Дифференциальные уравнения листок 3

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Задача 1

$\dot{x} = \sqrt[3]{x}, \quad x(0) = 0$

$\dfrac{dx}{dt} = \sqrt[3]{x} \to \dfrac{dx}{\sqrt[3]{x}} = dt \to \dfrac{3}{2}x^{\frac{2}{3}} = t + C \to x = \pm(\dfrac{2}{3}t + C)^{\frac{3}{2}} \to [x(0) = 0] \to C = 0 \to \mathbf{x} = \pm(\dfrac{\mathbf{2}}{\mathbf{3}}\mathbf{t})^{\frac{\mathbf{3}}{\mathbf{2}}}$

Однако, мы забыли про возможность сдвинуть и доопределить график, то есть для $\forall C \geq 0$:

$x(t) = 0, \quad t \leq C$

$x(t) = \pm(\dfrac{2}{3}(t - C))^{\frac{3}{2}}, \quad t > C$

Задача 3

(a) $y' = \dfrac{y(1 + xy)}{x(1 - xy)}$

$y' = \dfrac{y(1 + xy)}{x(1 - xy)} \to \left[z = xy, \quad z' = xy' + y, \quad y' = (xz' - z)/x^2\right] \to \dfrac{xz' - z}{x^2} = \dfrac{(1 + z)z}{(1 - z)x^2} \to$

$\to xz' = \dfrac{(1 + z)z}{(1 - z)} + z \to z' = \dfrac{2z}{(1 - z)x} \to \dfrac{dz(1 - z)}{2z} = \dfrac{dx}{x} \to \dfrac{ln|z| - z}{2} = \ln|x| + C \to$

$\to \dfrac{|\mathbf{xy}|}{\mathbf{e^{xy}}} = \mathbf{Ce^{x^2}}, \, C \in \mathbb{R}$

(b) $y' = -\dfrac{x + y + 1}{4x + 4y + 10}$

$y' = -\dfrac{x + y + 1}{4x + 4y + 10} \to [z = x + y + 1, \quad z' = y' + 1, \quad y' = z' - 1] \to z' - 1 = -\dfrac{z}{4z + 6} \to$

$\to z' = \dfrac{3z + 6}{4z + 6} \to \dfrac{dz(4z + 6)}{3z + 6} = dx \to \dfrac{4}{3}z - \dfrac{2}{3}\ln|z + 2| = x + C \to 4x + 4y + 4 - \ln(x + y + 3)^2 = 3x + C \to$

$\to \mathbf{x + 4y + 4 - \ln(x + y + 3)^2 = C}, \, C \in \mathbb{R}$

(c) $y' = \sin(x + y)$

$y' = \sin(x + y) \to [z = x + y, \quad z' = y' + 1, \quad y' = z' - 1] \to z' - 1 = \sin(z) \to z' = \sin(z) + 1 \to$

$\to \dfrac{dz}{\sin(z) + 1} = dx \to \dfrac{2\sin(z/2)}{\sin(z/2) + \cos(z/2)} = x + C \to \dfrac{-2}{\tan(z/2) + 1} = x + C \to \tan(z/2) = \dfrac{-2}{x + C} - 1 \to$

$\to z/2 = \arctan(\dfrac{-x - 2 - C}{x + C}) \to \mathbf{y = 2}\arctan(\dfrac{\mathbf{-x - 2 - C}}{\mathbf{x + C}}) - \mathbf{x}, \, C \in \mathbb{R}$

(d) $y' = \sqrt{4x + 2y - 1}$

$y' = \sqrt{4x + 2y - 1} \to [z = 4x + 2y - 1, \quad z' = 4 + 2y', \quad y' = z'/2 - 2] \to z'/2 - 2 = \sqrt{z} \to$

$\to z' = 4 + 2\sqrt{z} \to \dfrac{dz}{4 + 2\sqrt{z}} = dx \to \sqrt{z} - 2\ln(\sqrt{z} + 4) = x + C \to$

$\to \sqrt{\mathbf{4x + 2y - 1}} - \mathbf{2}\ln(\sqrt{\mathbf{4x + 2y - 1}} + \mathbf{4}) = \mathbf{x + C}, \, C \in \mathbb{R}$

