Advanced Classical Mechanics
And
Special Relativity

Università degli studi di Roma "La Sapienza"
Physics and Astrophysics BSc

Matteo Cheri

Notes on Advanced Classical Mechanics and Special Relativity

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Part I

Lagrangian Mechanics

1 Newtonian Mechanics

§ 1.1 Particle Mechanics

Before dwelving into Lagrangian mechanics it's always a good thing to check up again on Newtonian mechanics.
In this section we will suppose of working with a «single» particle of mass m with radius vector x^μ, with $\mu = 1, 2, 3$. We immediately define the following

Definition 1.1.1 (Velocity and Momentum)**.** Given a particle of mass m and radius vector $x^\mu(t)$, we define the «velocity» as follows

$$v^\mu(t) = \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \dot{r}^\mu(t) \tag{1.1}$$

From this we define the «momentum» as follows

$$p^\mu(t) = mv^\mu(t) \tag{1.2}$$

The «units» of both quantities are, since $[x^\mu] = L$, $[\mathrm{d}t] = T$, $[m] = M$ where $L,\ T,$ stand respectively for «length», «time» and «mass» are

$$[v^\mu] = \frac{L}{T}$$
$$[p^\mu] = \frac{ML}{T}$$

In the International System of units, therefore

$$[v^\mu] = \left[\frac{m}{s}\right]$$
$$[p^\mu] = \left[\frac{Kg \cdot m}{s}\right]$$

If the particle is subject to n forces $f^\mu_{(i)}$, and therefore a total force F^μ where

$$F^\mu = \sum_{i=1}^{n} f^\mu_{(i)}$$

We have, from Newton's second law of motion, that the motion of the particle is described as follows

$$F^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}t} = \dot{p}^\mu \tag{1.3}$$

Where, the units of force are

$$[F^\mu] = \left[\frac{\mathrm{d}p^\mu}{\mathrm{d}t}\right] = \frac{ML}{T^2} = \left[\frac{Kg\cdot m}{s^2}\right] = [N]$$

Where N stands for «Newtons» .
Or, if m is constant

$$F^\mu = m\frac{\mathrm{d}v^\mu}{\mathrm{d}t} = ma^\mu \tag{1.4}$$

Where a^μ is the «acceleration» and is defined as

$$a^\mu = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}t^2} = \ddot{r}^\mu$$

With units

$$[a^\mu] = \frac{L}{T^2} = \left[\frac{m}{s^2}\right]$$

From Newton's second law (1.3), we can define immediately a «conservation law» , which eases the solution of the differential equation

Theorem 1.1 (Conservation of Momentum)**.** ***If the sum of forces acting on a particle $F^\mu = 0$, then the momentum p^μ is constant***

Proof. The proof is immediate. Since we have $F^\mu = 0$, we insert it into Newton's second law and we get

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}t} = 0$$

Which implies that p^μ is constant. □

Having defined these quantities, we can define two new quantities

Definition 1.1.2 (Angular Momentum and Torque)**.** Given a particle with mass m and radius vector $x^\mu(t)$ we define the «angular momentum» as follows

$$L_\mu = \epsilon_{\mu\nu\gamma}x^\nu p^\gamma \tag{1.5}$$

Where $\epsilon_{\mu\nu\gamma}x^\nu p^\gamma \to \mathbf{x}\wedge\mathbf{p}$. Analoguously, we define the «torque» or «momentum of a force» as

$$\tau_\mu = \epsilon_{\mu\nu\gamma}x^\nu F^\gamma$$

The units of these quantities are

$$[L_\mu] = \frac{ML^2}{T} = [N\cdot m\cdot s] = [J\cdot s]$$
$$[\tau_\mu] = \frac{ML^2}{T^2} = [N\cdot m] = [J]$$

Where J stands for «Joules»

It's immediate then to demonstrate the following theorem

Theorem 1.2 (Conservation of Angular Momentum)**.** ***If the sum of torques acting on a particle*** $\tau_\mu = 0$***, then*** L_μ ***is constant***

Proof. The proof isn't as straightforward as for the conservation of momentum, but we begin from the definition of torque and applying the chain rule for wedge products

$$\tau_\mu = \epsilon_{\mu\nu\gamma} x^\nu \frac{\mathrm{d}p^\gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \epsilon_{\mu\nu\gamma} x^\nu p^\gamma - \epsilon_{\mu\nu\gamma} v^\nu p^\gamma$$

Since $v^\mu \parallel p^\mu$ the second term is null and therefore

$$\tau_\mu = \frac{\mathrm{d}}{\mathrm{d}t} \epsilon_{\mu\nu\gamma} x^\nu p^\gamma = \frac{\mathrm{d}L_\mu}{\mathrm{d}t}$$

Then, we have that if $\tau_\mu = 0$

$$\frac{\mathrm{d}L_\mu}{\mathrm{d}t} = 0$$

Which implies that L_μ is constant. □

After this, let's consider the work done by some external force F^μ in a path γ between two points a, b. By definition of Work, this is equal to the following equation

$$W_{ab} = \int_\gamma F^\mu \, \mathrm{d}x_\mu = \int_a^b F^\mu v_\mu \, \mathrm{d}t \tag{1.6}$$

For a constant mass, we have then

$$W_{ab} = m \int_a^b \frac{\mathrm{d}v^\mu}{\mathrm{d}t} v_\mu \, \mathrm{d}t = \frac{m}{2} \int_a^b \mathrm{d}v^2 = \frac{m}{2}(v_b^2 - v_a^2) \tag{1.7}$$

We define the following scalar quantity

Definition 1.1.3 (Kinetic Energy)**.** The «kinetic energy» of a particle is defined as the following scalar quantity

$$T = \frac{1}{2} m v^2(t) = \frac{1}{2} m v^\mu v_\mu \tag{1.8}$$

This, therefore gives that

$$W_{ab} = \Delta T = T_b - T_a \tag{1.9}$$

The units of kinetic energy, and therefore work, are

$$[T] = [W] = \frac{ML^2}{T^2} = [J]$$

If the work W_{ab} is independent of the path γ, the force field F^μ is said to be «conservative» . This implies that, for any path η, we have

$$\oint_\eta F^\mu \, \mathrm{d}x_\mu = 0 \tag{1.10}$$

Recalling how differential forms work, this implies that the differential form $\omega = F^\mu \, \mathrm{d}x_\mu$ is «exact» , and therefore, we can say that (noting that $g_{\mu\nu} = g^{\mu\nu} = \delta^\mu_\nu$ in cartesian coordinates)

$$F^\mu = -\partial^\nu \mathcal{U}(x^1, x^2, x^3) \tag{1.11}$$

The function $\mathcal{U}$ is called «potential energy» , and in terms of differential forms holds as from Stokes' theorem

$$\mathrm{d}\mathcal{U} = \omega = F^\mu \, \mathrm{d}x_\mu \tag{1.12}$$

Note the following assertion:

$$W_{ab} = \mathcal{U}_a - \mathcal{U}_b \tag{1.13}$$

This is immediate, since

$$W_{ab} = -\int_a^b \mathrm{d}\mathcal{U} = \mathcal{U}_a - \mathcal{U}_b \tag{1.14}$$

We define one last quantity, the «total energy»

Definition 1.1.4 (Total Energy)**.** Given a particle in a force field with potential $\mathcal{U}$, we define the total energy E of a system as follows

$$E = T + \mathcal{U} \tag{1.15}$$

Due to the previous definition of potential energy and kinetic energy, the units of total energy are therefore

$$[E] = \frac{ML^2}{T^2} = [J]$$

Where we will write directly, when convenient

$$\frac{ML^2}{T^2} = E$$

From the definition of the total energy, we have a new conservation law

Theorem 1.3 (Conservation of Energy)**.** ***Given a particle subject to a conservative force field, we have that in every path between two points a, b we have***

$$E_a = T_a + \mathcal{U}_a = T_b + \mathcal{U}_b = E_b$$

Proof. From the definition of work, we have

$$\begin{cases} T_b - T_a = W_{ab} \\ \mathcal{U}_b - \mathcal{U}_a = W_{ab} \end{cases}$$

This gives rise to the following equation

$$T_b - T_a = \mathcal{U}_a - \mathcal{U}_b$$

And therefore

$$(T_b + \mathcal{U}_b) - (T_a + \mathcal{U}_a) = 0$$

From the definition $E = T + \mathcal{U}$, we therefore have

$$E_b = E_a$$

Demonstrating the theorem. □

§ 1.2 System Mechanics

§§ 1.2.1 Momentum and Angular Momentum of a System of Particles

Now, in order to generalize everything to a system of n particles with masses m_i we need to make a distinction between the forces in play in this problem:

1. «Internal forces» $f^\mu_{(ij)}$, which are the interaction forces between the particles
2. «External forces» $F^\mu_{(i)}$, which are the forces that act on the system

The Newton equation for the $i-$th particle then becomes

$$\frac{\mathrm{d}p^\mu_{(i)}}{\mathrm{d}t} = \sum_j f^\mu_{(ji)} + F^\mu_{(i)} \tag{1.16}$$

For Newton's third law we have $f^\mu_{(ij)} = -f^\mu_{(ji)}$ and $f^\mu_{(ii)} = 0$ (the second comes from the fact that the particle doesn't exert force on itself). Finally, summing the effects for all particles, we have Newton's second law for a system of n particles

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sum_{i=1}^n m_i x^\mu_{(i)} = \sum_i F^\mu_{(i)} + \sum_{i\neq j} f^\mu_{(ij)} \tag{1.17}$$

In order to reduce the right hand side (RHS) of the equation, we define a new quantity

Definition 1.2.1 (Center of Mass)**.** Given a system of n particles with masses m_i, we can define a new "weighted" radius vector X^μ with the following equation

$$X^\mu - \frac{\sum m_i x^\mu_{(i)}}{\sum m_i} = \frac{1}{M}\sum_i m_i x^\mu_{(i)} \tag{1.18}$$

The end point of this vector is called «center of mass»

With the last definition, we have $f^{\mu}_{(ij)} = 0$ automatically, and the second Newton's law for a system of particles becomes, if $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$

$$M\frac{\mathrm{d}X^{\mu}}{\mathrm{d}t} = \sum_i F^{\mu}_{(i)} \tag{1.19}$$

The momentum of the whole system is therefore

$$P^{\mu} = \sum_i m_i \frac{\mathrm{d}x^{\mu}_{(i)}}{\mathrm{d}t} = M\frac{\mathrm{d}X^{\mu}}{\mathrm{d}t} = \sum_i p^{\mu}_{(i)} \tag{1.20}$$

The total angular momentum, is analogously

$$L_{\mu} = \sum_i l^{(i)}_{\mu} = \sum_i \epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} p^{\gamma}_{(i)} \tag{1.21}$$

Deriving it with respect to time we see also that

$$\frac{\mathrm{d}L_{\mu}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_i \epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} p^{\gamma}_{(i)} = \sum_i \epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} F^{\gamma}_{(i)} + \sum_{i\neq j} \epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} f^{\gamma}_{(ij)} \tag{1.22}$$

Where the last term on the RHS is intended as follows

$$\epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} f^{\gamma}_{(ij)} = \epsilon_{\mu\nu\gamma}(x^{\nu}_{(i)} - x^{\nu}_{(j)}) f^{\gamma}_{(ij)} = \epsilon_{\mu\nu\gamma} x^{\nu}_{(ij)} f^{\gamma}_{(ij)}$$

For Newton's third law we have therefore that the last product is null, and therefore we end up with the following equation

$$\frac{\mathrm{d}L_{\mu}}{\mathrm{d}t} = \sum_i \epsilon_{\mu\nu\gamma} x^{\nu}_{(i)} F^{\gamma}_{(i)} = T_{\mu} \tag{1.23}$$

With T_{μ} being the total torque applied on the system.
With these definitions we can now write two conservation theorems

Theorem 1.4 (Conservation of Linear and Angular Momentum)**.** *Given a system of particles with masses m_i, vector radius $x^{\mu}_{(i)}$ and center of mass X^{μ}, if*

1. *The sum of the total external forces $F^{\mu} = \sum_i F^{\mu}_{(i)} = 0$ then P^{μ} is conserved*
2. *The total torque $T_{\mu} = 0$ then L_{μ} is conserved*

Proof. For the first statement we have that

$$P^{\mu} = \sum_i m_i v^{\mu}_{(i)} = \sum_i p^{\mu}_{(i)}$$

And

$$\frac{\mathrm{d}P^{\mu}}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}p^{\mu}_{(i)}}{\mathrm{d}t} = \sum_i F^{\mu}_{(i)} = 0$$

And the first statement is demonstrated.
For the second, it's already obvious by the definition of total torque, since

$$\frac{\mathrm{d}L_\mu}{\mathrm{d}t} = \sum_i \frac{\mathrm{d}l_\mu^{(i)}}{\mathrm{d}t} = T_\mu = 0$$

□

Suppose now that the body is rotating and we choose a point $\mathcal{O}$ as a reference.
Let $\tilde{r}^\mu_{(i)}$ be the vector from the center of mass to the $i-$th particle, then, we can write, with x^μ the coordinates of the center of mass with respect to $\mathcal{O}$ and $x^\mu_{(i)}$ the new coordinates of the $i-$th particle

$$x^\mu_{(i)} = \tilde{r}^\mu(i) + x^\mu_{(i)} \qquad v^\mu_{(i)} = \tilde{v}^\mu_{(i)} + v^\mu$$

Therefore, the angular momentum of this system of particles with respect to $\mathcal{O}$ is

$$L_\mu = \sum_i \epsilon_{\mu\nu\gamma} x^\nu (m_i v^\gamma) + \sum_i \epsilon_{\mu\nu\gamma} \tilde{x}^\nu_{(i)} (m_i \tilde{v}^\gamma_{(i)}) + \epsilon_{\mu\nu\gamma} \left(\sum_i m_i \tilde{x}^\nu_{(i)} \right) v^\gamma + \epsilon_{\mu\nu\gamma} x^\nu \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i \tilde{x}^\gamma_{(i)}$$

Rearranging the terms and removing the null terms, we end up with this equation

$$L_\mu = \epsilon_{\mu\nu\gamma} X^\nu (M v^\gamma) + \sum_i \epsilon_{m\nu\gamma} \tilde{x}^\nu_{(i)} \tilde{p}^\gamma_{(i)} = L_\mu^{(cm)} + \tilde{l}_\mu \tag{1.24}$$

This equation tells us that the angular momentum of a system considered around a new axis passing through point $\mathcal{O}$ is given by the sum of the angular momentum of the center of mass, and the sum of the angular momentum given by the motion with respect to the center of mass of the single points

§§ 1.2.2 Energy of a System of Particles

In order to define the energy of a system of particles, we begin like we did for the case with a single particle defining the work of the system.
Letting F^μ being the following

$$F^\mu = \sum_i F^\mu_{(i)} + \sum_{i \neq j} f^\mu_{ij}$$

We have that the work for a system of particles is immediately defined using the properties of integrals.
Given a path γ therefore

$$W_\gamma = \int_\gamma F^\mu \,\mathrm{d}x_\mu = \sum_i \int_\gamma F^\mu_{(i)} \,\mathrm{d}x_\mu + \sum_{i \neq j} \int_\gamma f^\mu_{(ij)} \,\mathrm{d}x_\mu \tag{1.25}$$

The calculations are identical to the single particle case, and we therefore have $W_\gamma = T_b - T_a$ where T is the «total» kinetic energy of the system

$$T = \frac{1}{2} \sum_i m_i v_i^2 \tag{1.26}$$

Expressing everything in the coordinates of the center of mass we have

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\sum_i m_i v_i^2 \tag{1.27}$$

The construction of a potential energy for the external forces is also obvious.
Now consider the case that the forces $f^\mu_{(ij)}$ are also conservative. Therefore we can write

$$f^\mu_{(ij)} = -g^{\mu\nu}\partial^{(i)}_\nu V_{ij} = g^{\mu\nu}\partial^{(j)}_\nu V_{ij} = -f^\mu_{(ji)} \tag{1.28}$$

Where the labels in the parenthesis indicate whether we're considering the coordinates of the $i-$th or $j-$th particles when differentiating.
Integrating and using this "antisymmetry" property we see immediately that the internal forces' potential reduces to the following

$$-\int_\gamma \partial^{(ij)}_\mu V_{(ij)}\,\mathrm{d}x^\mu_{(ij)} \tag{1.29}$$

We can finally write a total potential $\mathcal{U}$ as follows

$$\mathcal{U}(x^1, x^2, x^3) = \sum_i u_i(x^1, x^2, x^3) + \frac{1}{2}\sum_{j\neq i} V_{(ij)}(x^1, x^2, x^3) \tag{1.30}$$

2 Lagrange Equations

§ 2.1 Constraints and Generalized Coordinates

In the previous chapter we analyzed in general terms how to solve problems in classical mechanics, reaching to the conclusion that they might be solved by simply forcing all data into the following differential equation

$$m_i \frac{\mathrm{d}^2 x^\mu_{(i)}}{\mathrm{d}t^2} = F^\mu_{(i)} - \sum_{i\neq j} f^\mu_{(ij)}$$

This way is clearly not the way to go, since we might also have «constraints» to the motion of the system, and therefore a new set of equations that constrain the motion.

Definition 2.1.1 (Constraints)**.** Given a system with a constrained motion, whose constraint is expressed with a function of the coordinates of the particles and time, we have two kinds of constraints:

- «Holonomic constraints»
- «Nonholonomic constraints»

The first kind might be defined with an equality as follows

$$f(x^\mu_{(1)}, x^\mu_{(2)}, \cdots, x^\mu_{(n)}, t) = 0 \tag{2.1}$$

Whereas the second kind is defined as an inequality, like

$$g(x^\mu_{(1)}, x^\mu_{(2)}, \cdots, x^\mu_{(n)}, t) \geq 0 \tag{2.2}$$

Example 2.1.1 (Constraints)**.** The simplest holonomic constraint one can choose is the «rigid body» , where the constraints take the following form

$$(x^\mu_{(i)} - x^\mu_{(j)})(x_\mu^{(i)} - x_\mu^{(j)}) - c^2_{(ij)} = 0 \tag{2.3}$$

An example of nonholonomic constraint can be seen as a particle constrained to the surface of a sphere with radius a, with the following constraint equations

$$x^2 - a^2 \geq 0 \tag{2.4}$$

These definitions immensely complicate the problem of solving the differential equations, since they introduce a dependence on the constraint equations of the particle's coordinates $x^\mu_{(i)}$.
If the constraints are Holonomic, one can introduce a new set of coordinates which reduces the dimensionality of the system to $N = 3n - k$ where n is the number of particles and k the number of constraints. N is called the «degrees of freedom» of the system. The coordinates of the single particles therefore become

$$x^\mu_{(i)} = x^\mu_{(i)}(q^\nu_{(1)} \cdots, q^\nu_{(N-k)}, t) \tag{2.5}$$

It's obvious that in this choice of coordinates automatically constrain the system, and can be seen as a parametric representation of x^μ.
A deeper understanding of how constraint work on the system can be given by deriving totally with respect to time the equation (2.1), we get, remembering that $x^\mu_{(i)}$ depends on time

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}x^\mu_{(i)}}{\mathrm{d}t}\frac{\partial f}{\partial x^\mu_{(i)}} + \frac{\partial f}{\partial t} = \tilde{v}^\mu_{(i)}\frac{\partial f}{\partial x^\mu_{(i)}} + \frac{\partial f}{\partial t} = 0 \tag{2.6}$$

From this we split a second equation, supposing that the constraint is fixed in time and therefore $\partial_t f = 0$

$$v^\mu_{(i)}\partial^{(i)}_\mu f = 0 \tag{2.7}$$

The velocities satisfying this equation are said to be «virtual velocities» . Note that these velocities «do not» necessarily coincide with the usual kinematic velocity, since we're taking it considering the constraint fixed in time. From this we define a new quantity

Definition 2.1.2 (Virtual Displacement)**.** Taken the virtual velocity $v^\mu_{(i)}$, by multiplying it with a quantity δt with units of time we get a displacement δx^μ called «virtual displacement» , i.e.

$$\delta x^\mu_{(i)} = v^\mu_{(i)}\delta t \tag{2.8}$$

This definition becomes fundamental in the following section, where we will use it for deriving Lagrange's equations that will let us "forget" the existence of constraints to the system

§ 2.2 Lagrange Equations

§§ 2.2.1 D'Alembert Principle

Suppose that we have now a constrained system for which at some t the total force applied on each particle $F^\mu_{(i)} = 0$. It's obvious that if we consider the virtual displacement $\delta x^\mu_{(i)}$ at this instant the dot product will be zero, therefore

$$\sum_i F^\mu_{(i)}\delta x^{(i)}_\mu = 0 \tag{2.9}$$

Decomposing the force term into an applied force $f^\mu_{(i)}$ and a vincular reaction force $r^\mu_{(i)}$ we get

$$\sum_i f^\mu_{(i)}\delta x^{(i)}_\mu + \sum_i r^\mu_{(i)}\delta x^{(i)}_\mu = 0$$

By definition of work, we define the «virtual work» W_v with the previous equations, so dividing into the applied forces' work and the constraint forces' work we get the previous equation to become the following

$$W_v^{(a)} + W_v^{(r)} = 0 \tag{2.10}$$

Restricting ourselves to constraints for which the reaction forces are perpendicular to the motion of the particles (i.e. «smooth» constraints) we get $W_v^{(r)} = 0$ and we finally get the following theorem

Theorem 2.1 (Principle of Virtual Work and D'Alembert)**.** ***Given a system constrained to a smooth constraint, the total work of the applied forces is null, i.e.***

$$W_v^{(a)} = \sum_i f^\mu_{(i)} \delta x^{(i)}_\mu = 0 \tag{2.11}$$

Note that in general, $f^\mu_{(i)} \neq 0$ ***when*** t ***is not fixed.***
A second expression of this theorem gives the «D'Alembert principle» which holds for the whole motion, where

$$\sum_i (f^\mu_{(i)} - \dot{p}^\mu_{(i)}) \delta x^{(i)}_\mu = 0 \tag{2.12}$$

Proof. In order to prove that in every situation (2.11) holds, we begin by making a step back. We have that

$$f^\mu_{(i)} = \frac{\mathrm{d}p^\mu_{(i)}}{\mathrm{d}t}$$

Therefore

$$f^\mu_{(i)} - \frac{\mathrm{d}p^\mu_{(i)}}{\mathrm{d}t} = 0$$

I.e. the system will be «always» in equilibrium with an "effective force" $-\dot{p}^\mu_{(i)}$ (the dot stands for time derivation). Writing our "new" applied force as

$$\tilde{f}^\mu_{(i)} = f^\mu_{(i)} - \frac{\mathrm{d}p^\mu_{(i)}}{\mathrm{d}t}$$

We finally have

$$\sum_i \tilde{f}^\mu_{(i)} \delta x^{(i)}_\mu = \sum_i \left(f^\mu_{(i)} - \dot{p}^\mu_{(i)} \right) \delta x^{(i)}_\mu = 0 \tag{2.13}$$

Therefore, in this case, the constraint forces «do not» appear □

The only problem with (2.12) is that, in general, the displacements $\delta x^\mu_{(i)}$ are not independent between eachother, then for holonomic constraints we can choose the generalized or Lagrangian coordinates we defined before and use a change of basis in order to define the displacements.
Therefore, we have for the virtual velocity, with $\nu = 1, \cdots, n_f$ where n_f are the degrees of freedom of the system

$$v^\mu_{(i)} = \frac{\mathrm{d}x^\mu_{(i)}}{\mathrm{d}t} = \frac{\partial x^\mu_{(i)}}{\partial q^\nu} \frac{\mathrm{d}q^\nu}{\mathrm{d}t} + \frac{\partial x^\mu_{(i)}}{\partial t} \tag{2.14}$$

Similarly for the virtual displacement $\delta x^\mu_{(i)}$ we have

$$\delta x^\mu_{(i)} = \frac{\partial x^\mu_{(i)}}{\partial q^\nu}\delta q^\nu \tag{2.15}$$

The first term of the virtual work in terms of generalized coordinates then becomes

$$\sum_i F^\mu_{(i)}\delta x^{(i)}_\mu = \sum_i F^\mu_{(i)}\frac{\partial x^{(i)}_\mu}{\partial q^\nu}\delta q^\nu \tag{2.16}$$

Defining the «generalized force» Q_ν as the forces expressed in the generalized coordinates, as follows

$$Q_\nu = \sum_i F^\mu_{(i)}\frac{\partial x^\mu_{(i)}}{\partial q^\nu} \tag{2.17}$$

Note that since q^ν «hasn't generally dimensions of length» , the generalized forces, generally, do not have dimension of force.
With this definition, we have that the virtual work of the forces applied to the system is

$$\sum_i F^\mu_{(i)}\delta x^{(i)}_\mu = Q_\nu \delta q^\nu \tag{2.18}$$

The second part of the virtual work, with these new coordinates becomes

$$\sum_i \dot{p}^\mu_{(i)} = \sum_i m_{(i)}\frac{\mathrm{d}^2 x^\mu_{(i)}}{\mathrm{d}t^2}\frac{\partial x^{(i)}_\mu}{\partial q^\nu}\delta q^\nu$$

Taking one time derivative and using the product rule we expand this to the following relation

$$\sum_i m_{(i)}\ddot{x}^\mu_{(i)}\frac{\partial x^{(i)}_\mu}{\partial q^\nu} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{(i)}\dot{x}^\mu_{(i)}\frac{\partial x^{(i)}_\mu}{\partial q^\nu}\right) + m_{(i)}\dot{x}^\mu_{(i)}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x^{(i)}_\mu}{\partial q^\nu}\right] \tag{2.19}$$

Where

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial x^\mu_{(i)}}{\partial q^\nu} = \frac{\partial}{\partial q^\nu}\frac{\mathrm{d}x^\mu_{(i)}}{\mathrm{d}t} = \frac{\partial v^\mu_{(i)}}{\partial q^\nu} = \frac{\partial^2 x^\mu_{(i)}}{\partial q^\nu \partial q^\gamma}\frac{\mathrm{d}q^\gamma}{\mathrm{d}t} + \frac{\partial^2 x^\mu_{(i)}}{\partial q^\nu \partial t} = -\frac{\partial v^\mu_{(i)}}{\partial q^\nu}$$

From (2.14) we also get that

$$\frac{\partial v^\mu_{(i)}}{\partial \dot{q}^\nu} = \frac{\partial^2 x^\mu_{(i)}}{\partial \dot{q}^\nu \partial q^\gamma}\dot{q}^\gamma + \frac{\partial x^\mu_{(i)}}{\partial q^\gamma}\frac{\partial \dot{q}^\gamma}{\partial \dot{q}^\nu} + \frac{\partial^2 x^\mu_{(i)}}{\partial \dot{q}^\nu \partial t}$$

Since the position $x^\mu_{(i)}$ is independent from the $\dot{q}^\nu$ and $\partial_{\dot{q}^\nu}\dot{q}^\gamma = \delta^\gamma_\nu$ we have also

$$\frac{\partial v^\mu_{(i)}}{\partial \dot{q}^\nu} = \frac{\partial x^\mu_{(i)}}{\partial q^\nu} \tag{2.20}$$

Substituting these results in (2.19) we get therefore

$$\sum_i m_{(i)}\ddot{x}^\mu_{(i)}\frac{\partial x^{(i)}_\mu}{\partial q^\nu} = \sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(m_{(i)}v^\mu_{(i)}\frac{\partial v^{(i)}_\mu}{\partial \dot{q}^\nu}\right) - m_{(i)}v^\mu_{(i)}\frac{\partial v^\mu_{(i)}}{\partial q^\nu}\right] \tag{2.21}$$

We can also easily say that

$$m_{(i)} v^{\mu}_{(i)} \frac{\partial v^{(i)}_{\mu}}{\partial \dot{q}^{\nu}} = \frac{\partial}{\partial \dot{q}^{\nu}} \left(\frac{1}{2} m_{(i)} v^2_{(i)} \right)$$

$$m_{(i)} v^{\mu}_{(i)} \frac{\partial v^{(i)}_{\mu}}{\partial q^{\nu}} = \frac{\partial}{\partial q^{\nu}} \left(\frac{1}{2} m_{(i)} v^2_{(i)} \right)$$

Therefore, finally, we can say that the second term on (2.12) becomes

$$\sum_i \left[\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}^{\nu}} \left(\frac{1}{2} m_{(i)} v^2_{(i)} \right) - \frac{\partial}{\partial q^{\nu}} \left(\frac{1}{2} m_{(i)} v^2_{(i)} \right) \right] = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^{\nu}} - \frac{\partial T}{\partial q^{\nu}} \tag{2.22}$$

Where T is the total kinetic energy of the system
And therefore, subtracting that from (2.18) we obtain d'Alembert's principle in general coordinates as follows

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^{\nu}} - \frac{\partial T}{\partial q^{\nu}} - Q_{\nu} \right) \delta q^{\nu} = 0 \tag{2.23}$$

Since all the coordinates δq^{ν} are chosen to be independent, D'Alembert's principle will be satisfied if and only if the terms inside the parenthesis are equal to 0, and we obtain what is known as the ««Lagrange equations» » , a set of n_f second order non linear partial differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} = Q_{\mu} \tag{2.24}$$

Note that if the forces F^{μ} are derivable from a potential function $\mathcal{U}$, we have that

$$\begin{aligned} &\sum_i F^{\mu} \frac{\partial x^{(i)}_{\mu}}{\partial q^{\nu}} \delta q^{\nu} = \sum_i \left(-g^{\mu\gamma} \frac{\partial \mathcal{U}}{\partial x^{\gamma}_{(i)}} \right) \frac{\partial x^{(i)}_{\mu}}{\partial q^{\nu}} \delta q^{\nu} = \\ &= -\sum_i g^{\mu\gamma} \frac{\partial \mathcal{U}}{\partial x^{\gamma}_{(i)}} \frac{\partial x^{(i)}_{\mu}}{\partial q^{\nu}} \delta q^{\nu} = -\sum_i \frac{\partial \mathcal{U}}{\partial x^{\gamma}_{(i)}} \frac{\partial x^{\gamma}_{(i)}}{\partial q^{\nu}} \delta q^{\nu} = -\frac{\partial \mathcal{U}}{\partial q^{\nu}} \delta q^{\nu} = Q_{\nu} \delta q^{\nu} \end{aligned} \tag{2.25}$$

Therefore, the generalized forces Q_{ν} can be rewritten in terms of a potential $\mathcal{U}(q^{\mu}, t)$, where

$$Q_{\mu} = -\partial_{\mu} \mathcal{U}(q^{\nu}, t)$$

Therefore, Lagrange's equations (2.24) become

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}^{\mu}} - \frac{\partial T}{\partial q^{\mu}} + \frac{\partial \mathcal{U}}{\partial q^{\mu}} = 0$$

Noting that $\mathcal{U}$ doesn't depend explicitly on the $\dot{q}^{\mu}$ coordinates, we can also write

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial (T - \mathcal{U})}{\partial \dot{q}^{\mu}} - \frac{\partial (T - \mathcal{U})}{\partial q^{\mu}} = 0$$

We can define a new function, called the «Lagrangian» of the system, as

$$\mathcal{L}(\dot{q}^{\mu}, q^{\mu}, t) = T(\dot{q}^{\mu}, q^{\mu}, t) - \mathcal{U}(q^{\mu}, t) \tag{2.26}$$

And Lagrange's equations take the usual well known shape

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}^{\mu}} - \frac{\partial \mathcal{L}}{\partial q^{\mu}} = 0 \tag{2.27}$$

§§ 2.2.2 Velocity Dependent Potentials and Non-conservative Forces

Definition 2.2.1 (Velocity Dependent Potentials)**.** Given a set of generalized forces Q_μ for which doesn't exist a potential function $\mathcal{U}(q^\mu, t)$ one can still define a «generalized potential» $\mathcal{V}$ which depends directly from the velocities $\dot{q}^\mu$, provided that

$$Q_\mu = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{V}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{V}}{\partial q^\mu} \tag{2.28}$$

This potential «explicitly» depends on the velocities.

