, actually means that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \gamma_n(x) f(x) \, \mathrm{d}x = f(0) = \delta_0(f) \quad \forall f \in \mathcal{K}$$

Then, letting $b(x) \in C^1(\mathbb{R})$ we can define the following quantity

$$\int_{\mathbb{R}} f(x)\delta(b(x)) \, \mathrm{d}x = \lim_{n\to\infty} \int_{\mathbb{R}} \gamma_n(b(x)) f(x) \, \mathrm{d}x \quad \forall f \in \mathcal{K}$$

Theorem 8.27. *Let $g \in C^1(\mathbb{R})$ be a function with isolated and simple zeros $Z_g := \{x_1, x_2, x_3, \cdots\}$, then $\forall f \in \mathcal{K}$ we have*

$$\lim_{n\to\infty} \int_{\mathbb{R}} \gamma_n(g(x)) f(x) \, \mathrm{d}x = \int_{\mathbb{R}} f(x)\delta(g(x)) \, \mathrm{d}x = \sum_{x_k \in Z_g} \frac{\delta_{x_k}(f)}{|g'(x_k)|} \tag{8.18}$$

Proof. Taken $g(x) \in C^1(\mathbb{R})$ such that $g(x) \geq g(y)\ \forall x \geq y$, supposing that $Z_g = \{x_0\}$ we have, since it's a simple zero, that $g'(x_0) \neq 0$
Taking in the previous integral the substitution $y = g(x)$, we have that $x = g^{-1}(y)$ and $\mathrm{d}x = \mathrm{D}\, g^{-1}(y) \, \mathrm{d}y$, therefore

$$I = \lim_{n\to\infty} \int_{\mathbb{R}} \gamma_n(g(x)) f(x) \, \mathrm{d}x = \lim_{n\to\infty} \int_{\inf(g)}^{\sup(g)} \gamma_n(y) \frac{f(g^{-1}(y))}{g'(g(y))} \, \mathrm{d}y$$

Evaluating the integral we have

$$I = \frac{f(g^{-1}(0))}{g'(g^{-1}(0))} = \frac{f(x_0)}{g'(x_0)} = \frac{1}{|g'(x_0)|} \delta_{x_0}$$

This is easily generalizable if there is more than one zero □

Example 8.2.9 (Composition with a Constant). Take $\delta(ax)$ with $g(x) = ax$ and $a \neq 0$. Since $Z_g = \{0\}$ using the previous formula we have $g'(0) = a$ and therefore

$$\delta(ax) = \frac{1}{|a|} \delta(x)$$

Example 8.2.10. Taken $g(x) = \text{atan}(x)$ we have that $Z_g := \{0\}$ and

$$g'(x) = \frac{1}{1+x^2} \quad g'(0) = 1$$

Therefore

$$\delta(\text{atan}(x)) = \delta(x)$$

Example 8.2.11 (Composition with a Polynomial). Take $g(x) = (x^2 - a^2)$, then $Z_g := \{-a, a\}$ and $g'(\pm a) = \pm 2a$, therefore, if $a \neq 0$

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\left(\delta(x-a) + \delta(x+a)\right)$$

Example 8.2.12 (Composition with an Exponential). Taken $g(x) = e^x$ we have that $Z_g = \{\}$ therefore $\delta(e^x) = 0$

Example 8.2.13 (Composition with a Trigonometric Function). Taken now $g(x) = \cos(x)$ we have that $Z_g = \{\, k \in \mathbb{Z} |\, k\pi/2\}$ and $g'(x) = -\sin(x)$, therefore

$$g'(x_k) = -\sin\left(\frac{k\pi}{2}\right) = (-1)^{k+1}$$

Therefore

$$\delta(\cos(x)) = \sum_{k\in\mathbb{Z}} \delta\left(x - \frac{k\pi}{2}\right) = \sum_{k=-\infty}^{\infty} \delta\left(x - \frac{k\pi}{2}\right)$$

Example 8.2.14 (Calculating an Integral with the Composite Delta). Take the following integral

$$\int_{\mathbb{R}} e^{-|x|}\delta(\sin(2x))\,\mathrm{d}x$$

In order for solving it we expand the composite delta. We have that $Z_g = \{\, k \in \mathbb{Z} |\, k\pi/2\}$ and $g'(x_k) = (-1)^k 2$, therefore

$$\int_{\mathbb{R}} e^{-|x|}\delta(\sin(2x))\,\mathrm{d}x = \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{-|x|}\delta\left(x - \frac{k\pi}{2}\right) = \frac{1}{2}\sum_{k=-\infty}^{\infty} e^{-\left|\frac{k\pi}{2}\right|}$$

Fixing the sum and summing, we have

$$\int_{\mathbb{R}} e^{-|x|}\delta(\sin(2x))\,\mathrm{d}x = \sum_{k\in\mathbb{N}} e^{-\frac{k\pi}{2}} - \frac{1}{2} = \frac{1}{1-e^{\frac{\pi}{2}}} - \frac{1}{2}$$

§ 8.3 Integral Representation of Distributions

§§ 8.3.1 Dirac Delta and Heaviside Theta

Theorem 8.28 (Integral Representation of the Delta Distribution)**.** *Taken the non-existent $\delta(x)$ function, we can represent it as follows*

$$\delta(x-y) = \frac{1}{2\pi}\iint_{\mathbb{R}^2} e^{ik(x-y)} f(x)\,\mathrm{d}k\,\mathrm{d}x \tag{8.19}$$

Which means

$$\delta_y(f) = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{ik(x-y)} f(x) \, \mathrm{d}k \, \mathrm{d}x$$

Theorem 8.29 (Integral Representation of the Theta Distribution)**.** *Taken $\vartheta(x)$ the Heaviside regular distribution we can write $\forall x \in \mathbb{R} \setminus \{0\}$*

$$\vartheta(x) = \frac{1}{2\pi i} \, \mathrm{PV} \int_{\mathbb{R}} \frac{e^{ikx}}{k - i\epsilon} \, \mathrm{d}k \tag{8.20}$$

Proof. This equation has two cases, one with $x > 0$ and one with $x < 0$. Therefore taking $x > 0$ we have

$$I(x) = \mathrm{PV} \int_{\mathbb{R}} \frac{e^{ikx}}{k - i\epsilon} \, \mathrm{d}k$$

This integral can be evaluated using the residue theorem. We have defining the function $f(k)$ as follows

$$f(k) = \frac{1}{k - i\epsilon}$$

The integral can be seen as follows

$$I(x) = \mathrm{PV} \int_{\mathbb{R}} e^{ikx} f(k) \, \mathrm{d}k$$

This can be written as a complex integral, writing the transformation $x \to z$ with $z \in \mathbb{C}$

$$I(x) \to J(x) = \lim_{R \to \infty} \oint_{\gamma_R} \frac{e^{izx}}{z - i\epsilon} \, \mathrm{d}z$$

Where the path is the following

$$\{\gamma_R\} = C_R^+ \cup [-R, R]$$

Writing it all out explicitly we have that

$$J(x) = \lim_{R \to \infty} \int_{C_R} \frac{e^{izx}}{z - i\epsilon} \, \mathrm{d}z + \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{izx}}{z - i\epsilon} \, \mathrm{d}z = J_R + I(\mathfrak{Re} z)$$

Due to the Jordan lemma we know for sure that $J_R = 0$ therefore we have that, writing $\mathfrak{Re}(z) = x$, thanks to the residue theorem

$$J(x) = I(x) = 2\pi i \sum_{z_k \in S_f} \operatorname*{Res}_{z = z_k} \frac{e^{izx}}{z - i\epsilon}$$

The set of singularities of f is formed by a single point $S_f = \{i\epsilon\}$ which is a simple pole, and therefore, since $i\epsilon \in \{\gamma_R\}^\circ$ we have

$$J(x) = 2\pi i \operatorname*{Res}_{z = i\epsilon} \frac{e^{izx}}{z - i\epsilon} = 2\pi i \lim_{z \to i\epsilon} (z - i\epsilon) \frac{e^{izx}}{z - i\epsilon} = e^{-\epsilon x}$$

Therefore, we get

$$J(x) = 2\pi i e^{-\epsilon x}$$

For getting back to the first integral, we have

$$\frac{1}{2\pi i}I(x) = \lim_{\epsilon \to 0^+} e^{-\epsilon x} = 1 = \vartheta(x) \quad x > 0$$

In the second case, for $x < 0$ we have that the new curve will be

$$\{\gamma_R^-\} = C_R^- \cup [-R, R]$$

Since $S_f \not\subset \{\gamma_R^-\}^\circ$ we have thanks to Cauchy-Goursat that

$$\frac{1}{2\pi i}I(x) = \frac{1}{2\pi i}J^-(z) = 0 = \vartheta(x) \quad x < 0$$

Proving our assumption □

§§ 8.3.2 Distributions in $\mathbb{R}^n$

Definition 8.3.1 (Scalar Test Fields)**.** Given a scalar field $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ one can define the space of scalar test fields $\mathcal{K}(\mathbb{R}^n)$ as follows

$$\mathcal{K}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n) \tag{8.21}$$

A function $f \in C_c^\infty(\mathbb{R}^n)$ is a function such that $\operatorname{supp} f \subset A \subset \mathbb{R}^n$ with A a compact set and $\exists \partial^\alpha f \quad \forall |\alpha| \in \mathbb{N}$

Definition 8.3.2 ($\mathcal{K}(\mathbb{R}^n)-$Convergence)**.** Given $f_n \in \mathcal{K}(\mathbb{R}^n)$ a sequence of scalar fields. f_n converges to $f \in \mathcal{K}(\mathbb{R}^n)$ if

1. $\exists A \subset \mathbb{R}^n$ compact set such that $f_n\left(A^c\right) = \{0\}$
2. $\forall \alpha \in \mathbb{N}^n$ multi-index $\partial^\alpha f_n \rightrightarrows \partial^\alpha f, \forall x \in A$

It's indicated as

$$f_n \rightharpoonup_{\mathcal{K}_\mathrm{n}} f$$

Definition 8.3.3 (Multidimensional Distribution)**.** A $n-$*dimensional distribution* is a continuous functional $\varphi : \mathcal{K}(\mathbb{R}^n) \longrightarrow \mathbb{C}$ such that

$$\forall g_n \rightharpoonup_{\mathcal{K}_\mathrm{n}} g \quad \varphi(g_n) \to \varphi(g) \tag{8.22}$$

Then $\varphi \in \mathcal{K}^\star(\mathbb{R}^n)$

Theorem 8.30 (Operations in $\mathcal{K}^\star(\mathbb{R}^n)$)**.** *We can define the usual distributional operations in $\mathcal{K}(\mathbb{R}^n)$. The usual operations defined for $\mathcal{K}^\star$ are defined in the same way, the only difference is given by the definition of the derivative Given $\alpha \in \mathbb{N}^n$, $f \in \mathcal{K}(\mathbb{R}^n)$, $\varphi \in \mathcal{K}^\star(\mathbb{R}^n)$*

$$\partial^\alpha \varphi(f) = (-1)^{|\alpha|}\varphi(\partial^\alpha f) \tag{8.23}$$

Example 8.3.1 (Laplacian of a Distribution). Taken $\partial^\alpha = \partial^2 = \partial_\mu \partial^\mu$

$$\partial_\mu \partial^\mu \varphi(f) = \varphi(\partial_\mu \partial^\mu f)$$

Remark (Local Integrability). The local integrability of some functions is different in $\mathbb{R}^n$. In fact taken the spherical $(n-1)$-dimensional coordinate transformation we have

$$d^n x = r^{n-1}\,\mathrm{d}r\,\mathrm{d}S_{n-1}$$

And taken the function $g(x^\nu) = \|x^\mu\|_\mu^{-a}$ we get

$$\int_{\mathbb{R}^n} \frac{f(x)}{\|x^\mu\|_\mu^a}\,\mathrm{d}^n x = \int_{\mathbb{R}} \int_{S_{n-1}} f(x)\,\mathrm{d}S_{n-1} \frac{r^{n-1}}{r^a}\,\mathrm{d}r$$

Which means that $\|x^\mu\|_\mu^{-a} \in L^1_{loc}(\mathbb{R}^n) \quad \forall a < n$

§ 8.4 Some Applications in Physics

§§ 8.4.1 Electrostatics

Example 8.4.1 (Point Charge). Taken q a point charge in the origin $(0,0,0)$ we define the potential generated by this charge as follows (in cgs)

$$\partial_\mu \partial^\mu V(x^\nu) = -4\pi\rho(x)$$

Where $\rho(x)$ is the charge density, defined as

$$\rho(x) = q\delta^3(x^\nu)$$

With δ^3 being the 3−d Dirac delta
Therefore

$$\partial_\mu \partial^\mu \left(\frac{1}{\|x^\mu\|_\mu} \right) = -4\pi\delta^3(x^\nu)$$

Which means actually, that $\forall f \in \mathcal{K}(\mathbb{R}^3)$

$$\iiint_{\mathbb{R}^3} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu}\,\mathrm{d}^3 x = -4\pi f(0)$$

Proof. Since $f \in \mathcal{K}(\mathbb{R}^3)$ we have that $\exists R > 0 \,:\, f(x^\mu) = 0 \forall x \notin B_{\frac{R}{2}}(0)$
Therefore we get that

$$\iiint_{\mathbb{R}^3} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu}\,\mathrm{d}^3 x = \lim_{\epsilon \to 0} \int_{\epsilon \le \|x^\mu\|_\mu \le R} \frac{\partial_\mu \partial^\mu f}{\|x^\mu\|_\mu}\,\mathrm{d}^3 x$$

Taken $A_{\epsilon R}(0) = \left\{ x^\mu \in \mathbb{R}^3 \middle| \, \epsilon \le \|x^\mu\|_\mu \le R \right\}$ We have that the second integral becomes, using Stokes' theorem

$$\iiint_{A_{\epsilon R}} f(x)\partial_\mu \partial^\mu \left(\frac{1}{\|x^\mu\|_\mu} \right) f(x)\,\mathrm{d}^3 x + \iint_{\partial A_{\epsilon R}} \left[\frac{1}{\|x^\mu\|_\mu} \partial_\mu f - f\partial_\mu \left(\frac{1}{\|x^\mu\|_\mu} \right) \right] n^\mu\,\mathrm{d}\sigma$$

Changing to spherical coordinates we have that

$$\partial_\mu \partial^\mu \left(\frac{1}{\|x^\mu\|_\mu} \right) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \left(\frac{1}{r} \right) \right) = 0$$

Therefore we get

$$\iiint_{A_{\epsilon R}} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, \mathrm{d}^3 x = - \iint_{\|x^\mu\|_\mu = \epsilon} \left(\frac{1}{r} \partial_\mu f - f \partial_\mu \left(\frac{1}{r} \right) \right) r^\mu \, \mathrm{d}S_2$$

Which, taken $\partial_\mu r^{-1} = -\hat{r}^\mu r^{-2}$ we have

$$\iiint_{A_{\epsilon R}} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, \mathrm{d}^3 x = - \iint_{\|x^\mu\|_\mu = \epsilon} \hat{r}^\mu \partial_\mu f r \, \mathrm{d}S_2 - \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, \mathrm{d}S_2$$

We have for the first integral that

$$\left\| \iint_{\|x^\mu\|_\mu = \epsilon} \hat{r}^\mu \partial_\mu f r \, \mathrm{d}S_2 \right\| \leq 4\pi\epsilon \sup_{\mathbb{R}^3} \|\partial_\mu f\|$$

And for the second

$$\left\| \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, \mathrm{d}S_2 - 4\pi f(0) \right\| \leq 4\pi \sup_{\|x^\mu\|_\mu = \epsilon} |f(x^\nu) - f(0)|$$

Since $\epsilon \to 0^+$ and since $f \in \mathcal{K}(\mathbb{R}^3)$ we have that the first and second integral converge to 0, and therefore

$$\iiint_{\mathbb{R}^3} \frac{1}{\|x^\mu\|_\mu} \partial_\mu \partial^\mu f \, \mathrm{d}^3 x = - \lim_{\epsilon \to 0} \iint_{\|x^\mu\|_\mu = \epsilon} f(x) \, \mathrm{d}S_2 = -4\pi f(0)$$

□

9 Ordinary Differential Equations

§ 9.1 Existence of Solutions

You might test that assumption at your convenience[1]

Definition 9.1.1 (Differential Equation)**.** Given $y \in C^m(I), \quad I \subseteq \mathbb{R}$ we define a *differential equation of order* m the following

$$F(x, y, y', \cdots, y^{(m)}) = 0$$

If the equation can be rewritten as follows

$$\frac{\mathrm{d}^m y}{\mathrm{d}x^m} = f(x, y, \cdots, y^{(m-1)})$$

It's called in *normal form.*
If only total derivatives appear, the differential equation is called *ordinary*, whereas if also partial derivatives appear, the differential equation is called *partial.*

