

Дифференциальные уравнения листок 3

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Задача 1

$$\dot{x} = \sqrt[3]{x}, \quad x(0) = 0$$

$$\frac{dx}{dt} = \sqrt[3]{x} \rightarrow \frac{dx}{\sqrt[3]{x}} = dt \rightarrow \frac{3}{2}x^{\frac{2}{3}} = t + C \rightarrow x = \pm\left(\frac{2}{3}t + C\right)^{\frac{3}{2}} \rightarrow [x(0) = 0] \rightarrow C = 0 \rightarrow \mathbf{x} = \pm\left(\frac{2}{3}t\right)^{\frac{3}{2}}$$

Однако, мы забыли про возможность сдвинуть и доопределить график, то есть для $\forall C \geq 0$:

$$x(t) = 0, \quad t \leq C$$

$$x(t) = \pm\left(\frac{2}{3}(t - C)\right)^{\frac{3}{2}}, \quad t > C$$

Задача 3

$$(a) \quad y' = \frac{y(1 + xy)}{x(1 - xy)}$$

$$\begin{aligned} y' &= \frac{y(1 + xy)}{x(1 - xy)} \rightarrow [z = xy, \quad z' = xy' + y, \quad y' = (xz' - z)/x^2] \rightarrow \frac{xz' - z}{x^2} = \frac{(1 + z)z}{(1 - z)x^2} \rightarrow \\ &\rightarrow xz' = \frac{(1 + z)z}{(1 - z)} + z \rightarrow z' = \frac{2z}{(1 - z)x} \rightarrow \frac{dz(1 - z)}{2z} = \frac{dx}{x} \rightarrow \frac{\ln|z| - z}{2} = \ln|x| + C \rightarrow \\ &\rightarrow \frac{|\mathbf{xy}|}{\mathbf{e}^{\mathbf{xy}}} = \mathbf{C}e^{\mathbf{x}^2}, \quad C \in \mathbb{R} \end{aligned}$$

$$(b) \quad y' = -\frac{x + y + 1}{4x + 4y + 10}$$

$$\begin{aligned} y' &= -\frac{x + y + 1}{4x + 4y + 10} \rightarrow [z = x + y + 1, \quad z' = y' + 1, \quad y' = z' - 1] \rightarrow z' - 1 = -\frac{z}{4z + 6} \rightarrow \\ &\rightarrow z' = \frac{3z + 6}{4z + 6} \rightarrow \frac{dz(4z + 6)}{3z + 6} = dx \rightarrow \frac{4}{3}z - \frac{2}{3}\ln|z + 2| = x + C \rightarrow 4x + 4y + 4 - \ln(x + y + 3)^2 = 3x + C \rightarrow \\ &\rightarrow \mathbf{x} + 4\mathbf{y} + 4 - \ln(\mathbf{x} + \mathbf{y} + 3)^2 = \mathbf{C}, \quad C \in \mathbb{R} \end{aligned}$$

$$(c) \quad y' = \sin(x + y)$$

$$\begin{aligned} y' &= \sin(x + y) \rightarrow [z = x + y, \quad z' = y' + 1, \quad y' = z' - 1] \rightarrow z' - 1 = \sin(z) \rightarrow z' = \sin(z) + 1 \rightarrow \\ &\rightarrow \frac{dz}{\sin(z) + 1} = dx \rightarrow \frac{2\sin(z/2)}{\sin(z/2) + \cos(z/2)} = x + C \rightarrow \frac{-2}{\tan(z/2) + 1} = x + C \rightarrow \tan(z/2) = \frac{-2}{x + C} - 1 \rightarrow \\ &\rightarrow z/2 = \arctan\left(\frac{-x - 2 - C}{x + C}\right) \rightarrow \mathbf{y} = 2\arctan\left(\frac{-\mathbf{x} - 2 - \mathbf{C}}{\mathbf{x} + \mathbf{C}}\right) - \mathbf{x}, \quad C \in \mathbb{R} \end{aligned}$$