(e) $(x + 2y)y' = 1, \quad y(0) = -1$

$(x + 2y)y' = 1 \to [z = x + 2y, \quad z' = 1 + 2y', \quad y' = (z' - 1)/2] \to z(z' - 1) = 2 \to z' = \dfrac{2}{z} + 1 \to$

$\to z' = \dfrac{2 + z}{z} \to \dfrac{zdz}{2 + z} = dx \to z - 2\ln(2 + z) = x + C \to \mathbf{x + 2y - 2}\ln(\mathbf{2 + x + 2y}) = \mathbf{x + C}, \, C \in \mathbb{R}$

(f) $xy' = x + y$

$$xy' = x + y \to y' = 1 + y/x \to \left[z = y/x, \quad z' = \frac{1}{x}\right] \to dz = dx/x \to z = \ln|x| + C \to$$

$$\to \mathbf{y} = \mathbf{x}(\ln|\mathbf{x}| + \mathbf{C}),\ C \in \mathbb{R}$$

Задача 4

(a) $xy' = y - xe^{y/x}$

$$xy' = y - xe^{y/x} \to y' = y/x - e^{y/x} \to \left[z = y/x, \quad z' = \frac{xy' - y}{x^2}, \quad z'x + y/x = y'\right] \to z'x + z = z - e^z \to$$

$$\to z' = e^z/x \to dze^{-z} = dx/x \to -e^{-z} = \ln|x| + C \to e^{-y/x} = \ln|x| + C \to$$

$$\to -y/x = \ln(\ln|x| + C) \to \mathbf{y} = -\mathbf{x}\ln(\ln|\mathbf{x}| + \mathbf{C}),\ C \in \mathbb{R}$$

(b) $(x^2 + y^2)y' = 2xy$

$$(x^2 + y^2)y' = 2xy \to \left[z = y/x, \quad z' = \frac{xy' - y}{x^2}, \quad z'x + y/x = y'\right] \to (1 + z^2)(z'x + z) = 2z \to$$