Theorem 9.1 (Reduction of Order)**.** *Given an ODE (ordinary differential equation) of order* m*, one can reduce the order of the equation through a mapping to* $\mathbb{R}^m$*, where*

$$(y, y', \cdots, y^{(m)}) \mapsto y^\mu(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(m)}(x) \end{pmatrix}$$

And $f(x, y, \cdots, y^{(m)}) \mapsto f^\mu(x, y^\mu)$*.*
The equation becomes a first order differential equation

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}x} = f^\mu(x, y^\mu)$$

[1] *Captain Picard, Star Trek: The Next Generation*

If f is linear, this system can be expressed in matrix form, where

$$\frac{\mathrm{d}y^\mu}{\mathrm{d}x} = A^\mu_\nu y^\mu$$

Definition 9.1.2 (Cauchy Problem)**.** A *Cauchy* or *initial value problem* is defined as the following system

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \tag{9.1}$$

Where $y_0, t_0 \in \mathbb{R}$ are known values

Theorem 9.2. *Given the Cauchy problem* (9.1) *we say that $y(x) \in C^1$ is a solution of the system if and only if is a continuous solution of the following integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,\mathrm{d}s \tag{9.2}$$

Proof. Supposing $y \in C^1$ we have that for the fundamental theorem of integral calculus

$$y(x) = y(x_0) + \int_{x_0}^{x} y'(s)\,\mathrm{d}s = y_0 + \int_{x_0}^{x} f(s, y(s))\,\mathrm{d}s$$

Instead, considering $f \circ y$ we have that the new composed function must be continuous, therefore we have for the fundamental theorem of integral calculus that $y(x) \in C^1$ and that

$$y(x_0) = y_0 + \int_{x_0}^{x_0} f(s, y(s))\,\mathrm{d}s = y_0$$

And therefore

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\int_{x_0}^{x} f(s, y(s))\,\mathrm{d}s = f(x, y(x))$$

Which gives back the thesis □

Theorem 9.3 (Picard-Lindelöf-Cauchy Existence)**.** *Let $A \subset \mathbb{R}^2$ with $f : A \longrightarrow \mathbb{R}$ a continuous function which defines an ODE* (9.1) *where*

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,\mathrm{d}s$$

Let $a, b > 0$ and take $R = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b] = I_a \times J_b \subset A$. Supposing f Lipschitz continuous on the second variable, i.e.

$$\exists L > 0 \, : \, |f(x, u) - f(x, w)| \leq L|u - w| \quad \forall (x, u), (x, w) \in R$$

Then, if $M = \max_R |f(x, y)|$

$$\exists \epsilon > 0 \, : \, \exists! y(x) \in C^1(B_\epsilon(y_0))$$

Where $\epsilon = \min\{a, b/M, 1/L\}$
Therefore, there is an unique local solution $\overline{y}(x)$ to the ODE

Proof. We firstly define a Banach space $(X, \|\cdot\|_u)$ where $X := \{y \in C(I_\epsilon) | \, \|y(x) - y_0\|_u \leq b\}$.
Define the *Picard operator* as follows

$$\hat{T} : X \longrightarrow X$$

Where

$$\hat{T}(w) = y_0 + \int_{x_0}^{x} f(s, w(s)) \, \mathrm{d}s = z(x)$$

We have

$$\left\|\hat{T}(y) - \hat{T}(z)\right\|_u = \left\|\hat{T}(y - z)\right\|_u = \left\|\int_{x_0}^{x} f(s, u(s)) - f(s, v(s)) \, \mathrm{d}s\right\|_u \leq L\epsilon\|u - v\|_u \quad \forall u, v, y, z \in X$$

Since $x \in I_\epsilon$ is completely arbitrary we have therefore, putting $d(x, y) = \|x - y\|_u$

$$d(y, z) = d(\hat{T}(y), \hat{T}(z)) \leq L\epsilon d(u, v)$$

Since $L\epsilon \in (0, 1)$, $\quad \hat{T}$ is a contractor and it has a single fixed point in X.
Taking this fixed point as $\overline{y}(x)$ we have

$$\hat{T}(\overline{y}) = y_0 + \int_{x_0}^{x} f(s, \overline{y}(s)) \, \mathrm{d}s = \overline{y}(x)$$

Which is our searched solution, and it's unique □

Definition 9.1.3 (Maximal and Global Solutions)**.** Given $y(x)$ a solution to the Cauchy problem (9.1) we define a *maximal solution* $y_m(x)$ a solution for which $y_m(x) = y(x) \quad \forall x \in I_\epsilon$ and which still solves the problem in a set $(a, b) \supset I_\epsilon$, and (a, b) is the biggest set for which $y_m(x)$ solves the ODE. If y_m is defined $\forall x \in \mathbb{R}$, the solution is called *global*

Theorem 9.4 (Prolungability of Solutions)**.** *Given the ODE* (9.1) *and let* $f : (a, b) \times \mathbb{R} \longrightarrow \mathbb{R}$*. Let* $c_1, c_2 \in K \subset (a, b)$ *with* K *a compact set, such that*

$$|f(x, y)| \leq c_1 + c_2|y(x)| \quad x \in K, \ \forall y \in \mathbb{R}$$

Then the solution can be extended in the whole set (a, b)*. The previous statement means that* f *is sublinear in the second variable*

Theorem 9.5. *Let* $y_m(x)$ *be a maximal solution to* (9.1) *defined on* (a, b)*. Then* $\forall K \subset A \quad \exists \delta > 0 : \forall x \notin (a + \delta, b - \delta), \ (x, y_m(x)) \notin K$

Theorem 9.6. *Taken* $y(x)$ *a solution to the ODE* (9.1)*. If* $\exists c > 0$ *such that*

$$|y(x)| \leq c \quad \forall t \in A$$

Then $\exists y_m(x) : (a, b) \longrightarrow \mathbb{R}$

Lemma 9.1.1 (Peano-Gronwall Inequality)**.** Let $\varphi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function $\varphi \in C(I)$, if

$$|\varphi(t)| \leq c + L\left|\int_{t_0}^{t} |\varphi(s)| \, \mathrm{d}s\right|$$

Then the following inequality holds

$$|\varphi(t)| \leq ce^{L|t - t_0|}$$

Therefore, if $y'(t) = f(t, y)$ is an ODE, taken $f(t, y)$ Lipschitz continuous on the second variable and chosen $\varphi(t) = u(t) - v(t)$ where u, v are two solutions, we have that

$$|f(t, u) - f(t, v)| \leq ce^{L|t - t_0|}$$

§ 9.2 Common Solving Methods

Definition 9.2.1 (Separable ODE)**.** Given $y'(t) = f(t, y(t))$ an ODE we define a *separable ODE* a differential equation such that

$$y'(t) = f(t, y(t)) = g(t)h(y)$$

Method 1 (Separation of Variables)**.** Suppose that we have the previous ODE, then, thanks to the Picard-Lindelöf-Cauchy theorem we can immediately say that

1. There is at least one local solution in $B_\epsilon(t_0)$ if $f \in C$, i.e. if $g, h \in C$
2. There is a unique local solution in $B_\epsilon(t_0)$ if $\partial_y f \in C$, i.e. if $h \in C^1$

The first thing to do is finding all constant solution to the ODE, called the *equilibrium solutions*. Then, a general solution can be found by direct integration, where we're integrating the following differential form

$$\frac{1}{h(y)}\,\mathrm{d}y = g(t)\,\mathrm{d}t$$

Integrating, and putting $H(t), G(t)$ as the two primitives, we can say that

$$H(y) = G(t) + H(y_0) - G(t_0)$$

Where $c = H(y_0) - G(t_0)$ is the integration constant in terms of the initial values, if given

Example 9.2.1. Take the following initial value problem

$$\begin{cases} w'(x) = w^2(x) \\ w(0) = 1 \end{cases}$$

We have $f(x, w) = w^2(x)$ i.e. the equation is separable, therefore

$$\int \frac{w'(x)}{w^2(x)}\,\mathrm{d}x = \int \mathrm{d}x - 1$$

Therefore, integrating the LHS

$$-\frac{1}{w(x)} = x - 1$$

And

$$w(x) = \frac{1}{1-x}$$

Definition 9.2.2 (Autonomous Differential Equation)**.** Given a differential equation in normal form $y'(x) = f(x, y(x))$, it's said to be an *autonomous differential equation* if

$$f(x, y(x)) = g(y(x))$$

Therefore

$$y'(x) = g(y(x)) \tag{9.3}$$

Note that it's a separable equation, and therefore, writing the primitive of $1/g(y) = G(y)$ we have that the general solution of this problem would be

$$G(y) = x + c$$

Where $c \in \mathbb{R}$ is a constant

Example 9.2.2. Take the following initial value problem

$$\begin{cases} y'(x) = t|y| \\ y(0) = y_0 \end{cases}$$

This is obviously a separable ODE, and we have $f(t, y(t)) = t|y|$.
We have that $f \in C(\mathbb{R}^2)$ therefore there exists at least one solution for this equation.
Now, applying Picard-Lindelöf-Cauchy we can see that, $\forall [\alpha, \beta] \subset \mathbb{R}$ compact set

$$|f(t, y) - f(t, z)| = |t|y| - t|z|| = |t|||y| - |z|| \leq \max \{|\alpha|, |\beta|\} |y - z|$$

Taken $L = \max \{|\alpha|, |\beta|\}$ we have that f is Lipschitz continuous in the second variable in every compact set $K \subset \mathbb{R}$ and therefore there exists a unique solution $y : K \longrightarrow \mathbb{R}$.
Successively we move on to integrate directly the differential equation. We have

$$\int_{y_0}^{y} \frac{\mathrm{d}y}{|y|} = \int_{t_0}^{t} t \, \mathrm{d}t$$

I.e.

$$\begin{aligned} \log \left| \frac{y}{y_0} \right| &= \frac{1}{2} \left(t^2 - t_0^2 \right) \\ \left| \frac{y}{y_0} \right| &= e^{\frac{1}{2} t_0^2} e^{\frac{1}{2} t^2} \end{aligned}$$

Since $t_0 = 0$ we get

$$|y|(t) = |y_0| e^{\frac{1}{2} t^2}$$

The solutions depend on the sign of y_0 and we finally get

$$y(t) = \begin{cases} y_0 e^{\frac{1}{2} t^2} & y_0 > 0 \\ y_0 e^{-\frac{1}{2} t^2} & y_0 < 0 \end{cases}$$

§§ 9.2.1 First Order Linear ODEs

Definition 9.2.3 (Integrating Factor)**.** An example of linear ODE is the following kind of equation

$$y'(t) = p(t) y(t) + q(t) \qquad p, q \in C(I), \ I \subseteq \mathbb{R} \tag{9.4}$$

This equation has a unique solution in all of $\mathbb{R}$, since $f(t, y)$ is Lipschitz continuous and sublinear in $\mathbb{R}$.
We immediately know that the solution $y \in C^1$ exists since $f \in C(\mathbb{R}^2)$ and by Cauchy-Picard-Lindelöf we have

$$|f(t, y) - f(t, z)| = |p(t) y(t) - p(t) z(t)| = |p(t)||y - z| \leq \|p\|_u |y - z|$$

Taken $L = \|p\|_u$ we see immediately that f satisfies the theorem and the solution is uniquely defined in every compact set $K \subset \mathbb{R}$.
We also see that f is sublinear, since

$$|f(t, y)| = |p(t) y(t) + q(t)| \leq \|p\|_u |y(t)| + \|q\|_u$$

Which means that the solution can be extended in $\mathbb{R}$, and $\exists! y : \mathbb{R} \longrightarrow \mathbb{R}$ solution to the ODE.
We now proceed to integrate the ODE, for which we will proceed with a different method than usual.
Let $\mu(t)$ be what we will call the *integrating factor*, defined as follows

$$\mu(t) = e^{-\int p(t)\,\mathrm{d}t}$$

Which has the property that $p(t)\mu(t) = \mu'(t)$

Method 2 (Integrating Factor)**.** Let $y'(t) = p(t)y(t) + q(t)$ be a linear ODE, we solve by finding the integrating factor $\mu(t)$ and then multiplying both sides of the equayion by the integrating factor and moving the $p(t)y(t)$ term from the RHS to the LHS

$$\mu(t)y'(t) - p(t)\mu(t)y(t) = q(t)\mu(t)$$

By definition of the integrating factor we have that $p(t)\mu(t) = -\mu'(t)$ and therefore

$$\mu(t)y'(t) + \mu'(t)y(t) = q(t)\mu(t)$$

Using the chain rule we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mu(t)y(t)\right) = \mu(t)q(t)$$

Integrating

$$\mu(t)y(t) = \int \mu(t)q(t)\,\mathrm{d}t + c$$

And therefore, the final general solution will be the following

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)q(t)\,\mathrm{d}t + \frac{c}{\mu(t)}$$

Example 9.2.3*.* Take the following initial value problem

$$\begin{cases} y'(t) = \dfrac{1}{t}y(t) + 3t^3 \\ y(-1) = 2 \end{cases}$$

We immediately see the previous definition, therefore we can immediately say that since f is Lipschitz continuous and sublinear the solution to the problem will be uniquely defined on all of $\mathbb{R}$.
For integrating this ODE we use the integrating factor method, taking $p(t) = t^{-1}$ and $q(t) = 3t^3$.
By definition, we have $\mu(t) = |t|$, and therefore, since $t < 0$, $\mu(t) = -t^{-1}$, which gives

$$-\frac{1}{t}y(t) = -3\int t^2\,\mathrm{d}t + c$$

Therefore

$$y(t) = t^4 - ct$$

Applying the initial condition

$$y(-1) = 1 + c = 2 \implies c = 1$$

And therefore the final, unique solution of the initial value problem is the following

$$y(t) = t^4 - t$$

Theorem 9.7 (General Solutions)**.** *Taken $y'(x)+p(x)y(x)=q(x)$ a linear non homogeneous ODE, we define the associated homogeneous ODE as the equation*

$$z'(x)+p(x)z(x)=0$$

The solution of the complete ODE will be given by the sum of the particular solution $\overline{y}(x)$ and the homogeneous solution $z(x)$

Proof. Take $y(x)$ a solution of the equation and $\overline{y}(x)$ the particular solution of the ODE, we then have, since both are solutions

$$\begin{cases} y''(x)+p(x)y(x)=q(x) \\ \overline{y}''(x)+p(x)\overline{y}(x)=q(x) \end{cases}$$

Subtracting term a term we have

$$(y''(x)-\overline{y}(x))+p(x)(y(x)-\overline{y}(x))=0$$

Chosen $y''(x)-\overline{y}(x)=z(x)$ we have that

$$z''(x)+p(x)z(x)=0$$

Therefore $z(x)$ is a solution, but it's also the solution to the associated homogeneous ODE, and therefore, inverting for the general solution $y(x)$, we have

$$y(x)=\overline{y}(x)+z(x)$$

□

Example 9.2.4. Taken the following initial value problem

$$\begin{cases} y'(x)+\dfrac{2}{x}y(x)=\dfrac{1}{x^2} \\ y(-1)=2 \end{cases}$$

We want to find the general integral of the equation, $y(x)$. We begin by solving the associated homogeneous equation

$$z'(x)+\frac{2}{x}z(x)=0$$

In order to integrate this we use the integrating factor $\mu(x)$, where

$$\mu(x)=\exp\left(2\int_{-1}^{x}\frac{1}{x}\,\mathrm{d}x\right)=\exp\left(2\log(|x|)\right)=\exp\left(\log\left(x^2\right)\right)=x^2$$

We multiply by the integrating factor, and get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2z(x)\right)=0$$

Therefore

$$x^2z(x)=c \implies z(x)=\frac{c}{x^2}$$

The particular solution $\overline{y}(x)$ will be

$$\overline{y}(x) = \frac{1}{\mu(x)} \int_{x_0}^{x} \frac{1}{x^2} \mu(x) \, \mathrm{d}x$$

Therefore

$$\overline{y}(x) = \frac{1}{x^2} \int_{-1}^{x} \mathrm{d}x = \frac{1}{x}$$

The general solution will be $y(x) = \overline{y}(x) + z(x)$, and we finally get

$$y(x) = \frac{c}{x^2} + \frac{1}{x}$$

Imposing the initial condition we have

$$y(-1) = c - 1 = 2 \implies c = 3$$

Therefore

$$y(x) = \frac{3}{x^2} + \frac{1}{x}$$

Note how we could have found the general integral directly using the formula

$$y(x) = \frac{c}{x^2} + \frac{1}{x^2} \int_{-1}^{x} \mathrm{d}x = \frac{3}{x^2} + \frac{1}{x^2} \int_{-1}^{x} \mathrm{d}x$$