Defined the generalized potential $\mathcal{V}$ one can still easily see that Lagrange's equations hold, if $\mathcal{L} = T - \mathcal{V}$.

Example 2.2.1 (Lorentz Force)*.* One example of such forces is the Lorentz force in electromagnetism, defined as follows

$$F_\mu = qE_\mu - q\epsilon_{\nu\gamma\delta}v^\gamma B^\delta \tag{2.29}$$

Where q is the charge of the particle and E_μ, B_μ are the electric and magnetic fields, which are defined from the potentials φ, A_μ as

$$\begin{cases} E_\mu = -\partial_\nu \varphi - \dfrac{\partial A_\mu}{\partial t} \\ B_\mu = \epsilon_{\nu\gamma\delta}\partial^\gamma A^\delta \end{cases} \tag{2.30}$$

The generalized potential of this force will be

$$\mathcal{V}(\dot{q}^\mu, q^\mu, t) = q\varphi - qA^\mu v_\mu \tag{2.31}$$

And the Lagrangian is

$$\mathcal{L}(\dot{q}^\mu, q^\mu, t) = \frac{1}{2}mv^\mu v_\mu - q\varphi + qA^\mu v_\mu \tag{2.32}$$

We have

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial v^\mu} = p_\mu + qA_\mu \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial v^\mu} = \dot{p}_\mu + q\dot{A}_\mu \\ \dfrac{\partial \mathcal{L}}{\partial x^\mu} = qv^\nu \partial_\mu A_\nu - q\partial_\mu \varphi \end{cases}$$

Where $mv_\mu = p_\mu$.
Note that

$$\dot{A}_\mu = v^\mu \partial_\mu A_\nu + \partial_t A_\mu$$

Putting them together we get

$$\begin{aligned} \dot{p}_\mu + q\left(v^\nu \partial_\nu A_\nu + \partial_t A_\mu\right) &= qv^\nu \partial_\mu A_\nu - q\partial_\mu \varphi \\ \dot{p}_\mu &= -q\left(\partial_\mu \varphi + \partial_t A_\mu\right) + q\left(v^\nu \partial_\mu A_\nu - v^\nu \partial_\nu A_\mu\right) \end{aligned} \tag{2.33}$$

Remembering the properties of the $\epsilon_{\mu\nu\gamma}$ symbol the second term can be rewritten as a double vector product, getting us

$$\dot{p}_\mu = q\left(\epsilon_{\mu\nu\gamma}v^\nu \epsilon_{\gamma\delta\sigma}\partial^\delta A^\sigma\right) - q\left(\partial_\mu \varphi + \partial_t A_\mu\right)$$

This gives us, finally, Newton's equation for a particle in an electromagnetic field, by simple substitution of the definitions of the E_μ, B_μ fields

$$\dot{p}_\mu = qE_\mu - q\epsilon_{\mu\nu\gamma}v^\nu B^\gamma = F_\mu$$

Note that if there are forces can't also be described by a potential $\mathcal{V}$, one can still write Lagrange's equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} = Q_\mu \tag{2.34}$$

A clear example comes from frictional forces, for which we can define a new function

Definition 2.2.2 (Rayleigh Dissipation Function)**.** Given a non-conservative force as drag acting on a system of N particles, one has that

$$F^\mu_{(i)} = -k_{(x,y,z)}v^\mu_{(i)}$$

Where $k_{(x,y,z)}$ are the drag coefficients on the 3 different directions.
We define the «Rayleigh dissipation function» as the function such that

$$F^\mu = -g^{\mu\nu}\frac{\partial \mathcal{F}}{\partial v^\nu}$$

The physical interpretation isn't hard to grasp. Taken a system moving on a surface with drag, we have that the work done by the system against friction is

$$\mathrm{d}W_f = -F^\mu_{(f)}\,\mathrm{d}x_\mu = -F^\mu v_\mu \mathrm{d}t = k_{(x,y,z)}v^\mu v_\mu \mathrm{d}t = 2\mathcal{F}\mathrm{d}t$$

The generalized force is therefore

$$Q_\nu = F^\mu_{(f)}\frac{\partial x_\mu}{\partial q^\nu} = -g^{\mu\nu}\frac{\partial \mathcal{F}}{\partial v^\mu}\frac{\partial x_\mu}{\partial q^\nu} = -\frac{\partial \mathcal{F}}{\partial v^\mu}\frac{\partial v^\mu}{\partial \dot{q}^\nu} = -\frac{\partial \mathcal{F}}{\partial \dot{q}^\nu}$$

With the previous definition of $\mathcal{F}(v^\mu, t)$ we have that the Lagrange equations (2.34) become

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} + \frac{\partial \mathcal{F}}{\partial \dot{q}^\mu} = 0 \tag{2.35}$$

§ 2.3 Euler-Lagrange Equations

§§ 2.3.1 Least Action Principle

Lagrange's equations can alternatively be derived using the calculus of variations on a functional. This functional is defined as follows

Definition 2.3.1 (Action Functional)**.** We define the «Action» of a system as the following functional

$$\mathcal{S}[q^\mu(t)] = \int_{t_1}^{t_2} \mathcal{L}(q^\mu, \dot{q}^\mu, t)\,\mathrm{d}t \tag{2.36}$$

It has obviously units of $E \cdot t$ therefore, in SI units $J \cdot s$

One might ask how do the equations of motion pop up from here, and we get the following, really important theorem

Theorem 2.2 (Hamilton's Least Action Principle)**.** ***Given a system with action $\mathcal{S}$ and Lagrangian $\mathcal{L}$, the equations of motion $q^\mu(t)$ will be satisfied if and only if the action has a minimal extrema for q^μ, and q^μ solves the «Euler-Lagrange equations» also known as Lagrange equations of the second kind***

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0 \tag{2.37}$$

This is true only if the «variation» of the functional along a small deviation of the path $q^\mu(t)$ is null

$$\delta \mathcal{S} = 0 \tag{2.38}$$

Proof. The variation of the path is defined as $q^\mu(t) + \delta q^\mu(t)$ where $\delta q^\mu(t) << 1$, $\forall t \in [t_1, t_2]$. We choose such variation of the path in order that $\delta q^\mu(t_1) = \delta q^\mu(t_2) = 0$, and we variate the action

$$\delta \mathcal{S}[q^\mu] = \mathcal{S}[q^\mu] - \mathcal{S}[q^\mu + \delta q^\mu] = 0$$

Plugging that into the definition of the action we get

$$\delta \mathcal{S} = \delta \int_{t_1}^{t_2} \mathcal{L}(q^\mu, \dot{q}^\mu, t)\,\mathrm{d}t = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial q^\mu}\delta q^\mu + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\delta \dot{q}^\mu \right) \mathrm{d}t = 0 \tag{2.39}$$

We integrate by parts the second term on the rightmost integral and we get

$$\delta \mathcal{S} = \left[\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\delta q^\mu \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} \right) \delta q^\mu\,\mathrm{d}t$$

Since we had as boundary conditions $\delta q^\mu(t_1) = \delta q^\mu(t_2) = 0$ the first term on the RHS is null, and we get

$$\delta \mathcal{S} = \int_{t_1}^{t_2} \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} \right) \delta q^\mu\,\mathrm{d}t = 0 \tag{2.40}$$

The integral will be zero if and only if the integrand is zero itself, thanks to the «fundamental theorem of the calculus of variations» and we finally get Euler-Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \frac{\partial \mathcal{L}}{\partial q^\mu} = 0 \tag{2.41}$$

Note that the variation acts only on the path q^μ, therefore terms depending explicitly on time won't participate in the final equations. □

Theorem 2.3 (Invariance of the Euler-Lagrange equations with respect to total differentials)**.** ***Given a Lagrangian $\mathcal{L}'$ defined as the following transformation***

$$\mathcal{L}' = \mathcal{L} + \frac{\mathrm{d}f}{\mathrm{d}t}$$

We have that

$$\delta \mathcal{S}' = \delta \mathcal{S} \tag{2.42}$$

Proof. The proof is almost immediate, we have

$$\mathcal{S}' = \int_{t_1}^{t_2} \mathcal{L}\,\mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t = \mathcal{S} + [f(q^\mu(t), t)]_{t_1}^{t_2}$$

By definiton of the variation, we have

$$\delta\mathcal{S}' = \delta\mathcal{S} + [f(q^\mu(t), t)]_{t_1}^{t_2} - [f(q^\mu(t) + \delta q^\mu(t), t)]_{t_1}^{t_2} = \delta\mathcal{S}$$

□

The Lagrangian we have defined is really versatile, especially when the system described is actually formed from two smaller interacting systems.
Suppose that you have the Lagrangian $\mathcal{L}$ for some closed system, which is formed from a system A which interacts with some system B. It will have obviously the following form

$$\mathcal{L}(q_A^\mu, \dot{q}_A^\mu, q_B^\mu, \dot{q}_B^\mu) = T_A(q_A^\mu, \dot{q}_A^\mu) + T_B(q_B^\mu, \dot{q}_B^\mu) - \mathcal{U}(q_A^\mu, q_B^\mu)$$

Variating the action and solving for q_B^μ we can substitute in the Lagrangian the result. It's easy to see that T_B depends explicitly on time and it won't affect the Euler-Lagrange equations and it can be ignored, and therefore we get

$$\mathcal{L} = T_A(q_A^\mu, \dot{q}_A^\mu) - \mathcal{U}(q_A^\mu, q_B^\mu(t))$$

Where our potential now actually depends explicitly from time.

§§ 2.3.2 Change of Coordinates

Suppose now that we have a free material point with mass m. Its Lagrangian will only be formed from the term $T(v)$ as follows

$$\mathcal{L} = \frac{1}{2}mv^2 \tag{2.43}$$

Euler-Lagrange equations immediately yeld Newton's second law for a free point $m\dot{v} = 0$ as they should, but suppose that we want to change coordinates quickly. given some infinitesimal line segment we can write

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{2.44}$$

In cartesian coordinates, it will be the following

$$\mathrm{d}s^2 = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z$$

And using a slight abuse of notation, we can write

$$v^2 = \frac{\mathrm{d}s^2}{\mathrm{d}t^2} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2$$

Which, if put back into the kinetic energy of the system gives back the already well known form in Cartesian coordinates, with the Lagrangian

$$\mathcal{L}(\dot{x}, \dot{y}, \dot{z}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Now suppose that we want to work in cylindrical coordinates, the line element will be

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + \mathrm{d}z^2$$

The velocity becomes

$$v^2 = \dot{r}^2 + r^2\dot{\theta} + \dot{z}^2$$

And the Lagrangian

$$\mathcal{L}\dot{r}, \dot{\theta}, \dot{z} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$$

Analogously, in spherical coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin\theta\,\mathrm{d}\varphi^2$$

The velocity is

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin(\theta)\dot{\varphi}^2$$

And our Lagrangian in spherical coordinates is

$$\mathcal{L}(\dot{r}, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin(\theta)\dot{\varphi}^2)$$

This brings us to the general law of transformation of coordinates. Suppose that we have some transformation from the coordinates $x^\mu \to q^\mu$, the transformation of the metric is

$$g_{\mu\nu}(q^\gamma) = g_{\delta\sigma}(x^\eta)\frac{\partial x^\delta}{\partial q^\mu}\frac{\partial x^\sigma}{\partial q^\nu}$$

Since, as we have seen we have that

$$v^2 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

the previous transformation corresponds to the change of basis of the metric, where we have, in the new coordinates q^μ

$$v^2 = g_{\eta\sigma}\dot{q}^\eta\dot{q}^\sigma$$

Or, in matrix form in 3D

$$v^2 = \begin{pmatrix}\dot{q}^1 & \dot{q}^2 & \dot{q}^3\end{pmatrix}\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33}\end{pmatrix}\begin{pmatrix}\dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3\end{pmatrix} \tag{2.45}$$

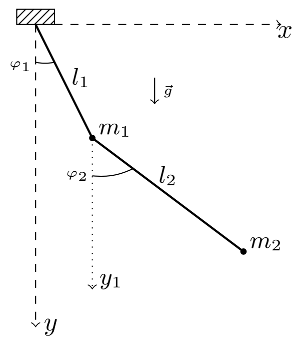
We finally have our general kinetic energy therefore as the following quadratic form

$$T(q^\mu, \dot{q}^\mu) = \frac{1}{2}m\begin{pmatrix}\dot{q}^1 & \dot{q}^2 & \dot{q}^3\end{pmatrix}\begin{pmatrix} g_{11}(q^\mu) & g_{12}(q^\mu) & g_{13}(q^\mu) \\ g_{21}(q^\mu) & g_{22}(q^\mu) & g_{23}(q^\mu) \\ g_{31}(q^\mu) & g_{32}(q^\mu) & g_{33}(q^\mu)\end{pmatrix}\begin{pmatrix}\dot{q}^1 \\ \dot{q}^2 \\ \dot{q}^3\end{pmatrix} \tag{2.46}$$

§§ 2.3.3 Solved Exercises on Euler-Lagrange Equations

One by now might ask why it's important to learn Lagrangian mechanics and all those mathematical theorems tied with it, and one easy answer is by its sheer power for solving physical problems immediately (without drag or other non-conservative forces) or with some corrections (accounting for Rayleigh's dissipation function) when dissipative forces are in action on the system.
These exercises are taken (and solved) from the book [LL69], Chapter 1, pg. 39-41

Exercise 2.3.1 (Planar Double Pendulum)**.** Take a quite hard problem to solve in Newtonian mechanics, the planar double pendulum.



Find the Lagrangian and the equations of motion of a fixed double planar pendulum with (rigid) lengths l_1 and l_2 with attached masses m_1, m_2

Figure 2.1: Double Pendulum

We choose the angles φ_1, φ_2 of the two pendulums as our Lagrangian variables, which have the following transformations (with (x_i, y_i) as the coordinates of the i-th particle)

$$\begin{cases} x_1 = l_1 \sin\varphi_1 \\ y_1 = l_1 \cos\varphi_1 \end{cases} \quad \begin{cases} \dot{x}_1 = l_1 \dot{\varphi}_1 \cos\varphi_1 \\ \dot{y}_1 = -l_1 \dot{\varphi}_1 \sin\varphi_1 \end{cases} \tag{2.47}$$

And

$$\begin{cases} x_2 = x_1 + l_2 \sin\varphi_2 \\ y_2 = y_1 + l_2 \cos\varphi_2 \end{cases} \quad \begin{cases} \dot{x}_2 = \dot{x}_1 + l_2 \dot{\varphi}_2 \cos\varphi_2 \\ \dot{y}_2 = \dot{y}_1 - l_2 \dot{\varphi}_2 \sin\varphi_2 \end{cases} \tag{2.48}$$

The kinetic energies of the two particles will be

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 l_1^2 \dot{\varphi}_1^2 \\ T_2 &= \frac{1}{2} m_2 l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \end{aligned} \tag{2.49}$$

And the kinetic energy of the system will therefore be

$$T(\varphi_{(i)}, \dot{\varphi}_{(i)}) = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \tag{2.50}$$

The potential energy of the two particles is

$$\begin{aligned} \mathcal{U}_1 &= -(m_1 + m_2) g y_1 = -(m_1 + m_2) g l_1 \cos\varphi_1 \\ \mathcal{U}_2 &= -m_2 g y_2 = -m_2 g l_2 \cos\varphi_2 \end{aligned} \tag{2.51}$$

The total Lagrangian of the system will therefore be

$$\begin{aligned}\mathcal{L}(\varphi_1,\varphi_2,\dot\varphi_1,\dot\varphi_2) = \frac{1}{2}(m_1+m_2)l_1\dot\varphi_1^2 + \frac{1}{2}m_2l_2^2\dot\varphi_2^2 + m_2l_1l_2\cos(\varphi_1-\varphi_2)\dot\varphi_1\dot\varphi_2 + \\ +(m_1+m_2)gl_1\cos(\varphi_1) + m_2gl_2\cos(\varphi_2)\end{aligned} \tag{2.52}$$

The derivatives of the Lagrangian are

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\dot\varphi_1} &= (m_1+m_2)l_1^2\dot\varphi_1 + m_2l_1l_2\cos(\varphi_1-\varphi_2)\dot\varphi_2 \\
\frac{\partial\mathcal{L}}{\partial\dot\varphi_2} &= m_2l_2^2\dot\varphi_2 + m_2l_1l_2\cos(\varphi_1-\varphi_2)\dot\varphi_1 \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\varphi_1} &= (m_1+m_2)l_1^2\ddot\varphi_1 + m_2l_1l_2\cos(\varphi_1-\varphi_2)\ddot\varphi_2 - m_2l_1l_2\dot\varphi_2(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot\varphi_2} &= m_2l_2\ddot\varphi_2 + m_2l_1l_2\cos(\varphi_1-\varphi_2)\ddot\varphi_1 - m_2l_1l_2\dot\varphi_1(\dot\varphi_1-\dot\varphi_2)\sin(\varphi_1-\varphi_2) \\
\frac{\partial\mathcal{L}}{\partial\varphi_1} &= -m_2l_1l_2\sin(\varphi_1-\varphi_2)\dot\varphi_1\dot\varphi_2 - (m_1+m_2)gl_1\sin\varphi_1 \\
\frac{\partial\mathcal{L}}{\partial\varphi_2} &= m_2l_1l_2\sin(\varphi_1-\varphi_2)\dot\varphi_1\dot\varphi_2 - m_2gl_2\sin\varphi_2
\end{aligned} \tag{2.53}$$

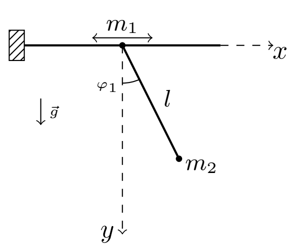
The equations of motion will be given by the Euler-Lagrange equation, where, taken $\varphi^\mu = (\varphi_1,\varphi_2)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\varphi^\mu} - \frac{\partial\mathcal{L}}{\partial\varphi^\mu} = 0$$

And therefore

$$\begin{cases}(m_1+m_2)l_1^2\ddot\varphi_1 + m_2l_1l_2\ddot\varphi_2\cos(\varphi_1-\varphi_2) - m_2l_1l_2\dot\varphi_2^2\sin(\varphi_1-\varphi_2) + (m_1+m_2)gl_1\sin\varphi_1 = 0 \\ m_2l_2^2\ddot\varphi_2 + m_2l_1l_2\ddot\varphi_1\cos(\varphi_1-\varphi_2) - m_2l_1l_2\dot\varphi_1^2\sin(\varphi_1-\varphi_2) + m_2gl_2\sin\varphi_2 = 0\end{cases} \tag{2.54}$$

Exercise 2.3.2 (Sliding Planar Pendulum)**.** Another similar problem is the planar pendulum, formed by two masses m_1 and m_2 which is constrained to slide on the x axis



Find the Lagrangian and the equations of motion of a pendulum formed by a mass m_1 constrained to move on the x axis, connected with a bar of length l with a mass m_2

Figure 2.2: Sliding Pendulum with Two Masses

The chosen Lagrangian coordinates are the angle φ formed between the vertical and the pendulum and the position x of the mass m_1. The coordinate transformations will be

$$\begin{cases} x_1 = x \\ y_1 = 0 \end{cases} \quad \begin{cases} \dot{x}_1 = \dot{x} \\ \dot{y}_1 = 0 \end{cases} \tag{2.55}$$

And

$$\begin{cases} x_2 = x + l\sin\varphi \\ y_2 = l\cos\varphi \end{cases} \quad \begin{cases} \dot{x}_2 = \dot{x} + l\dot{\varphi}\cos\varphi \\ \dot{y}_2 = -l\dot{\varphi}\sin\varphi \end{cases} \tag{2.56}$$

The kinetic energies of the two particles will be

$$\begin{aligned} T_1 &= \frac{1}{2}m_1\dot{x}^2 \\ T_2 &= \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\varphi}^2 + m_2l\dot{x}\dot{\varphi}\cos\varphi \end{aligned} \tag{2.57}$$

Therefore, the total kinetic energy will be

$$T(x,\varphi,\dot{x},\dot{\varphi}) = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\varphi}^2 + m_2l\dot{x}\dot{\varphi}\cos\varphi \tag{2.58}$$

The potential energy of the system is

$$\mathcal{U}(x,\varphi) = -m_2gl\cos\varphi$$

And the Lagrangian is, therefore

$$\mathcal{L}(x,\varphi,\dot{x},\dot{\varphi}) = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}m_2l^2\dot{\varphi}^2 + m_2l\dot{x}\dot{\varphi}\cos\varphi + m_2gl\cos\varphi \tag{2.59}$$

The derivatives of the Lagrangian are

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1+m_2)\dot{x} + m_2l\dot{\varphi}\cos\varphi \quad \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{x}} = (m_1+m_2)\ddot{x} + m_2l\ddot{\varphi}\cos\varphi - m_2l\dot{\varphi}^2\sin\varphi \\ &\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m_2l\dot{\varphi} + m_2l\dot{x}\cos\varphi \quad \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = m_2l\ddot{\varphi} + m_2l\ddot{x}\cos\varphi - m_2l\dot{x}\dot{\varphi}\sin\varphi \\ &\frac{\partial\mathcal{L}}{\partial x} = 0 \\ &\frac{\partial\mathcal{L}}{\partial\varphi} = -m_2l\dot{x}\dot{\varphi}\sin\varphi - m_2gl\sin\varphi \end{aligned} \tag{2.60}$$

Therefore, the equations of motion will be given from the Euler-Lagrange equations, set $x^\mu = (x,\varphi)$

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu} = 0$$

Which means

$$\begin{cases} (m_1+m_2)\ddot{x} + m_2l\ddot{\varphi}\cos\varphi - m_2l\dot{\varphi}^2\sin\varphi = 0 \\ m_2l\ddot{\varphi} + m_2l\ddot{x}\cos\varphi + m_2gl\sin\varphi = 0 \end{cases} \tag{2.61}$$

Exercise 2.3.3 (Planar Pendulum on a Circumference)**.** Another particular problem is finding the equations of motion of a pendulum on a vertical circumference with some added movements

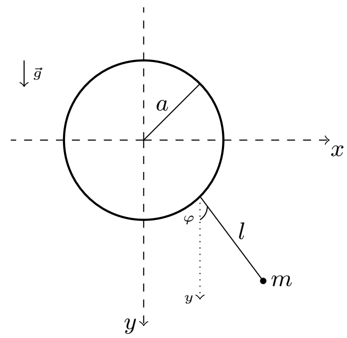


Figure 2.3: Pendulum constrained to a circumference with radius a

Find the Lagrangian and the equations of motion of a pendulum of length l with a mass m attached to the end, constrained to a circumference of radius a, where

1. The circumference rotates with constant frequency γ
2. The pendulum oscillates horizontally with law $x(t) = a\cos(\gamma t)$
3. The pendulum oscillates vertically with law $y(t) = a\cos(\gamma t)$

For all the problems, the chosen Lagrangian coordinate will be the angle φ between the vertical and the pendulum. The coordinate transformations for the first problem are

$$\begin{cases} x = a\cos(\gamma t) + l\sin\varphi \\ y = -a\sin(\gamma t) + l\cos\varphi \end{cases} \quad \begin{cases} \dot{x} = l\dot{\varphi}\cos\varphi - a\gamma\sin(\gamma t) \\ \dot{y} = -a\gamma\cos(\gamma t) - l\dot{\varphi}\sin\varphi \end{cases} \tag{2.62}$$

The sum of the squared dotted coordinates is

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\varphi}^2 + a^2\gamma^2 + 2al\gamma\dot{\varphi}\left(\sin(\gamma t)\cos(\varphi) - \cos(\gamma t)\sin(\varphi)\right)$$

Note that the last term of the RHS can be rewritten as follows

$$2al\gamma\dot{\varphi}\left(\cos(\gamma t)\sin(\varphi) - \sin(\gamma t)\cos(\varphi)\right) = 2al\gamma^2\sin(\varphi - \gamma t) + 2al\gamma\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos(\varphi - \gamma t)\right) \tag{2.63}$$

Due to the invariance of Euler-Lagrange equations to exact differentials and constants we can ignore the second and the last term therefore the kinetic energy of the pendulum will be

$$T(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\sin(\varphi - \gamma t) \tag{2.64}$$

The potential energy is

$$\mathcal{U}(\varphi) = -mgl\cos\varphi$$

And the searched Lagrangian is, ignoring all explicitly time-dependent terms

$$\mathcal{L}(\varphi, \dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\sin(\phi - \gamma t) + mgl\cos\varphi \tag{2.65}$$

The derivatives of the Lagrangian are

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\dot{\varphi} \quad \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\ddot{\varphi} \\ \dfrac{\partial\mathcal{L}}{\partial\varphi} = mal\gamma^2\cos(\varphi - \gamma t) - mgl\sin\varphi \end{cases} \tag{2.66}$$

And the Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = ml^2\ddot{\varphi} - mal\gamma^2\cos(\varphi - \gamma t) + mgl\sin\varphi = 0 \tag{2.67}$$

For the second problem the coordinate transformations are

$$\begin{cases} x = a\cos(\gamma t) + l\sin\varphi \\ y = l\cos\varphi \end{cases} \qquad \begin{cases} \dot{x} = l\dot{\varphi}\cos\varphi - a\gamma\sin(\gamma t) \\ \dot{y} = -l\dot{\varphi}\sin\varphi \end{cases} \tag{2.68}$$

The sum of the squared of the dotted coordinates is

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\varphi}^2 - 2al\gamma\dot{\varphi}\cos(\varphi)\sin(\gamma t) = l^2\dot{\varphi}^2 + 2al\gamma^2\sin(\varphi)\cos(\gamma t) - 2al\gamma\frac{\mathrm{d}}{\mathrm{d}t}(\sin(\varphi)\sin(\gamma t))$$

The potential and kinetic energy are

$$\begin{cases} T(\varphi,\dot{\varphi}) = \dfrac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\sin(\varphi)\cos(\gamma t) \\ \quad\mathcal{U}(\varphi) = -mgl\cos(\varphi) \end{cases} \tag{2.69}$$

Therefore the Lagrangian is

$$\mathcal{L}(\varphi,\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\sin(\varphi)\cos(\gamma t) + mgl\cos(\varphi) \tag{2.70}$$

The derivatives of the Lagrangian are

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\dot{\varphi} \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\ddot{\varphi} \\ &\frac{\partial\mathcal{L}}{\partial\varphi} = mal\gamma^2\cos(\varphi)\cos(\gamma t) - mgl\sin(\varphi) \end{aligned} \tag{2.71}$$

Therefore, the Euler-Lagrange equations are

$$ml^2\ddot{\varphi} - mal\gamma^2\cos(\varphi)\cos(\gamma t) + mgl\sin(\varphi) = 0 \tag{2.72}$$

For the third and last problem of this set we have that the coordinate transformations are

$$\begin{cases} x = l\sin\varphi \\ y = l\cos\varphi + a\cos(\gamma t) \end{cases} \qquad \begin{cases} \dot{x} = l\dot{\varphi}\cos\varphi \\ \dot{y} = -l\dot{\varphi}\sin\varphi - a\gamma\sin(\gamma t) \end{cases} \tag{2.73}$$

The sum of the squared of the dotted coordinates is

$$\dot{x}^2 + \dot{y}^2 = l^2\dot{\varphi}^2 + 2al\gamma^2\cos(\varphi)\cos(\gamma t) - 2al\gamma\frac{\mathrm{d}}{\mathrm{d}t}(\cos(\varphi)\sin(\gamma t))$$

Therefore, the kinetic and potential energies are

$$\begin{cases} T(\varphi,\dot{\varphi}) = \dfrac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\cos(\varphi)\cos(\gamma t) \\ \quad\mathcal{U}(\varphi) = -mgl\cos\varphi \end{cases} \tag{2.74}$$

And the searched Lagrangian is

$$\mathcal{L}(\varphi,\dot{\varphi}) = \frac{1}{2}ml^2\dot{\varphi}^2 + mal\gamma^2\cos(\varphi)\cos(\gamma t) + mgl\cos\varphi \tag{2.75}$$

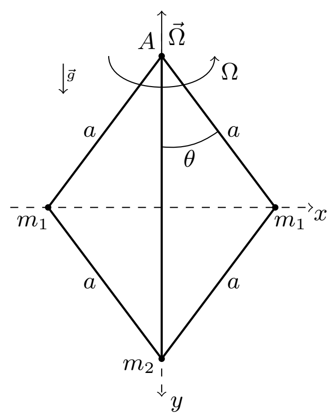
The derivatives of the Lagrangian are

$$\begin{cases} \dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\dot{\varphi} \quad \dfrac{\mathrm{d}}{\mathrm{dt}}\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}} = ml^2\ddot{\varphi} \\ \dfrac{\partial\mathcal{L}}{\partial\varphi} = -mal\gamma^2\sin(\varphi)\cos(\gamma t) - mgl\sin(\varphi) \quad = 0 \end{cases} \tag{2.76}$$

And the searched Euler-Lagrange equations are

$$ml^2\ddot{\varphi} + mal\gamma^2\sin(\varphi)\cos(\gamma t) + mgl\sin\varphi = 0 \tag{2.77}$$

Exercise 2.3.4 (Diamond-like Pendulum Spinning Top)**.** This last problem is definitely the most particular of the last ones, it's a set of coupled pendulums of length a, attached to a fixed point A and a massless rod that starts at a, at the end of the first two pendulums of lenght a there are two equal masses m_1 and they're both connected to a last mass m_2 which is fixed to move on the massless rod



Find the Lagrangian and equations of motion of a diamond-like spinning top formed by 3 masses and constrained as in figure (2.4), which rotates around the massless rod from A to m_2 with constant angular speed Ω

Figure 2.4: Diamond-like Spinning Pendulums

The first thing that might help in solving this problem is that the system is constrained from a sphere, and we might already use the generalized coordinates (θ,φ). Using the fact that T can be also written as

$$T = \frac{1}{2}mg_{\mu\nu}\dot{q}^\mu\dot{q}^\nu$$

We try to find the shape of T. The possible infinitesimal displacements for the masses of the system are

$$\begin{aligned} \mathrm{d}s_1^2 &= a^2\,\mathrm{d}\theta^2 + a^2\sin^2\theta\,\mathrm{d}\varphi \\ \mathrm{d}s_2^2 &= \mathrm{d}d(A,m_2)^2 = \mathrm{d}(2a\cos(\theta))^2 = 4a^2\sin^2\theta\mathrm{d}\theta \end{aligned} \tag{2.78}$$

Where with $\mathrm{d}d(A,m_2)$ we indicate the differential of the distance between the fixed point A and the mass m_2.