$$(d) \quad y' = \sqrt{4x + 2y - 1}$$

$$\begin{aligned} y' &= \sqrt{4x + 2y - 1} \rightarrow [z = 4x + 2y - 1, \quad z' = 4 + 2y', \quad y' = z'/2 - 2] \rightarrow z'/2 - 2 = \sqrt{z} \rightarrow \\ &\rightarrow z' = 4 + 2\sqrt{z} \rightarrow \frac{dz}{4 + 2\sqrt{z}} = dx \rightarrow 2\sqrt{z} - 8\ln(\sqrt{z} + 4) = x + C \rightarrow \\ &\rightarrow 2\sqrt{4\mathbf{x} + 2\mathbf{y} - 1} - 8\ln(\sqrt{4\mathbf{x} + 2\mathbf{y} - 1} + 4) = \mathbf{x} + \mathbf{C}, \quad C \in \mathbb{R} \end{aligned}$$

$$(e) \quad (x + 2y)y' = 1, \quad y(0) = -1$$

$$\begin{aligned} (x + 2y)y' &= 1 \rightarrow [z = x + 2y, \quad z' = 1 + 2y', \quad y' = (z' - 1)/2] \rightarrow z(z' - 1) = 2 \rightarrow z' = \frac{2}{z} + 1 \rightarrow \\ &\rightarrow z' = \frac{2 + z}{z} \rightarrow \frac{zdz}{2 + z} = dx \rightarrow z - 2\ln(2 + z) = x + C \rightarrow \mathbf{x} + 2\mathbf{y} - 2\ln(2 + \mathbf{x} + 2\mathbf{y}) = \mathbf{x} + \mathbf{C}, \quad C \in \mathbb{R} \end{aligned}$$

(f) $xy' = x + y$

$$xy' = x + y \rightarrow y' = 1 + y/x \rightarrow \left[z = y/x, \quad z' = \frac{1}{x} \right] \rightarrow dz = dx/x \rightarrow z = \ln|x| + C \rightarrow$$

$$\rightarrow \mathbf{y} = \mathbf{x}(\ln|\mathbf{x}| + \mathbf{C}), \quad C \in \mathbb{R}$$

Задача 4

(a) $xy' = y - xe^{y/x}$

$$xy' = y - xe^{y/x} \rightarrow y' = y/x - e^{y/x} \rightarrow \left[z = y/x, \quad z' = \frac{xy' - y}{x^2}, \quad z'x + y/x = y' \right] \rightarrow z'x + z = z - e^z \rightarrow$$

$$\rightarrow z' = e^z/x \rightarrow dze^{-z} = dx/x \rightarrow -e^{-z} = \ln|x| + C \rightarrow e^{-y/x} = \ln|x| + C \rightarrow$$

$$\rightarrow -y/x = \ln(\ln|x| + C) \rightarrow \mathbf{y} = -\mathbf{x} \ln(\ln|\mathbf{x}| + \mathbf{C}), \quad C \in \mathbb{R}$$

(b) $(x^2 + y^2)y' = 2xy$

$$(x^2 + y^2)y' = 2xy \rightarrow \left[z = y/x, \quad z' = \frac{xy' - y}{x^2}, \quad z'x + y/x = y' \right] \rightarrow (1 + z^2)(z'x + z) = 2z \rightarrow$$

$$\rightarrow z'x + z = \frac{2z}{1 + z^2} \rightarrow z' = \frac{2z - z - z^3}{(1 + z^2)x} \rightarrow \frac{(1 + z^2)dz}{z - z^3} = \frac{dx}{x} \rightarrow \ln|z| - \ln|1 - z^2| = \ln|x| + C \rightarrow$$

$$\rightarrow \ln\left(\frac{|z|}{|1 - z^2|}\right) = \ln|x| + C \rightarrow \frac{|z|}{|1 - z^2|} = C_1|x| \rightarrow \frac{|\mathbf{y}/\mathbf{x}|}{|\mathbf{1} - (\mathbf{y}/\mathbf{x})^2|} = \mathbf{C}_1|\mathbf{x}|$$