Method 3 (Variation of Constants)**.** Given a linear ODE $y'(t) + a(t)y = f(t)$, with $a, f \in C(I),\ I \subseteq \mathbb{R}$ suppose that $z(t)$ is the solution of the homogeneous equation, we have therefore

$$z(t) = \frac{c}{\mu(t)} = c \exp\left(\int a(t) \, \mathrm{d}t\right)$$

We can find a particular solution through *variation of constants*, i.e. we suppose $c = c(t)$, and we have

$$\overline{y}(t) = c(t) \exp\left(\int a(t) \, \mathrm{d}t\right)$$

Imposing this on the differential equation, we have

$$\overline{y}'(t) = c'(t) \exp\left(-\int a(t) \, \mathrm{d}t\right) - a(t)c(t) \exp\left(-\int a(t)\right) \mathrm{d}t$$

And therefore

$$\frac{1}{\mu(t)} \left(c'(t) - a(t)c(t)\right) + \frac{a(t)c(t)}{\mu(t)} = f(t)$$

Therefore

$$\frac{c'(t)}{\mu(t)} = f(t) \implies c(t) = \int f(t)\mu(t) \, \mathrm{d}t$$

Which implies

$$\overline{y}(t) = \frac{1}{\mu(t)} \int f(t)\mu(t) \, \mathrm{d}t$$

We then recover the previous formula by adding the particular and homogeneous solutions

$$y(t) = \frac{c}{\mu(t)} + \frac{1}{\mu(t)} \int f(t)\mu(t) \, \mathrm{d}t$$

§§ 9.2.2 Second Order Linear ODEs

Definition 9.2.4 (Initial Value Problem for 2nd Order Linear ODEs)**.** Given a second order linear differential equation $y''(x)+a(x)y'(x)+b(x)y(x)=f(x)$ we define the *initial value problem* for such differential equation as follows

$$\begin{cases} y''(x)+a(x)y'(x)+b(x)y(x)=f(x) \\ \hfill y'(x_0)=y'_0 \\ \hfill y(x_0)=y_0 \end{cases}$$

This kind of problem, if $a,b,f\in C(I)$ $I\subseteq\mathbb{R}$ is said to be *well defined*

Theorem 9.8 (Space of Solutions)**.** *Given a second order linear ODE, the solutions of the homogeneous equation span a vector space of dimension 2 $\mathcal{S}$, and the general solution of the homogeneous equation has the following form*

$$z_g(x)=c_1z_1(x)+c_2z_2(x)\in\mathcal{S}$$

I.e., if we define a matrix $W_{\mu\nu}(t)$ as follows

$$W_{\mu\nu}(t)=\begin{pmatrix} z_1(t) & z_2(t) \\ z_1'(t) & z_2'(t) \end{pmatrix}$$

If z_1,z_2 solve the differential equation, the matrix $W_{\mu\nu}(t)$ is non-singular

Proof. Suppose $z_1,z_2\in C^2(I)$ are two solutions. If they're linearly inddependent we have that they solve the following system with $c_1,c_2\in\mathbb{R}\setminus\{0\}$

$$\begin{cases} c_1z_1(x)+c_2z_2(x)=0 \\ c_1z_1'(x)+c_2z_2'(x)=0 \end{cases}$$

This system is solvable only if the determinant of the matrix $W_{\mu\nu}$ is non zero, and therefore the two functions must be linearly independent and span the vector space of solutions $\mathcal{S}$ $\square$

Method 4 (Characteristic Polynomial)**.** Take the following linear homogeneous ODE of order 2

$$z''(x)+az'(x)+bz(x)=0 \qquad a,b\in\mathbb{R}$$

We begin by "guessing" a solution in exponential form $z(x)=e^{\lambda x}$, where $\lambda\in\mathbb{C}$. Therefore, reinserting into the ODE, we have

$$e^{\lambda x}\left(\lambda^2+a\lambda+b\right)=0$$

Since the exponential is never zero $\forall x\in\mathbb{R},\ \forall\lambda\in\mathbb{C}$ we have that the polynomial must be zero, and therefore we get two roots

$$\lambda_{1,2}=-\frac{a}{2}\pm\frac{1}{2}\sqrt{a^2-4b}$$

We can now discern 3 cases.
Case 1), $a^2-4b>0$
There are two real roots $\lambda_1,\ \lambda_2$, and therefore we have

$$z(x)=c_1e^{\lambda_1x}+c_2e^{\lambda_2x}$$

Case 2), $a^2 - 4b < 0$
There are two complex conjugate roots $\lambda,\ \overline{\lambda}$, and the solution will be

$$z(x) = \mathfrak{Re}\left(c_1 e^{\lambda x} + c_2 e^{\overline{\lambda} x}\right) = c_1 e^{\mathfrak{Re}(\lambda)x}\cos(\mathfrak{Im}(\lambda)x) + c_2 e^{\mathfrak{Re}(\lambda)x}\sin(\mathfrak{Im}(\lambda)x)$$

Or

$$z(x) = Ae^{\mathfrak{Re}(\lambda)x}\cos(\mathfrak{Im}(\lambda) + \varphi)$$

With $A, \varphi \in \mathbb{R}$ Case 3) $a^2 - 4b = 0$
There is only one root $\lambda = a/2$ with multiplicity 2. In order to find the general integral $z(x)$ we impose the variation of constants, and we have

$$z(x) = c(x)e^{\lambda x}$$

And

$$\begin{aligned} z'(x) &= c'(x)e^{\lambda x} + \lambda c(x)e^{\lambda x} = e^{\lambda x}(c'(x) + \lambda c(x)) \\ z''(x) &= c''(x)e^{\lambda x} + \lambda c'(x)e^{\lambda x} + \lambda^2 c(x)e^{\lambda x} = e^{\lambda x}(c''(x) + \lambda c'(x) + \lambda^2 c(x)) \end{aligned}$$

Since λ is a root of the characteristic polynomial, we end up with

$$c''(x) = 0 \implies c(x) = c_2 + c_1 t$$

Therefore

$$z(x) = e^{\lambda t}(c_1 + c_2 t) = e^{-\frac{a}{2}t}(c_1 + c_2 t)$$

Method 5 (Similarity Method)**.** Another method for finding a particular solution is the similarity method. Given the following linear ODE

$$y''(x) + ay'(x) + by(x) = f(x)$$

The particular solution of this problem can be found using the shape of $f(x)$.
Case 1) $f(x) \in \mathbb{R}_n[x]$
In this case we will suppose that the solution will be a polynomial, and we can discern 3 cases

$$\overline{y}(x) = \begin{cases} q(x) & b \neq 0 \\ xq(x) & b = 0,\ a \neq 0 \\ x^2 q(x) & b = 0,\ a = 0 \end{cases}$$

Where $q(x) \in \mathbb{R}_n[x]$
Case 2) $f(x) = Ae^{\lambda x}$, $\lambda \in \mathbb{C}$
In this case we will suppose the particular solution as the real part of a complex exponential, where

$$\overline{y}(x) = \mathfrak{Re}\left(\tilde{y}(x)\right) = \gamma(x)\mathfrak{Re}\left(e^{\lambda x}\right)$$

Where $\gamma(x) \in C^2(I),\ I \subset \mathbb{R}$ is some unknown function. From this we will have, deriving twice

$$\begin{aligned} \tilde{y}'(x) &= e^{\lambda x}\left(\gamma'(x) + \lambda\gamma(x)\right) \\ \tilde{y}''(x) &= e^{\lambda x}\left(\lambda^2\gamma(x) + 2\lambda\gamma'(x) + \gamma''(x)\right) \end{aligned}$$

Substituting into the differential equation we obtain

$$\gamma''(x) + (2\lambda + a)\gamma'(x) + (\lambda^2 + b\lambda + a)\gamma(x) = A$$

We get again 3 different cases
a) $\lambda^2 + a\lambda + b = 0$
In this case we have

$$\gamma(x) = \frac{A}{\lambda^2 + a\lambda + b} \implies \tilde{y}(x) = \frac{A}{\lambda^2 + a\lambda + b} e^{\lambda x}$$

b) $\lambda^2 + a\lambda + b = 0,\ 2\lambda + a \neq 0$
In this gase instead we have

$$\gamma'(x) = \frac{A}{2\lambda + a} \implies \tilde{y}(x) = \frac{A}{2\lambda + a} x e^{\lambda x}$$

c) $\lambda^2 + a\lambda + b = 0,\ 2\lambda + a = 0$
Lastly we have

$$\gamma''(x) = A \implies \tilde{y}(x) = \frac{A}{2} x^2 e^{\lambda x}$$

The solution for the differential equation will lastly be either the real (or imaginary, if there is a sine in the RHS) part of the function we found.

Example 9.2.5. We will use the similarity method to solve the following differential equation

$$y''(x) + 2y'(x) + 3y(x) = 2e^{3x}$$

We see immediately that $f(x) = 2e^{3x}$ so we choose a particular solution $\overline{y}(x)$ with the following form

$$\overline{y}(x) = ke^{3x}$$

We therefore have

$$\overline{y}'(x) = 3ke^{3x}$$
$$\overline{y}''(x) = 9ke^{3x}$$

Substituting

$$18k = 2 \implies k = \frac{1}{9}$$

Therefore our searched solution will be the following

$$\overline{y}(x) = \frac{1}{9} e^{3x}$$

Example 9.2.6. Let's now solve the following differential equation

$$y''(x) + 2y'(x) - 3y(x) = 2e^{-3x}$$

Following the same approach, we see that the characteristic polynomial is null, and we end up with a contradiction, therefore we find a solution of the following kind

$$\overline{y}(x) = kxe^{-3x}$$

Deriving the particular solution, we have

$$\overline{y}'(x) = ke^{-3x}(1-3x)$$
$$\overline{y}''(x) = 3ke^{-3x}(3x-2)$$

Substituting

$$3k(3x-2) + 2k(1-3x) - 3kx = 2$$
$$-4k = 2$$
$$k = -\frac{1}{2}$$

Therefore the particular solution we're searching for is, finally

$$\overline{y}(x) = -\frac{x}{2}e^{-3x}$$

Method 6 (Variation of Constants)**.** Another method that might be used to solve a second order linear differential equation is the method of variation of constants.
Given the following differential equation and the associated homogeneous equation

$$y''(x) + ay'(x) + by(x) = f(x)$$
$$z''(x) + az'(x) + bz(x) = 0$$

Suppose that $z_1(x), z_2(x)$ are two linearly independent homogeneous solutions, and

$$z(x) = c_1 z_1(x) + c_2 z_2(x)$$

We search for a particular solution supposing that the two constants are actually functions, and we have

$$\overline{y}(x) = c_1(x)z_1(x) + c_2(x)z_2(x)$$

Due to the necessity of a non-singular Wronskian determinant, we impose the following condition

$$c_1'(x)z_1(x) + c_2'(x)z_2(x) = 0$$

Deriving two times the particular solution and then substituting in the original equation we end up with the following system

$$\begin{cases} c_1'(x)z_1(x) + c_2'(x)z_2(x) = 0 \\ c_1'(x)z_1'(x) + c_2'(x)z_2'(x) = f(x) \end{cases}$$

The necessity for the linear independence of the solutions is clear here, since the system is solvable if and only if the Wronskian determinant is zero.
We solve the system by substitution, and we have

$$\begin{cases} c_1'(x) = -c_2'(x)\dfrac{z_2(x)}{z_1(x)} \\ c_2'(x) = \dfrac{f(x) - c_1'(x)z_1'(x)}{z_2'(x)} \end{cases}$$

Therefore

$$\begin{cases} c_1'(x) = -\dfrac{z_1(x)}{z_1(x)z_2'(x) - z_2(x)z_1'(x)} f(x) \\ c_2'(x) = \dfrac{z_2(x)}{z_1(x)z_2'(x) - z_2(x)z_1'(x)} f(x) \end{cases}$$

Written in terms of the Wronskian determinant $\det_{\mu\nu} W_{\mu\nu}(x)$ we finally have as particular solution

$$\overline{y}(x) = z_2(x) \int \frac{z_2(x)}{\det_{\mu\nu} W_{\mu\nu}(x)} f(x) \, \mathrm{d}x - z_1(x) \int \frac{z_1(x)}{\det_{\mu\nu} W_{\mu\nu}(x)} f(x) \, \mathrm{d}x + c$$

And the general solution of the differential equation will be the following

$$y(x) = z_2(x) \int_{x_0}^{x} \frac{z_2(x)}{\det_{\mu\nu} W_{\mu\nu}(x)} f(x) \, \mathrm{d}x - z_1(x) \int_{x_0}^{x} \frac{z_1(x)}{\det_{\mu\nu} W_{\mu\nu}(x)} f(x) \, \mathrm{d}x + c_1 z_1(x) + c_2 z_2(x)$$

10 Fourier Calculus

§ 10.1 Bessel Inequality and Fourier Coefficients

Definition 10.1.1 (Fourier Coefficients)**.** Suppose $(u_k)_{k\in\mathbb{N}} = \mathcal{U} \subset \mathcal{V}$, with $(\mathcal{V}, \langle\cdot,\cdot\rangle)$ an euclidean space, taken $v \in \mathcal{V}$ we can define an operator $\hat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \mathbb{C}^{\mathbb{N}}$ such that

$$\forall v \in \mathcal{V} \quad \hat{\mathcal{F}}_{\mathcal{U}}(v) = c \in \mathbb{C}^{\mathbb{N}}$$

Where

$$c = (\langle v, u_1\rangle, \langle v, u_2\rangle, \cdots) \in \mathbb{C}^{\mathbb{N}}$$

The coefficients $\langle v, u_k\rangle \in \mathbb{C}$ are called the *Fourier coefficients* of the vector $v \in \mathcal{V}$

Theorem 10.1 (Bessel Inequality & Parseval's Theorem)**.** *Given $(u_k)_{k\in\mathbb{N}} = \mathcal{U} \subset \mathcal{V}$ an orthonormal system and $\mathcal{V}$ an euclidean space with $v \in \mathcal{V}$. Taken $\alpha_1, \cdots, \alpha_k \in \mathbb{C}$ some coefficients and defined the two following sums*

$$S_n = \sum_{k=1}^{n} \langle v, u_k\rangle u_k = \sum_{k=1}^{n} c_k u_k$$
$$S_n^\alpha = \sum_{k=1}^{n} \alpha_k u_k$$

Then

$$\|v - S_n\| \leq \|v - S_n^\alpha\|$$
$$\sum_{k=1}^{\infty} \|c_k\|^2 \leq \|v\|^2$$

The last inequality is known as Bessel's inequality
Lastly we also have Parseval's equality or Parseval's theorem, which states

$$v = \sum_{k=1}^{\infty} \langle v, u_k\rangle u_k = \sum_{k=1}^{\infty} c_k u_k \iff \sum_{k=1}^{\infty} \|c_k\|^2 = \|v\|^2$$

Due to this the operator $\hat{\mathcal{F}}_{\mathcal{U}}$ actually acts into $\ell^2(\mathbb{C})$, i.e.

$$\hat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \ell^2(\mathbb{C})$$

Proof. By definition of euclidean norm and using the bilinearity of the scalar product we have

$$\begin{aligned} 0 \leq \|v - S_n^\alpha\| &= \|v\|^2 - 2\mathfrak{Re}\left(\langle v, S_n^\alpha\rangle\right) + \|S_n^\alpha\|^2 = \\ &= \|v\|^2 - 2\mathfrak{Re}\left(\sum_{k=1}^{n}\langle v, u_k\rangle\overline{\alpha_k}\right) + \sum_{k=1}^{n}\|\alpha_k\|^2 \end{aligned}$$

Therefore

$$0 \leq \|v\|^2 - \sum_{k=1}^{n}\|c_k\|^2 + \sum_{k=1}^{n}\|\alpha_k - c_k\|^2$$

The minimum on the left is given for $\alpha_k = c_k$ and therefore, since $S_n^c = S_n$ we have

$$\|v - S_n\| \leq \|v - S_n^\alpha\|$$

And, using the non-negativity of the norm operator, putting $n \to \infty$ we have

$$0 \leq \|v - S_n\| = \|v\|^2 - \sum_{k=1}^{n}\|c_k\|^2 \implies \sum_{k=1}^{n}\|c_k\|^2 \leq \|v\|^2$$

Therefore

$$\sum_{k=1}^{\infty}\|\langle v, u_k\rangle\|^2 \leq \|v\|^2$$

Which means that the sum on the left converges uniformly, and therefore $c_k \in \ell^2(\mathbb{C})$. This demonstrates that $\hat{\mathcal{F}}_{\mathcal{U}} : \mathcal{V} \longrightarrow \ell^2(\mathbb{C})$ and Bessel's inequality.
This also gives Parseval's equality, since, for $n \to \infty$

$$\left\|v - \sum_{k=1}^{\infty}\langle v, u_k\rangle u_k\right\|^2 = 0 \iff \|v\|^2 = \sum_{k=1}^{\infty}\|\langle v, u_k\rangle\|^2$$