In this case we have that $g_{\mu\nu}$ is diagonal, and therefore, "dividing" by $\mathrm{d}t^2$ we can immediately write the kinetic energy of the system, remembering that the longitudinal velocity $\dot{\varphi} = \Omega$ is fixed.

$$T(\theta, \dot{\theta}) = 2T_{m_1} + T_{m_2} = m_1(a^2\dot{\theta}^2 + a^2\Omega^2 \sin^2\theta) + 2m_2 a^2\dot{\theta}^2 \sin^2\theta \tag{2.79}$$

The potential energy, evaluating from the point A is

$$\mathcal{U}(\theta) = -2(m_1 + m_2)ga \cos\theta \tag{2.80}$$

Therefore, the Lagrangian is

$$\mathcal{L}(\theta, \dot{\theta}) = m_1(a^2\dot{\theta}^2 + a^2\Omega^2 \sin^2\theta) + 2m_2 a\dot{\theta}^2 \sin^2(\theta) + 2(m_1 + m_2)ga \cos\theta \tag{2.81}$$

The derivatives of the Lagrangian exist only for the θ coordinate (the φ coordinate is called «cyclic») and we have

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2m_1 a^2\dot{\theta} + 4m_2 a^2\dot{\theta}\sin^2\theta \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{\theta}} = 2m_1 a^2\ddot{\theta} + 4m_2 a^2\ddot{\theta}\sin^2\theta + 8m_2 a^2\dot{\theta}^2 \sin\theta\cos\theta \\ \dfrac{\partial \mathcal{L}}{\partial \theta} = 4m_2 a^2\dot{\theta}^2 \sin\theta\cos\theta + 2m_1 a^2\Omega^2 \cos\theta\sin\theta - 2(m_1 + m_2)ga\sin\theta \end{cases} \tag{2.82}$$

The Euler-Lagrange equations therefore are

$$2a^2\ddot{\theta}(m_1 + 2m_2\sin^2\theta) + 4a^2 \sin\theta\cos\theta(m_2\dot{\theta}^2 - m_1\Omega^2) + 2(m_1 + m_2)ga\sin\theta = 0 \tag{2.83}$$

3 Equations of Motion

§ 3.1 Conservation Laws

§§ 3.1.1 Conservation of Generalized Energy and Generalized Momentum

All conservation laws can be determined again in Lagrangian mechanics using the symmetries of the Lagrangian. This comes from a really important theorem, called «Noether's theorem» which briefly states that for every continuously differentiable symmetry of the Lagrangian there exists a conservation law.

Definition 3.1.1 (Integral of Motion)**.** An «integral of motion» or «first integral» are all the quantities which depend only on the generalized coordinates and their derivatives $(q^\mu, \dot{q}^\mu)$.
Since they're time independent, we have that the integral $k_{(i)}$ is a conserved quantity, since

$$\frac{\mathrm{d}k_{(i)}}{\mathrm{d}t} = 0$$

We begin by stating the first two main simmetries that pop out from the properties of space and time itself, like isotropy and homogeneity.

Theorem 3.1 (Conservation of Energy)**.** ***Given an isolated system with Lagrangian*** $\mathcal{L}(q^\mu, \dot{q}^\mu)$ ***not subject to dissipative forces, due to the homogeneity of time, the generalized energy*** $E(q^\mu, \dot{q}^\mu, t)$ ***defined as***

$$E(q^\mu, \dot{q}^\mu) = \dot{q}^\mu \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \mathcal{L}(q^\mu, \dot{q}^\mu) \tag{3.1}$$

Is an integral of motion

Proof. We begin by finding the total time derivative of the Lagrangian. The differential of the Lagrangian is

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^\mu}\mathrm{d}q^\mu + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\mathrm{d}\dot{q}^\mu$$

Therefore

$$\dot{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial q^\mu}\dot{q}^\mu + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\ddot{q}^\mu$$

Substituting the first term on the RHS with the Euler Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}t}\dot{\partial}_\mu \mathcal{L} = \partial_\mu \mathcal{L}$, where with $\dot{\partial}_\mu$ we indicate the derivative with respect to the velocities, we have

$$\dot{\mathcal{L}} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\dot{q}^\mu + \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\ddot{q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^\mu \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\right)$$

Moving the terms around, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^\mu \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} - \mathcal{L}\right) = \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Which states that $E(q^\mu, \dot{q}^\mu)$ is an integral of motion since it's conserved. □

Note the previous definition of the energy. Since $\dot{q}^\mu \dot{\partial}_\mu \mathcal{L} = \dot{q}^\mu \dot{\partial}_\mu T = 2T$, we have

$$E(q^\mu, \dot{q}^\mu) = 2T(q^\mu, \dot{q}^\mu) - T(q^\mu, \dot{q}^\mu) + \mathcal{U}(q^\mu)$$

Therefore, we end up to the usual definition of energy

$$E(q^\mu, \dot{q}^\mu) = T(q^\mu, \dot{q}^\mu) + \mathcal{U}(q^\mu) \tag{3.2}$$

Another integral of motion can be determined from the properties of space

Theorem 3.2 (Conservation of Momentum)**.** *Due to the homogeneity of space, the quantity p_μ called «generalized momentum»*

$$p_\mu = \dot{\partial}_\mu \mathcal{L} \tag{3.3}$$

Is an integral of motion

Proof. The homogeneity of space implies that for a parallel translation of all the points of a system the Lagrangian is invariant. Therefore for an infinitesimal translation $q^\mu \to q^\mu + \delta q^\mu$ where $\delta q^\mu \le \epsilon$ we have

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^\mu}\delta q^\mu = 0$$

Therefore, using Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}^\mu} = \frac{\mathrm{d}p_\mu}{\mathrm{d}t} = 0$$

Which proves the theorem. This, also implies that, again using Euler-Lagrange

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}t} = \frac{\partial \mathcal{L}}{\partial q^\mu} = Q_\mu = 0$$

Which implies that the system is in equilibrium □

Exercise 3.1.1 (Application of Conservation Laws)**.** A quick way to show the importance of conservation laws is a simple exercise, that can be found at pg. 46 [LL69].
Suppose that a particle with mass m and initial velocity v_1 is moving in a potential field $\mathcal{U}_1$, which after passing a wall changes to $\mathcal{U}_2$.
Find the change in direction of the velocity.

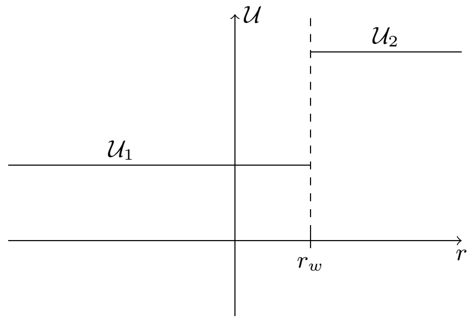


Figure 3.1: The piecewise potential $\mathcal{U}(r)$

Changing immediately to the orthogonal system of coordinates formed by the normal and tangent to the wall $q^\mu = (n, t)$ we have that

$$\partial_t \mathcal{U} = -\partial_t \mathcal{L} = 0$$

Therefore, using Euler-Lagrange equations we immediately know that in these coordinates

$$\dot{p}_t = 0$$

This implies therefore that the generalized momentum projected onto the tangent to the plane will be conserved going through the wall, and therefore

$$v_1^t = v_2^t$$

Now we might have two cases

1. The particle is already moving normally to the wall, hence $v_1^t = v_2^t = 0$
2. The particle is moving with an angle θ to the wall, hence $v_1 \sin\theta = v_2 \sin\eta$

The second case can be illustrated in this drawing

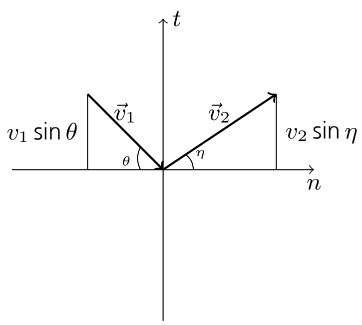


Figure 3.2: Incidence angle of the particle

Using the previous equation we have that

$$\frac{\sin\theta}{\sin\eta} = \frac{v_2}{v_1}$$

For determining the velocities we use the conservation of energy, which gives

$$T_1 + \mathcal{U}_1 = T_2 + \mathcal{U}_2$$

Therefore solving for T_2/T_1 we get

$$\frac{T_2}{T_1} = 1 + \frac{1}{T_1}(\mathcal{U}_1 - \mathcal{U}_2)$$

Substituting $T = mv^2/2$ we have finally

$$\frac{v_2^2}{v_1^2} = 1 + \frac{2}{mv_1^2}(\mathcal{U}_1 - \mathcal{U}_2)$$

I.e., taking the square root

$$\frac{\sin\theta}{\sin\eta} = \pm\sqrt{1 + \frac{2}{mv_1^2}(\mathcal{U}_1 - \mathcal{U}_2)}$$

This is finally the change in direction of the particle in passing through the wall

§§ 3.1.2 Change of Inertial Reference Frames

The behavior of Lagrangians and therefore action, in change of reference frames isn't hard to grasp. Suppose there is a system of particles with masses m_i and velocities $v^\mu_{(i)}$ in some reference frame K. Changing to a second reference frame $\tilde{K}$ moving with velocity V^μ we have that the velocities transform with the following law

$$v^\mu_{(i)} = \tilde{v}^\mu_{(i)} + V^\mu \tag{3.4}$$

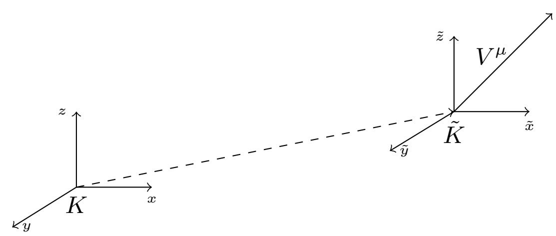


Figure 3.3: The reference frames K and $\tilde{K}$

If we write the total momentum of the system as $P^\mu = \sum m_i v^\mu_{(i)}$, we have that the momentum changes with the following law

$$P^\mu = \sum_i m_i \tilde{v}^\mu_{(i)} + V^\mu \sum_i m_i = \tilde{P}^\mu_{(i)} + MV^\mu \tag{3.5}$$

This tells us immediately that we can find a system such that $\tilde{P}^\mu = 0$, therefore, in this system

$$P^\mu = MV^\mu \tag{3.6}$$

The velocity of the system we're searching depends therefore on the masses, and we get

$$V^\mu = \frac{P^\mu}{M} \tag{3.7}$$

This can be seen as the total derivative of the radius vector of the center of mass.
Writing energy as $T + \mathcal{U}$ we have that it changes between inertial reference frames as follows

$$E = \frac{1}{2}\sum_i m_i(\tilde{v}^\mu_{(i)} + V^\mu)^2 + \mathcal{U} = \frac{1}{2}\sum_i m_i \tilde{v}^2 + V^\mu \tilde{P}_\mu + \frac{1}{2}MV^2 + \mathcal{U} \tag{3.8}$$

Writing $E_{int} = \tilde{T} + \mathcal{U}$ as the «internal energy» of the system, we have that, in the system of the center of mass

$$E = E_{int} + \frac{1}{2}MV^2 \tag{3.9}$$

Exercise 3.1.2 (Transformation of Action Between Inertial Frames)**.** A nice way for seeing these transformations in action is seeing how the action transforms between inertial reference frames, as in the problem at pg. 48 [LL69].
Since we know already that

$$T = \tilde{T} + \frac{1}{2}MV^2 + V^\mu \tilde{P}_\mu \tag{3.10}$$

We have that the Lagrangian $\mathcal{L} = T - \mathcal{U}$ changes as follows

$$\mathcal{L} = \tilde{T} + \frac{1}{2}MV^2 + V^\mu \tilde{P}_\mu - \mathcal{U} = \tilde{\mathcal{L}} + \frac{1}{2}MV^2 + V^\mu \tilde{P}_\mu \tag{3.11}$$

Integrating the Lagrangian with respect to time, we have

$$\mathcal{S} = \int \mathcal{L}\,\mathrm{d}t = \int \tilde{\mathcal{L}}\,\mathrm{d}t + V^\mu \int \tilde{P}_\mu\,\mathrm{d}t + \frac{1}{2}MV^2 \int \mathrm{d}t$$

Integrating directly, and putting $\tilde{P}^\mu = M\frac{\mathrm{d}}{\mathrm{d}t}\tilde{X}^\mu$, we have

$$\mathcal{S} = \tilde{\mathcal{S}} + MV^\mu \tilde{X}_\mu + \frac{1}{2}MV^2 t \tag{3.12}$$

§§ 3.1.3 Conservation of Angular Momentum

Theorem 3.3 (Conservation of Angular Momentum)**.** ***Due to the isotropy of space, the properties of a system are invariant to rotations, and the angular momentum L_μ defined as***

$$L_\mu = \epsilon_{\mu\nu\sigma} x^\nu p^\sigma$$

Proof. Take a system with radius vector x^μ and apply some rotation vector $\delta\varphi^\mu$, the system will rotate of an angle θ and have a linear displacement δx^μ as in the following figure

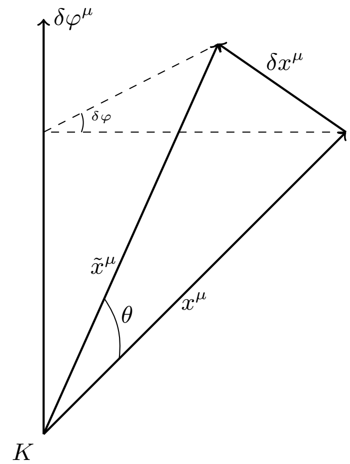


Figure 3.4: Uniform rotation of the system by an angle $\delta\varphi$

By the right hand rule we have

$$\|\delta x\| = \|x\| \|\delta\varphi\| \sin\theta \tag{3.13}$$

This implies, again by the right hand rule that

$$\delta x^\mu = g^{\mu\nu}\epsilon_{\nu\gamma\sigma}\delta\varphi^\gamma x^\sigma \tag{3.14}$$

Using the fact that $\delta\varphi^\mu$ is constant, we have that the variation of the velocity of the system is

$$\delta v^\mu = g^{\mu\nu}\epsilon_{\nu\gamma\sigma}\delta\varphi^\gamma v^\sigma \tag{3.15}$$

Shoving it all in the variation of the Lagrangian $\delta\mathcal{L}$ and imposing its invariance, we have

$$\begin{aligned}\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial v^\mu}\delta v^\mu \\ \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial x^\mu} g^{\mu\nu}\epsilon_{\nu\gamma\sigma}\delta\varphi^\gamma x^\sigma + \frac{\partial\mathcal{L}}{\partial v^\mu} g^{\mu\nu}\epsilon_{\nu\gamma\sigma}\delta\varphi^\gamma v^\sigma = 0\end{aligned} \tag{3.16}$$

Substituting $\partial_\mu\mathcal{L} = \dot{p}_\mu$ and $\dot{\partial}_\mu\mathcal{L} = p_\mu$ we have, after a cyclic exchange of indexes

$$\delta\varphi^\mu\left(\epsilon_{\mu\nu\gamma}x^\nu\dot{p}^\gamma + \epsilon_{\mu\nu\gamma}v^\nu p^\gamma\right) = 0 \tag{3.17}$$

Noting that inside there is actually a total derivative of a product, we can write

$$\delta\varphi^\mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\epsilon_{\mu\nu\gamma}x^\nu p^\gamma\right) = \frac{\mathrm{d}L_\mu}{\mathrm{d}t}\delta\varphi^\mu = 0 \tag{3.18}$$

Which is the searched integral of motion. □

In a change of inertial reference frames $K \to \tilde{K}$, where $\tilde{K}$ is moving with velocity V^μ, the angular momentum transforms as

$$L_\mu = \tilde{L}_\mu + \epsilon_{\mu\nu\sigma}X^\nu P^\sigma \tag{3.19}$$

Since

$$L_\mu = \sum_i m_i\epsilon_{\mu\nu\sigma}x^\nu_{(i)}\tilde{v}^\sigma_{(i)} + \sum_i m_i\epsilon_{\mu\nu\sigma}x^\nu_{(i)}V^\sigma$$

§§ 3.1.4 Noether's Theorem

In general all conservation laws can be condensed in one single theorem that works on the symmetries of the Lagrangian, this really important theorem is known as «Noether's theorem» and ties the invariance of the Lagrangian to the integrals of motion.

Theorem 3.4 (Noether's Theorem)**.** *Given any transformation of coordinates $q^\mu \to q^\mu + \delta q^\mu,\ \delta q^\mu \propto \epsilon$ which is related to a symmetry of the Lagrangian implies that the equation of motion are invariant and there exists a conserved quantity associated to the symmetry*

Proof. We begin the proof by applying the transformation to the Lagrangian. We obtain, using the fact that the transformation is infinitesimal

$$\delta\mathcal{L}(q^\mu + \delta q^\mu, \dot{q}^\mu + \delta\dot{q}^\mu, t) \simeq \delta\mathcal{L} + c_1 \delta q^\mu + c_2 \delta\dot{q}^\mu$$

If this transformation is a symmetry transformation of the Lagrangian we have that

$$\mathcal{L}(q^\mu + \delta q^\mu, \dot{q}^\mu + \delta\dot{q}^\mu, t) = \mathcal{L}(q^\mu, \dot{q}^\mu + \delta\dot{q}^\mu, t) \implies c_1 = 0$$

Therefore, integrating over time we get the associated variated action

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \delta\mathcal{L}\,\mathrm{d}t + \int_{t_1}^{t_2} c_2 \delta\dot{q}^\mu\,\mathrm{d}t$$

Using the usual boundary conditions of the least action theorem, and noting that the first term in the RHS must be zero in order to satisfy Euler-Lagrange equations, we have

$$\delta\mathcal{S} = \left[c_2 \delta q^\mu\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{c}_2 \delta q^\mu\,\mathrm{d}t = -\int_{t_1}^{t_2} \dot{c}_2 \delta q^\mu\,\mathrm{d}t = 0$$

Therefore, for the first theorem of variational calculus we have

$$\dot{c}_2 = 0$$

Which implies that c_2 is an integral of motion, derived from a symmetry transformation.
Note that if the transformation isn't a symmetry transformation we get

$$\dot{c}_2 = c_1$$

Therefore c_2 isn't generally an integral of motion □

§ 3.2 Mechanical Similarity

Due to the invariance of the Euler-Lagrange equations to the multiplication of constants, we might imagine to use this property to our advantage with some kind of coordinate transformation.
Suppose that the potential function $\mathcal{U}$ is k-homogeneous, which means

$$\mathcal{U}(\alpha q^\mu) = \alpha^k \mathcal{U}(q^\mu)$$

These kinds of potentials are actually common in physics as we will see later, but in order to showcase this method of mechanical similarity we apply the following transformation

$$\begin{cases} q^\mu \to \alpha\tilde{q}^\mu \\ t \to \beta\tilde{t} \end{cases} \tag{3.20}$$

This gives us that $\dot{q}^\mu \propto \frac{\alpha}{\beta}\dot{\tilde{q}}^\mu$, and due to the shape of the kinetic energy we get

$$T \propto \frac{\alpha^2}{\beta^2}\tilde{T}$$

Now, we have that the parameter α is fixed from the potential energy itself, while the second parameter β, for now, is free. In order to fix that parameter we must impose the invariance of the equation of motion, i.e. impose that

$$\mathcal{L} = \alpha^k\tilde{\mathcal{L}}$$

This is pretty easy to impose, we simply need that both the new modified kinetic energy and potential energy are multiplied by the same parameter α^k, and therefore

$$\frac{\alpha^2}{\beta^2} = \alpha^k \implies \beta = \alpha^{1-\frac{k}{2}} \tag{3.21}$$

This leaves the equations of motion unchanged, but also simplified. Other things that this brings is the possibility to write relations between quantities. We have, if we write the linear path length as l, that

$$\frac{\tilde{t}}{t} = \left(\frac{\tilde{l}}{l}\right)^{1-\frac{k}{2}} \tag{3.22}$$

Therefore, we also have for velocity, energy and length

$$\frac{\tilde{v}}{v} = \left(\frac{\tilde{l}}{l}\right)^{\frac{k}{2}} \qquad \frac{\tilde{E}}{E} = \left(\frac{\tilde{l}}{l}\right)^{k} \qquad \frac{\tilde{L}}{L} = \left(\frac{\tilde{l}}{l}\right)^{1+\frac{k}{2}} \tag{3.23}$$

Applying this to a Coulombian potential $\mathcal{U}(r) = k/r$, we have, if the field is attractive, i.e. $k = -1$

$$\frac{\tilde{t}}{t} = \left(\frac{\tilde{l}}{l}\right)^{\frac{3}{2}} \tag{3.24}$$

Which is simply the square root of Kepler's third law.

§§ 3.2.1 Virial Theorem

Using the techniques of mechanical similarity, there's a theorem which pops out quite easily, the virial theorem.

Theorem 3.5 (Virial Theorem)**.** *Suppose that the motion of a system happens in a limited space, therefore $\dot{x}^\mu \dot{\partial}_\mu T = 2T$. If $\tilde{\mathcal{U}} = k\mathcal{U}$ then*

$$\langle T \rangle = \frac{k}{2}\mathcal{U} \tag{3.25a}$$

And

$$\begin{cases} \langle T \rangle = \dfrac{k}{k+2} \langle E \rangle \\ \langle \mathcal{U} \rangle = \dfrac{2}{k+2} \langle E \rangle \end{cases} \tag{3.25b}$$

Proof. Since the motion is in a limited space, we have that $\dot{x}^\mu \dot{\partial}_\mu T = \dot{x}^\mu p_\mu = 2T$, and applying the time average, we have

$$\langle 2T \rangle = \lim_{\tau \to \infty} \frac{2}{\tau} \left(\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t} p_\mu x^\mu \, \mathrm{d}t - \int_0^\tau x^\mu \dot{p}_\mu \, \mathrm{d}t \right)$$

From Euler-Lagrange tho, we also have that $x^\mu \dot{p}_\mu = -x^\mu \partial_\mu \mathcal{U}$, and therefore, since the first integral is null

$$\langle 2T \rangle = \langle x^\mu \partial_\mu \mathcal{U} \rangle$$

If the potential is k-homogeneous, we have

$$\langle T \rangle = \frac{k}{2} \langle \mathcal{U} \rangle$$

Since the total energy in this case is constant and conserved, we must have

$$\langle T \rangle + \langle \mathcal{U} \rangle = E$$

Substituting the previous result, we have

$$\frac{k}{2} \langle \mathcal{U} \rangle + \langle \mathcal{U} \rangle = E$$

Solving for $\langle \mathcal{U} \rangle$ we have the first result

$$\langle \mathcal{U} \rangle = \frac{2}{k+2} E$$

Inserting again the first result

$$\langle T \rangle = \frac{k}{k+2} E$$

□

§ 3.3 Integration of the Equations of Motion

§§ 3.3.1 Unidimensional Motion

The integration of the equations of motion isn't always direct, but it's definitely easier when the considered system only has one degree of freedom $q(t)$. The most general Lagrangian of such system has the following shape

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2} a(q) \dot{q}^2 - \mathcal{U}(q)$$

If $q(t) = x(t)$ we have that $a(q) = m$, and therefore

$$\mathcal{L}(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - \mathcal{U}(x)$$

Without even writing the equations of motion we begin to write a differential equation starting from the first possible integral of motion, E.

$$E = \frac{1}{2} m \dot{x}^2 - \mathcal{U}(x)$$

Solving for $\dot{x}$ the result is a first-order separable differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2}{m} \left(E - \mathcal{U}(x)\right)} \tag{3.26}$$

From this we have two immediate solutions

$$\begin{aligned} x(t) &= \sqrt{\frac{2}{m}} \int_{t_0}^{t} \sqrt{E - \mathcal{U}(x)}\,\mathrm{d}t \\ t(x) &= \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{1}{\sqrt{E - \mathcal{U}(x)}}\,\mathrm{d}x \end{aligned} \tag{3.27}$$

Since $x(t)$ must be real, the motion is therefore possible only in the regions where $E \geq \mathcal{U}(x)$. The points where $E = \mathcal{U}(x)$ are called «stopping points» , i.e. where $\dot{x} = 0$.
If we have that $x(t)$ is limited between two stopping points x_{s_0} and x_{s_1}, the motion is said to be «limited» between these points, or «infinite» otherwise.
The nature of limited motion is inherently oscillatory, contained inside a potential hole or potential well. The period of the oscillation of the system from the first stopping point till the second is quite easy to calculate. Taking the second expression in equation (3.27) and fixing the integration extremes between the two stopping points and multiplying by 2, we have the period of the system in function to the total energy of the system

$$T(E) = \sqrt{2m} \int_{x_{s_1}}^{x_{s_2}} \frac{1}{\sqrt{E - \mathcal{U}(x)}}\,\mathrm{d}x \tag{3.28}$$

§§ 3.3.2 Motion in a Central Field

Definition 3.3.1 (Reduced Mass)**.** Suppose that two particles are moving in a field where $\mathcal{U} = \mathcal{U}(\|r_1^\mu - r_2^\mu\|_\mu)$, i.e. the field depends only on the distance between the two particles. It's possible then to reduce the problem with the following coordinate transformation

$$\begin{cases} r^\mu = r_1^\mu - r_2^\mu \\ \tilde{P}^\mu = m_1 r_1^\mu + m_2 r_2^\mu = 0 \end{cases} \tag{3.29}$$