Задача 5

```
from __future__ import print_function
import numpy as np
import math
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

import warnings
warnings.filterwarnings('ignore')

def func1(t, der = False):
    if (der):
        return (-np.sin(t), np.cos(t))
    return (np.cos(t), np.sin(t))

def func2(t, der = False):
    if (der):
        return (np.cos(t), -np.sin(t))
    return (np.sin(t), np.cos(t))

def func3(t, der = False):
    if (der):
        return ((np.cos(t) - t * np.sin(t)), (np.sin(t) + t * np.cos(t)))
    return (t*np.cos(t), t*np.sin(t))

def func4(t, der = False):
    if (der):
        return (1/(2 * np.sqrt(1 - t)), -1/(2 * np.sqrt(t)))
    return (np.sqrt(1 - t), np.sqrt(t))

def draw(func, data_t):
    fig = plt.figure(figsize = (12,12))
    ax = fig.add_subplot(111, projection='3d')
    ax.set_zlabel('T')
    ax.set_ylabel('Y')
    ax.set_xlabel('X')

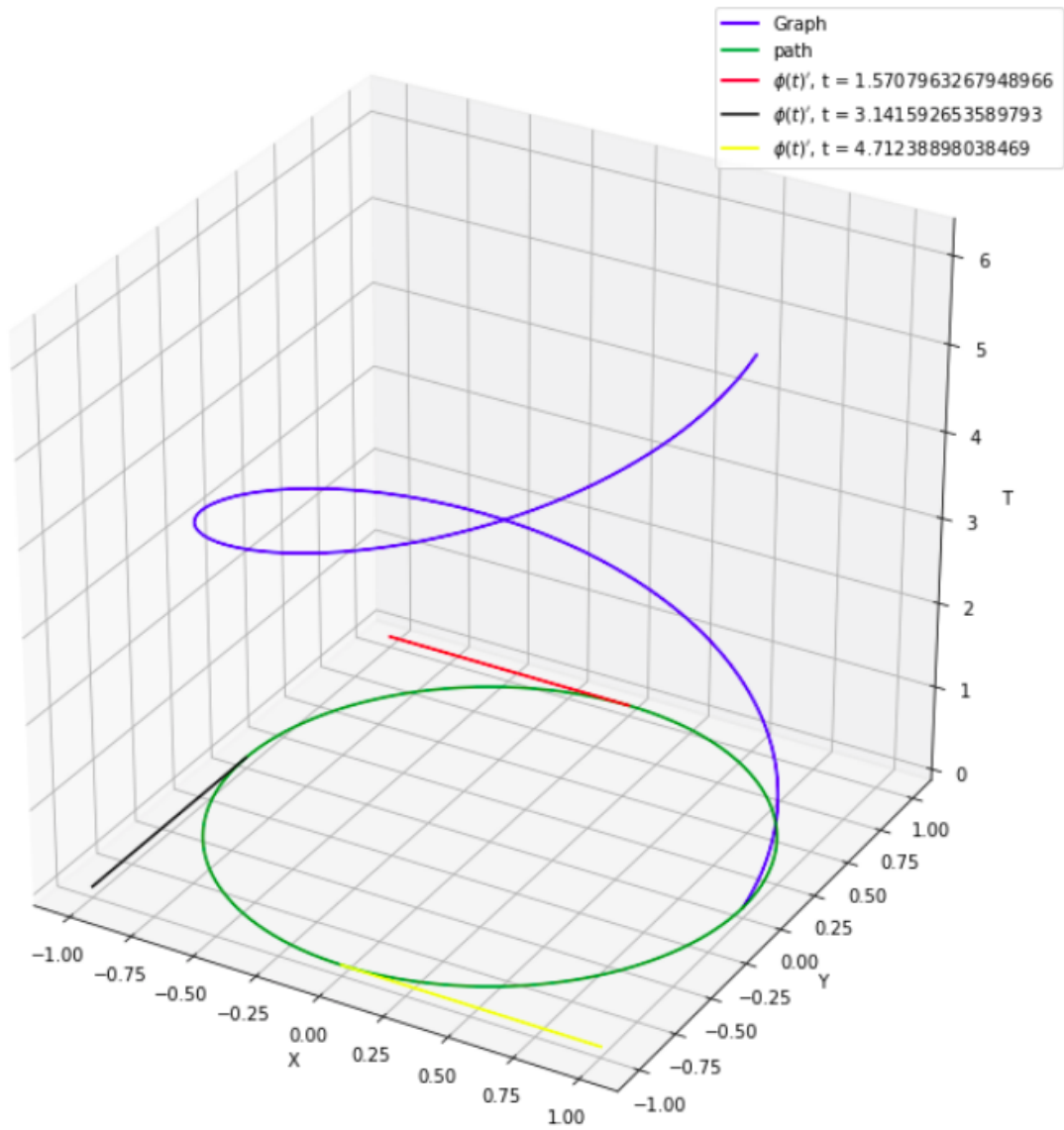
    data_x = []
    data_y = []

    for t in data_t:
        x, y = func(t)
        data_x.append(x)
        data_y.append(y)

    ax.plot(data_x, data_y, data_t, zdir = 'z', color = 'blue', linestyle = '-')
    plt.show()
```