Due to the uniform convergence in $\mathcal{V}$ we have therefore

$$\hat{\mathcal{F}}_{\mathcal{U}}(v) = \sum_{k=1}^{\infty}\langle v, u_k\rangle u_k$$

□

Definition 10.1.2 (Closed System)**.** An system $(u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ is said to be *closed* iff $\forall v \in \mathcal{V}$

$$\|v\|^2 = \sum_{k=1}^{\infty}\|\langle v, u_k\rangle\|^2$$

$$v = \sum_{k=1}^{\infty}\langle v, u_k\rangle u_k$$

Theorem 10.2 (Closeness and Completeness)**.** *Given an orthonormal system $\mathcal{U} = (u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ with $\mathcal{V}$ an euclidean space, we have that $\mathcal{U}$ is a complete set if and only if $\mathcal{U}$ is a closed system. If $\mathcal{U}$ is complete or closed, $\mathcal{V}$ is separable*

Proof. Defined S_n the partial sums of the Fourier representation of v (ndr the series that represents v with respect to the system (u_k)), we have that for the theorem to be true the following two things must hold

$$\lim_{n\to\infty} S_n = v \qquad \overline{\mathrm{span}(u_k)} = \mathcal{V}$$

I.e. $\forall \epsilon > 0\ \exists N \in \mathbb{N},\ \alpha_1, \cdots, \alpha_N \in \mathbb{C}$ such that $\|v - S_n^\alpha\| < \epsilon$. Using Bessel-Parseval we have

$$0 \le \|v - S_N\| \le \|v - S_N^\alpha\| < \epsilon$$

Proving the closure of the system if the space $\mathcal{V}$ is complete.
Taken $(u_k)_{k\in\mathbb{N}}$ a closed system, we have that $S_n \to v$, therefore $v \in \mathrm{ad}\left(\mathrm{span}(\mathcal{U})\right)$, which implies

$$v \in \overline{\mathrm{span}(\mathcal{U})} \implies \mathcal{V} = \overline{\mathrm{span}(\mathcal{U})}$$

The last implication is given by the fact that $v \in \mathcal{V}$ is arbitrary, and it implies the completeness of $\mathcal{U}$ and the separability of $\mathcal{V}$ □

Theorem 10.3 (Riesz-Fisher). *Given $\mathcal{V}$ a hilbert space and $\mathcal{U} = (u_k)_{k\in\mathbb{N}} \in \mathcal{V}$ an orthonormal system, therefore $\forall c \in \ell^2\ \exists v \in \mathcal{V} : \hat{\mathcal{F}}_{\mathcal{U}}[v] = c$ and*

$$\begin{aligned} c_k &= \langle v, u_k\rangle \\ \|v\|^2 &= \|c\|_2^2 = \sum_{k=1}^{\infty} \|c_k\|^2 \\ v &= \sum_{k=1}^{\infty} \langle v, u_k\rangle u_k \end{aligned}$$

Proof. Taken a sequence $(v_k) \in \mathcal{V}$ defined as follows

$$v_n = \sum_{k=1}^{m} c_k u_k$$

This sequence is a Cauchy sequence, therefore it converges to $v \in \mathcal{V}$, since

$$\begin{aligned} \|v_n - v_m\|^2 &= \left\| \sum_{k=n+1}^{m} c_k u_k \right\|^2 = \langle \sum_{k=n+1}^{m} c_k u_k, \sum_{k=n+1}^{m} c_k u_k \rangle = \\ &= \sum_{k=n+1}^{m} \sum_{i=n+1}^{m} c_k \overline{c_i} \langle u_i, u_k\rangle = \sum_{k=n+1}^{m} \|c_k\|^2 \end{aligned}$$

By definition, since $c \in \ell^2$, the sum on the right converges, therefore

$$\|v_n - v_m\|^2 \le \sum_{k=n+1}^{\infty} \|c_k\|^2 < \infty$$

Which means, that $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that

$$\|v_n - v_m\|^2 \le \sum_{k=n+1}^{\infty} \|c_k\|^2 < \epsilon \quad \forall n \ge N$$

Which implies that $v_n \to v$ and

$$v = \sum_{k=1}^{\infty} c_k u_k$$

We can now write $\langle v, u_k\rangle = \langle v_n, u_k\rangle + \langle v - v_n, u_k\rangle$.
We have

$$\forall n \geq k \ \langle v_n, u_k\rangle = \sum_{i=1}^{n} c_i \langle u_i, u_k\rangle = c_k$$

For Cauchy-Schwartz we also have that

$$\|\langle v - v_n, u_k\rangle\| \leq \|v - v_n\| \to 0$$

Which implies that $c_k = \langle v, u_k\rangle$ and therefore

$$v = \sum_{k=1}^{\infty} \langle v, u_k\rangle u_k = \sum_{k=1}^{\infty} c_k u_k$$

□

§ 10.2 Fourier Series

§§ 10.2.1 Fourier Series in $L^2[-\pi, \pi]$

Definition 10.2.1 (Fourier Series)**.** Given a function $f \in L^2[-\pi, \pi]$ we define the *Fourier series expansion* of this function the following expression

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{10.1}$$

Where

$$\begin{cases} a_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos(kx) \, \mathrm{d}x \\ b_k = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin(kx) \, \mathrm{d}x \end{cases} \tag{10.2}$$

The notation $\sim$ indicates that the Fourier series of the function *converges to* the function $f(x)$. Usually an abuse of notation is used, where the function is actually set as equal to the Fourier expansion.

Definition 10.2.2 (Trigonometric Polynomial)**.** A function $p \in L^2[-\pi, \pi]$ is said to be a *trigonometric polynomial* if, for some coefficients α_k, β_k we have

$$p(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} \alpha_k \cos(kx) + \beta_k \sin(kx) \tag{10.3}$$

Theorem 10.4 (Completeness of Trigonometric Functions)**.** *Given* $(u_k), (v_k) \in L^2[-\pi, \pi]$ *two sequences of functions, where*

$$\begin{cases} u_k(x) = \cos(kx) \\ v_k(x) = \sin(kx) \end{cases}$$

The set $\{u_k, v_k\}$ *is orthogonal and complete, i.e. a basis in* $L^2[-\pi, \pi]$

Remark. These trigonometric identities always hold, $\forall n, k \in \mathbb{N}$, $n \neq k$

$$\begin{aligned} \cos(nx)\cos(kx) &= \frac{1}{2}\left(\cos[(n+k)x] + \cos[(n-k)x]\right) \\ \sin(nx)\sin(kx) &= \frac{1}{2}\left(\cos[(n-k)x] - \cos[(n+k)x]\right) \\ \cos(nx)\sin(kx) &= \frac{1}{2}\left(\sin[(n+k)x] - \sin[(n-k)x]\right) \end{aligned} \tag{10.4}$$

Proof. We begin by demonstrating that the two function sequences u_k, v_k are orthogonal in $L^2[-\pi, \pi]$. Therefore, by explicitly writing the scalar product, we have, for $k \neq n$

$$\langle u_n, u_k \rangle = \int_{-\pi}^{\pi} \cos(nx)\sin(kx)\,\mathrm{d}x = \frac{1}{2}\int_{-\pi}^{\pi} \cos[(n+k)x] + \cos[(n-k)x]\,\mathrm{d}x$$

Therefore

$$\langle u_n, u_k \rangle = \frac{1}{2}\left[\frac{\sin[(n+k)x]}{n+k} + \frac{\sin[(n-k)x]}{n-k}\right]_{-\pi}^{\pi} = 0$$

Analogously

$$\langle v_n, v_k \rangle = \frac{1}{2}\left[\frac{\sin[(n-k)x]}{n-k} - \frac{\sin[(n+k)x]}{n+k}\right]_{-\pi}^{\pi} = 0$$

And, finally

$$\langle u_n, v_k \rangle = \frac{1}{2}\left[\frac{\cos[(n+k)x]}{n+k} - \frac{\cos[(n-k)x]}{n-k}\right]_{-\pi}^{\pi} = \frac{1}{2} - \frac{1}{2} = 0$$

Which demonstrates that, for $k \neq n$ $u_k \perp u_n,\ v_k \perp v_n,\ u_k \perp v_k$.
Now, taken a trigonometric polynomial $p(x) \in L^2[-\pi, \pi]$ we need to prove that $\overline{\operatorname{span}\{u_k, v_k\}} = L^2[-\pi, \pi]$, i.e.

$$\forall \epsilon > 0\ \forall f \in L^2[-\pi, \pi] \quad \|p - f\|_2 < \epsilon$$

We have already that $\overline{C[-\pi, \pi]} = L^2[-\pi, \pi]$ and that for a Weierstrass theorem (without proof), every periodic function with period 2π is the uniform limit of a trigonometric polynomial.
Using these two results, given $f \in L^2[-\pi, \pi]$, $\exists g \in C[-\pi, \pi] : \|f - g\|_2 < \epsilon/3$. Taken $\hat{g}(x)$ as the periodic extension of $g(x)$, for Weierstrass we have

$$\|g - \hat{g}\|_2 < \frac{\epsilon}{3} \quad \|p - \hat{g}\|_2 < \frac{\epsilon}{3} \implies \|p - \hat{g}\|_u < \frac{\epsilon}{3\sqrt{2\pi}}$$

Therefore, finally $\|f - p\|_2 < \epsilon$ ◻

Theorem 10.5 (Parseval Identity)**.** *Given* $f \in L^2[-\pi, \pi]$ *we have that*

$$\int_{-\pi}^{\pi} |f(x)|^2\,\mathrm{d}x = \frac{|a_0|^2}{2} + \sum_{k=1}^{\infty} |a_k|^2 + |b_k|^2$$

Proof. The proof is quite straightforward, since trigonometric polynomials form a basis for $L^2[-\pi,\pi]$ we have that this is simply the already known Parseval identity, since

$$\|f\|_2^2 = \sum_{k=0}^{\infty} \|c_k\|^2$$

Writing $c_k = a_k + b_k$ we have

$$\|f\|_2^2 = \frac{\|a_0\|^2}{2} + \sum_{k=1}^{\infty} \|a_k\|^2 + \|b_k\|^2$$

□

§§ 10.2.2 Fourier Series in $L^2[a,b]$

Definition 10.2.3 (Basis of the Space)**.** In order to define a trigonometric basis in $L^2[a,b]$ with $a \neq b$, we can use a simple coordinate transformation onto the $\{(u_k),(v_k)\}$ basis of the space $L^2[-\pi,\pi]$.
Therefore, taken

$$y(x) = \frac{\pi}{b-a}(2x-a-b)$$

The new basis on $L^2[a,b]$ will be

$$\begin{cases} (u_k(y(x))) = \cos(ky(x)) = \cos\left(\frac{k\pi}{b-a}(2x-a-b)\right) \\ (v_k(y(x))) = \sin(ky(x)) = \sin\left(\frac{k\pi}{b-a}(2x-a-b)\right) \end{cases}$$

The completeness of this basis is given by the fact that, this change of coordinates is a smooth diffeomorphism between $L^2[-\pi,\pi], L^2[a,b]$.

Definition 10.2.4 (General Fourier Series)**.** With the previous definition, the Fourier series of a function $f \in L^2[a,b]$ is given as follows

$$f(x) \sim \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} \tilde{a}_k \cos\left(\frac{k\pi}{b-a}(2x-a-b)\right) + \tilde{b}_k \sin\left(\frac{k\pi}{b-a}(2x-b-a)\right) \tag{10.5}$$

Where

$$\begin{cases} \tilde{a}_k = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\cos\left(\frac{k\pi}{b-a}(2x-b-a)\right)\mathrm{d}x \\ \tilde{b}_k = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\sin\left(\frac{k\pi}{b-a}(2x-b-a)\right)\mathrm{d}x \end{cases} \tag{10.6}$$

§§ 10.2.3 Fourier Series in Symmetric Intervals, Expansion in Only Sines and Cosines

Definition 10.2.5. We firstly begin finding the Fourier series of a function in $L^2[-l,l]$. Using the previous general case in $L^2[a,b]$ and setting $a=-l,\ b=l$ we have $\forall f \in L^2[-l,l]$

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right) \tag{10.7}$$

With coefficients

$$\begin{cases} a_k = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x \\ b_k = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\left(\frac{k\pi x}{l}\right)\mathrm{d}x \end{cases} \tag{10.8}$$

Theorem 10.6. *Taken the space $L^2[0,\pi]$ we have that both trigonometric sequences $(u_k(x))$ and $(v_k(x))$ are orthogonal bases in this space, and the following equalities hold.*
$\forall f \in L^2[0,\pi]$

$$f(x) \sim \frac{a'_0}{2} + \sum_{k=1}^{\infty} a'_k \cos(kx)$$

$$f(x) \sim \sum_{k=1}^{\infty} b'_k \sin(kx)$$

Where

$$a'_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(kx)\,\mathrm{d}x$$

$$b'_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,\mathrm{d}x$$

Proof. The proof of this theorem is straightforward, we firstly define the even and uneven extensions of the function $f(x)$ in $L^2[-\pi,\pi]$ as follows

$$f^e(x) = \begin{cases} f(x) & x \in [0,\pi] \\ f(-x) & x \in [-\pi, 0) \end{cases}$$

And

$$f^u(x) = \begin{cases} f(x) & x \in [0,\pi] \\ -f(-x) & x \in [-\pi, 0) \end{cases}$$

Expanding both these functions in $[-\pi,\pi]$ we have that, indicating the coefficients of each as $a_k^e,\ b_k^e,\ a_k^u,\ b_k^u$

$$\begin{aligned} a_k^e &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^e(x)\cos(kx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(kx)\,\mathrm{d}x = a'_k \\ b_k^e &= 0 \\ a_k^u &= 0 \\ b_k^u &= \frac{1}{\pi}\int_{-\pi}^{\pi} f^u(x)\sin(kx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)\,\mathrm{d}x = b'_k \end{aligned}$$

Therefore

$$f^e(x) \sim \frac{a'_0}{2} + \sum_{k=1}^{\infty} a'_k \cos(kx)$$

$$f^u(x) \sim \sum_{k=1}^{\infty} b'_k \sin(kx)$$

Which implies that

$$\|f^e - S_n\|_2^2 = 2\|f - S_n\|_{[0,\pi]}^2 \to 0$$

And

$$\|f^u - S_n\|_2^2 = 2\|f - S_n\|_{[0,\pi]}^2 \to 0$$

Proving the theorem. □

Example 10.2.1. Taken the function $f(x) = x^2 \quad x \in [-l, l]$ we want to find the Fourier expansion of this function.
Since x^2 is even, thanks to the previous theorem we know that the coefficients $b_k = 0$ in all the set of definition, therefore

$$x^2 \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{l}\right)$$

We firstly calculate the coefficient a_0 of the expansion

$$a_0 = \frac{1}{l}\int_{-l}^{l} x^2 \,\mathrm{d}x = \frac{2l^2}{3}$$

The coefficients a_k can be calculated using the fact that x^2 is even, and therefore we have

$$\begin{aligned} a_k &= \frac{1}{l}\int_{-l}^{l} x^2 \cos\left(\frac{k\pi x}{l}\right) \mathrm{d}x = \frac{1}{l}\left[x^2 \sin\left(\frac{k\pi x}{l}\right)\frac{l}{k\pi}\right]_{-l}^{l} - \frac{4}{k\pi}\int_0^l x \sin\left(\frac{k\pi x}{l}\right) \mathrm{d}x = \\ &= \frac{4l}{(k\pi)^2}\left[x \cos\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} - \frac{4l}{(k\pi)^2}\int_0^l \sin\left(\frac{k\pi x}{l}\right) \mathrm{d}x \end{aligned}$$

Since the last integral is 0 we have

$$a_k = \frac{4l}{(k\pi)^2}\left[x \cos\left(\frac{k\pi x}{l}\right)\right]_{-l}^{l} = \frac{(-1)^k l^2}{(k\pi)^2}$$

The searched Fourier expansion is therefore

$$x^2 \sim \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos\left(\frac{k\pi x}{l}\right)$$

Example 10.2.2 (Parseval's Equality). Having now found the Fourier expansion for x^2, we can use Parseval's equality in order to calculate the sum

$$\sum_{k=1}^{\infty}\frac{1}{k^2}$$

Thanks to Parseval, we therefore have

$$\left\|x^2\right\|_2^2 = \frac{1}{l}\int_{-l}^{l} x^4 \,\mathrm{d}x = \frac{2l^4}{9} + \frac{16l^2}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}$$

The integral on the left is obvious, and moving the terms around we finally have

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{16 l^5} \int_{-l}^{l} x^4 \,\mathrm{d}x - \frac{\pi^4}{36} = \frac{\pi^4}{16} \left(\frac{2}{5} - \frac{2}{9} \right)$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$$

§§ 10.2.4 Complex Fourier Series

Theorem 10.7 (Complex Exponential Basis)**.** *Taken the space $L^2[-\pi, \pi]$, and defining a system $(e_k)_{k \in \mathbb{Z}} = e^{ikx}$, this system is an orthogonal basis for the space.*