This reduces the Lagrangian to the motion of only the center of mass of the two particles, where

$$\mathcal{L}(r^\mu, \dot{r}^\mu) = \frac{1}{2} \mu \dot{r}^\mu \dot{r}_\mu - \mathcal{U}(r)$$

The parameter μ is known as «reduced mass» of the system, where

$$\mu = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}} = \frac{m_1 m_2}{m_1 + m_2}$$

The inverse transformation is immediately retrieved from the system (3.29), where

$$\begin{cases} r_1^\mu = \dfrac{m_2}{m_1 + m_2} r^\mu \\ r_2^\mu = -\dfrac{m_1}{m_1 + m_2} r^\mu \end{cases}$$

The notion of reduced mass is extremely useful in treating central force fields, i.e. forces that have the same direction of the radius of a circle. Considering a conservative force and polar coordinates we have that such force can be expressed as follows

$$F^\mu = -g^{\mu\nu}\partial_\nu \mathcal{U} = -\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}r}\frac{\hat{r}^\mu}{r}$$

In this case we also have that $L_\mu \perp r^\mu$, and the Lagrangian of such system has the following shape

$$\mathcal{L}(r, \dot{r}, \varphi, \dot{\varphi}) = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2) - \mathcal{U}(r)$$

The absence of a direct dependence of the Lagrangian from the coordinate φ gives an immediate conservation law, where

$$\partial_\varphi \mathcal{L} = \dot{p}_\mu = 0$$

This coordinate is called «cyclic» , and corresponds to another conservation law

$$\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = L_z$$

Which, written explicitly using Euler-Lagrange equations, gives

$$\dot{p}_\varphi = \partial_\varphi \mathcal{L} = \dot{L}_z = 0$$

Identifying both L_z and p_φ as integrals of motion.
The geometrical interpretation of this is quite easy. Define the area spanned by a particle moving in this field, going from r^μ to $r^\mu + \mathrm{d}r^\mu$, calling this area $\mathrm{d}A$ we have

$$\mathrm{d}A = \frac{1}{2} m r^2 \,\mathrm{d}\varphi \tag{3.30}$$

Linking that to our previous integral of motion, we have

$$L_\mu = 2m\dot{A} \tag{3.31}$$

This implies that also $2m\dot{A}$ must be constant, and it's simply «Kepler's second law of orbital motion» .
Looking back at the relation between $\dot{\varphi}$ and L_z we can say

$$\dot{\varphi}^2 = \frac{L_z^2}{m^2 r^4} \tag{3.32}$$

Which lets us rewrite the Lagrangian in a different fashion

$$\mathcal{L} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \mathcal{U}(r) \tag{3.33}$$

The integration of these equations of motion is pretty easy, and remembering that the "new" part that we added «isn't» a piece of the potential, gives the following energy expression.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L_z^2}{2mr^2} + \mathcal{U}(r) \tag{3.34}$$

Solving for $\dot{r}$ we have a slightly modified version of what we saw for general one dimensional motion

$$\begin{aligned} r(t) &= \int_{t_1}^{t_2} \sqrt{\frac{2}{m}(E - \mathcal{U}(r)) - \frac{L_z^2}{m^2r^2}} \, \mathrm{d}t \\ t(r) &= \int_{r_1}^{r_2} \frac{1}{\sqrt{\frac{2}{m}(E - \mathcal{U}(r)) - \frac{L_z^2}{m^2r^2}}} \, \mathrm{d}r \end{aligned} \tag{3.35}$$

Using (again) the cyclic variable, we have that $\mathrm{d}\varphi = \frac{L_z}{mr^2}\,\mathrm{d}t$ we can rewrite the second integral in a more convenient way, as

$$\varphi(r) = \int_{r_1}^{r_2} \frac{L_z}{mr^2} \frac{1}{\sqrt{\frac{2}{m}(E - \mathcal{U}(r)) - \frac{L_z^2}{m^2r^2}}} \, \mathrm{d}r = \int_{r_1}^{r_2} \frac{L_z}{r^2} \frac{\mathrm{d}r}{\sqrt{2m(E - \mathcal{U}(r)) - \frac{L_z^2}{r^2}}} \tag{3.36}$$

This is extremely convenient since it gives the general solution to our problem, and also the equation of the trajectory of the system in this central field.
Another way to treat this is by defining a "new" «effective potential» $\mathcal{U}_{eff}$ where

$$\mathcal{U}_{eff}(r) = \mathcal{U}(r) + \frac{L_z^2}{2mr^2} \tag{3.37}$$

The new piece we added to this potential is known as «centrifugal energy» . Note that the stopping points of this effective potential «do not» coincide with the points where the system completely stops, but instead coincides with the points where only $\dot{r}$ changes sign, effectively giving the farthest and closest points of the trajectory with respect to the center of the field.
Note that we have an orbit around the origin of the field if and only if both the closest and farthest point coexist. This doesn't assure us that this orbit is closed tho, for this to be true we need that in going from r_{min} to r_{max} and back the system is again at its starting point, which is true «iff»

$$\Delta\varphi = 2\int_{r_{min}}^{r_{max}} \frac{L_z}{r^2}\left(2m(E - \mathcal{U}(r)) - \frac{L_z^2}{r^2}\right)^{-\frac{1}{2}} \mathrm{d}r \quad \text{mod } 2\pi = 0 \tag{3.38}$$

This integral basically tells us that for having a closed orbit, the variation of the position in this trajectory after a whole revolution (i.e. going from r_{min} to r_{max} and back) must be 0 or an integer multiple of 2π (if we consider more than one revolution), therefore indicating that for a «whole orbit»

$$\Delta\varphi \propto 2\pi \implies \Delta\varphi = \frac{m}{n}2\pi, \quad m, n \in \mathbb{Z} \tag{3.39}$$

Center Falling

One question might arise immediately: is it possible for a system which is interacting with a central potential to fall «directly» towards the center? The answer is «not always» . In fact the centrifugal energy component makes it possible if and only if

$$\frac{1}{2}m\dot{r}^2 = E - \mathcal{U}(r) > \frac{L_z^2}{mr^2} \tag{3.40}$$

Which also implies that

$$r^2\mathcal{U}(r) + \frac{L_z^2}{2m} < Er^2$$

Or, written more conveniently, we have that a system can fall to the center, i.e. $r \to 0$ is possible, if and only if

$$r^2\mathcal{U}(r) < -\frac{L_z^2}{2m} \tag{3.41}$$

I.e. for a Coulombian potential $\mathcal{U}(r) = -\alpha/r^2$, we must have $\alpha > L_z^2/2mr^2$, or the potential is not Coulombian, where $\mathcal{U}(r) \propto r^{-n}$ with $n > 2$

§§ 3.3.3 Kepler's Problem and Orbital Mechanics

The best physical example of a problem where central potentials come into action is orbital mechanics in attractive fields where $\mathcal{U}(r) \propto r^{-1}$, the gravitational potential is exactly one of these. We have, in this case

$$\begin{aligned} \mathcal{U}(r) &= -\frac{\alpha}{r} \\ \mathcal{U}_{eff}(r) &= \mathcal{U}(r) + \frac{L^2}{2mr^2} \end{aligned} \tag{3.42}$$

This field is notorious for diverging at $r = 0$ and fading to zero as $r \to \infty$, and the minimum is

$$\min\{\mathcal{U}_{eff}\} = \mathcal{U}_{eff}\left(\frac{L^2}{\alpha m}\right) = -\frac{\alpha^2 m}{2L^2}$$

Due to the shape of this potential, the motion considered will be finite iff $E < 0$ and the trajectory equation is directly integrable

$$\varphi(r) = \int \frac{L}{r}\left(2mE - 2m\mathcal{U}(r) - \frac{L^2}{r^2}\right)^{-\frac{1}{2}} \mathrm{d}r = \operatorname{acos}\left(\frac{\frac{L}{r} - \frac{m\alpha}{L}}{\sqrt{2mE + \frac{m^2\alpha^2}{L^2}}}\right) + \varphi_0 \tag{3.43}$$

Writing $p = \frac{L^2}{m\alpha}$ and $e = \sqrt{1 + \frac{2EL^2}{m\alpha^2}}$ this becomes the equation of a conic, with p and e as the «parameter» and the «eccentricity» of the orbit, giving us, finally

$$\frac{p}{r} = 1 + e\cos(\varphi) \tag{3.44}$$

Note that with $E < 0$ and $e < 1$ this is an ellipse

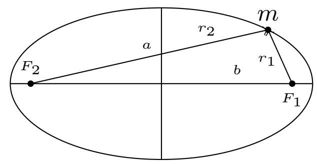


Figure 3.5: Orbit of a mass m in a Keplerian potential with $E < 0$ and $e < 1$.

The parameters of this ellipse can be directly tied to the actual parameters of motion, using the previous definition and a bit of geometry. We have

$$
\begin{aligned}
a &= \frac{p}{1-e^2} = \frac{\alpha}{2|E|} \\
b &= \frac{p}{\sqrt{1-e^2}} = \frac{L}{2m|E|}
\end{aligned}
\tag{3.45}
$$

The minimum of the energy is reached at the same time that the effective potential has its minimum (at $r = -\alpha^2 m/2L^2$). The minimum of the energy corresponds to this value iff $e = 0$ and the trajectory becomes that of a circle. The farthest and closest points from the center of the field (at one of the focuses of the ellipse) are

$$
r_{min} = \frac{p}{1+e} = a(1-e) \qquad r_{max} = \frac{p}{1-e} = a(1+e) \tag{3.46}
$$

Which, as said before, are the roots of the equation $\mathcal{U}_{eff} = E$, i.e. the "effective" stopping points. Since as we have seen before $2m\dot{A} = L$, we have that in a complete revolution $2mA = TL$ with A as the area of the ellipse. Rearranging everything into a system, we have

$$
\begin{cases}
2m\pi ab = TL \\
\qquad a = \dfrac{\alpha}{2|E|} \\
\qquad b = \dfrac{L}{\sqrt{2m|E|}}
\end{cases}
\tag{3.47}
$$

Substituting, we have that

$$
T = \frac{2m\pi\alpha}{2|E|\sqrt{2m|E|}}
$$

Rearranging everything and making a pop back again, we have

$$
T = 2\pi a^{\frac{3}{2}}\sqrt{\frac{m}{\alpha}} \tag{3.48}
$$

Squaring everything, we get back «Kepler's third law of orbital motion»

$$
T^2 = \frac{4m\pi a^3}{\alpha} \implies T^2 \propto a^3 \tag{3.49}
$$

This can be rearranged in a different way, we begin without trying to get a relation between the period T and the parameters of the ellipse (a, b, e, p) and we take the previous result and substitute what we found for a, giving us the period of the orbit with respect to the total energy of the system, as follows

$$T(|E|) = \alpha\pi\sqrt{\frac{m}{2|E|^3}} \tag{3.50}$$

Which is just a rephrasing of Kepler's third law.
Going back to the conic equation for the motion in a Keplerian field, we have that for $E > 0$ the motion is infinite (hyperbolic) with $e > 1$. The parameters of this orbit are

$$r_{min} = \frac{p}{e+1} = \frac{\alpha}{2|E|}(e+1) \tag{3.51}$$

Where $\alpha/2|E| = a$ is the axis of the hyperbola. A particular case of this hyperbolic motion is if $E = 0$ and therefore $e = 1$, we have

$$r_{min} = \frac{p}{2} = \frac{L^2}{2m\alpha} \tag{3.52}$$

Hyperbolic paths can be illustrated as follows

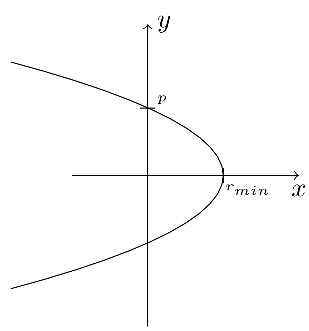


Figure 3.6: Hyperbolic orbit with orbital parameters, note that $r_{min} = a(e+1)$

4 Classical Collisions

§ 4.1 Disintegration of Particles

The most simple case of collision that we might begin to investigate, is the decay of a particle with mass M into two particles with masses m_1, m_2. Firstly considering the center of mass system (that we will call the «cm» system), we immediately apply the conservation of energy and momentum to such case, and writing $p_0 = \|p_1^\mu\|_\mu = \|p_2^\mu\|_\mu$, we must have

$$E_{int} = E_{1,int} + E_{2,int} + \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} \tag{4.1}$$

Where $p_0^2/2m_i$ is the kinetic energy of the i-th particle, and E_{int} is the "internal" energy of the particles. This process is possible only if the internal energy of the first particle is greater than the sum of the internal energies of the two particles, we call this the «disintegration energy» ϵ, and we must have

$$\epsilon = E_{int} - E_{1,int} - E_{2,int} > 0 \tag{4.2}$$

Subsituting into (4.1) we have a different definition for this energy

$$\epsilon = \frac{p_0^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{p_0^2}{2\mu} \tag{4.3}$$

Where the reduced mass μ pops directly from the definition, where $\mu^{-1} = m_1^{-1} + m_2^{-1}$. The velocities of the particles in the cm system are really easy to find, and since $m_i v_i = p_0$ we have

$$\begin{cases} v_1 = \dfrac{p_0}{m_1} \\ v_2 = \dfrac{p_0}{m_2} \end{cases} \tag{4.4}$$

Obviously they'll be directed in opposite directions.
We now change to a new reference system where our main particle with mass M moves with some velocity V, we will call this the «laboratory» system (indicated as «l»).
Considering only the first of the two resulting particles we have that in this transformation, if v_0^μ is the velocity in the cm system

$$v^\mu = V^\mu + v_0^\mu \implies v_0^2 = v^2 + V^2 + 2vV\cos\theta \tag{4.5}$$

Where the $\cos\theta$ pops out from a scalar product, and indicates the angle between the direction of the velocity of the main particle and the decayed particle.
An easy and intriguing way that can be used to solve this kind of collision was given by L. D. Landau and E. M. Lifshits in the book [LL69], where we use geometry to get everything about the system.

1. Begin by drawing a circle with radius v_0, draw a vector from the origin till the circle at an angle θ_0, this will represent the vector v_0 and the angle in the cm system. The point where v_0 will touch the circumference will be the point C
2. Draw a line of length V from the origin and call the arrival point A, make this a vector by adding the verse, strictly from A to the origin
3. Draw a vector from A to the point C, this will represent the vector v in the l system. The angle between v and V is the angle θ

This gives immediately two cases. If $V < v_0$ the vector V will be completely contained in the circle and the point C can lay in every point of the circumference, i.e. $\theta, \theta_0 \in [0, 2\pi]$, instead if $V > v_0$ this is not true anymore, and there will be some θ_{max} that can be reached.

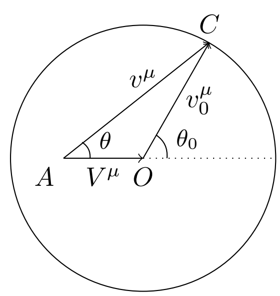


Figure 4.1: Collision circle for the case $V < v_0$

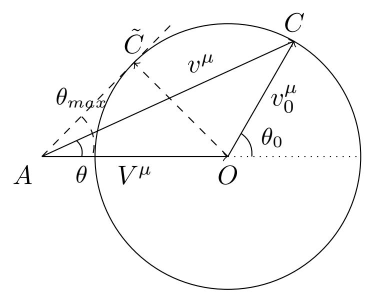


Figure 4.2: Collision circle for the case $V > v_0$. Note the presence of a maximum possible angle.

The maximum angle possible for the second case, where $V > v_0$ is calculated easily through trigonometry. We have

$$V \sin(\theta_{max}) = v_0 \implies \sin(\theta_{max}) = \frac{v_0}{V} \tag{4.6}$$

Using the same circle it's possible to find the relations between the flight angles θ, θ_0 between the l system and the cm system. We have

$$\begin{cases} v \sin\theta = v_0 \sin\theta_0 \\ v \cos\theta = v_0 \cos\theta_0 + V \end{cases} \tag{4.7}$$

Thus giving us the following relation

$$\tan\theta = \frac{v_0 \sin\theta_0}{v_0 \cos\theta_0 + V} \tag{4.8}$$

§§ 4.1.1 Disintegration Into Multiple Particles

In physics, while studying particle decays and disintegrations, it's common to actually have the disintegration of one particle into many particles. This problem in the cm system is obvious since resulting particles of the same kind have the same energy and their flight angle distribution is therefore isotropic in space.
This implies that the number of particles passing through some solid angle $\mathrm{d}\Omega_0$ is proportional to the linear dimension of this element, $\mathrm{d}\Omega_0/4\pi$. We therefore can write

$$4\pi\,\mathrm{d}N_p = 2\pi\sin\theta_0\mathrm{d}\theta_0 = \mathrm{d}\Omega_0 \implies \mathrm{d}N_p = \frac{1}{2}\sin\theta_0\mathrm{d}\theta_0 \tag{4.9}$$

Applying the transformation $v^\mu = v_0^\mu + V^\mu$ we move into the l system, and using equation (4.5), we get

$$\mathrm{d}(\cos\theta_0) = \sin\theta_0\mathrm{d}\theta_0 = \frac{\mathrm{d}(v^2)}{2v_0V} \tag{4.10}$$

Introducing the kinetic energy T we can rewrite the previous equation as follows

$$\sin\theta_0\mathrm{d}\theta_0 = \frac{\mathrm{d}T}{2mv_0V} \tag{4.11}$$

This gives us a constraint on the possible magnitude of the kinetic energy, which must be contained between these two values

$$\begin{cases} T_{min} = \frac{1}{2}m\left(v_0 - V\right)^2 \\ T_{max} = \frac{1}{2}m\left(v_0 + V\right)^2 \end{cases} \tag{4.12}$$

Note that this gives us only an interval of possible energies in the l system, whereas in the cm system we end up having complete arbitrary values for T.
Consider now all resulting particles, minus one with mass m_1, as a whole system with mass m_2. Using the conservation of momentum we can say that the energy of particle 1 must be

$$T_{10} = \frac{p_0^2}{2m_1} \tag{4.13}$$

Using the definition of ϵ, (4.3), we can write this quantity differently

$$T_{10} = \frac{M - m_1}{M}\left(\frac{1}{M + m_1} + \frac{1}{m_1}\right)\frac{p_0^2}{2} = \frac{M - m_1}{M}\left(E_{int} - E_{1int} - E_{2int}\right) = \frac{M - m_1}{M}\epsilon \tag{4.14}$$

Where E_{int}, M are respectively the internal energy and mass of the main particle «before» the decay.

§ 4.2 Elastic Collisions

A collision between two particles is called elastic if there are no changes in their states after the collision. The internal energy of the particles can be ignored thanks to the conservation of energy.

Indicating with the symbol $\square_0$ the quantities in the cm system, thanks to the conservation of momentum the particles will move with the following velocities in the cm system

$$\begin{cases} v_{10}^{\mu} = \dfrac{m_2}{m_1+m_2} v^{\mu} \\ v_{20}^{\mu} = -\dfrac{m_1}{m_1+m_2} v^{\mu} \end{cases} \tag{4.15}$$

Where $v^{\mu} = v_1^{\mu} - v_2^{\mu} = v\hat{n}_0^{\mu}$ is directed as the first particle's velocity.
Switching to the l system and accounting for the added velocity V^{μ} of the system we get, considering directly the new momentum of the two particles

$$\begin{cases} \tilde{p}_1^{\mu} = \mu v \hat{n}_0^{\mu} + \dfrac{m_1}{m_1+m_2}(p_1^{\mu} + p_2^{\mu}) \\ \tilde{p}_2^{\mu} = -\mu v \hat{n}_0^{\mu} + \dfrac{m_2}{m_1+m_2}(p_1^{\mu} + p_2^{\mu}) \end{cases} \tag{4.16}$$

Where the unsigned momentum is the momentum of the particle before the collision, and μ is the reduced mass
As for the disintegration of particles, we can represent an elastic collision in a collision circle as the following

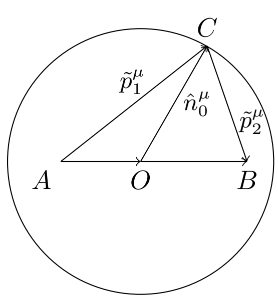


Figure 4.3: Example of a collision circle for an elastic collision

$$\begin{cases} \overline{OC} = \mu v^{\mu} \\ \overline{AO} = \dfrac{m_1}{m_1+m_2}(p_1^{\mu} + p_2^{\mu}) \\ \overline{OB} = \dfrac{m_2}{m_1+m_2} \\ \overline{AB} = p_1^{\mu} \\ \overline{AC} = \tilde{p}_1^{\mu} \\ \overline{CB} = \tilde{p}_2^{\mu} \\ \dfrac{\|\overline{AO}\|}{\|\overline{OB}\|} = \dfrac{m_1}{m_2} \end{cases} \tag{4.17}$$

As in the figure the vector $\hat{n}_0^{\mu}$ lays on the direction of the segment OC. The vectors going from the points A to the point C and from C to B represent respectively the impulses of the two particles after the collision. If both are fixed, so must be the radius of the circumference and both points A, B, while the point C can instead lay wherever on the circumference

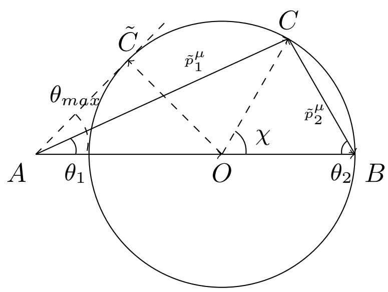


Figure 4.4: Collision circle for an elastic collision between a moving particle and a particle at rest $m_1 > m_2$.

A second case is if one of the two particles is at rest before the collision. Supposing that m_2 is at rest, we have that the point B must lay on the circumference, since the segment AB must coincide with the impulse p_1^μ.

There can be two main cases here, one where $m_1 < m_2$ as in fig. (4.4) and therefore the point A lays on the inside of the circle and C is free, and a second where $m_1 > m_2$ as in fig. (4.5) and A is outside the circumference

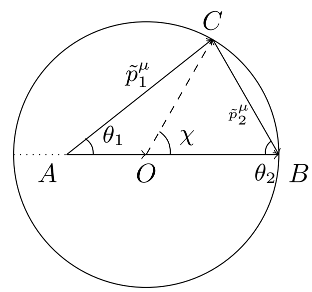


Figure 4.5: Collision circle for the same collision where $m_1 < m_2$

The angles θ_1, θ_2 depicted in both figures represent the deflection of both particles from the initial direction of the incoming particle. The angle χ, analogously to the angle θ_0 represents the deflection of the incoming particle in the system cm. Both angles θ_1, θ_2 can be expressed in terms of χ using trigonometry, yielding

$$\tan\theta_1 = \frac{m_2 \sin\chi}{m_1 + m_2\cos\chi} \qquad \theta_2 = \frac{\pi - \chi}{2} \tag{4.18}$$

The magnitudes of both velocities can also be expressed using the angle χ, as

$$\tilde{v}_1 = \frac{1}{m_1 + m_2}\sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos\chi} \qquad \tilde{v}_2 = \frac{2m_1 v}{m_1 + m_2}\sin\left(\frac{\chi}{2}\right) \tag{4.19}$$

From these two diagrams we can also extrapolate another constraint to the angle $\theta = \theta_1 + \theta_2$, we must have therefore

$$\begin{cases} \theta > \dfrac{\pi}{2} & m_1 < m_2 \\ \theta < \dfrac{\pi}{2} & m_1 > m_2 \end{cases} \tag{4.20}$$

Another case possible is that the particles collide head on, where $\chi = \pi$, forcing the new momenta of each particle to be equal and opposite or either in the same direction between points A and O. Looking back at the dependence between the velocities and χ, this is the maximum value possible for

$\tilde{v}_2$. The maximum possible energy of such collision is therefore

$$\tilde{E}_{2,max} = \frac{m_2 \tilde{v}_{2,max}^2}{2} = \frac{4\mu}{m_1 + m_2} E_1 \tag{4.21}$$

Where we used the law of conservation of energy in order to tie this with the energy of the incoming projectile E_1.
The maximum angle possible θ_{max} shown in the diagram for the collision with a projectile with mass $m_1 > m_2$ is calculated directly with trigonometry as follows

$$\sin\theta_{max} = \frac{OC}{OA} = \frac{m_1}{m_2} \tag{4.22}$$

The final and most simple case of elastic collision is the one between the projectile and target of both equal mass. Both points A, B lay on the circumference and the angles θ_1, θ_2 are mutually orthogonal

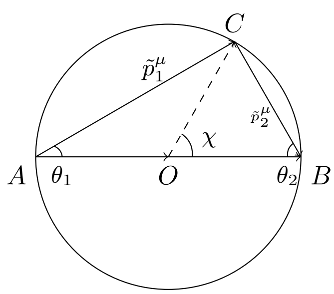


Figure 4.6: Collision circle for an elastic collision where $m_1 = m_2$

$$\left[\begin{array}{l} \theta_1 = \dfrac{\chi}{2} \\ \tilde{v}_1 = v\cos\left(\dfrac{\chi}{2}\right) \\ \theta_2 = \dfrac{\pi - \chi}{2} \\ \tilde{v}_2 = v\sin\left(\dfrac{\chi}{2}\right) \end{array}\right.$$

§ 4.3 Particle Scattering

As seen in the previous section, for getting the angle χ and therefore a complete definition of the problem it's necessary to solve the equations of motion with some interaction law between the particles. It's possible to consider an equivalent, and general, case where a particle of mass m gets deflected from a central force field $\mathcal{U}(r)$ (as if it was centered in the center of mass of the 2 colliding particles). As already known, the trajectory of the particles in a central field are symmetric to a line from the center to the closest point of the particle, and the asymptotes of the orbit intersecate this line at the same angles φ_0. The angle of scattering χ will therefore be

$$\chi = |\pi - 2\varphi_0| \tag{4.23}$$

The angle φ_0 is determined by the integral

$$\varphi_0 = \int_{r_{min}}^{\infty} \frac{1}{r^2\sqrt{2m(E - \mathcal{U}(r)) - \frac{L^2}{r^2}}}\,\mathrm{d}r \tag{4.24}$$

Where r_{min} is the closest distance from the center, found as

$$\sqrt{2m(E-\mathcal{U}(r_{min})-\frac{L^2}{r_{min}^2}}=0 \tag{4.25}$$

For an infinite motion, it's useful to consider directly the speed at ∞, v_∞ and a «collision parameter» ρ, where

$$\begin{cases} v_\infty = \sqrt{\dfrac{2E}{m}} \\ \rho = \dfrac{L}{mv_\infty} \end{cases} \tag{4.26}$$

Therefore, rewriting the integral (4.24)

$$\varphi_0(\rho, v_\infty)=\int_{r_{min}}^{\infty}\frac{\rho}{r^2\sqrt{1-\frac{\rho^2}{r^2}-\frac{2\mathcal{U}(r)}{mv_\infty^2}}}\mathrm{d}r \tag{4.27}$$

Which gives us the dependence of χ to those two parameters.
In the case that there is actually a beam of particles getting scattered, this gets a bit more difficult since each particle of the beam has its own ρ_i.
Call $\mathrm{d}N$ the density of particles getting scattered with some angle in $[\chi, \chi+\mathrm{d}\chi]$. We define the «scattering cross-section» $\mathrm{d}\sigma$ as

$$\mathrm{d}\sigma=\frac{\mathrm{d}N}{n} \tag{4.28}$$

Where n is the number of particles passing through a section of the beam per unit time. Note that $[\mathrm{d}\sigma]=L^2$ therefore it's rightfully called a "section".
This cross-section is completely determined by the force field and it's the most important parameter in particle beam scattering.
Supposing that the relation between χ and ρ is bijective in $(\chi, \chi+\mathrm{d}\chi)$ the particles that get scattered will have collision parameters with values between ρ and $\rho+\mathrm{d}\rho$, therefore, for a cylindrical beam

$$\mathrm{d}N=n\mathrm{d}S=2\pi n\rho(\chi)\mathrm{d}\rho$$

Which means

$$\mathrm{d}\sigma=2\pi\rho\mathrm{d}\rho$$

Rewriting and adding an absolute sign to the derivative we have

$$\mathrm{d}\sigma=2\pi\rho(\chi)\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right|\mathrm{d}\chi \tag{4.29}$$

Considering instead a solid angle $\mathrm{d}\Omega=2\pi\sin\chi\mathrm{d}\chi$ we have

$$\mathrm{d}\sigma=\frac{\rho(\chi)}{\sin\chi}\left|\frac{\mathrm{d}\rho}{\mathrm{d}\chi}\right|\mathrm{d}\Omega \tag{4.30}$$

§§ 4.3.1 Rutherford Scattering

Having laid out the framework for evaluating scattering in central force fields, the first step we might do is to use a specific potential, a «Coulombian potential» , the scattering in such potentials is also known as «Rutherford Scattering» .
Having defined this potential the integral (4.27) is directly solvable (after some tedious algebra) and the result is

$$\varphi_0 = \text{acos}\left(\frac{\alpha}{mv_\infty^2\rho\sqrt{1+\left(\frac{\alpha}{mv_\infty\rho}\right)^2}}\right) \tag{4.31}$$

From which we can determine the collision parameter ρ

$$\rho^2 = \frac{\alpha^2}{m^2v_\infty^4}\tan^2\varphi_0 = \frac{\alpha^2}{m^2v_\infty^4}\cot^2\left(\frac{\chi}{2}\right)$$

I.e.

$$\rho(\chi) = \frac{\alpha}{mv_\infty^2}\cot\left(\frac{\chi}{2}\right)$$

Where we substituted inside $\varphi_0 = (\pi - \varphi_0)/2$.
Substituting the formula for the cross-section (4.29) and (4.30) we get

$$\begin{cases} \mathrm{d}\sigma = \pi\left(\frac{\alpha}{mv_\infty^2}\right)^2\cot\left(\frac{\chi}{2}\right)\csc^2\left(\frac{\chi}{2}\right)\mathrm{d}\chi \\ \mathrm{d}\sigma = \left(\frac{\alpha}{2mv_\infty^2}\right)^2\csc^4\left(\frac{\chi}{2}\right)\mathrm{d}\Omega \end{cases} \tag{4.32}$$

This last equation is known as «Rutherford's formula» . It's evident that this cross-section is independent from the sign of α, therefore in this case both attractive and repulsive fields' cross-sections are described by the same equations.
This formula tho is valid only in the cm system, and substituting $\chi = \pi - 2\theta_2$ we obtain the cross-section formula for the beam of scattered particles in the system l

$$\begin{cases} \mathrm{d}\sigma_2 = 2\pi\left(\frac{\alpha}{mv_\infty^2}\right)^2\tan\theta_2\sec^2\theta_2\mathrm{d}\theta_2 \\ \mathrm{d}\sigma_2 = \left(\frac{\alpha}{mv_\infty^2}\right)^2\sec^3\theta_2\mathrm{d}\Omega_2 \end{cases} \tag{4.33}$$

Where θ_2 is the deflection angle of the outgoing particles.
It's not easy to find the same formula for the incoming particles, but it can be easily derived in three main cases

1. $m_2 >> m_1$, then $\chi \approx \theta_1$ and $\mu \approx m_1$ therefore

$$\mathrm{d}\sigma_1 = \left(\frac{\alpha}{4E_1}\right)^2\csc^4\left(\frac{\theta_1}{2}\right)\mathrm{d}\Omega_1$$

   Where $E_1 = m_1v_\infty^2/2$ is the energy of the projectiles

2. $m_1 = m_2$, then $\chi \approx 2\theta_1$ and $\mu = m_1/2$, therefore

$$\begin{cases} \mathrm{d}\sigma_1 = 2\pi \left(\frac{\alpha}{E_1}\right)^2 \cot\theta_1 \csc^2\theta_1 \mathrm{d}\theta_1 \\ \mathrm{d}\sigma_1 = \left(\frac{\alpha}{E_1}\right)^2 \cot\theta_1 \csc^3\theta_1 \mathrm{d}\Omega_1 \end{cases}$$

3. Both projectiles and targets are identical, therefore there's no way to discern targets from projectiles. Summing $\theta = \theta_1 + \theta_2$ and $\mathrm{d}\sigma = \mathrm{d}\sigma_1 + \mathrm{d}\sigma_2$ we have a general cross-section

$$\mathrm{d}\sigma = \left(\frac{\alpha}{E_1}\right)^2 \left(\csc^4\theta + \sec^4\theta\right) \cos\theta \mathrm{d}\Omega$$

It's also possible to use Rutherford's formula for finding the distribution of scattered particles in function of the lost energy.
For an arbitrary fraction of m_1 and m_2 the velocity taken by the target can be expressed in terms of χ in the cm system as

$$\tilde{v}_{20} = \frac{2m_1}{m_1 + m_2} v_\infty \sin\left(\frac{\chi}{2}\right)$$

Therefore, the associated lost energy is

$$\varepsilon(\chi) = \frac{m_2 \tilde{v}_{20}^2}{2} = \frac{2m_1^2}{m_2} v_\infty^2 \sin^2\left(\frac{\chi}{2}\right)$$

Inverting this function and inserting $\mathrm{d}\chi(\varepsilon)$ into the definition of Rutherford's cross section we have

$$\mathrm{d}\sigma = \frac{2\pi\alpha^2}{m_2 v_\infty^2} \frac{\mathrm{d}\varepsilon}{\varepsilon^2} \tag{4.34}$$

5 Small Oscillations

§ 5.1 Stability and Free Oscillations

One of the most important topics of mechanics is the idea of stability and small oscillations around these "stable points". The idea of stable and unstable points stems from the main idea of equilibrium.