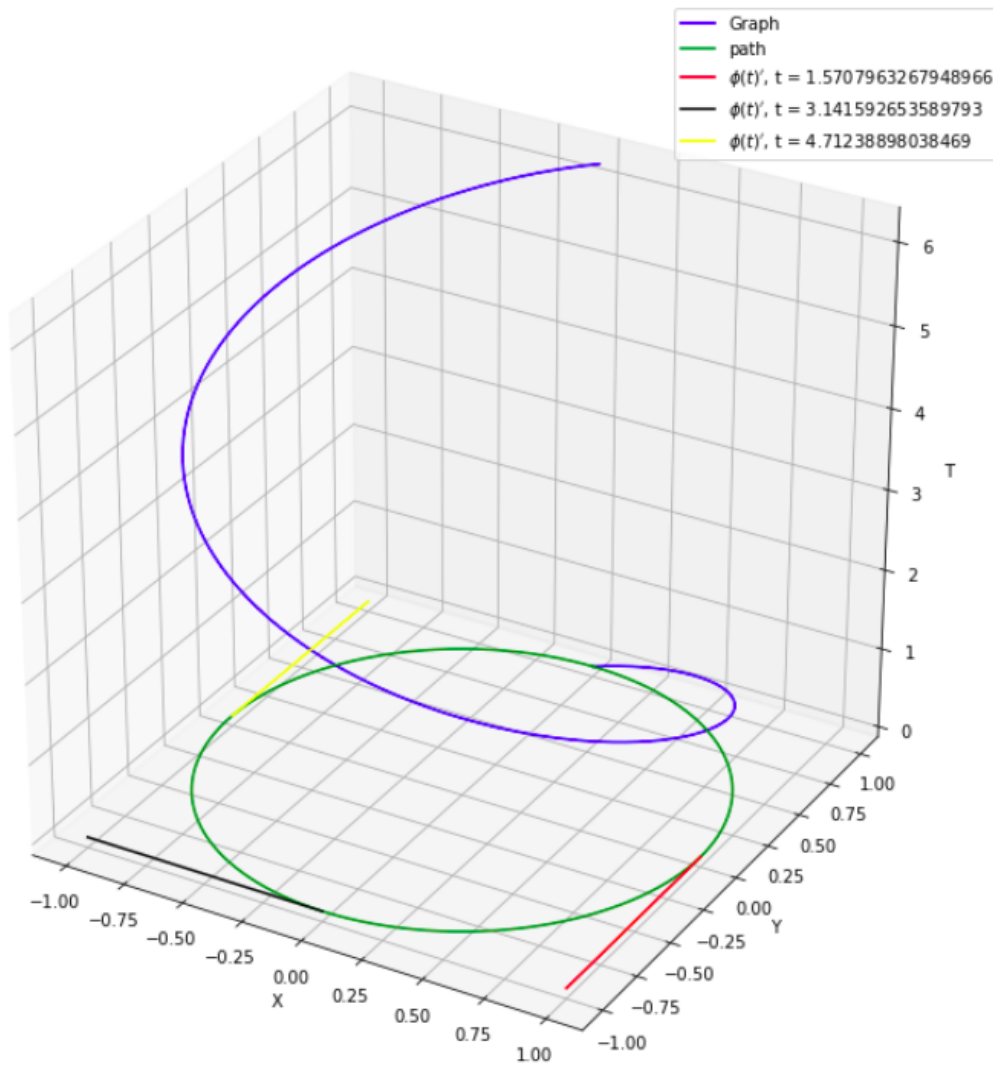
```
data_t1 = np.linspace(0, 2 * np.pi, 10000)
```

```
draw(func1, data_t1)
```