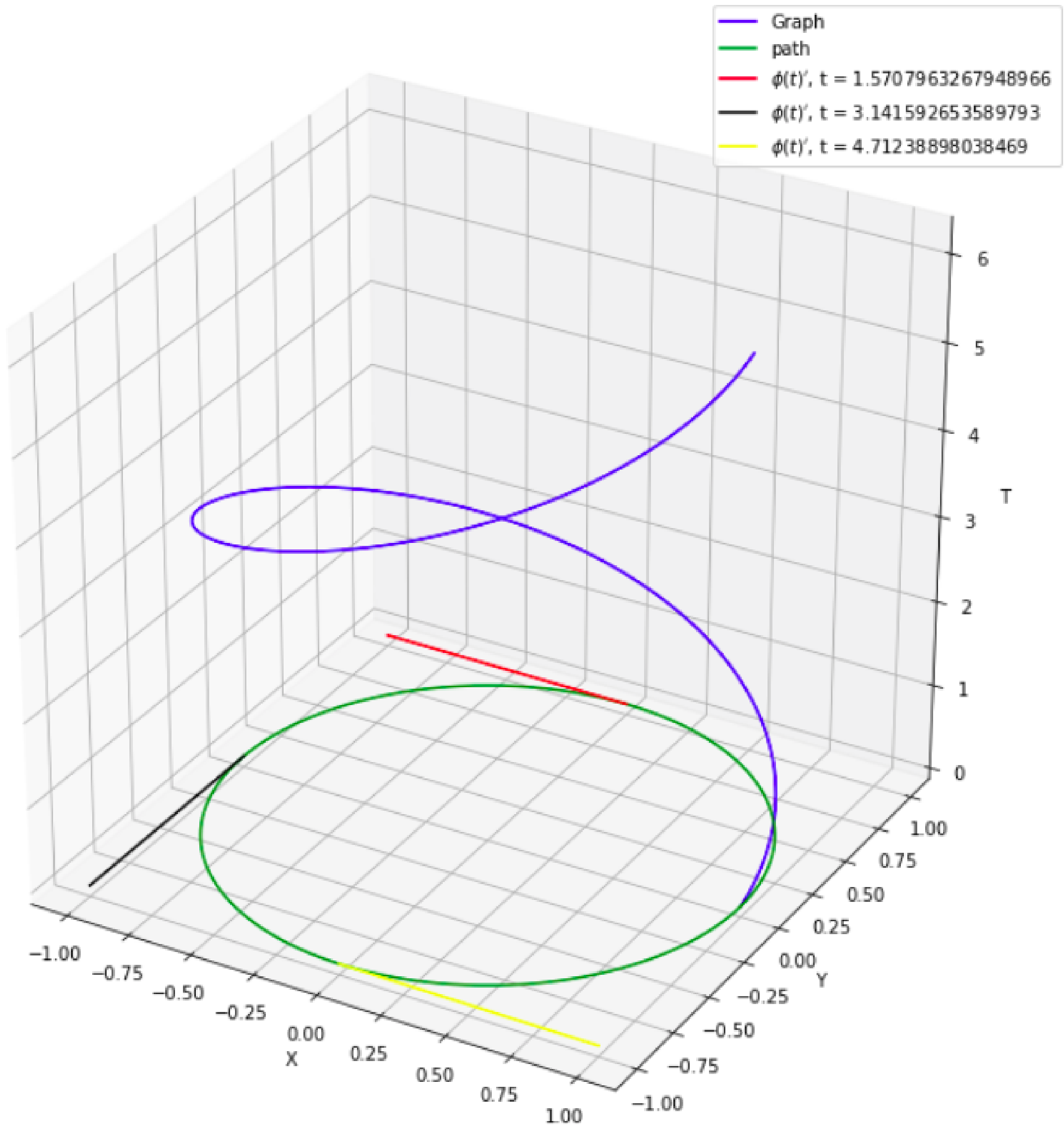
$$\to z'x + z = \frac{2z}{1 + z^2} \to z' = \frac{2z - z - z^3}{(1 + z^2)x} \to \frac{(1 + z^2)dz}{z - z^3} = \frac{dx}{x} \to \ln|z| - \ln|1 - z^2| = \ln|x| + C \to$$

$$\to \ln\left(\frac{|z|}{|1 - z^2|}\right) = \ln|x| + C \to \frac{|z|}{|1 - z^2|} = C_1|x| \to \frac{|\mathbf{y}/\mathbf{x}|}{|\mathbf{1} - (\mathbf{y}/\mathbf{x})^{\mathbf{2}}|} = \mathbf{C_1}|\mathbf{x}|$$