Definition 5.1.1 (Mechanical Equilibrium, Stability and Instability)**.** Given a system interacting with a certain potential $\mathcal{U}$, equilibrium points of such are defined as the «critical points» of the potential $\mathcal{U}$. The stability of such points is then defined by their nature, where minimums of the potential function are said to be «stable equilibrium points» and maximums as «unstable equilibrium points» of the system.

In order to evaluate the stability of equilibrium points, given the definition, immediately uses optimization theory applied on the potential function.
With this, for a system with n degrees of freedom, given a critical point q_0^μ of $\mathcal{U}(q^\mu)$, and having defined the Hessian matrix of $\mathcal{U}$ as $\partial^2_{\mu\nu}\mathcal{U} = \mathcal{U}_{\mu\nu}$ one has two main results.

1. If $\mathcal{U}_{\mu\nu}(q_0^\mu)$ is positive definite, the critical point is a minimal and therefore a stable equilibrium point

2. If $\mathcal{U}_{\mu\nu}(q_0^\mu)$ is negative definite, then it's a maximal and therefore an unstable equilibrium point

In case that the Hessian matrix is undefined, nothing can be derived through this criterion.
Note that using Sylvester's criterion for determining the signature of the eigenvalues of a matrix, in two dimensions this reduces to evaluating sign of the determinant of $\mathcal{U}_{\mu\nu}$ and the sign of the first entry of the matrix, $\mathcal{U}_{11}$, where

1. $|\mathcal{U}_{\mu\nu}| > 0$ and $\mathcal{U}_{11} > 0$ imply a stable equilibrium point

2. $|\mathcal{U}_{\mu\nu}| > 0$ and $\mathcal{U}_{11} < 0$ imply an unstable equilibrium point

3. $|\mathcal{U}_{\mu\nu}| < 0$ imply a saddle (unstable) point of equilibrium

4. $|\mathcal{U}_{\mu\nu}| = 0$ implies that $\mathcal{U}_{\mu\nu}$ is indefinite and therefore no clear conclusion can be given on the type of equilibrium at the considered point

§§ 5.1.1 Free Oscillations in 1 Dimension

Suppose having some system with one degree of freedom interacting in a field with potential $\mathcal{U}(q)$, which has q_0 as a local minimum point. Therefore we have

$$\left(\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}q}\right)_{q_0} = 0 \tag{5.1}$$

For evaluating small oscillations we expand with a power series to the second order this potential around q_0

$$\mathcal{U}(q) \simeq \mathcal{U}(q_0) + \frac{1}{2}\left(\frac{\mathrm{d}^2\mathcal{U}}{\mathrm{d}q^2}\right)_{q_0}(q-q_0)^2 + \mathcal{O}\big((q-q_0)^3\big) \tag{5.2}$$

Imposing $\mathcal{U}(q_0) = 0$ and writing $\mathcal{U}''(q_0) = k$ we have that our potential can be approximated to a harmonic oscillator potential up to second order, yielding

$$\mathcal{U}(q) \approx \frac{1}{2}k(q-q_0)^2 \tag{5.3}$$

Substituting $q - q_0 = x$ we can then write the approximate harmonic Lagrangian as the usual harmonic oscillator Lagrangian

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{5.4}$$

Deriving the Lagrangian we get the usual Euler-Lagrange equations for the harmonic oscillator, and substituting $\omega^2 = k/m$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = \ddot{x} + \omega^2 x = 0 \tag{5.5}$$

This differential equation has two possible solutions

$$\begin{aligned} x(t) &= c_1\cos(\omega t) + c_2\sin(\omega t) \\ x(t) &= A\cos(\omega t + \phi) \end{aligned} \tag{5.6}$$

Where $c_1, c_2, A, \phi \in \mathbb{R}$. The second solution can be obtained using the formula

$$\cos(\omega t + \phi) = \cos(\omega t)\cos\phi - \sin(\omega t)\sin\phi$$

Which gives us the expression of A and ϕ in terms of the two constants c_1, c_2 as follows

$$A = \sqrt{c_1^2 + c_2^2} \qquad \tan\phi = -\frac{c_2}{c_1} \tag{5.7}$$

A is known as teh «amplitude» of the motion, while ϕ is the initial value of the phase with frequency ω.
Writing the mechanical energy of an oscillatory system and substituting the solution $x(t)$ we have

$$E = \frac{1}{2}m\omega^2 A^2 \tag{5.8}$$

Which gives us $E \propto A^2$. Noting also how $x(t)$ is shaped, we can also write it using Euler's identity as the real part of a complex function, as

$$x(t) = \mathfrak{Re}\left\{ae^{i\omega t}\right\} \tag{5.9}$$

Where we set $a = Ae^{i\phi}$ as the complex amplitude, which has the amplitude as modulus and the phase as argument.

§ 5.2 Free Oscillations in n Degrees of Freedom

Working analogously in n dimensions, for studying these oscillations we choose a certain critical point q_0^μ of the potential, using $x^\mu = q^\mu - q_0^\mu$ we approximate the potential as

$$\mathcal{U}(x^\mu) \approx \frac{1}{2}\partial^2_{\mu\nu}\mathcal{U}(0)x^\mu x^\nu = \frac{1}{2}k_{\mu\nu}x^\mu x^\nu \tag{5.10}$$

The matrix $k_{\mu\nu}$ is the Hessian matrix of $\mathcal{U}$, and therefore, since $\mathcal{U} \in C^2(\mathbb{R}^n)$, $k_{\mu\nu} = k_{\nu\mu}$.
Analogously, for the kinetic energy we have

$$T = \frac{1}{2}a_{\mu\nu}(q_0^\gamma)\dot{q}^\mu\dot{q}^\nu \tag{5.11}$$

Writing $a_{\mu\nu}(q_0^\gamma)$ as $m_{\mu\nu}$ and using the fact that $T \in C^2(\mathbb{R}^n)$ we have the linearized Lagrangian for a system with n degrees of freedom

$$\mathcal{L}(\dot{x}^\mu, x^\mu) = \frac{1}{2}\left(m_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - k_{\mu\nu}x^\mu x^\nu\right) \tag{5.12}$$

For finding the equations of motion of this system we write the total differential of the Lagrangian, getting

$$\mathrm{d}\mathcal{L} = m_{\mu\nu}\dot{x}^\mu\mathrm{d}\dot{x}^\nu - k_{\mu\nu}x^\mu\mathrm{d}x^\nu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\nu}\mathrm{d}\dot{x}^\nu + \frac{\partial\mathcal{L}}{\partial x^\nu}\mathrm{d}x^\nu \tag{5.13}$$

Therefore, the searched equations of motion will be a system of ODEs, which will be

$$m_{\mu\nu}\ddot{x}^\mu + k_{\mu\nu}x^\mu = 0 \tag{5.14}$$

Keeping in mind what we found before for oscillating system we suppose the solution as a complex vector where

$$x^\mu(t) = A^\mu e^{i\omega t} \quad A^\mu \in \mathbb{C}^n$$

Deriving and substituting back this function we get a linear system with A^μ as our unknown vector

$$\left(-\omega^2 m_{\mu\nu} + k_{\mu\nu}\right)A^\mu = 0 \tag{5.15}$$

The solution of this system boils down to searching simultaneous eigenvectors and eigenvalues for the two matrices, and therefore this is possible if and only if

$$\det\left|-\omega^2 m_{\mu\nu} + k_{\mu\nu}\right| = 0 \tag{5.16}$$

Since both $m_{\mu\nu}, k_{\mu\nu}$ are symmetric matrices we must have that $\omega^2 \in \mathbb{R}$, i.e. that all the eigenvalues of the combined system are real. This also implies that, if Δ^μ_a is the $a-$th minor of the composite matrix $d_{\mu\nu} = -\omega^2 m_{\mu\nu} + k_{\mu\nu}$, then $A^\mu \propto \Delta^\mu_a$, which implies that the general solution therefore is

$$x^\mu_a(t) = \Delta^\mu_a C_a e^{i\omega_a t} \tag{5.17}$$

Where x^μ_a is the solution associated with the $a-$th eigenvalue ω_a. The complete solution will therefore be a real superposition of the previous one, which boils down to the following function

$$x^\mu(t) = \sum_{a=1}^{n}\Delta^\mu_a \mathfrak{Re}\left\{C_a e^{i\omega_a t}\right\} = \sum_{a=1}^{n}\Delta^\mu_a\Theta_a(t) \tag{5.18}$$

Where with $\Theta_a(t)$ we have indicated a simple periodic oscillation with frequency ω_a. These simple oscillations are called the «normal oscillations» of the system and they are linearly independent from one another, forming an orthogonal basis in which the matrix $d_{\mu\nu}$ is diagonal. Obviously they also solve the ODE

$$\ddot{\Theta}_a + \omega_a^2 \Theta_a = 0 \tag{5.19}$$

With a basis transformation, the linearized Lagrangian (5.12) becomes as follows

$$\mathcal{L}\left(\dot{\Theta}, \Theta\right) = \sum_a \frac{1}{2} m_a \left(\dot{\Theta}_a^2 - \omega^2 \Theta_a^2\right) \tag{5.20}$$

These basis vectors can be immediately normalized to a new basis, where $Q_a = \sqrt{m_a \Theta_a}$, which changes slightly our Lagrangian to the new version

$$\mathcal{L}(\dot{Q}, Q) = \frac{1}{2} \sum_{a=1}^{n} \left(\dot{Q}_a^2 - \omega_a^2 Q_a^2\right) \tag{5.21}$$

6 Rigid Bodies

§ 6.1 Angular Velocity

Definition 6.1.1 (Rigid Body)**.** In mechanics, a rigid body can be seen as a system of points for which the mutual distance between two points is fixed.
For describing such system there can be defined two coordinate systems. A fixed system XYZ and a mobile xyz system constrained to the body. Calling R^μ the radius vector from the system XYZ to the system xyz we have that the motion of a rigid body can be described via the 3 components of R^μ and the angles $\theta_1, \theta_2, \theta_3$ between the xyz axes and the XYZ axes, making the rigid body a system with 6 degrees of freedom.

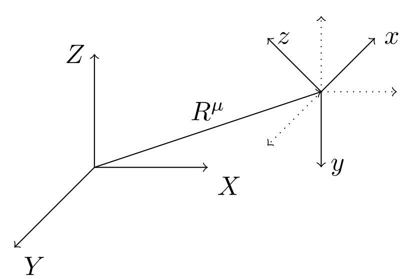


Figure 6.1: An example of the two reference systems of a rigid bodies, the angles θ_i, not drawn, are the angles between the dotted arrows and the axes of the mobile system xyz

Consider now an infinitesimal movement of the rigid body, this amounts to a translation of the center of mass of such and an infinitesimal rotation of the axes xyz (which we choose fixed at the center of mass). Calling r^μ the radius vector of some point P of the body with respect to the moving frame and with $\tilde{r}^\mu$ the radius vector of the same point with respect to the fixed frame XYZ we have, if $\mathrm{d}R^\mu$ is the infinitesimal displacement of the center of mass

$$\mathrm{d}\tilde{r}^\mu = \mathrm{d}R^\mu + \epsilon^{\mu\nu\gamma}\mathrm{d}\varphi_\nu r_\gamma \tag{6.1}$$

Dividing this by $\mathrm{d}t$ we get three velocities

$$\begin{aligned}\frac{\mathrm{d}\tilde{r}^\mu}{\mathrm{d}t} &= \tilde{v}^\mu \\ \frac{\mathrm{d}R^\mu}{\mathrm{d}t} &= V^\mu \\ \frac{\mathrm{d}\varphi_\nu}{\mathrm{d}t} &= \Omega_\nu\end{aligned} \tag{6.2}$$

Which are respectively the translation velocity of the point, the translation velocity of the rigid body and the «rotation velocity» also known as «angular velocity» of the whole body (the points are fixed). This gives

$$\tilde{v}^\mu = V^\mu + \epsilon^{\mu\nu\gamma}\Omega_\nu r^\gamma$$

Therefore with these two velocities that we have defined we can determine the state of every point in the solid

§ 6.2 Inertia Tensor

Let the moving frame coincide with the center of mass of the body. The kinetic energy of the full body will therefore be

$$T = \sum_{k=1}^{n} \frac{1}{2} m_i v^2 \tag{6.3}$$

We develop it using the definition of v^μ, and we get

$$v^2 = V^2 + \epsilon_{\mu\nu\sigma}\epsilon^{\mu\delta\gamma}\Omega_\delta r_\gamma \Omega^\nu r^\sigma + V^\mu \epsilon_{\mu\nu\sigma}\Omega^\nu r^\sigma$$

Applying the properties of the epsilon symbol, we have

$$v^2 = V^2 + V^\mu \epsilon_{\mu\nu\sigma}\Omega^\nu r^\sigma + \left(\delta^\delta_\nu \delta^\gamma_\sigma - \delta^\gamma_\nu \delta^\delta_\sigma\right)\Omega_\delta r_\gamma \Omega^\nu r^\sigma$$

Using the permutation properties of the second term and contracting the indices, we finally get

$$T = T_d + T_{rot} = \frac{1}{2}MV^2 + \sum_{k=1}^{n} m_i \left[\Omega^2 r^2 - (\Omega_\mu r^\mu)^2\right] \tag{6.4}$$

Where we used $\sum m = M$ and $\sum m r^\mu = 0$.
The second term, called «rotational kinetic energy» can be simplified even more rewriting it as follows

$$T_{rot} = \frac{1}{2}\sum m\left(r^2 \Omega_\mu \Omega^\nu \delta^\mu_\nu - \Omega_\mu r^\mu \Omega^\nu r_\nu\right)$$

Bringing outside the parenthesis the dyadic $\Omega_\mu \Omega^\nu$ we have

$$T_{rot} = \frac{1}{2}\Omega_\mu \Omega^\nu \sum m\left(r^2 \delta^\mu_\nu - r^\mu r_\nu\right) = \frac{1}{2} I^\mu_\nu \Omega_\mu \Omega^\nu \tag{6.5}$$

Where the last part of the multiplication is called the «inertia tensor» , where

$$I^\mu_\nu = \sum m\left(r^2 \delta^\mu_\nu - r^\mu r_\nu\right) \tag{6.6}$$

The most compact way of writing a Lagrangian of a rigid body then becomes the usual $T - \mathcal{U}$, giving

$$\mathcal{L} = \frac{MV^2}{2} + \frac{1}{2} I^\mu_\nu \Omega_\mu \Omega^\nu - \mathcal{U} \tag{6.7}$$

Doing all calculations, the matrix representation of this tensor, is

$$I^\mu_\nu = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum myx & \sum m(x^2+z^2) & -\sum mxz \\ -\sum mzx & -\sum mzy & \sum m(x^2+y^2) \end{pmatrix}^\mu_\nu \tag{6.8}$$

In case the body is continuous, we apply the limit $\sum m \to \int_V \rho \mathrm{d}V$ where V is the volume of the continuous solid and ρ its density, obtaining the following formula

$$I^\mu_\nu = \int_V \rho(x,y,z) \left(r^2 \delta^\mu_\nu - r^\mu r_\nu \right) \mathrm{d}^3 x \tag{6.9}$$

Since the tensor is symmetric of rank 2 it's possible to find a diagonal representation of such, where

$$I^\mu_\nu \Omega^\mu \Omega_\nu = I_1 \Omega_1 + I_2 \Omega_2 + I_3 \Omega_3 \tag{6.10}$$

Where, by definition, we must have

$$I_1 + I_2 \geq I_3$$

The eigenvalues I_i are known as the «principal moments» of inertia, which let us distinguish between 3 different kinds of rigid bodies

- If $I_1 \neq I_2 \neq I_3$ the body is known as an asymmetric top
- If $I_1 = I_2 \neq I_3$ the body is known as a symmetric top
- If $I_1 = I_2 = I_3$ the body is known as a spherical top

The properties of the body also help finding the principal axes of inertia (the eigenvectors) by using the symmetries of the body itself, as an example take a body with a plane of symmetry. The center of mass will lay on such plane as will two of the principal axes of inertia, while the third will be orthogonal to the plane.
An idea of this can be given by a system of particles lying on the $x^1 x^2$ plane. Using $x^3 = 0$ we have immediately

$$I_1 = \sum m x_2^2, \; I_2 = \sum m x_2^1, \; I_3 = \sum m(x_1^2 + x_2^2) = I_1 + I_2$$

If instead the body has an axis of symmetry of whatever order, the center of mass and one principal axis must lay on this axis, and due to the indistinguishability of the other two axes we must have

$$I_1 = I_2 = \sum m x_3^2, \; I_3 = 0$$

This system is a symmetric top and is also known as a «rotator» . Note that $I_3 = 0$ since the rotation around this axis has no physical meaning and can't be expressed.
Another property of the inertia tensor is that it's dependent on the choice of origin of the coordinates. Take a new origin $\tilde{O}$ distant a^μ from O. The new radius vector will be

$$\tilde{r}^\mu = r^\mu - a^\mu$$

Inserting this into the definition of the inertia tensor and expanding $\tilde{r}^\mu$ we have

$$\tilde{I}^\mu_\nu = I^\mu_\nu + 2\sum m\left(r^\gamma a_\gamma \delta^\mu_\nu - r^\mu a_\nu\right) + \sum m\left(a^2\delta^\mu_\nu - a^\mu a_\nu\right) \tag{6.11}$$

Using $\sum m r^\mu = 0$ we have that the second addend is zero, therefore, if $\sum m = M$

$$\tilde{I}^\mu_\nu = I^\mu_\nu + M\left(a^2\delta^\mu_\nu - a^\mu a_\nu\right) \tag{6.12}$$

Note that if we take a^μ as a parameter, $\min\left(\tilde{I}^\mu_\nu(a^\gamma)\right) = I^\mu_\nu(0)$, i.e. when $a^\mu = 0$ and $\tilde{O} = O$, with O as our center of mass

§§ 6.2.1 Angular Momentum of a Rigid Body

The angular momentum of a rigid body has a straightforward definition, with a simple generalization

$$L_\mu = \sum m\epsilon_{\mu\nu\sigma} r^\nu v^\sigma \tag{6.13}$$

Substituting $v^\mu = \epsilon^{\mu\nu\sigma}\Omega_\nu r_\sigma$ we have

$$L_\mu = \sum m\epsilon_{\mu\nu\sigma}\epsilon^{\sigma\delta\gamma} r^\nu \Omega_\gamma r_\delta$$

Imposing the properties of the epsilon, we have

$$L_\mu = \sum m\left(r^2\Omega_\mu - r_\mu\left(r^\nu\Omega_\nu\right)\right)$$

Using the same trick we did for I^μ_ν we can insert a Kronecker delta and bring outside Ω_ν, giving us

$$L_\mu = \sum m\left(r^2\delta^\nu_\mu - r^\nu r_\mu\right)\Omega_\nu = I^\mu_\nu\Omega_\nu \tag{6.14}$$

Note that if the tensor is diagonal, the components of the angular momentum will be proportional to the same component of the angular velocity, and in case that the body is a spherical top we have $I_1 = I_2 = I_3$ i.e.

$$L_\mu = I\Omega_\mu \tag{6.15}$$

This symmetry simplifies everything, since the plain conservation of angular momentum imposes that Ω_μ is constant, and everything reduces to the study of an uniformly rotating body around an axis defined by L_μ.

For a rotator, taking the rotation axis as x^3 we still have $L_\mu = I\Omega_\mu$ and everything again reduces, via the conservation of L_μ to an uniform rotation on the plane orthogonal to the rotation axis.

For a symmetric top, we can again use the conservation of angular momentum to simplify the problem. Using the arbitrariness of the choice of the x^1, x^2 axes ($I_1 = I_2$) we can take x_2 be orthogonal to the plane defined by $L_\mu x^3$, and therefore we must have $L_2 = \Omega_2 = 0$, implying that both vectors are coplanar.

Calling this plane π we have that $v^\mu \perp \pi$ for all points situated in the axis of symmetry, or in other words, the axis x^3 precedes around L_μ in a regular manner, describing a spherical cone.

Calling θ the angle of such cone, we have using the definitions, that

$$\begin{aligned}\Omega_3 &= \frac{L_3}{I_3} = \frac{L}{I_3}\cos\theta \\ \Omega_1 &= \frac{L_1}{I_1} = \frac{L}{I_1}\sin\theta\end{aligned} \tag{6.16}$$

Writing Ω_p as the module of the precession velocity, we have $\Omega_1 = \Omega_p \sin\theta$, and therefore we can also write

$$\Omega_p = \frac{L}{I_1} \tag{6.17}$$

§ 6.3 Equations of Motion

The determination of the equations of motion of a rigid body are strictly tied to its nature. Since it has 6 degrees of freedom, 3 linear and 3 rotational, we expect to have 3 differential equations for the total momentum P^μ and another 3 for the total angular momentum L_μ. For a system of particles, we have by definition

$$P^\mu = \sum p^\mu \tag{6.18}$$

Which, implies, via linearity of the derivative, that

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}t} = \sum \frac{\mathrm{d}p^\mu}{\mathrm{d}t} = \sum f^\mu = F^\mu \tag{6.19}$$

Where F^μ is the sum of the external forces applied on the body.
Note that if f^μ_{ij} is the force between the $i-$th and $j-$th particle of the body, $f^\mu_{ij} = -f^\mu_{ji}$ as for Newton's third law, making it irrelevant in the previous equation. Note that if that wouldn't be true, then the body can't be a rigid body.
As usual, if the forces F^μ are conservative, we can write a potential $\mathcal{U}(R^\mu)$, where R^μ is the radius vector to the center of mass of the body, for which

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}t} = F^\mu = -g^{\mu\nu}\frac{\partial\mathcal{U}}{\partial R^\mu}$$

Or, rewriting a Lagrangian $\mathcal{L}(R^\mu, V^\mu)$, with $V^\mu = \dot{R}^\mu$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial V^\mu} = \frac{\mathrm{d}P_\mu}{\mathrm{d}t} = F_\mu = \frac{\partial\mathcal{L}}{\partial R^\mu} \tag{6.20}$$

Which are the first Euler Lagrange equations for a rigid body in the coordinate system of its center of mass. The other equations can be derived using Galilean invariance and finding an inertial frame such that the center of mass is at rest for some t, where, by definition

$$\frac{\mathrm{d}L_\mu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum \epsilon_{\mu\nu\sigma} r^\nu p^\sigma = \sum \epsilon_{\mu\nu\sigma} r^\nu \dot{p}^\sigma$$

Having used this "rest frame" we have that $\dot{r}^\mu = v^\mu = 0$, and substituting $\dot{p}^\mu = f^\mu$ we have

$$\frac{\mathrm{d}L_\mu}{\mathrm{d}t} = \sum \epsilon_{\mu\nu\sigma} r^\nu f^\sigma = \tau_\mu \tag{6.21}$$

Where τ_μ is the already known «torque» of the body, also known as the «momentum of forces» .
Note that the torque is frame-independent. Taking a new origin distant a^μ from the previous one we have

$$\tau_\mu = \tilde{\tau}_\mu + \epsilon_{\mu\nu\sigma} a^\nu F^\sigma \tag{6.22}$$

Supposing that $F^\sigma = F_1^\sigma + F_2^\sigma$ and it's a force couple ($F_1^\sigma = -F_2^\sigma$) then the second part is zero and $\tau_\mu = \tilde{\tau}_\mu$.
Imposing "rotational coordinates" $\varphi_\mu, \dot{\varphi}_\mu$ to our Lagrangian, with the constraint $\dot{\varphi}_\mu = \Omega_\mu$, we can write EL equations for the rotational degrees of freedom as follows

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \Omega^\mu} = \frac{\partial \mathcal{L}}{\partial \varphi^\mu} \tag{6.23}$$

Writing the general Lagrangian for a rigid body we have

$$\frac{\partial \mathcal{L}}{\partial \Omega^\mu} = I_\mu^\nu \Omega_\nu = L_\mu \tag{6.24}$$

Whereas, for evaluating the other derivative, we have that for an infinitesimal displacement δR^μ given by some rotation $\delta\varphi^\mu$, we must have

$$\delta \mathcal{U} = \frac{\partial \mathcal{U}}{\partial R^\mu}\delta R^\mu = -\sum f_\mu \delta R^\mu = -\sum f^\mu \epsilon_{\mu\nu\sigma}\delta\varphi^\nu r^\sigma = -\sum \delta\varphi^\mu \epsilon_{\mu\nu\sigma} f^\nu r^\sigma$$

Or, inserting the definition of torque

$$\delta \mathcal{U} = -\tau_\mu \delta\varphi^\mu$$

Which implies $\partial_{\varphi^\mu}\mathcal{U} = -\tau_\mu$, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \Omega^\mu} = \frac{\mathrm{d}L_\mu}{\mathrm{d}t} = \tau_\mu = \frac{\partial \mathcal{L}}{\partial \varphi^\mu} \tag{6.25}$$

Which complete the two sets of differential equation for a rigid body, in complete accordance with the usual Newtonian counterpart.