Proof. Using Euler's formula for complex exponentials we have

$$(e_k)_{k \in \mathbb{Z}} = e^{ikx} = \cos(kx) + i \sin(kx) = u_k(x) + i v_k(x)$$

Therefore, due to the linearity of the scalar product, these functions are orthogonal to each other, and due to linearity we also have

$$\mathrm{span}\{e^{ikx}\} = \mathrm{span}\{\cos(kx), \sin(kx)\}$$

Which, implies

$$\overline{\mathrm{span}\{e^{ikx}\}} = L^2[-\pi, \pi]$$

Note that

$$\left\| e^{ikx} \right\|_2^2 = 2\pi$$

□

Definition 10.2.6 (Complex Fourier Series)**.** Given $f \in L^2[-\pi, \pi]$ we can now define a Fourier expansion in complex exponentials as follows

$$f(x) \sim \sum_{k=-\infty}^{\infty} \frac{\langle f(x), e^{ikx} \rangle}{\| e^{ikx} \|_2^2} e^{ikx} = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \tag{10.9}$$

Where, the coefficients will be

$$c_k = \frac{\langle f(x), e^{ikx} \rangle}{\| e^{ikx} \|_2^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} \,\mathrm{d}x \tag{10.10}$$

Note that if $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ we have

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{ikx} \,\mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{f(x) e^{ikx}} \,\mathrm{d}x = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \,\mathrm{d}x = c_{-k}$$

Therefore, for a real valued function

$$f(x) \sim c_0 + \sum_{k=1}^{\infty} c_k e^{ikx} + c_{-k} e^{-ikx} = c_0 + 2 \sum_{k=1}^{\infty} \mathfrak{Re} \left\{ c_k e^{ikx} \right\} \tag{10.11}$$

Example 10.2.3. Taken the function $f(x) = e^x$, $x \in [-\pi, \pi]$, we want to find the Fourier series in terms of complex exponentials. Since $f(x)$ is a real valued function, we have

$$e^x \sim c_0 + 2\sum_{k=1}^{\infty} \mathfrak{Re}\left\{c_k e^{ikx}\right\}$$

The coefficients will be

$$\begin{aligned} c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \, \mathrm{d}x = \frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right) = \frac{\sinh(\pi)}{\pi} \\ c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x} \, \mathrm{d}x = \frac{1}{2\pi}\left[\frac{1}{1-ik}\left(e^{\pi(1-ik)} - e^{-\pi(1-ik)}\right)\right] \end{aligned}$$

The second expression can be seen as follows

$$c_k = \frac{1}{2\pi(1-ik)}\left(e^{ik\pi}e^{\pi} - e^{-ik\pi}e^{-\pi}\right) = \frac{(-1)^k}{\pi(1-ik)}\sinh(\pi)$$

The final expansion will then be given from finding the real part of this coefficient times the basis vector, i.e.

$$\mathfrak{Re}\left\{\frac{(-1)^k \sinh(\pi)}{\pi(1-ik)}e^{ikx}\right\} = \frac{(-1)^k \sinh(\pi)}{1+k^2}\mathfrak{Re}\left((1+ik)(\cos(kx) + i\sin(kx))\right)$$

The last calculation is obvious, and we therefore have

$$\mathfrak{Re}\left\{\frac{(-1)^k \sinh(\pi)}{\pi(1-ik)}e^{ikx}\right\} = \frac{(-1)^k \sinh(\pi)}{1+k^2}\left(\cos(kx) - k\sin(kx)\right)$$

And the final solution will be

$$e^x \sim \frac{\sinh(\pi)}{\pi} + \frac{2\sinh(\pi)}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{1+k^2}\left(\cos(kx) - k\sin(kx)\right)$$

§§ 10.2.5 Piecewise Derivability, Pointwise and Uniform Convergence of Fourier Series

Definition 10.2.7 (One Sided Derivatives)**.** Let $f : [a,b] \longrightarrow \mathbb{C}$ be a piecewise continuous function and let $x \in [a,b)$. $f(x)$ is said to be *right (or left) derivable*, if the following limits exist

$$\begin{aligned} f'_+(x) &= \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x^+)}{\epsilon} \\ f'_-(x) &= \lim_{\epsilon \to 0^-} -\frac{f(x+\epsilon) - f(x^+)}{\epsilon} \end{aligned}$$

Where

$$f(x^{\pm}) = \lim_{y \to x^{\pm}} f(y)$$

Example 10.2.4. Take the following function

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{2} & x = 0 \\ 1 - x & x > 0 \end{cases}$$

We have

$$f'_+(0) = \lim_{\epsilon \to 0^+} \frac{f(\epsilon) - f(0^+)}{\epsilon} = \frac{1 - \epsilon - 1}{\epsilon} = -1$$
$$f'_-(0) = \lim_{\epsilon \to 0^+} \frac{f(\epsilon) - f(0^+)}{-\epsilon} = \frac{0}{\epsilon} = 0$$

It's important to see how the right and left derivatives might not coincide with the right and left limits of the derivative, as explained in the following theorem

Theorem 10.8. *Let $f(x) : [a, b] \longrightarrow \mathbb{C}$ be a piecewise differentiable function, then given $x \in [a, b)$ we have*

$$f'_+(x) = f'(x^+)$$
$$f'_-(x) = f'(x^-)$$

Proof. Since $f(x)$ is piecewise differentiable, we have that $\exists \gamma > 0$ such that $f(x)$ is differentiable $\forall x \in (x, x + \gamma)$ and we can define $f'(x^+)$
Therefore $\forall \alpha > 0,\ \exists \epsilon_1 > 0$ such that

$$\forall y \in (x, x + \epsilon_1) \quad \left| f'(y) - f'(x^+) \right| < \alpha$$

Thanks to the Lagrange theorem, and the definition of one sided limit, we have, given $0 < \delta < \epsilon \leq \epsilon_1$

$$\left| \frac{f(x + \epsilon) - f(x + \delta)}{\epsilon - \delta} - f'(x^+) \right| < \alpha$$

Which implies therefore

$$\lim_{\epsilon \to 0^+} \lim_{\delta \to 0^+} \left| \frac{f(x + \epsilon) - f(x^+)}{\epsilon} - f'(x^+) \right| = 0$$

Which also implies, by definition

$$f'_+(x) = \lim_{\epsilon \to 0^+} \frac{f(x + \epsilon) - f(x^+)}{\epsilon} = f'(x^+)$$

□

Definition 10.2.8 (Periodic Extension)**.** Given a function $f : [-\pi, \pi] \longrightarrow \mathbb{C}$, non periodic, we define the periodic extension $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{C}$ as

$$\tilde{f}(x) = f(x + 2k\pi) \quad k \in \mathbb{Z},\ x + 2k\pi \in (-\pi, \pi]$$

Note that it coincides with the same function, given that $x \in (-\pi, \pi]$ and therefore the periodic extension has discontinuities of the first kind at the points $x_k = (2k + 1)\pi$, $k \in \mathbb{Z}$

Lemma 10.2.1 (Riemann-Lebesgue)**.** Let $f \in C[a,b]$ be a function such that f' is piecewise continuous (also holds $\forall f \in L^1[a,b]$), then

$$\lim_{s\to\infty}\int_a^b f(x)\sin(sx)\,\mathrm{d}x = 0$$

Proof. The proof comes directly from the calculus of the integral

$$\int_a^b f(x)\sin(sx)\,\mathrm{d}x = -\frac{1}{s}\left[f(x)\cos(sx)\right]_a^b + \frac{1}{s}\int_a^b f'(x)\sin(sx)\,\mathrm{d}x$$

Since $\|f\|_u = M$ and $\|f'\|_u = M'$, we have

$$\left|\int_a^b f(x)\sin(sx)\,\mathrm{d}x\right| \le \frac{1}{|s|}\left(2M + |b-a|M'\right) \to 0$$

□

Definition 10.2.9 (Dirichlet Kernel)**.** We define the *Dirichlet kernel* as the following function

$$D_n(z) = \frac{1}{2\pi}\left(\frac{\sin\left(\frac{2n+1}{2}z\right)}{\sin\left(\frac{z}{2}\right)}\right) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^n \cos(kz)$$

Lemma 10.2.2. Given $f : \mathbb{R} \longrightarrow \mathbb{C}$ a piecewise continuous function, and $x \in \mathbb{R}$ such that exists $f'_+(x)$ we have that

$$\begin{cases} \displaystyle\lim_{n\to\infty}\int_0^\pi f(x+z)D_n(z)\,\mathrm{d}z = \frac{1}{2}f(x^+) \\ \displaystyle\lim_{n\to\infty}\int_{-\pi}^0 f(x+z)D_n(z)\,\mathrm{d}z = \frac{1}{2}f(x^-) \end{cases}$$

Theorem 10.9. *Given $f : \mathbb{R} \longrightarrow \mathbb{C}$ a $2\pi-$periodic piecewise continuous function and letting S_n be the n-th partial sum of the Fourier series expansion of f and letting $x \in \mathbb{R}$ such that exist both the left and right derivative in the point, we have that*

$$\lim_{n\to\infty} S_n(x) = \begin{cases} f(x) & f \text{ is continuous in } x \\ \frac{1}{2}\left(f(x^+) + f(x^-)\right) & f \text{ is not continuous in} x \end{cases}$$

Or also

$$\lim_{n\to\infty} S_n(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

Proof. By definition of the Fourier expansion, we have that, in the trigonometric basis

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^n a_k\cos(kx) + b_k\sin(kx)$$

Inserting the usual definitions of the Fourier coefficients and using the fact that the sum is finite, hence it converges, we have

$$S_n(x) = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(s)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos(kx)\cos(ks) + \sin(kx)\sin(ks)\right) \mathrm{d}s\right]$$

Simplifying

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos\left(k(s-x)\right)\right) \mathrm{d}s$$

Rearranging the second term we see that it's the Dirichlet kernel, and using a transformation $z = s - x$, we have

$$S_n(x) = \int_{-\pi}^{\pi} f(x+z)D_n(z)\,\mathrm{d}z$$

Note that the extremes of integration are the same since both these functions are $2\pi-$periodic. Using the definition of the integral between f and the Dirichlet kernel, we have finally the statement of the theorem, in the case that the function has a discontinuity at the point x

$$\lim_{n\to\infty} S_n(x) = \frac{1}{2}\left(f(x^+) + f(x^-)\right)$$

□

Theorem 10.10 (Pointwise Convergence of the Fourier Series)**.** *Given a piecewise continous $2\pi-$periodic function $f : \mathbb{R} \longrightarrow \mathbb{C}$, we have that the Fourier series converges pointwise to the following two cases, in case the function is continous or not in the point $x \in \mathbb{R}$*

Theorem 10.11 (Uniform Convergence of the Fourier Series)**.** *Given a $2\pi-$periodic function $f \in C(\mathbb{R})$, such that its derivative is piecewise continuous, we have that*

$$S_n \rightrightarrows f$$

Proof. Since $f(x) \in C(\mathbb{R})$ for the previous theorem we have that $S_n(x) \to f(x) \quad \forall x \in \mathbb{R}$, therefore

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k\cos(kx) + b_k\sin(kx) \quad x \in \mathbb{R}$$

Using the Weierstrass M-test we have that, taken the following sequence

$$c_k = a_k\cos(kx) + b_k\sin(kx)$$

The sequence is limited as follows

$$|c_k| \le |a_k||\cos(kx)| + |b_k||\sin(kx)| \le |a_k| + |b_k| = M_k$$

In order to check that the sum of this extremum is convergent we find the Fourier expansion of the derivative of f

$$f'(x) \sim \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k'\cos(kx) + b_k'\sin(kx)$$

It's not hard to prove that

$$\begin{cases} a'_k = -\dfrac{b_k}{k} \\ b'_k = \dfrac{a_k}{k} \end{cases}$$

Therefore

$$\sum_{k=1}^{\infty} |a_k| + |b_k| = \sum_{k=1}^{\infty} \frac{1}{k} \left(|a'_k| + |b'_k| \right) \leq \frac{1}{2} \sum_{k=1}^{\infty} |a'_k|^2 + |b'_k|^2 + \frac{2}{k^2} < \infty$$

Since $a_k, a'_k, b_k, b'_k \in \ell^2(\mathbb{R})$ Therefore, the sum converges and for Weierstrass' M-test it converges uniformly ☐

§§ 10.2.6 Solving the Heat Equation with Fourier Series

Definition 10.2.10 (Heat Equation)**.** In physics, the equation governing heat transfer is the *heat equation* a partial differential equation of order 2 in space and of order 1 in time.
The equation is the following

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} \tag{10.12}$$

Example 10.2.5 (Heat Equation with Von Neumann Boundary Conditions)*.* We firstly write the heat equation with Von Neumann boundary conditions

$$\begin{cases} \partial_t u = \lambda \partial_x^2 u \\ \partial_x u(0,t) = \partial_x u(l,t) = 0 \\ u(x,0) = f(x) \end{cases} \tag{10.13}$$

Where $x \in [0,l], \ t > 0$
We suppose that $u(x,t)$ is expressible as a uniformly convergent Fourier series of only sines or cosines.
Since we want the derivative on the x to vanish in order to satisfy immediately the boundary conditions, we suppose the following expansion

$$u(x,t) = \frac{a_0(t)}{2} + \sum_{k=1}^{\infty} a_k(t) \cos\left(\frac{k\pi x}{l} \right)$$

Deriving, we have

$$\begin{aligned} \partial_t u &= \frac{a'_0(t)}{2} + \sum_{k=1}^{\infty} a'_k(t) \cos\left(\frac{k\pi x}{l} \right) \\ \partial_x u &= -\frac{\pi}{l} \sum_{k=1}^{\infty} k a_k(t) \sin\left(\frac{k\pi x}{l} \right) \\ \partial_x^2 u &= -\frac{\pi^2}{l^2} \sum_{k=1}^{\infty} k^2 a_k(t) \cos\left(\frac{k\pi x}{l} \right) \end{aligned}$$

It's immediate to see that the boundary conditions are immediately satisfied, and therefore, reinserting it back into the differential equation, we get

$$-\frac{\pi^2\lambda}{l^2}\sum_{k=1}^{\infty}k^2a_k(t)\cos\left(\frac{k\pi x}{l}\right)=\frac{a_0'(t)}{2}+\sum_{k=1}^{\infty}a_k'(t)\cos\left(\frac{k\pi x}{l}\right)$$

Therefore, we end up with the following *infinite* system of ODEs of order 1

$$\begin{cases} a_0'(t)=0 \\ a_k'(t)=-\lambda\left(\dfrac{k\pi}{l}\right)^2a_k(t) \end{cases}$$

With $k>1,\ k\in\mathbb{N}$. Therefore, integrating we get

$$a_0=a_0 \qquad a_k(t)=a_ke^{-\lambda\left(\frac{k\pi}{l}\right)^2t}$$

Reinserting into the second boundary condition $u(x,0)=f(x)$ we end up determining the coefficients as the cosine-Fourier coefficients of the function $f(x)$

$$\begin{aligned} a_k&=\frac{2}{l}\int_0^l f(x)\cos\left(\frac{k\pi x}{l}\right)\mathrm{d}x \\ a_0&=\frac{1}{l}\int_0^l f(x)\,\mathrm{d}x \end{aligned}$$

Therefore, the complete solution to the PDE is

$$u(x,t)=\frac{1}{2l}\int_0^l f(x)\,\mathrm{d}x+\frac{2}{l}\sum_{k=1}^{\infty}e^{-\left(\frac{k\pi}{l}\right)^2\lambda t}\cos\left(\frac{k\pi x}{l}\right)\int_0^l f(s)\cos\left(\frac{k\pi s}{l}\right)\mathrm{d}s$$

Example 10.2.6 (Dirichlet Boundary Conditions). Let's now take the heat equation with different boundary conditions, namely

$$\begin{cases} \partial_t u=\lambda\partial_x^2u \\ u(x,0)=f(x) \\ u(0,t)=u(l,t)=0 \end{cases}$$

Where $(x,t)\in[0,l]\times\mathbb{R}^+\setminus\{0\}$ Since the first derivative doesn't appear in the boundary conditions we choose a particular form of $u(x,t)$ in terms of an only sine Fourier expansion (assuming uniform convergence). We have therefore

$$u(x,t)=\sum_{k=1}^{\infty}b_k(t)\sin\left(\frac{k\pi x}{l}\right)$$

Deriving, we get therefore

$$\begin{aligned} \partial_t u&=\sum_{k=1}^{\infty}b_k'(t)\sin\left(\frac{k\pi x}{l}\right) \\ \partial_x^2u&=-\frac{\pi^2}{l^2}\sum_{k=1}^{\infty}k^2b_k(t)\sin\left(\frac{k\pi x}{l}\right) \end{aligned}$$