Задача 5

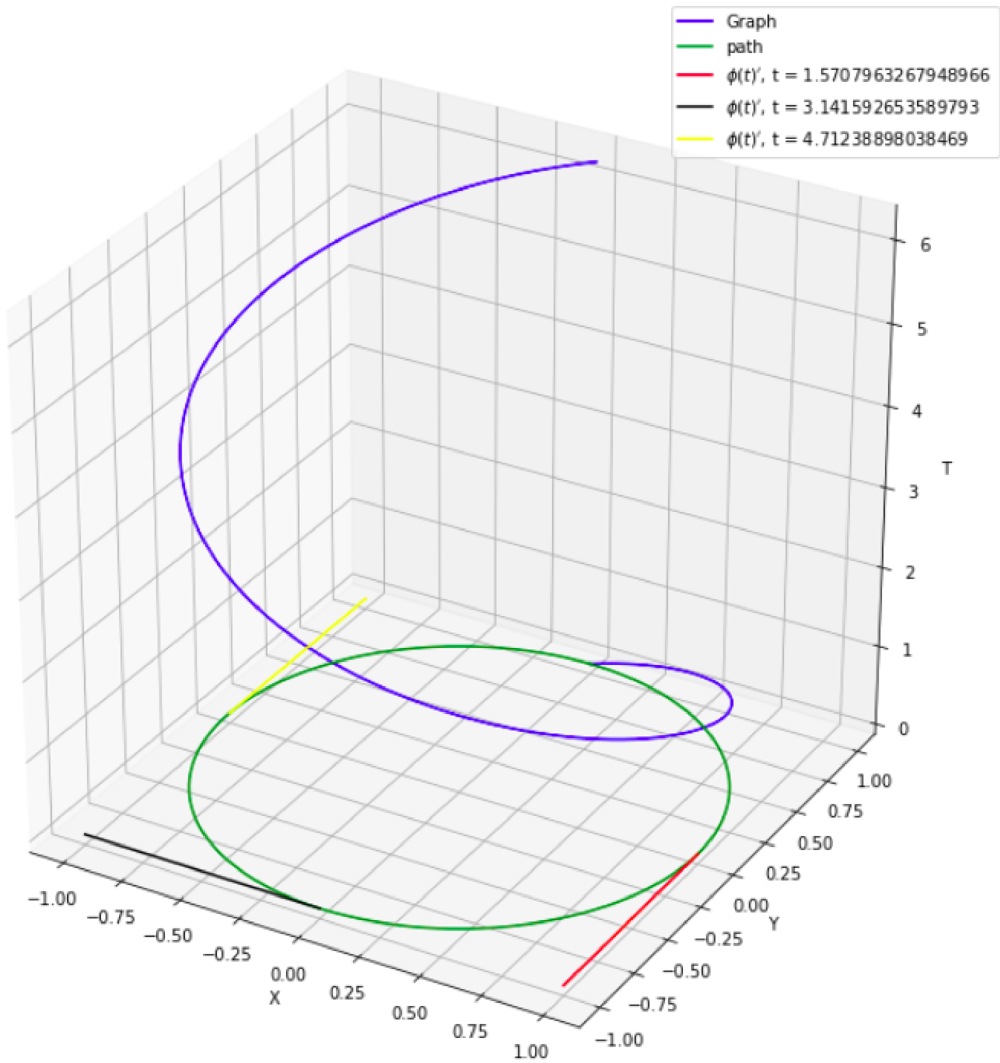
```
from __future__ import print_function
import numpy as np
import math
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

import warnings
warnings.filterwarnings('ignore')

def func1(t, der = False):
    if (der):
        return (-np.sin(t), np.cos(t))
    return (np.cos(t), np.sin(t))

def func2(t, der = False):
    if (der):
        return (np.cos(t), -np.sin(t))
    return (np.sin(t), np.cos(t))

def func3(t, der = False):
    if (der):
        return ((np.cos(t) - t * np.sin(t)), (np.sin(t) + t * np.cos(t)))
    return (t*np.cos(t), t*np.sin(t))

def func4(t, der = False):
    if (der):
        return (1/(2 * np.sqrt(1 - t)), -1/(2 * np.sqrt(t)))
    return (np.sqrt(1 - t), np.sqrt(t))

def draw(func, data_t):
    fig = plt.figure(figsize = (12,12))
    ax = fig.add_subplot(111, projection='3d')
    ax.set_zlabel('T')
    ax.set_ylabel('Y')
    ax.set_xlabel('X')

    arrows_t = np.linspace(min(data_t), max(data_t), 5)
    ts = [arrows_t[1], arrows_t[2], arrows_t[3]]

    data_x = []
    data_y = []
    for t in data_t:
        x, y = func(t)
        data_x.append(x)
        data_y.append(y)
    ax.plot(data_x, data_y, data_t, zdir = 'z', color = 'blue',
      linestyle = '-', label = "Graph")
    ax.plot(data_x, data_y, color = 'green', label = "path")

    colors = ['red', 'black', 'yellow']
    for t in ts:
        x, y = func(t)
        dx, dy = func(t, True)
        data_x = [x, x+dx]
        data_y = [y, y + dy]
        ax.plot(data_x, data_y, color = colors[ts.index(t)],
         label = "$\phi(t)'$, " + "t = " + str(t))
    plt.legend()
    plt.show()
```

```
data_t1 = np.linspace(0, 2 * np.pi, 10000)

draw(func1, data_t1)
```