Part II

Hamiltonian Mechanics

7 Canonical Equations of Motion

§ 7.1 Canonical Variables

The idea behind the reformulation of mechanics comes directly from the theory of differential equations. Take a second order ODE as the one following

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = f(y, y', t)$$

This equation can be reduced of order by imposing the transformation $u(t) = \dot{y}(t)$, which reduces the previous problem to a system of 2 ODEs of the first order

$$\begin{cases} \dot{u}(t) = f(y, u, t) \\ \dot{y}(t) = u(t) \end{cases}$$

This process can also be applied to Euler-Lagrange equations, where the N differential equations of the second order can be reduced to a system of $2N$ differential equations of the first order.
Since $\det \dot{\partial}_{\mu\nu}\mathcal{L} \neq 0$ we know for sure that the following differential equation can be solved

$$\dot{q}^\mu = f^\mu(q^\nu, \dot{q}^\nu, t) \tag{7.1}$$

The space of dinamical configurations of the system can be described by the couple $(q^\mu, \dot{q}^\mu)$, or using $\dot{\partial}_\mu \mathcal{L} = p_\mu$ and its independece with respect to q^μ, we can define a new space, called the «phase space» , spanned by the couple (q^μ, p_μ). This space is of dimension $2n$, and it's denoted here as Γ^{2n}.
The two variables q^μ, p_μ are known as the «canonical variables» of the system, and will describe a motion in this phase space via a curve $\gamma^\mu(t)$ which will be determined by the solution of the appropriate equations of motion.

§ 7.2 Canonical Equations of Motion

In order to solve the previous problem and actually reduce the Euler-Lagrange equations to a lower order, we begin by differentiating the Lagrangian

$$\mathrm{d}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\mu}\mathrm{d}\dot{q}^\mu + \frac{\partial \mathcal{L}}{\partial q^\mu}\mathrm{d}q^\mu = \dot{p}_\mu \mathrm{d}q^\mu + p_\mu \mathrm{d}\dot{q}^\mu \tag{7.2}$$

Rewriting $p_\mu \mathrm{d}\dot{q}^\mu = \mathrm{d}\left(p_\mu \dot{q}^\mu\right) - \dot{q}^\mu \mathrm{d}p_\mu$, where we treat p_μ as an independent variable, we have

$$\mathrm{d}\left(p_\mu \dot{q}^\mu - \mathcal{L}\right) = \dot{q}^\mu \mathrm{d}p_\mu - \dot{p}_\mu \mathrm{d}q^\mu \tag{7.3}$$

The function on the left is known as «Hamiltonian» of the system, and corresponds to the generalized energy in canonical coordinates. It's indicated as $\mathcal{H}$, and differentiating we get ***Hamilton's equations of motion*** also known as the ***canonical equations of motion***

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial p_\mu} = \dot{q}^\mu \\ \dfrac{\partial \mathcal{H}}{\partial q^\mu} = -\dot{p}_\mu \end{cases} \tag{7.4}$$

Integrating the differential on the left we can write

$$\mathcal{H}(p_\mu, q^\mu, t) = p_\mu \dot{q}^\mu - \mathcal{L}(q^\mu, q^\mu, t) \tag{7.5}$$

Where $\dot{q}^\mu = f^\mu(p_\mu, q^\mu, t)$. This process is called the «Legendre transformation» of the Lagrangian with respect to $\dot{q}^\mu$.
The previous equations (7.4) define the motion of the system in the phase space and are the searched reduction of the Euler-Lagrange equation from n ODEs of the second order to $2n$ ODEs of the first order.
Note that since in the phase space the Hamiltonian corresponds to the mechanical energy of the system, we can rewrite some theorems in a different way

Theorem 7.1 (Conservation of Energy)**.** ***The mechanical energy of the system E is conserved if the Hamiltonian function is independent from time, i.e.***

$$\frac{\partial \mathcal{H}}{\partial t} = 0 \implies \frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Proof. By definition, the Hamiltonian function of the system corresponds to the energy in the phase space, so we can immediately write its total derivative with respect to time

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial p_\mu}\dot{p}_\mu + \frac{\partial \mathcal{H}}{\partial q^\mu}\dot{q}^\mu \tag{7.6}$$

Substituting the canonical equations inside the expression we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial t}$$

Therefore

$$\frac{\partial \mathcal{H}}{\partial t} = 0 = \frac{\mathrm{d}E}{\mathrm{d}t}$$

□

Exercise 7.2.1 (Hamiltonians)**.** Find the Hamiltonian of a particle in

1. Cartesian coordinates
2. Cylindrical coordinates
3. Spherical coordinates

1) We begin by writing explicitly the Lagrangian for a particle in Cartesian coordinates.

$$\mathcal{L}(x^\mu, \dot{x}^\mu, t) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \mathcal{U}(x, y, z) \tag{7.7}$$

The canonical coordinates will be defined by taking the derivative with respect to the dotted coordinates, giving

$$\dot{\partial}_\mu \mathcal{L} = m\dot{x}_\mu \implies \dot{x}_\mu(p_\mu) = \frac{p_\mu}{m} \tag{7.8}$$

The kinetic counterpart transforms as $\dot{x}^\mu(p_\mu)\dot{x}_\mu(p_\mu)$, getting

$$\dot{x}^\mu \dot{x}_\mu = \frac{1}{m^2} p^\mu p_\mu$$

And the Hamiltonian will be

$$\mathcal{H}(p_\mu, q^\mu, t) = p_\mu \frac{p^\mu}{m} - \frac{m}{2}\frac{1}{m^2} p^\mu p_\mu + \mathcal{U}(x^\mu) \tag{7.9}$$

Which, simplified becomes the searched Hamiltonian

$$\mathcal{H}(p_\mu, q^\mu, t) = \frac{1}{2m} p^\mu p_\mu + \mathcal{U}(x^\mu) = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + \mathcal{U}(x, y, z) \tag{7.10}$$

2) Analogously, for cylindrical coordinates we have

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) - \mathcal{U}(r, \theta, z) \tag{7.11}$$

The conjugated coordinates will therefore be

$$\dot{\partial}_\mu \mathcal{L} = \begin{pmatrix} m\dot{r} & mr^2\dot{\theta} & m\dot{z} \end{pmatrix} \implies \dot{x}_\mu = \frac{1}{m}\begin{pmatrix} p_r & \frac{p_\theta}{r^2} & p_z \end{pmatrix} \tag{7.12}$$

The Hamiltonian will be

$$\mathcal{H} = \frac{1}{m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) - \frac{m}{2}\left(\frac{p_r^2}{m^2} + \frac{r^2 p_\theta^2}{m^2 r^4} + \frac{p_z^2}{m^2}\right) + \mathcal{U}(r, \theta, z)$$

I.e.

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + p_z^2\right) + \mathcal{U}(r, \theta, z) \tag{7.13}$$

3) The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2\right) - \mathcal{U}(r, \theta, \varphi) \tag{7.14}$$

The canonical coordinates are

$$p_\mu = \begin{pmatrix} m\dot{r} & mr^2\dot{\theta} & mr^2\sin^2\theta\dot{\varphi} \end{pmatrix}, \implies \dot{x}_\mu = \frac{1}{m}\begin{pmatrix} p_r & \frac{p_\theta}{r^2} & \frac{p_\varphi}{r^2\sin^2\theta} \end{pmatrix} \tag{7.15}$$

Substituting into the Legendre transform we have

$$\frac{1}{m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta}\right)-\frac{1}{2}m\left(\frac{p_r^2}{m^2}+\frac{r^2p_\theta^2}{m^2r^4}+\frac{r^2\sin^2\theta p_\varphi^2}{m^2r^4\sin^4\theta}\right)+\mathcal{U}(r,\theta,\varphi) \tag{7.16}$$

And therefore the Hamiltonian is

$$\mathcal{H}=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\varphi^2}{r^2\sin^2\theta}\right)+\mathcal{U}(r,\theta,\varphi) \tag{7.17}$$

§ 7.3 Hamilton-Jacobi Equation and Hamilton's Principle in Γ^{2n}

The principle of least action can be reformulated in Hamiltonian mechanics in a particular manner changing the boundary conditions for the variational principle, and considering the action as a function of coordinates.

Begin by considering that the path $q^\mu(t)$ will start from a fixed point $q^\mu(t_1)=q_1^\mu$ and ends in some unknown point $q^\mu(t_2)$. The boundary conditions for the variational principle will therefore be

$$\begin{cases}\delta q^\mu(t_1)=0\\ \delta q^\mu(t_2)=\delta q^\mu\end{cases}$$

Where $\delta q^\mu(t_1)=0$ since q_1^μ is a constant vector.

The variation of the action integral will be, as usual

$$\delta\mathcal{S}=\left[\frac{\partial\mathcal{L}}{\partial\dot{q}^\mu}\delta q^\mu\right]_{t_2}^{t_1}+\int_{t_1}^{t_2}\left(\frac{\partial\mathcal{L}}{\partial q^\mu}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q}^\mu}\right)\delta q^\mu\mathrm{d}t$$

Imposing the obvious condition that $q^\mu(t)$ must represent a physical motion, the integral must be 0, since the Euler-Lagrange equations are automatically solved. Evaluating the term on the left we obtain the variation of the action as

$$\delta\mathcal{S}=\frac{\partial\mathcal{L}}{\partial\dot{q}^\mu}\delta q^\mu=p_\mu\delta q^\mu \tag{7.18}$$

This implies immediately that

$$\frac{\partial\mathcal{S}}{\partial q^\mu}=p_\mu$$

Now, considering $\mathcal{S}=\mathcal{S}(q^\mu,t)$ we also must have

$$\mathrm{d}\mathcal{S}=\frac{\partial\mathcal{S}}{\partial q^\mu}\mathrm{d}q^\mu+\frac{\partial\mathcal{S}}{\partial t}\mathrm{d}t=\mathcal{L}\mathrm{d}t$$

Or, substituting, we have

$$\mathrm{d}\mathcal{S}=p_\mu\mathrm{d}q^\mu+\frac{\partial\mathcal{S}}{\partial t}=\mathcal{L}\mathrm{d}t$$

Dividing by $\mathrm{d}t$, we have

$$\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t}=p_\mu\dot{q}^\mu+\frac{\partial\mathcal{S}}{\partial t}=\mathcal{L}$$

And rearranging in terms of $\partial_t \mathcal{S}$

$$\frac{\partial \mathcal{S}}{\partial t} = \mathcal{L} - p_\mu \dot{q}^\mu$$

Substituting inside the definition of the Hamiltonian function, we have

$$\frac{\partial \mathcal{S}}{\partial t} = -\mathcal{H} \tag{7.19}$$

This equation is called the ***Hamilton-Jacobi equation***. Rewriting the differential of the action, we have

$$\mathrm{d}\mathcal{S}(q^\mu, t) = p_\mu \mathrm{d}q^\mu - \mathcal{H}\mathrm{d}t$$

The previous Hamilton-Jacobi equation, if solved, imposes that the action as a function of (q^μ, t) must be a total differential. With this consideration, one can reformulate the principle of least action in Hamiltonian mechanics in a new and elegant way, where now the variation is made on a path in Γ^{2n}, the phase space.

$$\mathcal{S}[q^\mu(t)] = \int_{t_1}^{t_2} \left(p_\mu \mathrm{d}q^\mu - \mathcal{H}(p_\mu, q^\mu, t)\mathrm{d}t\right) \tag{7.20}$$

Imposing the usual conditions on the variation of the coordinates q^μ that were used already in the chapter on Lagrangian mechanics, we have

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \left(\delta p_\mu \mathrm{d}q^\mu + p_\mu \mathrm{d}\delta q^\mu - \delta\mathcal{H}\mathrm{d}t\right)$$
$$\delta\mathcal{S} = \left[p_\mu \delta q^\mu\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\delta p_\mu \mathrm{d}q^\mu + \delta q^\mu \mathrm{d}p_\mu - \frac{\partial \mathcal{H}}{\partial p_\mu}\delta p_\mu - \frac{\partial \mathcal{H}}{\partial q^\mu}\delta q^\mu\right)$$

Rearranging the terms and noting that the first term goes to zero we have

$$\delta\mathcal{S} = \int_{t_1}^{t_2} \delta p_\mu \left(\mathrm{d}q^\mu - \frac{\partial \mathcal{H}}{\partial p_\mu}\mathrm{d}t\right) - \int_{t_1}^{t_2} \delta q^\mu \left(\mathrm{d}p_\mu + \frac{\partial \mathcal{H}}{\partial q^\mu}\right) \tag{7.21}$$

The condition $\delta\mathcal{S} = 0$ imposes that both the integrals must be zero simultaneously, and since $\delta p_\mu, \delta q^\mu \neq 0$ in general, we must have

$$\begin{cases} \mathrm{d}q^\mu - \dfrac{\partial \mathcal{H}}{\partial p_\mu}\mathrm{d}t = 0 \\ \mathrm{d}p_\mu + \dfrac{\partial \mathcal{H}}{\partial q^\mu}\mathrm{d}t = 0 \end{cases} \tag{7.22}$$

Which are Hamilton's equations of motion. Note that dividing by $\mathrm{d}t$ and rearranging, we obtain the usual form of the equations

$$\begin{cases} \dfrac{\partial \mathcal{H}}{\partial p_\mu} = \dot{q}^\mu \\ \dfrac{\partial \mathcal{H}}{\partial q^\mu} = -\dot{p}_\mu \end{cases}$$

§§ 7.3.1 Maupertuis' Principle

A particular case of the previous variation was given by Maupertuis, where he stated the following theorem

Theorem 7.2 (Maupertuis Principle)**.** *Defined the «abbreviated action» of a system $\mathcal{S}_0$ as*

$$\mathcal{S}_0 = \int_{t_1}^{t_2} p_\mu \mathrm{d}q^\mu$$

Then, the equations of motion can be derived by finding an extremal of $\mathcal{S}_0$ if and only if energy is conserved.

Proof. The proof is similar to the previous derivation and quick. Since energy is conserved we have $\partial_t \mathcal{H} = 0$ and $\mathcal{H} = E$. Integrating directly the action $\mathcal{S}$ we have

$$\mathcal{S} = \int_{t_1}^{t_2} p_\mu \mathrm{d}q^\mu - E(t_2 - t_1)$$

Therefore

$$\mathcal{S} = \mathcal{S}_0 + E(t_2 - t_1)$$

Variating and imposing the least action principle, we have Maupertuis' principle

$$\delta\mathcal{S} = \delta\mathcal{S}_0 = 0$$

□

8 Phase Space Formulation

§ 8.1 Canonical Transformations

Given a physical system that solves Hamilton's canonical equation, since the canonical variables (p_μ, q^ν) don't hold any intrinsic meaning, it's possible to find a new canonical coordinate set (P_μ, Q^ν) that represent the same state (we take t as a parameter during the coordinate transformation).
By definition, we must have a diffeomorphism between these two coordinate systems, therefore

$$\det\left|\frac{\partial(P_\mu, Q^\nu)}{\partial(p_\mu, q^\nu)}\right| \neq 0 \tag{8.1}$$

Where the reversibility condition must be satisfied. Note that in general, the system in the new coordinates (P_μ, Q^ν) is not Hamiltonian.

Definition 8.1.1 (Canonical Transformation)**.** A canonical transformation is a coordinate transformation such that

$$\mathcal{H}(p_\mu, q^\nu, t) \to \tilde{\mathcal{H}}(P_\mu, Q^\nu, t) \tag{8.2}$$

Where

$$\begin{cases} \dfrac{\partial \tilde{\mathcal{H}}}{\partial P_\mu} = \dot{Q}^\mu \\ \dfrac{\partial \tilde{\mathcal{H}}}{\partial Q^\mu} = -\dot{P}_\mu \end{cases} \tag{8.3}$$

In general $\mathcal{H} \neq \tilde{\mathcal{H}}$, but if $\mathcal{H} = \tilde{\mathcal{H}}$ the transformation is said to be «fully canonical»

§§ 8.1.1 Generating Functions of Canonical Transformation

Theorem 8.1 (Lie Condition)**.** ***Given an invertible transformation*** $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$***, the transformation is canonical if and only if, given*** $\lambda \in \mathbb{R},\ F, \psi : \Gamma^{2n} \to \mathbb{R}$***, then it also maps*** $p_\mu \mathrm{d}q^\mu$ ***as follows***

$$p_\mu \mathrm{d}q^\mu \to \lambda P_\mu \mathrm{d}Q^\mu + \psi(P_\mu, Q^\nu, t)\mathrm{d}t + \mathrm{d}F \tag{8.4}$$

Or, in other words, it's necessary to verify that the following differential form is exact

$$p_\mu \mathrm{d}q^\mu - \lambda P_\mu \mathrm{d}Q^\mu = \psi \mathrm{d}t + \mathrm{d}F \tag{8.5}$$

Using Lie's condition, it's possible to define 4 different generating functions of canonical transformations, F_1, F_2, F_3, F_4, tied between themselves via Legendre transforms.

1. Generator of the 1st kind $F_1(q^\mu, Q^\nu, t)$. Lie's conditions becomes

$$p_\mu \mathrm{d}q^\mu - P_\mu \mathrm{d}Q^\mu = \psi \mathrm{d}t + \frac{\partial F_1}{\partial q^\mu}\mathrm{d}q^\mu + \frac{\partial F_1}{\partial Q^\mu}\mathrm{d}Q^\mu + \frac{\partial F_1}{\partial t}\mathrm{d}t \tag{8.6}$$

   In order to satisfy the theorem, it must hold that

$$\frac{\partial F_1}{\partial q^\mu} = p_\mu, \quad \frac{\partial F_1}{\partial Q^\mu} = -P_\mu, \quad \frac{\partial F_1}{\partial t} = -\psi \tag{8.7}$$

2. Generator of the 2nd kind $F_2(q^\mu, P_\nu, t)$

$$p_\mu \mathrm{d}q^\mu + Q^\mu \mathrm{d}P_\mu = \psi \mathrm{d}t + \frac{\partial F_2}{\partial q^\mu}\mathrm{d}q^\mu + \frac{\partial F_2}{\partial P_\mu}\mathrm{d}P_\mu + \frac{\partial F_2}{\partial t}\mathrm{d}t \tag{8.8}$$

   I.e.

$$\frac{\partial F_2}{\partial q^\mu} = p_\mu, \quad \frac{\partial F_2}{\partial P_\mu} = Q^\mu, \quad \frac{\partial F_2}{\partial t} = -\psi \tag{8.9}$$

3. Generator of the 3rd kind $F_3(p_\mu, Q^\nu, t)$

$$q^\mu \mathrm{d}p_\mu + P_\mu \mathrm{d}Q^\mu = -\psi \mathrm{d}t - \frac{\partial F_3}{\partial p_\mu}\mathrm{d}p_\mu - \frac{\partial F_3}{\partial Q^\mu}\mathrm{d}Q^\mu - \frac{\partial F_3}{\partial t}\mathrm{d}t \tag{8.10}$$

   Therefore

$$\frac{\partial F_3}{\partial p_\mu} = -q^\mu, \quad \frac{\partial F_3}{\partial Q^\mu} = -P_\mu, \quad \frac{\partial F_3}{\partial t} = -\psi \tag{8.11}$$

4. Generator of the 4th kind $F_4(p_\mu, P_\nu, t)$

$$q^\mu \mathrm{d}p_\mu - Q^\mu \mathrm{d}P_\mu = -\psi \mathrm{d}t - \frac{\partial F_4}{\partial p_\mu}\mathrm{d}p_\mu - \frac{\partial F_4}{\partial P_\mu}\mathrm{d}P_\mu - \frac{\partial F_4}{\partial t}\mathrm{d}t \tag{8.12}$$

   Which means

$$\frac{\partial F_4}{\partial p_\mu} = -q^\mu, \quad \frac{\partial F_4}{\partial P_\mu} = -Q^\mu, \quad \frac{\partial F_4}{\partial t} = -\psi \tag{8.13}$$

Since $\tilde{\mathcal{H}} = \mathcal{H} - \psi$, it's obvious that if the generator function is stationary $\partial_t F_i = 0$, then the transformation is fully canonical.
A better way to see the previous list is as a series of Legendre transforms from (p_μ, q^ν) till (P_μ, Q^μ). In fact, we can write

$$\begin{aligned} F_2 &= F_1 + P_\mu Q^\mu \\ F_3 &= F_1 - p_\mu q^\mu \\ F_4 &= F_1 + P_\mu Q^\mu - p_\mu q^\mu \end{aligned} \tag{8.14}$$

§ 8.2 Poisson Brackets and Liouville's Theorem

Definition 8.2.1 (Poisson Brackets)**.** The space Γ^{2n} comes equipped with a bilinear transformation called the «Poisson brackets» .
Consider a function $f : \Gamma^{2n} \to \mathbb{R}$, then its total derivative with respect to time will be

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial p_\mu}\dot{p}_\mu + \frac{\partial f}{\partial q^\mu}\dot{q}^\mu \tag{8.15}$$

Substituting Hamilton's equations we have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} = \frac{\partial f}{\partial t} + \{\mathcal{H}, f\} \tag{8.16}$$

Where we defined the poisson brackets as

$$\{\mathcal{H}, f\} = \frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial f}{\partial q^\mu} - \frac{\partial \mathcal{H}}{\partial q^\mu}\frac{\partial f}{\partial p_\mu} \tag{8.17}$$

This operator is obviously bilinear and antisymmetric, in fact

$$\{f, \mathcal{H}\} = \frac{\partial f}{\partial p_\mu}\frac{\partial \mathcal{H}}{\partial q^\mu} - \frac{\partial f}{\partial q^\mu}\frac{\partial \mathcal{H}}{\partial p_\mu} = -\left(\frac{\partial \mathcal{H}}{\partial p_\mu}\frac{\partial f}{\partial q^\mu} - \frac{\partial \mathcal{H}}{\partial q^\mu}\frac{\partial f}{\partial p_\mu}\right) = -\{\mathcal{H}, f\}$$

Through this quick definition of this operator, one can immediately say, that if f is an integral of motion, one must have

$$\frac{\partial f}{\partial t} + \{\mathcal{H}, f\} = 0 \tag{8.18}$$

This operator can be directly generalized to two functions $g, h : \Gamma^{2n} \to \mathbb{R}$ as follows

$$\{g, h\} = \frac{\partial g}{\partial p_\mu}\frac{\partial h}{\partial q^\mu} - \frac{\partial g}{\partial q^\mu}\frac{\partial h}{\partial p_\mu} \tag{8.19}$$

Note that applying this operator to the canonical coordinates we obtain the two main properties of such

$$\begin{cases} \{q^\mu, q^\nu\} = \{p_\mu, p_\nu\} = 0 \\ \{p_\mu, q^\nu\} = \delta^\nu_\mu \end{cases} \tag{8.20}$$

It's also possible to derive the following identity through iteration, called the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \tag{8.21}$$

Applying canonical transformations to the definition of Poisson brackets it's possible to find more direct approaches for determining whether a transformation is canonical or not, using the following theorems

Theorem 8.2 (Invariance of Poisson Brackets)**.** *Given two stationary functions $f, g : \Gamma^{2n} \to \mathbb{R}$ and a canonical transformation $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$, such that*

$$\begin{aligned}\tilde{f}(P_\mu, Q^\nu) &= f\left(p_\mu(P, Q), q^\mu(P, Q)\right)\\ \tilde{g}(P_\mu, Q^\nu) &= q\left(p_\mu(P, Q), q^\mu(P, Q)\right)\end{aligned}$$

Then, if we define $\{\cdot, \cdot\}_{PQ}$ as the Poisson brackets in the new coordinate system, then

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \{f, g\} \tag{8.22}$$

I.e. Poisson brackets are invariant to canonical transformations.

Proof. Supposing that g is the Hamiltonian of some system, we can write

$$\{f, g\} = \frac{\mathrm{d}f}{\mathrm{d}t}$$

This implies that $\tilde{g}$ is the transformed Hamiltonian, therefore

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}t}$$

Since canonical transformation preserve Hamilton's equations we must have

$$\frac{\mathrm{d}\tilde{f}}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$$

Which implies the statement of the theorem

$$\left\{\tilde{f}, \tilde{g}\right\}_{PQ} = \{f, g\}$$

This also proves that

$$\{Q^\mu, Q^\nu\}_{PQ} = \{P_\mu, P_\nu\}_{PQ} = 0$$

And

$$\{P_\nu, Q^\mu\}_{PQ} = \delta^\mu_\nu$$

□

Theorem 8.3 (Lie Condition on Poisson Brackets)**.** *Given a transformation $(p_\mu, q^\nu) \to (P_\mu, Q^\nu)$, it is canonical if and only if*

$$\begin{cases}\{Q^\mu, Q^\nu\} = \{P_\mu, P_\nu\} = 0\\ \{P_\nu, Q^\mu\} = \delta^\mu_\nu\end{cases} \tag{8.23}$$

Another theorem that can be inferred is the so-called Liouville theorem, which states that an infinitesimal volume in the phase space is invariant to canonical transformations

Theorem 8.4 (Liouville)**.** *Given an infinitesimal volume in Γ^{2n}, $\mathrm{d}\Gamma = \mathrm{d}^n p\,\mathrm{d}^n q$, then applying a canonical transformation we must have*

$$\mathrm{d}\tilde{\Gamma} = \mathrm{d}^n P\,\mathrm{d}^n Q = \mathrm{d}^n p\,\mathrm{d}^n q = \mathrm{d}\Gamma$$

Where if J is the determinant of the Jacobian, we must have $J = 1$

Proof. In order for the theorem to be demonstrated we must prove that

$$\int \mathrm{d}\tilde{\Gamma} = \int J\mathrm{d}\Gamma, \quad J = 1$$

By definition, we can write the determinant of the Jacobian as follows

$$J = \det\left|\frac{\partial(P_\mu, Q^\nu)}{\partial(p_\mu, q^\nu)}\right|$$

From here we write two intermediate canonical transformations and write the new Jacobian matrix as the product of the two matrices of the intermediate transformations.
We have, choosing the transformations $(p_\mu, q^\nu) \to (P_\mu, q^\nu) \to (P_\mu, Q^\nu)$, that our Jacobian can be written as follows

$$J = \det\left|\frac{\partial(P_\mu, Q^\nu)}{\partial(P_\mu, q^\nu)}\frac{\partial(P_\mu, q^\nu)}{\partial(p_\nu, q^\mu)}\right|$$

Simplifying the equal rows we have that

$$J = \det\left|\frac{\partial Q^\mu}{\partial q^\nu}\frac{\partial P_\nu}{\partial p_\mu}\right|$$

Imposing that the transformation is canonical we must have that it comes from a F_2 generating function, so that, using (8.9)

$$\frac{\partial Q^\mu}{\partial q^\nu} = \frac{\partial^2 F_2}{\partial q^\mu \partial P_\nu}, \quad \frac{\partial p_\mu}{\partial P_\nu} = \frac{\partial^2 F_2}{\partial P_\nu \partial q^\mu}$$

Imposing that the determinant of the Hessian of the generating function is some number d, we have

$$J = d/d = 1$$

Therefore

$$\int \mathrm{d}\tilde{\Gamma} = \int J\mathrm{d}\Gamma = \int \mathrm{d}\Gamma$$

□

§§ 8.2.1 Poincaré Recurrence

This theorem gives rise to a paradox known as ***Poincaré's recurrence theorem***. Basically this theorem states, against common sense, that an autonomous (time-independent) Hamiltonian system with some initial conditions (p^0_μ, q^ν_0) will evolve till returning to the initial conditions at some finite time t. Technically we have

Theorem 8.5 (Poincaré Recurrence Theorem)**.** *Given an autonomous Hamiltonian system confined in a subset $\Lambda \subset \Gamma^{2n}$ with some initial condition $x_0^\mu \in \Lambda$, if we evolve $x_0^\mu \to x^\mu(t)$ then*

$$\forall \tau \in \mathbb{R} \ \exists t^\star > \tau \ : \ \forall \epsilon > 0 \ B_\epsilon(x^\mu(t^\star)) \cap B_\epsilon(x_0^\mu) \neq \{\}$$

I.e.

$$\forall \epsilon > 0 \ x(t^\star) \in B_\epsilon(x_0)$$

Where $B_\epsilon(x^\mu)$ is the open ball centered in x^μ with radius ϵ

Proof. Begin by defining a sequence of times t_n such that $x^\mu(t_n) = x_n^\mu$, then

$$\exists n_1 \neq n_2 \in \mathbb{N} \ : \ B_\epsilon(x_{n_1}^\mu) \cap B_\epsilon(x_{n_2}^\mu) = \{\}$$

Defining a measure μ on the phase space we must have, that after n iterations, the total path measure will be

$$\mu\left(\bigsqcup_{i=1}^{n} B_\epsilon(x_i^\mu)\right) = \sum_{i=1}^{n} \mu\left(B_\epsilon(x_i^\mu)\right)$$

Considering time as a completely canonical transformation, we have for Liouville's theorem that

$$\forall i \neq j = 1, \cdots, n \quad \mu\left(B_\epsilon(x_i^\mu)\right) = \mu\left(B_\epsilon(x_j^\mu)\right)$$

Which implies that for $n \to \infty$

$$\mu\left(\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu)\right) = \sum_{i=1}^{\infty} \mu\left(B_\epsilon(x_i^\mu)\right) \to \infty$$

This cannot be true, since by hypothesis we have that the motion of the system is confined in a set $\Lambda \subset \Gamma^{2n}$, therefore

$$\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu) \subseteq \Lambda$$

In terms of measures this means

$$\mu\left(\bigsqcup_{i=1}^{\infty} B_\epsilon(x_i^\mu)\right) = \sum_{i=1}^{\infty} \mu\left(B_\epsilon(x_i^\mu)\right) \leq \mu\left(\Lambda\right) < \infty$$

Which implies that there must exists some set such that the intersection is not null, therefore there must exist, for some $n_1, n_2, k \in \mathbb{N}$

$$\mu\left(B_\epsilon(x_{n_1}^\mu) \cap B_\epsilon(x_{n_2}^\mu)\right) = \mu\left(B_\epsilon(x_{n_1-k}^\mu) \cap B_\epsilon(x_{n_2-k}^\mu)\right) \neq 0$$

Choosing $k = \min\{n_1, n_2\} = n_2$, where we supposed $n_2 < n_1$ we have

$$\mu\left(B_\epsilon(x_{n_1}^\mu) \cap B_\epsilon(x_{n_2}^\mu)\right) = \mu\left(B_\epsilon(x_{n_1-n_2}^\mu) \cap B_\epsilon(x_0^\mu)\right) \neq \{\}$$

Since $x_{n_1-n_2}^\mu = x^\mu(t_{n_1-n_2})$ and choosing $t^\star = t_{n_1-n_2}$ we have that for $t = t^\star$ the system will find itself in a ball of radius $\epsilon > 0$ from the initial value x_0^μ

$$B_\epsilon(x^\mu(t^\star)) \cap B_\epsilon(x_0^\mu) \neq \{\}$$

□

§ 8.3 Hamilton-Jacobi Method

A really good use for the canonical transformation is to find a quick and trivial solution to Hamilton-Jacobi's equation.
The differential equation we intend to solve is the following

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(\frac{\partial \mathcal{S}}{\partial q^{\mu}}, q^{\mu}, t\right) = 0 \tag{8.24}$$

The complete solution of this equation can be inferred to be a function of the coordinates and $n+1$ parameters corresponding to the independent variables of the system, including time. Therefore we might write

$$\mathcal{S}(q^{\mu}, t) = f\left(q^{\mu}, t; \alpha_1, \cdots, \alpha_n\right) + A \tag{8.25}$$

Where $t, \alpha_1, \cdots, \alpha_n, A \in \mathbb{R}$ are our parameters.
We can choose now a canonical transformation to a new set of variables $(\alpha_{\mu}, \beta^{\mu})$ that give the following relations

$$\frac{\partial f}{\partial q^{\mu}} = p_{\mu}, \quad \frac{\partial f}{\partial \alpha_{\mu}} = \beta^{\mu}, \quad \tilde{\mathcal{H}} = \mathcal{H} + \frac{\partial f}{\partial t} \tag{8.26}$$

Note that for our hypothesis f is a complete solution of Hamilton-Jacobi, therefore the last relation gives

$$\tilde{\mathcal{H}} = 0$$

Basically, with this canonical transformation, we mapped our Hamiltonian to a null Hamiltonian, for which the equations of motion are trivial, giving in the new variables $\alpha_{\mu}, \beta^{\mu} =$constant.
Using the definition of β^{μ} via the transformation, and using the reversibility of such, we can determine the q^{μ} and the analytical form of our complete solutions.