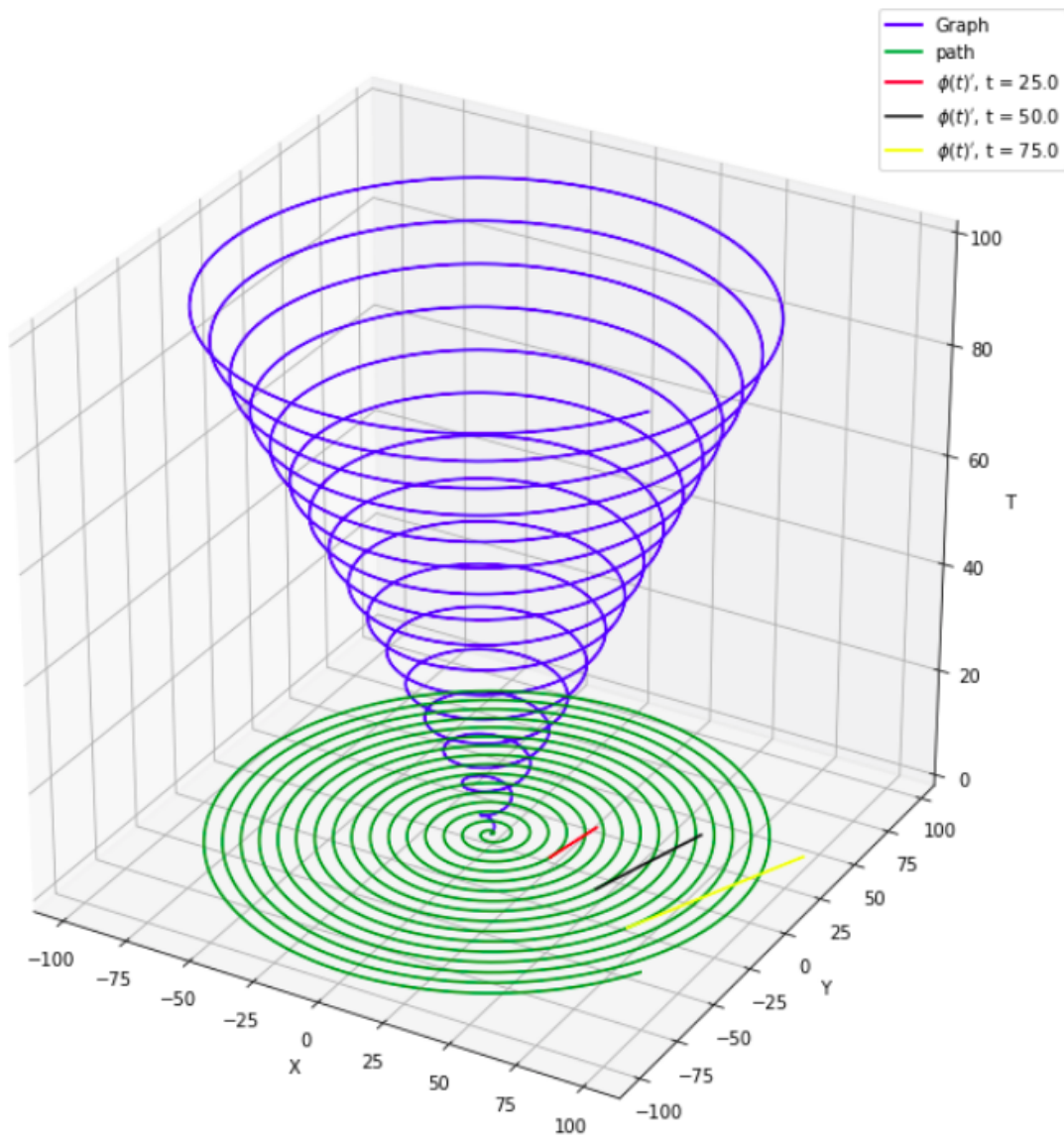
```
data_t2 = np.linspace(0, 2 * np.pi, 10000)
```

```
draw(func2, data_t2)
```



```
data_t3 = np.linspace(0, 100, 100000)
```

```
draw(func3, data_t3)
```



```
data_t4 = np.linspace(0, 1, 10000)
```

```
draw(func4, data_t4)
```