Reinserting into the differential equation, we have

$$\sum_{k=1}^{\infty} b_k'(t)\sin\left(\frac{k\pi x}{l}\right) + \frac{\lambda\pi^2 k^2}{l^2} b_k(t)\sin\left(\frac{k\pi x}{l}\right) = 0$$

Therefore, equating the coefficients for the infinite ODEs we get

$$b_k'(t) = \left(\frac{k\pi}{l}\right)^2 b_k(t)$$
$$b_k(t) = b_k e^{-\lambda\left(\frac{k\pi}{l}\right)^2 t}$$

And our particular solution will be, therefore

$$u(x,t) = \sum_{k=1}^{\infty} b_k e^{-\frac{\lambda k^2\pi^2}{l^2}t}\sin\left(\frac{k\pi x}{l}\right)$$

Imposing the last condition we get

$$u(x,t) = \sum_{k=1}^{\infty} e^{-\frac{\lambda k^2\pi^2}{l^2}t}\sin\left(\frac{k\pi x}{l}\right)\int_0^l f(x)\sin\left(\frac{k\pi s}{l}\right)\mathrm{d}s$$

§ 10.3 Fourier Transform

§§ 10.3.1 Fourier Integrals, Translations, Dilations

Proposition 16 (Extending the Fourier Series)**.** Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a non periodic function and $f_l : [-l, l] \subset \mathbb{R} \longrightarrow \mathbb{C}$ be a function with periodic extension that converges to $f(x)$ for $l \to \infty$.
We have, using the complex exponential basis that

$$f_l(x) \sim \frac{1}{2l}\sum_{k\in\mathbb{Z}} e^{\frac{-ik\pi x}{l}}\int_{-l}^{l} f_l(s)e^{\frac{ik\pi s}{l}}\,\mathrm{d}s$$

Sending $l \to \infty$ we have that the sum of coefficients behaves like a Riemann sum and converges to the following integral

$$\int_{\mathbb{R}} g(\lambda)e^{ikx}\,\mathrm{d}x = f(x)$$

Where the last equality is given by the fact that $f_l(x) \to f(x)$.
We define the integral used for finding these "coefficients" the *Fourier Integral Transform* of f

$$g(\lambda) = \hat{\mathcal{F}}[f](\lambda) = \hat{f}(\lambda)\frac{1}{2\pi}\int_{\mathbb{R}} f(x)e^{-i\lambda x}\,\mathrm{d}x$$

Definition 10.3.1 (Parity, Translation and Dilation Operators)**.** Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be some function.
We define the following operators

$$\hat{P}[f](x) = f(-x) \qquad \text{Parity}$$
$$\hat{T}_a[f](x) = f(x-a) \qquad \text{Translation}$$
$$\hat{\Phi}_a[f](x) = f(ax) \qquad \text{Dilation}$$

Definition 10.3.2 (Fourier Operator)**.** Given a function $f \in L^1(\mathbb{R})$ we define the *Fourier operator* $\hat{\mathcal{F}}[f]$ as follows

$$\hat{\mathcal{F}}[f] = \int_{\mathbb{R}} f(x) e^{-i\lambda x} \, \mathrm{d}x \quad \lambda \in \mathbb{R}$$

Which is basically the Fourier transform $\hat{f}$ of the function f.
Note that $\hat{\mathcal{F}} : L^1(\mathbb{R}) \to L^1(\mathbb{R})$, since

$$\left\| \hat{\mathcal{F}}[f] \right\| = \int_{\mathbb{R}} \left| f(x) e^{-i\lambda x} \right| \mathrm{d}x = \int_{\mathbb{R}} |f(x)| \, \mathrm{d}x$$

Theorem 10.12 (Properties of the Fourier Transform)**.** *Given $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{C}$ we hafe that*

1. $\hat{\mathcal{F}}[af + bg] = a\hat{\mathcal{F}}[f] + b\hat{\mathcal{F}}[g]$
2. $\hat{\mathcal{F}}[\overline{f}](\lambda) = \overline{\hat{\mathcal{F}}[f]}(-\lambda)$
3. $\mathfrak{Im}(f) = 0 \implies \hat{\mathcal{F}}[f](\lambda) = \overline{\hat{\mathcal{F}}[f](-\lambda)}$
4. $\mathfrak{Im}(f) = 0,\ f$ *even* $\implies \mathfrak{Im}\left(\hat{\mathcal{F}}[f]\right) = 0,\ \hat{\mathcal{F}}[f]$ *even*
5. $\mathfrak{Im}(f) = 0,\ f$ *uneven* $\implies \mathfrak{Re}\left(\hat{\mathcal{F}}[f]\right) = 0,\ \hat{\mathcal{F}}[f]$ *uneven*

Theorem 10.13 (Action of the Dilation, Parity and Translation Operators on the Fourier Operator)**.** *Given $f \in L^1(\mathbb{R})$ and $a \in \mathbb{R} \setminus \{0\}$ we have*

1. $\hat{\mathcal{F}}\hat{P} = \hat{P}\hat{\mathcal{F}}$
2. $\hat{\mathcal{F}}\hat{T}_a = e^{-i\lambda a}\hat{\mathcal{F}}$
3. $\hat{T}_a\hat{\mathcal{F}} = \hat{\mathcal{F}}[e^{iax} f(x)]$
4. $\hat{\mathcal{F}}\hat{\Phi}_a = |a|^{-1}\hat{\Phi}_{a^{-1}}\hat{\mathcal{F}}$
5. $\hat{\mathcal{F}}\Phi_a\hat{T}_b = |a|^{-1}e^{-i\lambda b}\hat{\Phi}_{a^{-1}}\hat{\mathcal{F}}$
6. $\hat{\mathcal{F}}\hat{T}_b\hat{\Phi}_a = |a|^{-1}e^{-i\lambda b/a}\hat{\Phi}_{a^{-1}}\hat{\mathcal{F}}$
7. $\hat{\mathcal{F}}\hat{D} = i\lambda\hat{\mathcal{F}}$

Example 10.3.1 (Fourier Transform of the Set Index Function). Let $f(x) = \mathbb{1}_{[-a,a]}(x)$ be the index function of $[-a, a]$, we have

$$\hat{\mathcal{F}}[\mathbb{1}_{[-a,a]}](\lambda) = \int_{\mathbb{R}} \mathbb{1}_{[-a,a]}(x) e^{-i\lambda x} \, \mathrm{d}x = \int_{-a}^{a} e^{-i\lambda x} \, \mathrm{d}x = \frac{i}{\lambda}[\mathrm{e}^{-i\lambda x}]_{-a}^{a} = \frac{2}{\lambda}\sin(\lambda a)$$

For $a = 1/2$ we have $\mathbb{1}_{-1/2,1/2}(x) = \mathrm{rect}(x)$ and therefore

$$\hat{\mathcal{F}}[\mathrm{rect}(x)](\lambda) = \frac{\sin(\lambda/2)}{\lambda/2} = \mathrm{sinc}\left(\frac{\lambda}{2\pi}\right)$$

Example 10.3.2 (Fourier Transform of the Triangle Function). We define the triangle function $\mathrm{tri}(x) = \max\{1-|x|, 0\} = \mathrm{rect}(x/2)(1-|x|)$. We then have

$$\hat{\mathcal{F}}[\max\{1-|x|,0\}] = \int_{\mathbb{R}} \max\{1-|x|,0\}\, e^{-i\lambda x}\,\mathrm{d}x = \int_{-1}^{1} (1-|x|)e^{-i\lambda x}\,\mathrm{d}x$$

Using the properties of the absolute value and using a change in coordinayes we have

$$\hat{\mathcal{F}}[\max\{1-|x|,0\}] = \int_0^1 (1-x)\left(e^{i\lambda x} + e^{-i\lambda x}\right)\mathrm{d}x = 2\int_0^1 (1-x)\cos(\lambda x)\,\mathrm{d}x$$

By direct integration, we therefore get

$$\hat{\mathcal{F}}[\max\{1-|x|,0\}] = \frac{2}{\lambda}\int_0^1 \sin(\lambda x)\,\mathrm{d}x = \frac{2}{\lambda^2}\left(1-\cos(\lambda)\right) = \frac{4}{\lambda^2}\sin^2(\lambda/2) = \mathrm{sinc}^2\left(\frac{\lambda}{2\pi}\right)$$

Example 10.3.3 (A Couple Fourier Transforms More). 1) $f(x) = H(x)e^{-ax}$ where $a \in \mathbb{R},\ a > 0$ This one is quite straightforward. We have

$$\hat{\mathcal{F}}[H(x)e^{-ax}] = \int_{\mathbb{R}} H(x)e^{-(a+i\lambda)x}\,\mathrm{d}x = \int_{\mathbb{R}+} e^{-(a+i\lambda)x} = -\left[\frac{e^{-(a+i\lambda)x}}{a+i\lambda}\right]_0^{+\infty} = \frac{1}{a+i\lambda}$$

2) $f(x) = 1/(1+x^2)$ For this we immediately choose to use the residue theorem, so, we firstly suppose that $\lambda < 0$ and we choose a path enclosing the region $\mathfrak{Im}(z) < 0$, for which the only singularity is given by $\tilde{z} = -i$

$$\hat{\mathcal{F}}\left[\frac{1}{1+x^2}\right] = \lim_{R\to\infty}\int_{\gamma_R^-} \frac{e^{-i\lambda z}}{1+z^2}\,\mathrm{d}z = -2\pi i \operatorname*{Res}_{z=-i}\left(\frac{e^{-i\lambda z}}{1+z^2}\right)$$

Therefore

$$\hat{\mathcal{F}}\left[\frac{1}{1+x^2}\right] = -2\pi i \lim_{z\to -i}\left[(z+i)\frac{e^{-i\lambda z}}{(z+i)(z-i)}\right] = \pi e^{\lambda}$$

Analogously, for $\lambda > 0$ we choose a curve enclosing the region $\mathfrak{Im}(z) > 0$, and therefore, noting that the only encompassed singularity is $\tilde{z} = i$

$$\hat{\mathcal{F}}\left[\frac{1}{1+x^2}\right] = \lim_{R\to\infty}\int_{\gamma_R^+} \frac{e^{-i\lambda z}}{1+z^2}\,\mathrm{d}z = 2\pi i \operatorname*{Res}_{z=i}\left(\frac{e^{-i\lambda z}}{1+z^2}\right)$$

Therefore

$$\hat{\mathcal{F}}\left[\frac{1}{1+x^2}\right] = 2\pi i \lim_{z\to i}\left[(z-i)\frac{e^{-i\lambda z}}{(z+i)(z-i)}\right] = \pi e^{-\lambda}$$

Uniting both cases, i.e. for $\lambda \in \mathbb{R}$, we have finally

$$\hat{\mathcal{F}}\left[\frac{1}{1+x^2}\right] = \pi e^{-|\lambda|}$$

3) $f(x) = e^{-a|x|}$ Last but not least, we can calculate this Fourier transform using the properties of the Fourier operator. Firstly

$$e^{-a|x|} = H(x)e^{-ax} + H(-x)e^{ax}$$

We can also write this as follows

$$e^{-a|x|} = H(x)e^{-ax} + \hat{P}[H(x)e^{-ax}]$$

Therefore, using the linearity of the Fourier transform and the behavior of it under parity transformations, we have

$$\hat{\mathcal{F}}[e^{-a|x|}] = \frac{1}{a+i\lambda} + \hat{P}\left[\frac{1}{a+i\lambda}\right] = \frac{1}{a+i\lambda} + \frac{1}{a-i\lambda} = \frac{2a}{a^2+\lambda^2}$$

Example 10.3.4 (A Particular Way of Solving a Fourier Integral). Take now the function $f(x) = e^{-x^2}$, using the properties of this function under derivation we can build easily a differential equation

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -2xf(x)$$

Applying the Fourier operator on both sides we get

$$\hat{\mathcal{F}}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = -2\hat{\mathcal{F}}\left[xf(x)\right]$$

$$i\lambda\hat{\mathcal{F}}[f(x)] = -2i\frac{\mathrm{d}}{\mathrm{d}\lambda}\hat{\mathcal{F}}[f(x)]$$

Therefore, we've built a differential equation in the Fourier domain, where the searched function is actually the Fourier transform of the initial equation. Solving, we get

$$\frac{\mathrm{d}\log}{\mathrm{d}\lambda}\hat{\mathcal{F}}[f(x)] = -\frac{\lambda}{2}$$

Therefore, imposing the condition of integration on all $\mathbb{R}$ and remembering that $\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$, we have

$$\hat{\mathcal{F}}[f(x)] = \sqrt{\pi}e^{-\frac{\lambda^2}{4}} = \hat{\mathcal{F}}[e^{-x^2}]$$

§§ 10.3.2 Behavior of Fourier Transforms

Theorem 10.14. *Let $f \in L^1(\mathbb{R})$ be some function. Taken $\hat{f}(\lambda) = \hat{\mathcal{F}}[f](\lambda)$, we have that $\hat{f} \in C_0(\mathbb{R})$*

Proof. Firstly, we need to demonstrate that $\hat{f} \in C(\mathbb{R})$. Therefore, by definition of continuity we have

$$\left|\hat{f}(\lambda+\epsilon) - \hat{f}(\lambda)\right| = \left|\int_{\mathbb{R}} f(x)e^{-i\lambda x}(e^{-i\epsilon x} - 1)\,\mathrm{d}x\right|$$

Using the properties of the modulus operator, we have that

$$\left|\hat{f}(\lambda+\epsilon) - \hat{f}(\lambda)\right| \le \int_{\mathbb{R}} |f(x)|\left|e^{-i\epsilon x} - 1\right|\mathrm{d}x$$

For some $a \in \mathbb{R}$, we also have that

$$\int_{\mathbb{R}} |f(x)|\left|e^{-i\epsilon x} - 1\right|\mathrm{d}x \le \int_{|x|\le a} |f(x)|\left|e^{-i\epsilon x} - 1\right|\mathrm{d}x + 2\int_{|x|>a} |f(x)|\left|e^{-i\epsilon x} - 1\right|\mathrm{d}x$$

And for $x \leq a$ we can also say

$$\left|e^{-i\epsilon x} - 1\right| = (1 - \cos(\epsilon x))^2 - \sin^2(\epsilon x) = 4\sin^2\left(\frac{\epsilon x}{2}\right) \leq (\epsilon x)^2 \leq (\epsilon a)^2$$

Letting $\|f\|_1$ be the usual p integral norm on $L^1(\mathbb{R})$, we have therefore

$$\int_{|x|\leq a} |f(x)|\left|e^{-i\epsilon x} - 1\right| \mathrm{d}x + 2\int_{|x|>a} |f(x)|\left|e^{-i\epsilon x} - 1\right| \mathrm{d}x \leq \epsilon|a|\|f\|_1 + 2\int_{|x|>a} |f(x)|\left|e^{-i\epsilon x} - 1\right| \mathrm{d}x$$

The last integral goes to 0 for $a = \epsilon^{-1/2}$, therefore

$$\left|\hat{f}(\lambda + \epsilon) - \hat{f}(\lambda)\right| \leq \epsilon|a|\|f\|_1$$

Proving that $\hat{f} \in C(\mathbb{R})$.
Instead, for proving that $\hat{f}(\lambda) \to 0$ for $\lambda \to \infty$, we have

$$\left|\hat{f}(\lambda)\right| \leq \left|\int_{|x|\leq a} f(x)e^{-i\lambda x}\,\mathrm{d}x\right| + 2\left|\int_{|x|>a} f(x)e^{-i\lambda x}\,\mathrm{d}x\right|$$

We have then

$$\left|\hat{f}(\lambda)\right| \leq \left|\int_{|x|\leq a} f(x)e^{-i\lambda x}\,\mathrm{d}x\right| + \epsilon$$

For the Riemann-Lebesgue lemma we therefore have

$$\lim_{\lambda\to\infty}\left|\hat{f}(\lambda)\right| = \lim_{\lambda\to\infty}\left|\int_{|x|\leq a} f(x)e^{-i\lambda x}\,\mathrm{d}x\right| = 0$$

□

Theorem 10.15. *Given $f \in C^{p-1}(\mathbb{R})$ a function, such that $\partial^p f(x)$ is piecewise continuous, and $\partial^k f(x) \in L^1(\mathbb{R})$ for $k = 1, \cdots, p$. If this holds, we have that*

1. $\hat{\mathcal{F}}[\partial^k f](\lambda) = (i\lambda)^k \hat{\mathcal{F}}[f]$*, for* $k = 1, \cdots, p$
2. $\lim_{\lambda\to\pm\infty} \lambda^p \hat{\mathcal{F}}[f](\lambda) = 0$

Proof. Through integration by parts of the definition of the Fourier transform we have

$$\hat{\mathcal{F}}[f'](\lambda) = [f(x)e^{-i\lambda x}]_{\mathbb{R}} + i\lambda\hat{\mathcal{F}}[f](\lambda)$$