Part III

Special Relativity

9 Lorentz Transformations and Kinematics

§ 9.1 Principle of Relativity

The principle of relativity states a quite simple but deep affirmation: «All interaction propagate at a constant speed independent from the chosen frame of reference» . This speed is usually denoted as c and it's informally known as the speed of light, which has the following value (in SI units)

$$c = 2.998 \times 10^8 \text{ m/s} \tag{9.1}$$

In the part on classical mechanics we always intended between the lines that all interactions are instantaneous and therefore we'd have $c \to \infty$ formally. This can be interpreted as taking classical mechanics as an approximation of Einstein's relativity for which $v/c << 1$, which is the case for our really slow classical particles.
Note that this constant speed of propagation precludes that time isn't universal, and it is frame dependent. In order to understand this it's useful to get two coordinate frames K and $\tilde{K}$, where one is moving with respect to the other with a constant speed V.
Suppose now that a point A emits a signal towards two other points B and C

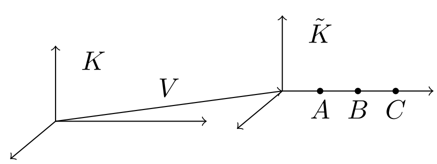


Figure 9.1: The two frames K and $\tilde{K}$

In the frame $\tilde{K}$, where A is at rest, we see that the signal reaches both points at the same time, but the same CANNOT be true for the other system, since the relativity principle would be violated. Thinking in a different way, suppose that you're standing at the origin of the K system. If the velocity of the signal is constant in all reference frames we can for sure say that it's so where we're standing, therefore we end up seeing B moving towards the signal and C moving away from it, both with speed V. In this system we therefore must see a delay in when the two points receive such signal.
Although counterintuitive we're experimentally more than sure that this is actually a better approximation of nature than our beloved Newtonian mechanics.

§ 9.2 Spacetime

Since time it's not anymore an universal thing and behaves itself as a coordinate, we can now think of our universe as a $4D$ manifold with time as a new coordinate. This is known as «Minkowsky Spacetime» or in short as «Spacetime» . This new definition follows:

Definition 9.2.1 (Event)**.** Given a spacetime with coordinates (ct, x, y, z) with c the speed of light, we define a point in spacetime as an «event» in such.
Since time only "flows" one way, we have that for every particle corresponds a wordline which connects all the events pertaining to such. Note that events are also known as «universe points»

Given the principle of relativity one might also ask rightfully how to formulate mathematically all of this, bringing out some invariants that might help with further derivations. Take again the previous system and call l the distance traveled by the signal after being emitted from A. Calling t_1 and t_2 the emission time and the arrival time respectively, we have that for obvious reasons

$$l = c(t_1 - t_2) \tag{9.2}$$

But, we can also write as follows

$$l = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{9.3}$$

With (x_1, y_1, z_1) being the departure coordinates and (x_2, y_2, z_2) the arrival coordinates in K In $\tilde{K}$, analogously we have

$$\begin{aligned} \tilde{l} &= c(\tilde{t}_2 - \tilde{t}_1) \\ \tilde{l} &= \sqrt{(\tilde{x}_2 - \tilde{x}_1)^2 + (\tilde{y}_2 - \tilde{y}_1)^2 + (\tilde{z}_2 - \tilde{z}_1)^2} \end{aligned} \tag{9.4}$$

Tying up both equations we end with the following result

$$\begin{cases} c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 = 0 \\ c^2(\tilde{t}_2 - \tilde{t}_1)^2 - (\tilde{x}_2 - \tilde{x}_1)^2 + (\tilde{y}_2 - \tilde{y}_1)^2 + (\tilde{z}_2 - \tilde{z}_1)^2 = 0 \end{cases} \tag{9.5}$$

In "layman" words this basically means, that the following quantity

$$s_{12}^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2 \tag{9.6}$$

Called, «interval» , is a «relativistic invariant» , and therefore invariant with respect to changes of coordinate frames in the context of special relativity.
From (9.5) we have that if the two points are infinitesimally close to eachother we can define the infinitesimal interval as

$$\mathrm{d}s^2 = c^2\mathrm{d}t - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{9.7}$$

The invariance of such differential quantity is easy to show considering the previous case we stated where $\mathrm{d}s = \mathrm{d}\tilde{s} = 0$ we have, using basic intuition that

$$\mathrm{d}s^2 = a(V)\mathrm{d}\tilde{s}^2 \tag{9.8}$$

Where $a(V)$ is a function of the relative velocity between the two considered frames. It cannot depend on direction due to the isotropy of space.

Consider now three inertial reference frames K, K_1, K_2, and let V_1, V_2 be the velocities of the frames K_1, K_2. We can therefore say, using (9.8) that

$$\begin{aligned} \mathrm{d}s^2 &= a(V_1)\mathrm{d}s_1^2 = a(V_2)\mathrm{d}s_2^2 \\ \mathrm{d}s_1^2 &= a(V_{12})\mathrm{d}s_2^2 \end{aligned} \tag{9.9}$$

Where we defined the velocity between K_1, K_2 as V_{12}. Rewriting the equation we have

$$\mathrm{d}s^2 = a(V_1)a(V_{12})\mathrm{d}s_2^2 = a(V_2)\mathrm{d}s_2^2$$

Equating the coefficients of the differential $\mathrm{d}s_2$, we have

$$a(V_{12}) = \frac{a(V_2)}{a(V_1)} \tag{9.10}$$

The previous equation then might be true if and only if $a(V_{12})$ depends only on the angle between the velocities V_1, V_2. This cannot be true due to the isotropy of spacetime, as we stated for the previous problem, and therefore $a(V)$ might only be a constant function. Taking $a(V_{12}) = 1$ for consistency between frames of reference, we have finally demonstrated that the differential spacetime interval is invariant

$$\mathrm{d}s = \mathrm{d}\tilde{s} \tag{9.11}$$

This definition of $\mathrm{d}s$ gives rise to three kinds of intervals:

1. «Spacelike intervals» if $s_{12}^2 < 0$
2. «Timelike intervals» if $s_{12}^2 > 0$
3. «Light-like intervals» if $s_{12}^2 = 0$

These three distinctions let us answer two previously impossible questions: is it possible to find a reference frame where two events happen at the same time or at the same place in our three-dimensional perception?. The answer is surprisingly ***yes***. It depends on the kind of the interval between the two points.

Let's work with the first assumption, taken two events in spacetime E_1, E_2, defined $t_{12} = t_2 - t_1$ and l_{12} as our usual 3D distance between the events, we have

$$s_{12}^2 = c^2 t_{12}^2 - l_{12}^2$$

Let's now search a system where $l'_{12} = 0$. In order to have this, using that $s_{12} = s'_{12}$ we have

$$s_{12} = c^2 t_{12} - l_{12}^2 = c^2 t_{12}'^2 = s_{12}'^2 > 0$$

I.e. the spacetime interval between the frame of reference at rest with respect to the two events and the new unknown frame of reference is timelike.

Analogously, if we wanted to find a new system where the two events happen at the same time, we might have set $t'_{12} = 0$, therefore getting

$$s_{12} = c^2 t_{12} - l_{12}^2 = l_{12}'^2 = s_{12}'^2 < 0 \tag{9.12}$$

§§ 9.2.1 Spacetime Diagrams

The idea of spacetime and absoluteness of the velocity of interactions can be described well by a 2D spacetime diagram. Taken an origin for our system of coordinates (ct, x) we have that, considering v as the slope of a constant wordline, that $|v| < c$.

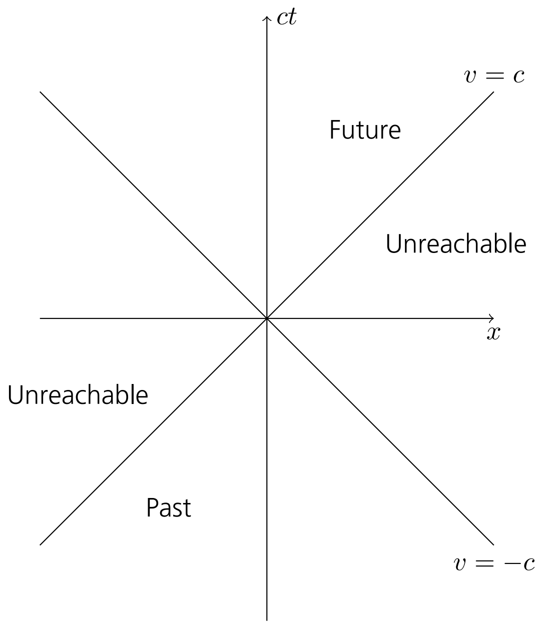


Figure 9.2: Simple spacetime diagram. Note how all the events beyond the asymptote (or «horizon») $v = \pm c$ are inaccessible from 0

Thought in higher dimensions we have that all the past and future of an event are enclosed inside a cone bordered by our horizon $|v| = c$ which separates physical impossibilities from the actual physical past and future of what we're considering.
Note that if $v = \pm c$ we must have $x = \pm ct$, giving us a spacelike interval for our diagram.
Considering instead past and future it's also easy to see that the past is always spacelike, since $c^2t^2 - x^2 < 0$, and that the future is always timelike. Note also that past and future must be absolute

§ 9.3 Proper Time

Since time is not a relativistic invariant, we need to search for a good substitute of it. Given a clock fixed at the origin of some inertial frame K'. After some time $\mathrm{d}t$, the clock has moved (in our system) by the following quantity

$$\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$$

By definition, in K' this clock is at rest, therefore we have

$$\mathrm{d}x' = \mathrm{d}y' = \mathrm{d}z' = 0$$

Imposing the invariance of intervals we have that

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = c^2\mathrm{d}t'^2 \tag{9.13}$$

Therefore, it must be true that

$$\mathrm{d}t' = \mathrm{d}t\sqrt{1 - \frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}{c^2\mathrm{d}t^2}} \tag{9.14}$$

This is the expression for the passing of time in the system where the clock is at rest, and it's called the «proper time» of the clock, usually indicated with τ. Writing the sum of differentials as $\mathrm{d}r^2$ and using the definition of v^2, we have that

$$\mathrm{d}\tau = \mathrm{d}t\sqrt{1 - \frac{v^2}{c^2}} = \frac{\mathrm{d}s}{c} \tag{9.15}$$

Integrating and using the fundamental theorem of calculus, we have that a given time interval will be "felt" differently by the clock, where

$$\Delta\tau = \int_{\tau_1}^{\tau_2} \sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t < \Delta t \tag{9.16}$$

This tells us that a moving clock will tick slower than a clock at rest (note also on how this definition depends directly on the chosen frame).
This difference of measured time is known as «time dilation» .

§ 9.4 Formalization of the Principle of Relativity

All of what we found before can be crammed into the most fundamental element of relativity: coordinate transformations.
Consider two reference frames K, (ct, x, y, z) and K', (ct, x, y, z). Mathematically, what we call interval is the usual 4D distance in a seminegative definite metric, and due to its invariance we must have that all coordinate transformations between these two systems must be rototraslations (isometries).
Translations can be immediately ignored since they only move the origin of the system, and therefore we choose our faithful rotations in order to find these coordinate transformation laws.
All the possible rotations are between the planes xy, xz, yz and tx, ty, tz. All rotations xy, xz, yz are our usual 3D rotations and are of no use, therefore we choose the rotations tx, ty, tz. Taking tx as the chosen one we have that the spacetime interval is

$$s^2 = c^2t^2 - x^2$$

Therefore, all searched rotations ***must preserve*** this relationship. The first idea one might have is to look at the symmetry of the system and deduce immediately that such rotation must be hyperbolic in nature. We therefore define the following

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\psi & \sinh\psi \\ \sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{9.17}$$

Taking $x' = 0$ it all reduces to this single equation

$$\frac{x}{ct} = \frac{V}{c} = \tanh\psi \tag{9.18}$$

It's common to indicate such value with the pure number β, called the «Lorentz Boost» , where

$$\beta = \frac{V}{c}$$

Solving (9.18) we have that

$$\beta = \frac{\sinh\psi}{\sqrt{1+\sinh^2\psi}} = 0 \implies \sinh^2\psi = \frac{\beta}{\sqrt{1-\beta^2}} \tag{9.19}$$

And

$$\cosh^2\psi = 1 + \sinh^2\psi \implies \cosh\psi = \frac{1}{\sqrt{1-\beta^2}} = \gamma \tag{9.20}$$

Where γ is known as the «Lorentz/Gamma Factor» .
Substituting back into (9.18) we have back our searched transformations

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{9.21}$$

Note that the inverse transformation is simply given imposing $\beta \to -\beta$.
The complete transformation between the two reference frames will finally be a 4D linear system as follows

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{9.22}$$

These transformations are known as «Lorentz Transformations» and are the fundamental transformations between frames of reference in special relativity. These transformations formalize the principle of relativity. For $v << c$ these transformations bring back the usual Galilean transformations corrected by a first order factor in c, as we expected

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} \tag{9.23}$$

§§ 9.4.1 Length Contraction and Time Dilation

Using Lorentz transformations it's possible to mathematically formalize all relativistic effects. One of such is known as «length contraction» , where the measured length of an object depends on the chosen reference frame.
As a matter of example take a "rigid" rod in a system K, long Δx, and consider the system K' where the rod is at rest. In this system we have

$$\Delta x' = x'_1 - x'_2 = \gamma(x_2 - x_1) - \gamma\beta c(t_2 - t_1) = \gamma\Delta x - \gamma\beta c\Delta t \tag{9.24}$$

Since we're measuring the length directly, we can say without problems that $\Delta t = 0$, and we get

$$\Delta x' = \gamma \Delta x = \frac{\Delta x}{\sqrt{1-\beta^2}} = \frac{\Delta x}{\sqrt{1-\frac{v^2}{c^2}}} \tag{9.25}$$

Therefore, for $\beta \neq 0$ we have $\Delta x' < \Delta x$. We call $\Delta x = l_0$ as the proper lenght of this rod.
Note that a major consequence of this is that a rigid body in the classical sense of the term cannot be conceived in Special Relativity.
A second effect that we stated before and didn't formalize properly is that of time dilation. Taken a clock at rest in a system K' and two events happening at some coordinate (x', y', z') of K'. We have that the time elapsed between the two events will be $\Delta t' = t_2' - t_1'$, and therefore, using Lorentz transformations we get, in K

$$\Delta t = \gamma \left(t_1' + \frac{\beta}{c} \Delta x' \right) \tag{9.26}$$

Imposing that the events happen at the same place (x', y', z') we have $\Delta x' = 0$ and therefore

$$\Delta t = \gamma \Delta t' \tag{9.27}$$

Therefore, the clock in the still frame is measuring smaller time intervals, and the time measured is dilated.

§§ 9.4.2 Velocity Transformations

As we have seen velocities have an upper bound which is the speed of light. It's possible to find the transformations of velocities from the transformations (9.21) and applying them to differentials.
We have

$$\begin{pmatrix} \mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix} = \begin{pmatrix} \gamma & \frac{\beta\gamma}{c} & 0 & 0 \\ \frac{\beta\gamma}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix} \tag{9.28}$$

Rearranging the terms we have finally

$$\begin{cases} v_x = \dfrac{v_x' + \beta c}{1 + \frac{\beta}{c} v_x'} \\ v_y = \dfrac{v_y'}{\gamma \left(1 + \frac{\beta}{c} v_x'\right)} \\ v_z = \dfrac{v_z'}{\gamma \left(1 + \frac{\beta}{c} v_x'\right)} \end{cases} \tag{9.29}$$

Approximating for $v << c$ we get the usual velocity composition formula with an added relativistic correction

$$\begin{cases} v_x \approx v_x' + V \left(1 - \dfrac{v_x^{2'}}{c^2}\right) \\ v_y \approx v_y' - v_x' v_y' \dfrac{\beta}{c} \\ v_z \approx v_z' - v_x' v_z' \dfrac{\beta}{c} \end{cases} \tag{9.30}$$

Or, in vector form

$$v^i = v^{i'} + V^i - \frac{v^{i'}}{c^2}(V^i v'_i) \tag{9.31}$$

Note how v and v' are tied asymetrically in the transformation. Consider now a simple planar motion in the xy plane, where $v^i = (v_x, v_y, 0)$, we can find the law of transformation of angles considering that v^i can be rewritten in polar coordinates, as follows

$$\begin{cases} v_x = v\cos\theta \\ v_y = v\sin\theta \\ v_z = 0 \end{cases}$$

Applying the transformations, we have

$$\begin{cases} v\cos\theta = \dfrac{v'\cos\theta' + \beta c}{1 + \frac{\beta}{c}v'\cos\theta'} \\ v\sin\theta = \dfrac{v'\sin\theta'}{\gamma\left(1 + \frac{\beta}{c}v'\cos\theta'\right)} \end{cases} \tag{9.32}$$

Where we used that the motion in the new system will be still planar.
Rewritten in other terms, we have

$$\tan\theta = \frac{\frac{v'\sin\theta'}{\gamma\left(1+\frac{\beta}{c}v'\cos\theta'\right)}}{\frac{v'\cos\theta'+\beta c}{1+\frac{\beta}{c}v'\cos\theta'}} = \frac{v'\sin\theta'}{\gamma\left(v'\cos\theta' + \beta c\right)} \tag{9.33}$$

Which explicitates the change of direction of velocity between different coordinate systems.

§ 9.5 4-Vectors

[1]

As we have already suggested before, the 4-tuple $x^\mu = (ct, x, y, z)$ can be seen as a set of coordinates in spacetime, or as a radius vector. The square of vectors in spacetime can be seen as a non-euclidean scalar product as follows

$$x^\mu x_\mu = g_{\mu\nu}x^\mu x^\nu = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{9.34}$$

Where $g_{\mu\nu}$ is the metric tensor of spacetime

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9.35}$$

From what we wrote for special relativity itself, we have a new definition

[1] From here on, all greek indexes $(\mu, \nu, \sigma, \cdots)$ are to be intended as spacetime indexes, and latin indexes $(i, j, k, \cdots)$ as usual 3D indexes if not otherwise stated

Definition 9.5.1 (4-Vector)**.** A «4-vector» is a 4-tuple that transforms between coordinate frames using Lorentz transformations, as

$$a^\mu = \Lambda^\mu_\nu a^\nu \tag{9.36}$$

Where λ^μ_ν is the already defined transformation matrix of the Lorentz transformations.

Using the metric tensor one can transforms between covariant vectors and contravariant vectors using $a_\mu = g_{\mu\nu} a^\nu$, and due to the semidefinite signature of the metric one has that $a^i = -a_i$, where a^i is the spatial part of the vector. Note also that inserting it into the formula for a scalar product ($a^\mu b_\mu$) one gets back what we had defined before.
It's also possiible to define 4-scalars, which are relativistic invariants. One of such 4-scalars is the square of a 4-vector or the scalar product between 2 4-vectors.
Another way of writing 4-vectors is with a tuple composed as follows

$$a^\mu = (a^0, a^i) \tag{9.37}$$

Where the first component is known as the «polar» component of the 4-vector, and the second is known as the «axial» component of the 4-vector. Therefore we can write

$$\begin{aligned} x^\mu &= (ct, x^i) \\ x_\mu &= (ct, -x_i) \end{aligned} \tag{9.38}$$

§§ 9.5.1 4-Velocity and 4-Acceleration

It's possible to define a 4-vector analogue to the velocity of a particle. Indicating with τ the proper time we define the 4-velocity u^μ as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{9.39}$$

Since $\mathrm{d}\tau = \frac{c}{\gamma}\mathrm{d}t$ we have

$$u^\mu = \frac{\gamma}{c}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$$

In other words

$$u^\mu = \left(\gamma, \frac{\gamma}{c} v^i\right)$$

Note that the square of u^μ is a relativistic invariant and special in nature due to its unitary value, in fact

$$u^\mu u_\mu = \gamma^2 - \gamma^2 \frac{v^2}{c^2} = 1$$

The 4-acceleration w^μ is defined analogously derivating again with respect to the proper time, hence

$$w^\mu = \frac{\gamma}{c}\mathrm{d}u^\mu t = \left(\frac{\gamma}{c}\frac{\mathrm{d}\gamma}{\mathrm{d}t}, \frac{\gamma}{c^2}\frac{\mathrm{d}\gamma v^i}{\mathrm{d}t}\right) \tag{9.40}$$

Deriving with respect to time we have firstly that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{v^i a_i}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{3}{2}}} = \frac{\gamma^3}{c^2} v^i a_i$$

And therefore

$$w^\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\gamma}{c}\left(\frac{\gamma^3}{c^2}v^i a_i, \frac{\gamma}{c^3}v^j a_j v^i + \frac{\gamma}{c}a^i\right) \tag{9.41}$$

It's possible to demonstrate that $w^\mu u_\mu = 0$, i.e. that 4-velocity and 4-acceleration are always mutually orthogonal. In fact

$$\frac{\mathrm{d}}{\mathrm{d}\tau}u^\mu w_\mu = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}u_\mu + \frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}u^\mu = 2u^\mu w_\mu = 0$$

§ 9.6 Exercises

Exercise 9.6.1 (Uniformly Accelerated Motion)**.** Solve the motion of an uniformly accelerated particle in the context of Special Relativity.
Consider that the 4-acceleration is constant only in the frame comoving with the particle.
S O L U T I O N
We have that in the comoving frame $\gamma = 1$ and $v = 0$, therefore

$$w^\mu = \left(0, \frac{\dot{v}^i}{c^2}\right)$$

Since a is constant we rotate the 3D system in order to get $a \parallel x$, therefore getting

$$w^\mu = \left(0, \frac{a}{c^2}, 0, 0\right)$$

Note that we can also define a 4-scalar

$$w^\mu w_\mu = -\frac{a^2}{c^2}$$

Changing to the fixed frame of reference, we have

$$w^{\mu'} = \frac{\gamma}{c}\left(\frac{\gamma^3}{c^2}v^i\dot{v}_i, \frac{\gamma^3}{c^2}v^j\dot{v}_j v^i + \frac{\gamma}{c}\dot{v}^i\right) = \frac{\gamma^4}{c^2}\left(\frac{v^i\dot{v}_i}{c}, \frac{v^2}{c^2}\dot{v}^i + \frac{\dot{v}^i}{\gamma^2}\right)$$

Using that

$$\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right)\dot{v}^i = \dot{v}^i$$

We end up with the following simplified result

$$w^{\mu'} = \frac{\gamma^4}{c^2}\left(\frac{1}{c}\dot{v}^i v_i\right)$$

Which gives us the following differential equation

$$w^\mu w_\mu = \frac{\gamma^8}{c^4}\left(\frac{1}{c^2}(v^i\dot{v}_i)^2\right) - \frac{\gamma^8}{c^4}\dot{v}^2 = -\frac{a^2}{c^4}$$

Simplifying the LHS we get

$$\frac{\gamma^8}{c^4}\left(\frac{v^2}{c^2}\dot{v}^2 - \dot{v}^2\right) = \frac{\gamma^8}{c^4}\left(\frac{v^2}{c^2} - 1\right) = -\frac{\gamma^6}{c^4}\dot{v}^2$$

Therefore, putting it back into the first equation, we get

$$-\gamma^6\dot{v}^2 = -a^2 \implies \gamma^3\frac{\mathrm{d}v}{\mathrm{d}t} = a$$

Note that using the derivative of γ with respect to time we can rewrite the LHS as the derivative of a product, in fact

$$\frac{\mathrm{d}(\gamma v)}{\mathrm{d}t} = \frac{\gamma^3}{c^2}v^2\dot{v} + \gamma\dot{v} = \dot{v}\left(\frac{\gamma^3}{c^2}v^2 + \gamma\right) = \gamma^3\dot{v}\left(\frac{v^2}{c^2} + \frac{1}{\gamma^2}\right) = \gamma^3\dot{v}$$

Therefore, finally

$$\frac{\mathrm{d}(\gamma t)}{\mathrm{d}t} = a \implies \gamma v(t) = at + c$$

Imposing that $v(0) = 0$ we get $c = 0$ and therefore, solving for $v(t)$, we have

$$\frac{v(t)}{\sqrt{1 - \frac{v^2}{c^2}}} = at \implies v^2 = a^2t^2 - \frac{a^2t^2}{c^2}v^2 \implies v^2 = a^2t^2\left(1 + \frac{a^2t^2}{c^2}\right)^{-1}$$

Therefore

$$v(t) = \frac{at}{\sqrt{1 + \frac{a^2t^2}{c^2}}}$$

Then, by direct integration we can find $x(t)$

$$x(t) = \int \frac{at}{\sqrt{1 + \frac{a^2t^2}{c^2}}}\mathrm{d}t = \frac{c^2}{2a}\int \frac{1}{\sqrt{1 + w^2}}\mathrm{d}w = \frac{c^2}{2a}\left(2\sqrt{1 + w} + k\right)$$

Where we used the substitution $w = \frac{a^2t^2}{c^2}$. Imposing the initial condition that $x(0) = 0$ we get $k = -1$, and therefore

$$x(t) = \frac{c^2}{a}\left(2\sqrt{1 + \frac{a^2t^2}{c^2}} - 1\right)$$

The proper time of the particle is

$$\tau = \frac{1}{c}\int_{s_0}^{s}\mathrm{d}s = \int_{t_0}^{t}\frac{1}{\gamma}\mathrm{d}t = \int_0^t\sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t$$

From the definition of $v(t)$ we have that

$$\gamma = \frac{1}{1 - \frac{a^2t^2}{c^2\left(1 + \frac{a^2t^2}{c^2}\right)}}$$

Therefore our integral becomes

$$\tau=\int_0^t\sqrt{1-\frac{a^2t^2}{c^2\left(1+\frac{a^2t^2}{c^2}\right)}}\mathrm{d}t=\frac{a}{c}\int_0^{\frac{a}{c}t}\sqrt{1-\frac{z^2}{1+z^2}}\mathrm{d}z=\frac{a}{c}\int_0^{\frac{a}{c}t}\frac{1}{\sqrt{1+z^2}}\mathrm{d}z=\frac{a}{c}\,\mathrm{arcsin}\left(\frac{at}{c}\right)$$

Where we used the substitution $\frac{at}{c}=z$

10 Relativistic Mechanics

§ 10.1 Relativistic Least Action Principle

In order to use a relativistic version of the least action principle, we need to impose the principle of relativity into its formulation. This is done via imposing that $\mathcal{S}$ must be a relativistic invariant.
It's already obvious that the simplest invariant differential in special relativity is $\mathrm{d}s$ therefore we must have that $\mathrm{d}\mathcal{S} \propto \mathrm{d}s$, and we can rewrite our action as follows

$$\mathcal{S} = -\alpha \int_a^b \mathrm{d}s \tag{10.1}$$

Where $\alpha \in \mathbb{R}$ is a parameter that depends directly on the properties of the particle. The minus sign added there is needed in order to make sure that $\mathcal{S}$ as an extremal in $[s(a), s(b)]$.
We now proceed to find a Lagrangian as per usual, therefore we need to transform the wordline integral into a time integral. Since time is NOT a relativistic invariant we can only choose one "time", which is the proper time of the reference frame, which is also proportional to the interval differential.
Using $\mathrm{d}s = \frac{c}{\gamma}\mathrm{d}t$ we have

$$\mathcal{S} = -\alpha c \int_{\tau_1}^{\tau_0} \frac{1}{\gamma}\mathrm{d}t = \int_{\tau_0}^{\tau_1} -\alpha c\sqrt{1 - \frac{v^2}{c^2}}\mathrm{d}t \tag{10.2}$$

What we have inside the integral symbol on the RHS is the Lagrangian of the system, which depends on the parameter α.
In order to determine this parameter we need to have that for $v/c << 1$ our relativistic Lagrangian must become our known classical Lagrangian. We have for $\beta \to 0$

$$\mathcal{L} = -\frac{\alpha c}{\gamma} = -\alpha c\sqrt{1 - \beta^2} \approx -\alpha c + \frac{\alpha v^2}{2c} = \frac{mv^2}{2} \tag{10.3}$$

Comparing the terms with v^2 we have that $\alpha = mc$, and therefore we have

$$\mathcal{S} = -mc \int_{\tau_0}^{\tau_1} \frac{1}{\gamma}\mathrm{d}t \implies \mathcal{L} = -\frac{mc}{\gamma} \tag{10.4}$$

Where γ is the already known Lorentz factor.