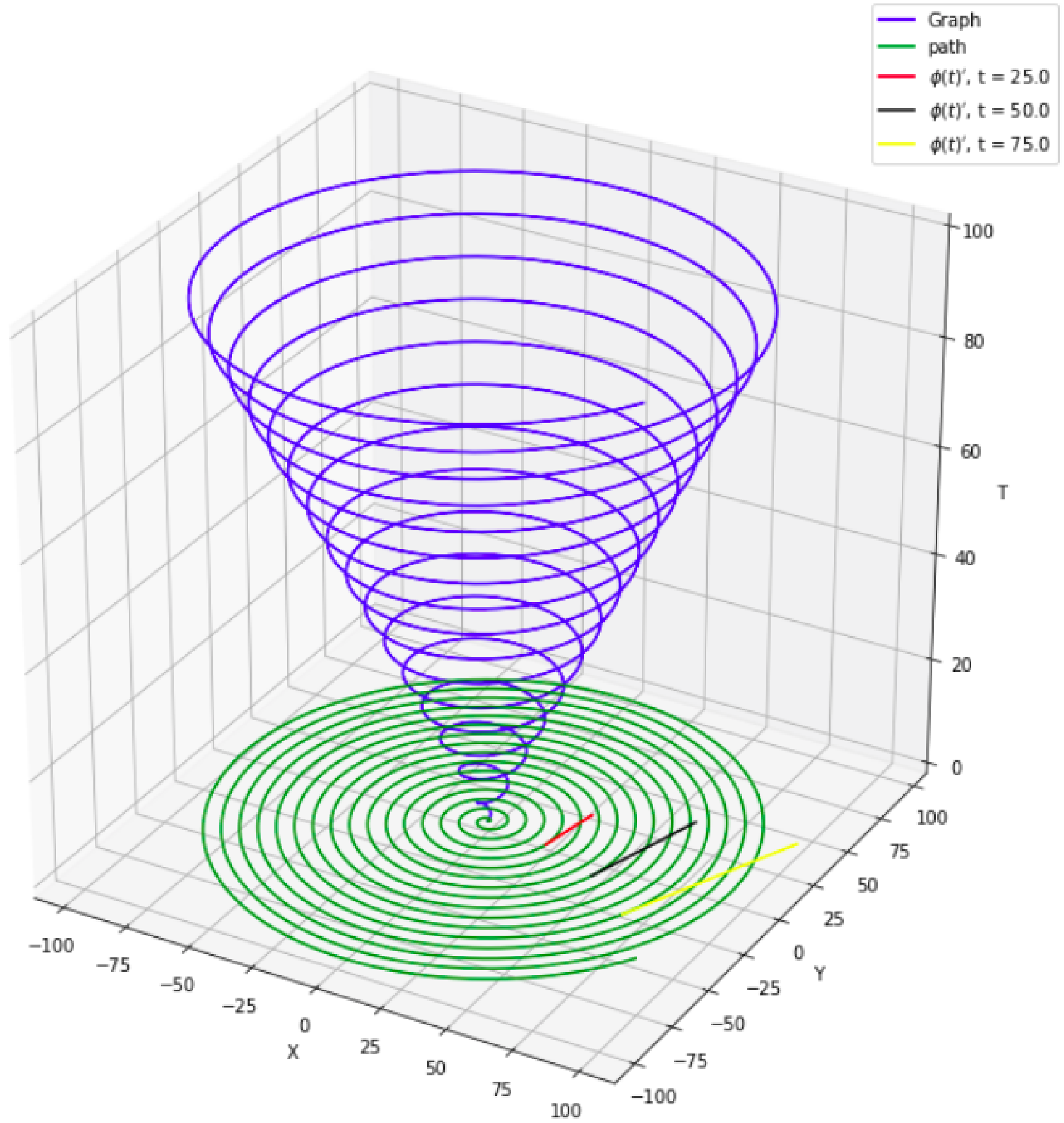


```
data_t2 = np.linspace(0, 2 * np.pi, 10000)
```

```
draw(func2, data_t2)
```



```
data_t3 = np.linspace(0, 100, 100000)
draw(func3, data_t3)
```



```
data_t4 = np.linspace(0, 1, 10000)
draw(func4, data_t4)
```