If the evaluation of $f(x)e^{-i\lambda x}$ on all $\mathbb{R}$ gives back 0 we have that the first part of the theorem is demonstrated through iteration.
Using that $f' \in L^1(\mathbb{R})$ tho, we can define using the fundamental theorem of calculus

$$f(x) = f(0) + \int_0^x f'(s)\,\mathrm{d}s$$

Also, since $f' \in L^1(\mathbb{R})$ we must have that the limits at $\pm\infty$ of $f(x)$ must be finite, therefore

$$\lim_{x\to\pm\infty} f(x)e^{-i\lambda x} = 0$$

And

$$\hat{\mathcal{F}}[f'](\lambda) = i\lambda\hat{\mathcal{F}}[f](\lambda)$$

Through this, we therefore have by iteration that

$$\lambda^p\hat{\mathcal{F}}[f](\lambda) = \frac{1}{i^p}\hat{\mathcal{F}}[f^{(p)}](\lambda)$$

Which, thanks to Riemann-Lebesgue, gives

$$\lim_{\lambda\to\infty}\lambda^p\hat{\mathcal{F}}[f](\lambda) = \frac{1}{i^p}\lim_{\lambda\to\infty}\hat{\mathcal{F}}[f](\lambda) = 0$$

□

Theorem 10.16. *Given $f \in L^1(\mathbb{R})$ such that $x^k f \in L^1(\mathbb{R})$ for $k = 1, \cdots, p$, we have that $\hat{\mathcal{F}}[f] \in C^p(\mathbb{R})$, and*

$$\partial^k\hat{\mathcal{F}}[f](\lambda) = \hat{\mathcal{F}}[(-ix)^k f(x)](\lambda)$$

Proof. In order to see if it's true, we start for the first derivative and apply the definition. Therefore, given $\hat{f}(\lambda) = \hat{\mathcal{F}}[f](\lambda)$

$$\left|\frac{1}{\epsilon}(\hat{f}(\lambda+\epsilon) - \hat{f}(\lambda)) - \int_{\mathbb{R}}(-ix)f(x)e^{-i\lambda x}\,\mathrm{d}x\right|$$

Using the triangle inequality and the definition of $\hat{f}(\lambda)$ we have

$$\int_{\mathbb{R}}|f(x)|\left|\frac{e^{-i\epsilon x}-1}{\epsilon} + ix\right|\mathrm{d}x = \int_{\mathbb{R}}|xf(x)|\left|\frac{e^{-i\epsilon x}-1}{\epsilon x} + i\right|\mathrm{d}x$$

Dividing the improper integral around a point $a \in \mathbb{R}$ we have that everything is lesser or equal to the following quantity

$$\int_{|x|\le a}|xf(x)|\left|\frac{e^{-i\epsilon x}-1}{\epsilon x} + i\right|\mathrm{d}x + 2\int_{|x|>a}|xf(x)|\left|\frac{e^{-i\epsilon x}-1}{\epsilon x} + i\right|\mathrm{d}x$$

We also have that

$$\left|\frac{e^{-i\epsilon x}-1}{\epsilon x} + i\right| \le \left|\frac{\cos(\epsilon x)-1}{\epsilon x} + i\frac{\epsilon x\sin(\epsilon x)}{\epsilon x}\right| \le \frac{|\cos(\epsilon x)-1|}{|\epsilon x|} + \frac{|\epsilon x - \sin(\epsilon x)|}{|\epsilon x|}$$

Using the Taylor expansions for the cosine and the sine we get, approximating, that

$$|\cos\theta - 1| \le \frac{1}{2}\theta^2$$
$$|\sin(\theta) - \theta| = \frac{1}{3!}|\theta|^3$$

Therefore

$$\left|\frac{e^{-i\epsilon x}-1}{\epsilon x}+i\right| \leq \frac{1}{2}|\epsilon x|^2+\frac{1}{6}|\epsilon x|^3$$

Putting that back into the integral, we have that it must be smaller or equal to the following quantity

$$\int_{|x|\leq a}|xf(x)|\left(\frac{|\epsilon x|}{2}+\frac{|\epsilon x|^2}{6}\right)\mathrm{d}x+2\int_{|x|>a}|xf(x)|\left(\frac{\epsilon x}{2}+\frac{|\epsilon x|^2}{6}\right)\mathrm{d}x$$

Using in the first integral that $|\epsilon x|\leq\epsilon|a|$ we get, that all the quantity will be surely less than the supremum of such, and therefore

$$\left(\frac{|\epsilon a|}{2}+\frac{|\epsilon a|}{6}\right)\|xf(x)\|_1+2\int_{|x|>a}|xf(x)|\,\mathrm{d}x$$

Therefore, imposing as before $a=\epsilon^{-1/2}$, we get

$$\lim_{\epsilon\to 0}\left|\frac{\hat{f}(\lambda+\epsilon)-\hat{f}(\lambda)}{\epsilon}-\int_{\mathbb{R}}(-ix)f(x)e^{-i\lambda x}\,\mathrm{d}x\right|=0$$

Proving the thesis of the theorem □

Theorem 10.17 (Invariance of the Schwartz Space under Fourier Transforms)**.** *Given* $f\in\mathcal{S}(\mathbb{R})\subset L^1(\mathbb{R})$*, then* $\hat{\mathcal{F}}[f]\in\mathcal{S}(\mathbb{R})$*.*

Proof. We will prove a weaker assumption. Since $\mathcal{S}(\mathbb{R})=\left(C^\infty(\mathbb{R}),\|\cdot\|_{j,k}\right)$ where the seminorm $\|\cdot\|_{j,k}$ is defined as follows

$$\|f\|_{j,k}=\left\|x^j\partial^k f\right\|_u<\infty \qquad j,k\in\mathbb{N}$$

We have, $\forall f\in\mathcal{S}(\mathbb{R})$ that for a given constant $C_{a,k}\in\mathbb{R}$

$$\left|\partial^k f\right|\leq\frac{C_{a,k}}{(x^2+1)^{\frac{a}{2}}} \qquad a,k\in\mathbb{N},\ x\in\mathbb{R}$$

Therefore, taken $a=j+2$

$$\int_{\mathbb{R}}\left|x^j\partial^k f(x)\right|\mathrm{d}x\leq C_{j+2,k}\int_{\mathbb{R}}\frac{|x|^j}{(x^2+1)^{\frac{j}{2}+1}}\,\mathrm{d}x\leq C_{j+2,k}\int_{\mathbb{R}}\frac{1}{x^2+1}\,\mathrm{d}x<\infty$$

Therefore, we have that $x^j\partial^k f\in L^1(\mathbb{R})$ $\forall j,k\in\mathbb{N}$. But we can also write the following result using Leibniz's rule

$$\partial^j\left(x^k f(x)\right)=\sum_{m=0}^{j}\binom{j}{m}\partial^m x^j\partial^{j-m}f(x)\in L^1(\mathbb{R})$$

Therefore, applying the Fourier transform and using the previous property, we have

$$(i\lambda)^j(\partial^k\hat{f}(\lambda))=(i\lambda)^j(-i)^k\hat{\mathcal{F}}[x^k f](\lambda)=(-i)^k\hat{\mathcal{F}}[\partial^j(x^k f)](\lambda)\in C_0(\mathbb{R})$$

Which finally gives

$$\left\|\hat{f}\right\|_{j,k}=\left\|\hat{\mathcal{F}}[\partial^j(x^k f)](\lambda)\right\|_{j,k}<\infty \qquad \forall j,k\in\mathbb{N}$$

Which finally gives

$$f\in\mathcal{S}(\mathbb{R})\implies\hat{\mathcal{F}}[f]\in\mathcal{S}(\mathbb{R})$$

□

§§ 10.3.3 Inverse Fourier Transform

Definition 10.3.3 (Fourier Antitransform)**.** Let $f \in L^1(\mathbb{R})$, we define the *Fourier Antitransform* as $\hat{\mathcal{F}}^a[f](x)$ the following computation

$$\hat{\mathcal{F}}^a[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(\lambda)e^{i\lambda x}\,\mathrm{d}\lambda \tag{10.14}$$

Note that, taken the transformation $\lambda = -u$ we get

$$\hat{\mathcal{F}}^a[f](x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(-u)e^{-iux}\,\mathrm{d}u = \frac{1}{2\pi}\hat{\mathcal{F}} \circ \hat{P}[f]$$

This brings to the definition of the following theorem

Theorem 10.18 (Inversion Formula)**.** *Given $f \in \mathcal{S}(\mathbb{R})$, we have that $\hat{\mathcal{F}}^a\hat{\mathcal{F}}[f] = \hat{\mathcal{F}}\hat{\mathcal{F}}^a[f] = \hat{\mathbb{1}}[f] = f$, therefore $\hat{\mathcal{F}}^a = \hat{\mathcal{F}}^{-1}$ and the Fourier transform is a bijective map in $\mathcal{S}(\mathbb{R})$*

$$\hat{\mathcal{F}} : \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R})$$

Proof. Weakening the statement, we can say that taken $f \in \mathcal{K}$ such that $f(x) = 0$ for $|x| > a \in \mathbb{R}$, and taken $\epsilon \in (0, 1/a)$, we have that, Fourier expanding the function in $[-\epsilon^{-1}, \epsilon]$, we get

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ik\pi\epsilon x}, \quad c_k = \frac{2}{\epsilon}\int_{-\frac{1}{\epsilon}}^{\frac{1}{\epsilon}} f(x)e^{-ik\pi\epsilon x}\,\mathrm{d}x \to \frac{\epsilon}{2}\hat{\mathcal{F}}[f](k\pi\epsilon)$$

Therefore, we can immediately write

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{\mathcal{F}}[f](k\pi\epsilon)e^{ik\pi\epsilon x}$$

Letting $\lambda_{k,\epsilon} = k\pi\epsilon$ we have

$$f(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{\mathcal{F}}[f](\lambda_{k,\epsilon})e^{i\lambda_{k,\epsilon}}\Delta\lambda_{k,\epsilon} \to \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\mathcal{F}}[f](\lambda)e^{i\lambda x}\,\mathrm{d}x = \hat{\mathcal{F}}^a\hat{\mathcal{F}}[f]$$

Where we let $\epsilon \to 0^+$ in the Fourier series, which written in that way gives a Riemann-Lebesgue sum converging to the integral of the antitransform, therefore proving that in $\mathcal{S}(\mathbb{R})$ $\hat{\mathcal{F}}^a = \hat{\mathcal{F}}^{-1}$ □

Theorem 10.19 (Plancherel)**.** *Given $f, g \in \mathcal{S}(\mathbb{R})$, and let $\langle\cdot,\cdot\rangle$ be the usual scalar product defined as follows*

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x$$

Then, we have

$$\begin{aligned} \left\|\hat{\mathcal{F}}[f]\right\|_2 &= \sqrt{2\pi}\|f\|_2 \\ \langle\hat{\mathcal{F}}[f], \hat{\mathcal{F}}[g]\rangle &= 2\pi\langle f, g\rangle \end{aligned} \tag{10.15}$$

Proof. For Parseval we have

$$\int_{\mathbb{R}} \|f(x)\|^2 \, \mathrm{d}x = \frac{\epsilon}{2} \sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left|\hat{\mathcal{F}}[f](\lambda_{k,\epsilon})\right| \Delta\lambda_{k,\epsilon}$$

Taking the limit $\epsilon \to 0^+$ the sum on the right converges to the following value

$$\int_{\mathbb{R}} |f(x)|^2 \, \mathrm{d}x = \frac{1}{2\pi} \int_{\mathbb{R}} \left|\hat{\mathcal{F}}[f]\right|^2 \mathrm{d}\lambda \implies \|f\|_2 = \sqrt{2\pi}\|f\|_2$$

Then for the polarization identity, and this result, we get the final proof of the theorem □

Theorem 10.20 (Continuity Expansion)**.** *Let $p, q \in C(\mathbb{R})$, then if $p(x) = q(x) \quad \forall x \in \mathbb{Q}$ we have that*

$$p(x) = q(x) \quad \forall x \in \mathbb{R}$$

This holds for any dense subset of $\mathbb{R}$

Proof. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we can take a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{Q}$ such that $x_n \to x \in \mathbb{R}$. Therefore, using the continuity of p, q we have

$$p(x) = p\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} q(x_n) = q\left(\lim_{n\to\infty} x_n\right) = q(x)$$

□

Theorem 10.21 (Extension of the Inversion Formula)**.** *Let $f, g \in \mathcal{S}(\mathbb{R})$. Taken a metric $d_{\mathcal{S}}(\cdot,\cdot) : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathbb{R}$ defined as follows*

$$d_{\mathcal{S}}(f, g) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{2^{j+k}} \frac{\|f-g\|_{j,k}}{1 + \|f-g\|_{j,k}} \quad j, k \in \mathbb{N}$$

Where $\|\cdot\|_{j,k}$ is the Schwartz seminorm.
Since $\mathcal{K} \subset \mathcal{S}(\mathbb{R})$ is dense with this norm, we have that we can extend continuously the Fourier inversion formula $\forall f \in \mathcal{S}(\mathbb{R})$, we have

$$\hat{\mathcal{F}}^a \hat{\mathcal{F}} = \hat{\mathcal{F}} \hat{\mathcal{F}}^a = \hat{\mathbb{1}} \implies \hat{\mathcal{F}}^a = \hat{\mathcal{F}}^{-1} \quad \forall f \in \mathcal{S}(\mathbb{R})$$

Theorem 10.22. *Given $f \in L^1(\mathbb{R})$, then, since we might have that $\hat{\mathcal{F}}[f] \notin L^1(\mathbb{R})$, using the Cauchy principal value*

$$\frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \hat{\mathcal{F}}[f](\lambda) e^{i\lambda x} \, \mathrm{d}\lambda = \frac{1}{2}\left(f(x^-) + f(x^+)\right) \quad \forall x \in \mathbb{R} \tag{10.16}$$

And, if f is continuous in x, we have that

$$\frac{1}{2\pi} \operatorname{PV} \int_{\mathbb{R}} \hat{\mathcal{F}}[f](\lambda) e^{i\lambda x} \, \mathrm{d}\lambda = \hat{\mathcal{F}}^{-1} \hat{\mathcal{F}}[f] = \hat{\mathbb{1}}[f(x)] = f(x)$$

Theorem 10.23 (New Calculation Rules)**.** *With what we added so far, in operatorial form, we can write down two new calculation rules. Supposing the inversion formula holds, and therefore $\hat{\mathcal{F}}^a = \hat{\mathcal{F}}^{-1}$*

$$\begin{aligned} \hat{\mathcal{F}}^{-1} &= \frac{1}{2\pi}\left(\hat{\mathcal{F}}\hat{P}\right) \\ \hat{\mathcal{F}}^{-1}\hat{\mathcal{F}} &= 2\pi\hat{\mathbb{1}} \end{aligned} \tag{10.17}$$

Example 10.3.5. Let's calculate the Fourier transform of the following function $f : \mathbb{R} \longrightarrow \mathbb{C}$

$$f(x) = \frac{1}{(x+i)^3}$$

Acting symbolically on this we know already that $\hat{\mathcal{F}}\left[H(x)e^{-x}\right] = (1+i\lambda)^{-1}$, therefore

$$\hat{\mathcal{F}}^{-1}\left[\frac{1}{1+ix}\right](\lambda) = 2\pi H(-\lambda)e^{\lambda}$$

Applying the parity operator and multiplying the inverse-transformed function by $-i$ we obtain

$$\hat{\mathcal{F}}^{-1}\left[\frac{1}{x+i}\right] = -2i\pi H(\lambda)e^{-\lambda}$$

Lastly, we can derive it twice and divide the result by two, obtaining

$$\hat{\mathcal{F}}^{-1}\left[\frac{1}{(x+i)^3}\right] = i\pi\lambda^2 H(\lambda)e^{-\lambda}$$

§§ 10.3.4 Convolution Product

Definition 10.3.4 (Convolution Product)**.** Given $f, g \in L^1$ two bounded functions, we define the *convolution* of these two functions as follows

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}y \tag{10.18}$$

Theorem 10.24. *Defined the convolution product of two bounded functions, we have that*

$$* : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$$

And

$$\begin{aligned}\|f * g\|_u &\leq \|f\|_u \|g\|_1 \\ \|f * g\|_1 &\leq \|f\|_1 \|g\|_1\end{aligned}$$

Proof. The proof of the first result is direct

$$\left|\int_{\mathbb{R}} f(y)g(x-y)\,\mathrm{d}x\right| \leq \|f\|_u \int_{\mathbb{R}} |g(x-y)|\,\mathrm{d}y = \|f\|_u \|g\|_1$$