§§ 10.1.1 Relativistic Energy and Momentum

The easiest way possible to define relativistic energy and relativistic momentum is by using our well known identities in Lagrangian mechanics. We must have therefore, for the 3-momentum

$$p_i = \frac{\partial \mathcal{L}}{\partial v^i} \tag{10.5}$$

Since we know that $\mathcal{L} = -mc/\gamma$ the calculation is straightforward. We have

$$\frac{\partial \mathcal{L}}{\partial v^i} = -mc\frac{\partial}{\partial v^i}\sqrt{1-\frac{v^2}{c^2}} = \frac{mv^i}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m v^i$$

As usual, using the Lagrangian for defining our energy we have that $p^i v_i - \mathcal{L} = E$, i.e.

$$E = \gamma m v^2 + \frac{mc^2}{\gamma} = \gamma\left(mv^2 + \frac{mc^2}{\gamma^2}\right) = \gamma\left(mv^2 + mc^2 - mv^2\right) = \gamma mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \tag{10.6}$$

Note that due for this definition, if $v = 0$, $E \neq 0$. We have $E_0 = mc^2$, and this is known as the «rest energy» of the given particle.
This shape for our energy also demonstrates that mass is not conserved anymore in relativity. In fact, given a body comprised of multiple particles we have that E_0^b will be, if the mass of the body is M

$$E_0^b = Mc^2 \tag{10.7}$$

But, since in the rest energy of the single particles composing the body we also have to add all the interaction energies between the body, we will therefore have

$$E_0^b = Mc^2 \neq \sum_i m_i c^2 \tag{10.8}$$

Therefore, finally $M \neq \sum_i m_i$, which makes our previous point.
After defining Lagrangian and energy, the next step we can make is to find the Hamiltonian of a relativistic particle. From the definition of energy and momentum we can write the following system

$$\begin{cases} E = \gamma mc^2 \\ p^i = \gamma m v^i \end{cases} \tag{10.9}$$

Manipulating γ and the whole system, we have

$$\gamma(p) = \sqrt{1 + \frac{p^2}{m^2c^2}} \tag{10.10}$$

Plugging it into the definition of E we have

$$E = \gamma(p) mc^2 = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \tag{10.11}$$

By definition of Hamiltonian function this is our $\mathcal{H}$. Rewriting in a different way we have

$$\mathcal{H} = \gamma(p)mc^2 = mc^2\sqrt{1+\frac{p^2}{m^2c^2}} = \sqrt{m^2c^4+p^2c^2} = c\sqrt{m^2c^2+p^2} \tag{10.12}$$

Again, for $\frac{o^2}{m^2c^2} = \beta(p) << 1$ we get the classical counterpart, plus the relativistic rest energy.
Note how, using the previous equations we have that for $v \to c$ $E \to \infty$ if $m \neq 0$, and defining the existence of massless particles is not obvious. Using the Hamiltonian formulation we immediately see that if $m = 0$ we have

$$\mathcal{H} = E = pc \tag{10.13}$$

It's also obvious from this that the only velocity such particle can have is $v = c$. These kinds of particles are known as ultrarelativistic particles, and photons are one example of such.
Note that it's possible to approximate the energy of massive particles with their ultrarelativistic counterparts in case where the rate between the rest energy and the total energy of the particle is small enough for the needs. I.e.

$$mc^2 << E \implies E \approx pc \tag{10.14}$$

§ 10.2 Relativistic Hamilton Jacobi Equation and 4-Vector Formulation

It's possible to rewrite the relativistic least action principle using 4-vector notation. For what we wrote in the previous section in the part on 4-vectors, we can imagine to define the infinitesimal interval as a 4-scalar via the definition of the infinitesimal 4-radius vector $\mathrm{d}x^\mu = (c\mathrm{d}t, \mathrm{d}r^i)$. We have

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \implies \mathrm{d}s = \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{10.15}$$

We add to this the boundary conditions for the action in spacetime, as $\delta x^\mu(a) = \delta x^\mu(b) = 0$, and we get, for our least action principle

$$\mathcal{S} = -mc\int_a^b \mathrm{d}s = -mc\int_a^b \sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} \tag{10.16}$$

Variating the action we have, firstly

$$\delta\sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu} = \frac{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}\delta x^\nu}{\sqrt{g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu}} = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\delta\mathrm{d}x^\nu = u_\mu\delta\mathrm{d}x^\mu$$

Then

$$\delta\mathcal{S} = -mc\int_a^b u_\mu\delta\mathrm{d}x^\mu$$

Using as usual integration by parts in order to move the differentials we have, implicitly using the boundary conditions

$$\delta\mathcal{S} = mc\int_a^b \frac{\mathrm{d}u_\mu}{\mathrm{d}s}\delta x^\mu\mathrm{d}s \tag{10.17}$$

Imposing lastly the least action principle we get the relativistic equation of motion for a free particle

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}s} = 0 \tag{10.18}$$

Considering instead the second condition $\delta x^\mu(b) = \delta x^\mu$ nonzero we get the usual definition of action as a function of (spacetime) coordinates

$$\delta\mathcal{S} = -mcu_\mu \delta x^\mu \implies \frac{\partial \mathcal{S}}{\partial x^\mu} = -mcu_\mu$$

By comparation with the classical definition, the derivative of the action with respect to the coordinates is defined as the generalized momentum of the system. Since in this case we're using 4-vectors and the derivative of the action with respect to the 4-position is a 4-vector itself (not really in general relativity, but in SR it's true) we have a new definition for the momentum, the «4-momentum» of a relativistic system

$$p_\mu = \partial_\mu \mathcal{S} = mcu_\mu \tag{10.19}$$

Using the definition of 4-velocity as $u_\mu = \gamma\left(1, -\frac{1}{c}v^i\right)$ we have, remembering that $E = \gamma mc^2$

$$p_\mu = \left(\frac{E}{c}, -\gamma m v^i\right) = \left(\gamma mc, -p^i\right) \tag{10.20}$$

Using instead that $\partial_0 = c^{-1}\partial_t$ we have

$$p_0 = \partial_0 \mathcal{S} = \frac{1}{c}\frac{\partial \mathcal{S}}{\partial t} = \frac{E}{c} \tag{10.21}$$

Where we used the classical conclusion that $\partial_t \mathcal{S} = E$. Using the already known Lorentz transformations we have that energy and momentum, since they're tied by a 4-vector, they aren't invariants and transform as follows

$$\begin{cases} E = \gamma\left(E' + \beta c p'_x\right) \\ p_x = \gamma\left(p'_x + \dfrac{\beta}{c}E'\right) \\ p_y = p'_y \\ p_z = p'_z \end{cases} \tag{10.22}$$

Using that $u^\mu u_\mu = 1$ we can define a relativistic invariant for 4-momentum, as follows

$$p^\mu p_\mu = m^2c^2 u^\mu u_\mu = m^2c^2 \tag{10.23}$$

We can also find a 4-force definition by deriving with respect to proper time.

$$f^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{\gamma}{c}\frac{\mathrm{d}p^\mu}{\mathrm{d}t} = \frac{\gamma}{c}\left(\dot{\gamma}mc, \frac{\mathrm{d}p^i}{\mathrm{d}t}\right) \tag{10.24}$$

Using the already known derivative of the Lorentz factor we have, finally

$$f^\mu = \frac{\gamma}{c}\left(\frac{\gamma^3}{c}mv^i a_i, f^i\right) = \left(\frac{\gamma}{c^2}f^i v_i, \frac{\gamma}{c}f^i\right) = \left(\frac{\gamma}{c^2}W, \frac{\gamma}{c}f^i\right) \tag{10.25}$$

Where W is the already well known work of the force.
The Hamilton-Jacobi equation can be defined from (10.23), and it simply becomes

$$\partial_\mu \mathcal{S} \partial^\mu \mathcal{S} = m^2 c^2 \tag{10.26}$$

Or, in explicit form

$$\frac{1}{c^2}\left(\frac{\partial \mathcal{S}}{\partial t}\right)^2 - (\nabla \mathcal{S})^2 = m^2 c^2 \tag{10.27}$$

11 Relativistic Collisions

[1]

§ 11.1 Laboratory and Center of Mass Reference Frames

We have already proven that in special relativity there are no preferred reference frames, therefore we might already choose the ones that ease our calculations.
In the field of relativistic particle collisions (like particle physics) we have two main choices of reference frames

1. The laboratory reference frame
2. The center of mass reference frame

The first one is defined as the reference frame of the resting observer of the event, while the second is the reference frame of the center of mass of the system of particles interacting.[2]
For understanding properly consider the collision of two particles m_1, m_2 where the second is a target particle at rest in the lab frame. We have, before the collision

$$\begin{aligned} p_1^\mu &= \left(\frac{E_1}{c}, p_1^i\right) \\ p_2^\mu &= (m_2 c, 0) \end{aligned} \tag{11.1}$$

And after the collision

$$P^\mu = p_1^\mu + p_2^\mu = \left(\frac{E_1}{c} + m_2 c, p_1^i\right) \tag{11.2}$$

By definition of center of mass, we have that this reference frame will be the one for which $P^i = 0$, i.e., going back to our two particles pre-collision

$$p_1^{\mu^\star} = \left(\frac{E_1^\star}{c}, p^{i^\star}\right), \qquad p_2^{\mu^\star} = \left(\frac{E_2^\star}{c}, -p^{i^\star}\right) \tag{11.3}$$

I.e.

$$P^{\mu^\star} = \left(\frac{1}{c}\left(E_1^\star + E_2^\star\right), 0\right) \tag{11.4}$$

[1] NOTE: this chapter deals mostly with concepts useful in the study of particle physics
[2] I will use a star indicating the center of mass r.f. the rest will be intended as being in the laboratory reference frame

§ 11.2 Invariant Mass

Consider a system of n particles with momentum $p^\mu_{(k)} = (E_k, p^i_{(k)})$ and let the sum of all 4-momentums be P^μ[3].

Definition 11.2.1 (Invariant Mass)**.** Given the previous system of particles, the relativistic invariant of the total 4-momentum is defined as «invariant mass» $\sqrt{s}$ of the system, so

$$\sqrt{s} = \sqrt{P^\mu P_\mu} = \sqrt{\left(\sum_k E_k\right)^2 - \left(\sum_k p_k\right)^2} = \sum_k E^\star_k = E^\star = M^\star \tag{11.5}$$

Where we used that $\sum_k p^i_{(k)} = 0$ in the $\star-$system, also known before as the center of mass system

Consider now the case of a particle colliding into a target particle. We have

$$p^\mu_1 = \left(E_1, p^i_1\right), \quad p^\mu_2 = (m_2, 0), \quad P^\mu = \left(E_1 + m_2, p^i_1\right) \tag{11.6}$$

And

$$P^\mu P_\mu = (E_1 + m_2)^2 - p^2_1 = E^2_1 + m^2_2 + 2E_1 m_2 - p^2_1 = P^{\mu^\star} P_{\mu^\star} \tag{11.7}$$

Note that $E^2_1 = m^2_1$. Therefore, putting it all together, for a particle colliding into a target, the invariant mass will be

$$\sqrt{s} = \sqrt{P^\mu P_\mu} = \sqrt{m^2_1 + m^2_2 + 2E_1 m_2} = E^\star = M^\star \tag{11.8}$$

Note that if we have $m_1, m_2 << E_1$, ie $\beta \approx 1$ and the particles are ultrarelativistic, then the invariant mass formula can be approximated as follows

$$\sqrt{s} \approx \sqrt{2E_1 m_2} \tag{11.9}$$

In case that both particle have $p^i_{(k)} \neq 0$ in the lab system, we have

$$p^\mu_1 = \left(E_1, p^i_1\right), \quad p^\mu_2 = \left(E_2, p^i_2\right), \quad P^{\mu^\star} = (E^\star_1 + E^\star_2, 0) \tag{11.10}$$

Therefore

$$\begin{aligned} P^\mu P_\mu &= E^2_1 + E^2_2 + 2E_1E_2 - \left(p^2_1 + p^2_2 + 2p_1p_2 \cos\theta\right) \\ P^{\mu^\star} P_{\mu^\star} &= \left(E^\star_1 + E^\star_2\right)^2 = P^\mu P_\mu \end{aligned}$$

Or in simpler terms

$$\sqrt{s} = \sqrt{m^2_1 + m^2_2 + 2\left(E_1E_2 - p_1p_2\cos\theta\right)} = E^\star_1 + E^\star_2 \tag{11.11}$$

If $m_1, m_2 << E_1$, so in the ultrarelativistic case, we have

$$\sqrt{s} \approx \sqrt{2E_1E_2\left(1 - \cos\theta\right)} \tag{11.12}$$

Note that in cases like particle colliders, like the LHC, SuperKamiokande or Fermilab, we have that the collision is frontal, i.e. $\theta = \pi$ and therefore the invariant mass formula becomes extremely easier to remember, especially if the particles are of the same kind ($E_1 = E_2$)

$$\sqrt{s} \approx 2\sqrt{E_1E_2} = 2E \tag{11.13}$$

[3] From now on we will work in God-given units, where $c = 1$

§§ 11.2.1 Transformations of the Invariant Mass

Let's go back to what we had defined before for the invariant mass. We have that $\sqrt{s} = E^\star$, therefore, considering the total 4-momentum in the $\star-$system, we can write without problems

$$P^{\mu^\star} = \left(\sqrt{s}, 0\right) \tag{11.14}$$

This, as every 4-momentum, transforms with Lorentz transformations. Consider the boost along the x-axis without loss of generality, and transform towards the lab system.

$$\begin{pmatrix} \sqrt{s} \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sum_k E_k \\ \sum_k p_k \\ 0 \\ 0 \end{pmatrix} \tag{11.15}$$

Expanding the system and keeping only the two nonzero lines we have

$$\begin{cases} \gamma \sum_k E_k - \beta\gamma \sum_k p_k = \sqrt{s} \\ \gamma \sum_k p_k - \beta\gamma \sum_k E_k = 0 \end{cases} \tag{11.16}$$

From the second row we have

$$\sum_k p_k = \beta \sum_k E_k \implies \beta = \frac{\sum_k p_k}{\sum_k E_k} = \frac{P}{E} \tag{11.17}$$

And, therefore

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-\left(\frac{P}{E}\right)^2}} = \sqrt{\frac{E^2}{E^2-P^2}} = \frac{E}{\sqrt{s}} \tag{11.18}$$

Which, wrapped up, gives a different way to interpret the Lorentz boost and Lorentz factor

$$\begin{cases} \gamma = \dfrac{E}{\sqrt{s}} \\ \beta = \dfrac{P}{E} \end{cases} \tag{11.19}$$

§ 11.3 Transverse Momentum and Transformation of Angles

Consider now a particle moving along the z axis. Transforming from the lab system to the $\star-$system we have, considering spherical polar coordinates

$$\begin{pmatrix} E \\ p \sin\theta \cos\varphi \\ p \sin\theta \sin\varphi \\ p \cos\theta \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E^\star \\ p^\star \cos\theta^\star \cos\varphi^\star \\ p^\star \sin\theta^\star \sin\varphi^\star \\ p \cos\theta^\star \end{pmatrix} \tag{11.20}$$

Where γ, β are the Lorentz factors of the lab system..

Definition 11.3.1 (Transverse Momentum)**.** We define the «transverse momentum» $p_\perp$ as the 3-momentum orthogonal to the z axis.
In general it's defined as the 2-vector $p_\perp = (p_x, p_y)$, i.e.

$$p_\perp = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p \sin\theta \cos\varphi \\ p \sin\theta \sin\varphi \end{pmatrix} \tag{11.21}$$

Applying the transformation it's obvious that $p_\perp = p_\perp^\star$, therefore $p_\perp$ is a relativistic invariant.

From this last relativistic invariant, taking the square we therefore must have

$$p_\perp^2 = \left(p_\perp^\star\right)^2 \implies p^2 \sin\theta = (p^\star)^2 \sin^2\theta^\star \tag{11.22}$$

Note how φ disappears from the calculations, giving $\varphi = \varphi^\star$, this means that the azimuthal angle is another relativistic invariant of motion.
Applying now the transformation on p_z we have

$$p_z = p \cos\theta = \gamma\left(\beta E^\star + p^\star \cos\theta^\star\right) \tag{11.23}$$

Using that p_y, φ are relativistic invariants we can write the following system

$$\begin{aligned} p_y &= p \sin\theta \sin\varphi = p^\star \sin\theta^\star \cos\varphi^\star \\ p_z &= p \cos\theta = \gamma\left(\beta E^\star + p^\star \cos\theta^\star\right) \end{aligned} \tag{11.24}$$

Which, solving for θ gives

$$\frac{p_y}{p_z} = \tan\theta = \frac{\sin\theta^\star}{\gamma\left(\beta E^\star + p^\star \cos\theta^\star\right)} \tag{11.25}$$

Rewriting the denominator we get

$$\tan\theta = \frac{\sin\theta^\star}{\gamma_0\left(\beta_0 \frac{E^\star}{p^\star} - \cos\theta^\star\right)} = \frac{\sin\theta^\star}{\gamma_0\left(\frac{\beta_0}{\beta^\star} - \cos\theta^\star\right)} = \tag{11.26}$$

Where we defined the boost of the center of mass with respect to the $\star$ energy-momentum as follows

$$\beta^\star = \frac{p^\star}{E^\star} \tag{11.27}$$

From this it's possible to define 3 major cases for the transformation of angles between the lab and the $\star$ system after a collision.
1) $\beta > \beta^\star$
If $\beta > \beta^\star$ we have

$$\frac{\beta}{\beta^\star} - \cos\theta^\star > 0$$

Which implies that $\forall\theta^\star \in [0, \pi]$, $\theta \in [0, \pi/2]$.
This means that the particle after the collision, in the lab system will be observed as moving forwards with a flight angle θ between $0, \pi/2$ with respect to the initial motion.

Since also $\theta = 0$ for $\theta^\star = 0, \pi$, we have that there must exist a maximum flight angle $\theta_{max} < \pi/2$. Deriving the previous equation with respect to $\theta^\star$ we get

$$\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta^\star} = \frac{1 + \frac{\beta}{\beta^\star}\cos\theta^\star}{\left(\frac{\beta}{\beta^\star} - \cos\theta^\star\right)^2} = 0$$

Which gives

$$\cos\theta^\star_{max} = -\frac{\beta^\star}{\beta} \tag{11.28}$$

Shoving it back into the equation for the tangent, we have

$$\tan\theta_{max} = \frac{\beta^\star}{\gamma\sqrt{\beta^2 - (\beta^\star)^2}} \tag{11.29}$$

Using $\gamma E^\star + \beta\gamma p^\star \cos\theta^\star = E$ we also have

$$E(\theta_{max}) = \gamma\left(E^\star - \beta^\star p^\star\right) = \gamma\left(\frac{m^2 - (p^\star)^2}{E^\star}\right) = \gamma\frac{m^2}{E^\star} = m\frac{\gamma}{\gamma^\star} \tag{11.30}$$

Where we defined

$$\gamma^\star = \frac{1}{\sqrt{1 - (\beta^\star)^2}} = \frac{m}{E^\star} \tag{11.31}$$

2) $\beta < \beta^\star$
In this case the velocity of the particle in the lab system can stop the center of mass, giving $\theta \geq \pi/2$. Note that there is no maximum angle since the derivative of the tangent is always positive.
3) $\beta = \beta^\star$
In this case $\cos\theta^\star = -1$ and it corresponds to a single possible angle $\theta = \pi/2$. Here the particle moves opposite to the center of mass ($\theta^\star = \pi$), while in the lab it's at rest.

§ 11.4 N-body Decays and Threshold Energy

Consider now a particle colliding with a target at rest, such that after the collision n particles are produced.

Definition 11.4.1 (Threshold Energy)**.** The «threshold energy» of the reaction is defined as the minimal kinetic energy T_{th} that the projectile needs in order to produce all the n particles at rest in the $\star$ system.

In order to find this T_{th} we have that the invariant mass in the final state is

$$\sqrt{s} = \sum_{f=1}^{n}\left(T^\star_f + m_f\right) \tag{11.32}$$

Whereas in the initial state

$$s = (E_i + m_T)^2 - p^i p_i = 2m_T E_i + E_i^2 + m_p^2 \tag{11.33}$$

Where m_T is the mass of the target and m_p is the mass of the projectile.
Writing $T_i = (\gamma - 1)E_i = E_i - m_i$ We can rewrite the invariant mass before the collision as follows

$$s = 2m_T T_i + 2m_T m_i + (m_i^2 + m_T^2)$$

Which gives

$$\sqrt{s} = \sqrt{2m_T T_i + (m_i + m_T)^2} \tag{11.34}$$

Equating $\sqrt{s}$ before and after the collision (it's a relativistic invariant) we have the following equality

$$\left(\sum_{f=1}^{n} \left(T_f^\star + m_f \right) \right)^2 = 2m_T T_i + (m_i + m_T)^2 \tag{11.35}$$

Solving for T_i we get

$$T_i = \frac{\left(\sum_{f=1}^{n} (T_f^\star + m_f) \right)^2 - (m_i + m_T)^2}{2m_T} \tag{11.36}$$

Setting $T_f^\star = 0$ we find the value of T_i such that the particles are produced at rest in the center of mass, i.e. the threshold energy for the reaction.

$$T_{th} = \frac{\left(\sum_{f=1}^{n} m_f \right)^2 - (m_i + m_T)^2}{2m_T} \tag{11.37}$$

Note that if $T_i < T_{th}$ the reaction is ***kinematically impossible***.

§ 11.5 Elastic Scattering

In the event of elastic scattering between particles, the classical conservation of energy and momentum in special relativity translattes into the conservation of the 4-momentum, i.e.

$$p_1^\mu + p_2^\mu = p_1^{\mu'} + p_2^{\mu'} \tag{11.38}$$

Consider now the scattering between an electron e^- and the nucleus of an atom A. We have that if in the lab A is at rest, we can write

$$p_{e^-}^\mu = \left(E_{e^-}, p_{e^-}^i \right), \quad p_A^\mu = (M, 0) \tag{11.39}$$

After the collision we get

$$p_{e^-}^{\mu'} = \left(E'_{e^-}, p_{e^-}^{i'} \right), \quad p_A^{\mu'} = \left(E_A, p_A^{i'} \right) \tag{11.40}$$

We must have $P^\mu = P^{\mu'}$, therefore, using $P^\mu P_\mu = P^{\mu^\star} P_{\mu^\star}$ we have

$$P^\mu P_\mu = p_{e^-}^\mu p_\mu^{e^-} + p_A^\mu p_\mu^A + 2p_A^\mu p_\mu^{e^-} = m_{e^-}^2 + M^2 + 2p_{e^-}^{\mu'} p_{\mu'}^{e^-} \tag{11.41}$$

Note that experimentally only the electron gets measured after the scattering, therefore it's convenient to write the following

$$p_A^{\mu'} = p_{e^-}^\mu + p_A^\mu - p_{e^-}^{\mu'}$$

Therefore

$$p^{\mu}_{e^-} p^A_{\mu} = p^{\mu'}_{e^-} \left(p^{e^-}_{\mu} + p^A_{\mu} - p^{e^-}_{\mu'} \right) = p^{\mu'}_{e^-} p^{e^-}_{\mu} + p^{\mu'}_{e^-} p^A_{\mu} - m^2_{e^-}$$

In the lab system we have that the previous equation becomes

$$E_{e^-} M = E'_{e^-} E_{e^-} + E'_{e^-} M - m^2_{e^-} - p^{i'}_{e^-} p^{e^-}_i$$

If the electron is ultrarelativistic we have

$$EM = E'E + E'M - pp' \cos\theta \implies E' = \frac{E}{1 + \frac{E}{M}(1 - \cos\theta)} \tag{11.42}$$

Where θ is the final scattering angle

§ 11.6 N-body Decays

Consider a particle of mass M decaying in n particles with masses m_i. Considering the reference system where M is at rest, we can say using the conservation of energy, that

$$M = \sqrt{s} \sum_{i=1}^{n} E^\star_i = \sum_{i=1}^{n} \sqrt{(p^\star_i)^2 + m_i^2} \geq \sum_{i=1}^{n} m_i \tag{11.43}$$

Note that this defines also a threshold energy for the decay

$$M \geq \sum_{i=1}^{n} m_i$$

Consider now, without loss of generality, a 2 particle decay $a \to b + c$. We have

$$\begin{aligned} &M = \sqrt{s} = E^\star_b + E^\star_c \\ &p^\star_c = p^\star_b \end{aligned} \tag{11.44}$$

Using (11.43) we have that

$$M = \sqrt{p^2 + m_a^2} + \sqrt{p^2 + m_b^2}$$

Therefore, juggling a bit with the equation, we end up with

$$\begin{aligned} M^2 + m_b^2 - m_c^2 &= 2M \sqrt{(p^\star)^2 + m_b^2} \\ M^4 + (m_b^2 - m_c^2) + 2M^2(m_b^2 - m_c^2) &= 4M^2 \left((p^\star)^2 + m_b^2 \right) \end{aligned}$$

Solving for $p^\star$ we get

$$p^\star = \frac{1}{2M} \sqrt{M^4 + (m_b^2 - m_c^2)^2 - 2M^2(m_b^2 + m_c^2)} \tag{11.45}$$

Using the dispersion relation for $E^\star$ we get

$$E^\star_b = \frac{M^2 + (m_b^2 - m_c^2)}{2M} \tag{11.46}$$

And using $\sqrt{s} = M$, we have

$$E_b^\star = \frac{s + m_b^2 - m_c^2}{2\sqrt{s}} \tag{11.47}$$

The calculation is completely analogous for $E_c^\star$, and we get

$$E_c^\star = \frac{s - m_b^2 - m_c^2}{2\sqrt{s}} \tag{11.48}$$

It's obvious that this decay has only one possible decay energy, and hence it's called «monoenergetic» . This is not true for decays with $n \geq 2$.
In the $\star$ system the decay is isotropic due to the conservation of 3-momentum, but the direction of one particle with respect to the other is fixed, where

$$p_b^\star = -p_c^\star \implies \theta_{bc}^\star = \pi$$

In this case $\star$ coincides with the lab system, therefore $\theta_{bc} = \pi$.
Consider now another case, where the particle decays in flight, with $\beta \neq 0$. In this case the angle measured in the lab system will be obviously different from the one in the $\star$ system.
Using what we got before for defining $\beta, \beta^\star, \gamma$ using momentum and energy, we have our usual transformation of angles for the first particle at $\theta^\star$ and the second at $\pi - \theta^\star$

$$\tan\theta_{bc} = \frac{\sin\theta^\star}{\gamma_a\left(\beta_a\beta_{bc}^\star \pm \cos\theta^\star\right)} \tag{11.49}$$

In general, with n particles, we have three cases
1) $\beta_a > \beta_i$
In the lab system the i-th child particle is emitted forwards, with a maximum angle $\theta_{max} < \pi/2$, corresponding to the angle $\theta_i^\star = \arccos\left(-\beta/\beta_i^\star\right)$
2) $\beta_a < \beta_i$
The i-th child particle is emitted forwards in the lab system if and only if $\cos\theta_i^\star > -\beta_a/\beta_i^\star$. At that value the particle gets emitted at $\theta_i = \pi/2$, while if $\cos(\theta_i^\star) < -\beta_a/\beta_i^\star$ it gets emitted backwards.

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