For the second proof, we begin taking a compact set $[a, b] \subset \mathbb{R}$ and we move forward sending $a \to -\infty$ and $b \to \infty$. Therefore

$$\int_a^b |(f * g)(x)|\,\mathrm{d}x \leq \int_{\mathbb{R}} |f(y)| \int_{a-y}^{b-y} |g(u)|\,\mathrm{d}u \leq \iint_{\mathbb{R}^2} |f(y)||g(u)|\,\mathrm{d}y\,\mathrm{d}u = \|f\|_1 \|g\|_1$$

Therefore we have that

$$* : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$$

And that the convolution product is bounded $\square$

Theorem 10.25 (Properties of the Convolution Product). *Given $f, g, h \in L^1(\mathbb{R})$ three bounded functions, then*

$$\begin{gathered} f * g = g * f \\ (f * g) * h = f * (g * h) = f * g * h \end{gathered} \tag{10.19}$$

These properties are easily demonstrable using the properties of the integral

Theorem 10.26 (Derivation of a Convolution). *Given $f \in L^1(\mathbb{R})$ a bounded function, and $g \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$ a bounded function, we have that*

$$\frac{\mathrm{d} f * g}{\mathrm{d} x} = f * \frac{\mathrm{d} g}{\mathrm{d} x} = f * g' \tag{10.20}$$

Proof. Written $A(x,t) = f(t)g(x-t)$ we have that

$$(f * g)(x) = \int_{\mathbb{R}} A(x,t) \,\mathrm{d}t$$

And therefore

$$\frac{\mathrm{d} f * g}{\mathrm{d} x} = \frac{\partial}{\partial x} \int_{\mathbb{R}} A(x,t) \,\mathrm{d}t$$

Since we also have that, due to the boundedness of g', that

$$\left| \frac{\partial A}{\partial x} \right| = |f(t) g'(x-t)| \leq M |f(t)|$$

Due to the fact that $f(t) \in L^1(\mathbb{R})$ the integral is well defined, and using Leibniz's derivation rule, we have

$$\frac{\mathrm{d} f * g}{\mathrm{d} x} = \int_{\mathbb{R}} \frac{\partial A}{\partial x} \,\mathrm{d}t = \int_{\mathbb{R}} f(t) g'(x-t) \,\mathrm{d}t = (f * g')(x)$$

□

Theorem 10.27 (Fourier Transform of a Convolution). *Given $f, g \in L^1(\mathbb{R})$ two bounded functions, we have that*

$$\hat{\mathcal{F}}[f * g] = \hat{\mathcal{F}}[f] \hat{\mathcal{F}}[g] \tag{10.21}$$

Proof. The proof of this is quite direct, we have that

$$\begin{aligned} \hat{\mathcal{F}}[f * g](\lambda) = \int_{\mathbb{R}} (f * g)(x) e^{-i\lambda x} \,\mathrm{d}x = \int_{\mathbb{R}} e^{-i\lambda x} \int_{\mathbb{R}} f(y) g(x-y) \,\mathrm{d}y \,\mathrm{d}x = \\ = \int_{\mathbb{R}} f(y) e^{-i\lambda y} \int_{\mathbb{R}} g(x-y) e^{-i\lambda(x-y)} \,\mathrm{d}x \,\mathrm{d}y = \hat{\mathcal{F}}[f] \hat{\mathcal{F}}[g] \end{aligned}$$

Where on the last equality we used the Fubini-Tonelli theorem □

Example 10.3.6 (Convolution of Two Set Functions). Given a set $A = [-a, a]$ and a set $B = [-b, b]$, we know that

$$\begin{gathered} \mathbb{1}_A(x) \mathbb{1}_B(x) = \mathbb{1}_{A \cap B}(x) \\ \mathbb{1}_{[-a,a]}(x+y) = \mathbb{1}_{[-a+y, a+y]}(x) \end{gathered}$$

Suppose we need to calculate the convolution product $\mathbb{1}_A * \mathbb{1}_A$. The calculation is quite easy with those properties

$$(\mathbb{1}_A * \mathbb{1}_A)(x) = \int_{\mathbb{R}} \mathbb{1}_A(y)\mathbb{1}_A(x-y)\,\mathrm{d}y$$

Taken $C = [-a+x, a+x]$, we have

$$(\mathbb{1}_A * \mathbb{1}_A)(x) = \int_{\mathbb{R}} \mathbb{1}_{A\cap C}(y)\,\mathrm{d}y = \mu\,(A \cap C)$$

Where, we know already that

$$\mu\,(A\cap C) = \begin{cases} 0 & x < 2a,\ x > 2a \\ 2a - x & x \in [0, 2a] \\ 2a + x & x \in [2a, 0] \end{cases}$$

Summarized

$$\mu\,(A\cap C) = \max\{2a - |x|, 0\} = (\mathbb{1}_A * \mathbb{1}_A)(x)$$

Remembering the definition of the $\mathrm{rect}(x)$, $\mathrm{tri}(x)$ functions, we get that

$$(\mathrm{rect} * \mathrm{rect})(x) = \max\{1 - |x|, 0\} = \mathrm{tri}(x)$$

Example 10.3.7 (Two Gaussians). Taken $\alpha, \beta \in \mathbb{R}$ we might want to calculate the following convolution product

$$e^{-\alpha x^2} * e^{-\beta x^2} = \int_{\mathbb{R}} e^{-\alpha y^2} e^{-\beta(x-y)^2}\,\mathrm{d}x$$

The direct calculation of the integral is immediate, but we might want to test here the power of the last theorem that was stated. Hence we have

$$\hat{\mathcal{F}}^{-1}\left[\hat{\mathcal{F}}[e^{-\alpha x^2} * e^{-\beta x^2}]\right] = \hat{\mathcal{F}}^{-1}\left[\hat{\mathcal{F}}[e^{-\alpha x^2}]\hat{\mathcal{F}}[e^{-\beta x^2}]\right]$$

Using that

$$\hat{\mathcal{F}}[e^{ax^2}](\lambda) = \sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$$

We have immediately that the searched convolution will be

$$e^{-\alpha x^2} * e^{-\beta x^2} = \frac{\pi}{\sqrt{\alpha\beta}}\hat{\mathcal{F}}^{-1}\left[e^{-\frac{\lambda^4}{4}\left(\frac{\alpha+\beta}{\alpha\beta}\right)}\right](x)$$

Which gives

$$\frac{\pi}{\sqrt{\alpha\beta}}\hat{\mathcal{F}}^{-1}\left[e^{-\frac{\lambda^2}{4}\left(\frac{\alpha+\beta}{\alpha\beta}\right)}\right](x) = \frac{\pi}{\sqrt{\alpha\beta}}\frac{1}{\sqrt{\pi}}\sqrt{\frac{\alpha\beta}{\alpha+\beta}}e^{-\left(\frac{\alpha\beta}{\alpha+\beta}\right)x^2}$$

With some rearrangement of the constant terms, we get finally the expected result

$$e^{-\alpha x^2} * e^{-\beta x^2} = \sqrt{\frac{\pi}{\alpha+\beta}}e^{-\left(\frac{\alpha\beta}{\alpha+\beta}\right)x^2}$$

§§ 10.3.5 Solving the Heat Equation with Fourier Transforms

The power of Fourier Calculus, as we seen in the Fourier series section, also comes when dealing with differential equations. In this section, we will consider once again the heat equation, but instead of considering a finite rod, we will suppose that the rod is actually infinite. We therefore get the following partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R},\ t \in \mathbb{R}^+ \setminus \{0\} \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

We begin by applying the Fourier transform on the solution as follows. The transformed variable will be indicated as an index

$$v(\lambda, t) = \hat{\mathcal{F}}_x[u(x,t)](\lambda) = \int_{\mathbb{R}} u(x,t)e^{-i\lambda x}\,\mathrm{d}x \tag{10.22}$$

Using the properties of the Fourier transform, we have

$$\hat{\mathcal{F}}_x[\partial_x^2 u](\lambda) = (i\lambda)^2\hat{\mathcal{F}}_x[u(x,t)](\lambda) = -\lambda^2 v(\lambda,t)$$

Supposing also that $v(x,t)$ is derivable with respect to t (i.e. Leibniz's rule holds), we have

$$\frac{\partial v}{\partial t} = \hat{\mathcal{F}}[\partial_t u(x,t)](\lambda) = \int_{\mathbb{R}} \frac{\partial u}{\partial t} e^{-i\lambda x}\,\mathrm{d}x$$

And the heat equation, becomes after the Fourier transformation

$$\begin{cases} \frac{\partial v}{\partial t} = -k\lambda^2 v(\lambda,t) & \lambda \in \mathbb{R},\ t \in \mathbb{R}^+ \setminus \{0\} \\ v(\lambda,0) = \hat{\mathcal{F}}[u_0(x)](\lambda) & \lambda \in \mathbb{R} \end{cases} \tag{10.23}$$

The solution is almost immediate, and we therefore get

$$v(\lambda,t) = v(\lambda,0)e^{-k\lambda^2 t} = \hat{\mathcal{F}}[u_0(x)](\lambda)e^{-k\lambda^2 t}$$

Note that

$$e^{-k\lambda^2 t} = \frac{1}{2\sqrt{\pi k t}}\hat{\mathcal{F}}_x\left[e^{-\frac{x^2}{4kt}}\right](\lambda)$$

Therefore, remembering that the product of transforms gives the transform of the convolution

$$v(\lambda,t) = \hat{\mathcal{F}}_x[u(x,t)] = \frac{1}{2\sqrt{\pi k t}}\hat{\mathcal{F}}_x\left[u_0 * e^{-\frac{x^2}{4kt}}\right](\lambda)$$

And, the searched solution will therefore be the following

$$u(x,t) = \frac{1}{2\sqrt{\pi k t}} u_0 * e^{-\frac{x^2}{4kt}}$$

Appendices

A Useful Concepts

§ A.1 Multi-Index Notation

In order to ease various calculations one can utilize more abstract index constructions. One of these is the *multi-index* notation, where instead of having an index $i \in \mathbb{N}$ or $j \in \mathbb{Z}$, one constructs a "vector" of indices, like $\alpha = (a_1, \cdots, a_n) \in \mathbb{N}^n$ or $\beta = (b_1, \cdots, b_n) \in \mathbb{Z}^n$.
This notation includes a set of operations on such multi-indexes, defined as follows

Theorem A.1 (Operations on Multi-indexes)**.** *Given a multi-index $\alpha \in \mathbb{N}^n$, one can define the following operations on them*

$$\begin{aligned} |\alpha| &= \sum_{i=1}^{n} a_i \\ \alpha! &= \prod_{i=1}^{n} a_i! \end{aligned} \tag{A.1}$$

Given $x \in \mathbb{R}^n$ and the del operator ∂ one can also write

$$\begin{aligned} x^\alpha &= \prod_{i=1}^{n} x_i^{a_i} \\ \partial^\alpha &= \prod_{i=1}^{n} \partial_i^{a_i} = \frac{\partial^{|\alpha|}}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}} = \frac{\partial^{|\alpha|}}{\partial x^\alpha} \end{aligned} \tag{A.2}$$

§ A.2 Properties of the Fourier Transform

Here's a list of the properties of the Fourier transform, useful for dealing with calculations in operatorial form

$$
\begin{aligned}
\hat{\mathcal{F}}\hat{P} &= \hat{P}\hat{\mathcal{F}} \\
\hat{\mathcal{F}}\hat{T}_a &= e^{-i\lambda a}\hat{\mathcal{F}} \\
\hat{T}_a\hat{\mathcal{F}} &= \hat{\mathcal{F}}\left[e^{-iax}f(x)\right] \qquad a\in\mathbb{R},\ f\in L^1(\mathbb{R}) \\
\hat{\mathcal{F}}\hat{\Phi}_a &= \frac{1}{|a|}\hat{\Phi}_{\frac{1}{a}}\hat{\mathcal{F}} \qquad a\in\mathbb{R}\setminus\{0\} \\
\hat{\mathcal{F}}\hat{\Phi}_a\hat{T}_b &= \frac{e^{-i\lambda b}}{|a|}\hat{\Phi}_{\frac{1}{a}}\hat{\mathcal{F}} \qquad a\in\mathbb{R}\setminus\{0\},\ b\in\mathbb{R} \\
\hat{\mathcal{F}}\hat{T}_b\hat{\Phi}_a &= \frac{e^{-i\lambda\frac{b}{a}}}{|a|}\hat{\Phi}_{\frac{1}{a}}\hat{\mathcal{F}} \qquad a\in\mathbb{R}\setminus\{0\},\ b\in\mathbb{R} \\
\hat{\mathcal{F}}\hat{\partial} &= i\lambda\hat{\mathcal{F}} \\
\hat{\partial}\hat{\mathcal{F}} &= \hat{\mathcal{F}}\left[-ixf(x)\right] \qquad f\in L^1(\mathbb{R}) \\
\hat{\mathcal{F}}^{-1} &= \frac{1}{2\pi}\hat{\mathcal{F}}\hat{P} = \frac{1}{2\pi}\hat{P}\hat{\mathcal{F}} \\
\hat{\mathcal{F}}^{-1}\hat{\mathcal{F}} &= \hat{\mathcal{F}}\hat{\mathcal{F}}^{-1} = 2\pi\hat{\mathbb{1}}
\end{aligned}
\tag{A.3}
$$

B Common Fourier Transforms

§ B.1 Common Fourier Transforms

In this appendix, you'll find a table of some particular Fourier transforms that might pop up in Fourier calculus, and might also be helpful for calculating more complex transforms.
Note that

$$\hat{\mathcal{F}}[f](\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}\,\mathrm{d}x \tag{B.1}$$

$f(x)$	Conditions	$\hat{\mathcal{F}}[f](\lambda)$
$\mathbb{1}_{[-a,a]}(x)$		$\dfrac{2\sin(\lambda a)}{\lambda}$
$\mathrm{rect}(x)$		$\mathrm{sinc}(\lambda/2\pi)$
$\mathrm{tri}(x)$		$\mathrm{sinc}^2(\lambda/2\pi)$
$\mathrm{sinc}(x)$		$\mathrm{rect}(\lambda/2\pi)$
$\cos(x)\mathbb{1}_{[-a,a]}$		$\dfrac{\sin[(\lambda+1)a]}{\lambda+1} + \dfrac{\sin[(\lambda-1)a]}{\lambda-1}$

$f(x)$	Conditions	$\hat{\mathcal{F}}[f](\lambda)$
$e^{-\|x\|}$		$\frac{2a}{\lambda^2+a^2}$
$H(x)e^{-ax}$	$a>0$	$\frac{1}{a+i\lambda}$
$H(ax)e^{-ax}$		$\frac{\mathrm{sgn}(a)}{a+i\lambda}$
$\frac{1}{a^2+x^2}$		$\frac{\pi}{a}e^{-a\|\lambda\|}$
$\frac{1}{a^2x^2+bx+c}$	$\beta=\frac{\sqrt{4ac-b^2}}{2\|a\|}>0$	$\frac{\pi}{\beta a}e^{-\beta\|\lambda\|+i\frac{\lambda}{2a}}$
$\frac{1}{(x^2+1)^2}$		$\frac{\pi}{2}e^{-\|\lambda\|}(\|\lambda\|+1)$
$\frac{1}{1+x^4}$		$\pi e^{-\frac{\|\lambda\|}{\sqrt{2}}}\sin\left(\frac{\lambda}{2}+\frac{\pi}{4}\right)$
e^{-ax^2}		$\sqrt{\frac{\pi}{a}}e^{-\frac{\lambda^2}{4a}}$
e^{ix^2}		$\sqrt{\frac{\pi}{2}}(1+i)e^{-\frac{i\lambda^2}{4}}$
e^{-ix^2}		$\sqrt{\frac{\pi}{2}}(1-i)e^{\frac{i\lambda^2}{4}}$
$\cos(x^2)$		$\sqrt{\pi}\cos\left[\frac{\lambda^2-\pi}{4}\right]$
$\sin(x^2)$		$-\sqrt{\pi}\sin\left[\frac{\lambda^2-\pi}{4}\right]$
$\mathrm{sech}(x)$		$\pi\,\mathrm{sech}\left(\frac{\pi\lambda}{2}\right)$
$\frac{e^{-\frac{x^2}{2}}}{\sqrt{\sqrt{\pi}2^n n!}}H_n(x)$		$\frac{(-i)^n\sqrt{2\pi}e^{\frac{\lambda^2}{2}}}{\sqrt{\sqrt{\pi}2^n n!}}H_n(\lambda)$

Remember that:

$$\begin{cases} \operatorname{sinc}\left(\dfrac{x}{2\pi}\right) = \dfrac{2}{x}\sin\left(\dfrac{x}{2}\right) \\ \operatorname{rect}(x) = \mathbb{1}_{[-\frac{1}{2},\frac{1}{2}]}(x) \\ \operatorname{tri}(x) = \max\{1-|x|, 0\} \\ \operatorname{sech}(x) = \dfrac{1}{\cosh(x)} \end{cases} \tag{B.2}$$

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