4. Conditional Probability and Independence

4.1 Conditional probability

We know that two events are **independent** of one another if the occurrence of first event in no way affects the outcome of the second event. On the other hand, two events are said to be **dependent** when the occurrence of the first event changes the probability of the occurrence of the second event.

The conditional probability of an event B in relationship to an event A is defined as the probability that event B occurs after event A has already occurred.

- Conditional probability provides us with a way to reason about the outcome of an experiment, based on partial information.
- If the occurrence of one event has an effect on the occurrence of the other event then the two events are conditional or dependent events.
- The conditional probability of an event is a probability obtained with the additional information that some other event has already occurred.

Conditional probability and ….

The conditional probability of an event A given that B has already occurred, denoted by P(A|B) is

$$P(A/B) = \frac{P(A \cap B)}{P(B)},\ where\ P(B) > 0$$

The conditional probability of an event B in relationship to an event A is defined as the probability that event B occurs after event A has already occurred.

$$P(B/A) = \frac{P(A \cap B)}{P(A)},\ where\ P(A) > 0$$

The notation P(B|A) does not mean that B is divide by A; rather, it means the probability that event B occurs given that event A has already occurred.

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Conditional …

Example 1

A box contains 80 candles, 30 of which are defective. Suppose we take two candles from this box in sequence with and without replacement. Defining the two events as follows, find the probabilities for the events under the two situations.

A={the first candle is defective} B={the second candle is defective}

a) with replacement

P(A)= $\frac{30}{80} = \frac{3}{8}$

P(B)= $\frac{30}{80} = \frac{3}{8}$

b) without replacement

P(A)= $\frac{30}{80} = \frac{3}{8}$ P(B)=? This depends whether A did or did not occur.

If A did not occur, P(B)= $\frac{30}{79}$

If A did occur, on the 2nd draw there are only 79 candles left of which 29 are defective.

Here P(B)= P(B/A)= $\frac{29}{79}$

P(B|A) is the conditional probability of B given that A has already occurred.

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Conditional probability …

Example 2

A box contains black balls and white balls. A person selects two balls **without replacement.** If the probability of selecting a black ball *and* a white ball is $\frac{15}{56}$, and the probability of selecting a black ball on the first draw is $\frac{3}{8}$ find the probability of selecting a white ball on the second draw, *given* that the first ball selected was a black ball.

Soln

Let *B*= selecting a black ball *W*= selecting a white ball

Given: P(B and W) $=\frac{15}{56}$ P(B) = $\frac{3}{8}$

Then by the conditional probability relationship $P(B|A)=\frac{P(A \text{ and } B)}{P(A)}$

$$P(W|B)=\frac{P(B \text{ and } W)}{P(B)}=\frac{15/56}{3/8}=\frac{15}{56}\div\frac{3}{8}=\frac{15}{56}\times\frac{8}{3}=\frac{5}{7}$$

Five over seven is the probability of selecting a white ball on the second draw given that the first ball selected was black.

Conditional probability …

Exercise 1

A random car is chosen among all those passing through the gate of CNS on a certain day. The probability that the car is yellow is 3/100: the probability that the driver is female is 1/5; and the probability that the car is yellow and the driver is female is 1/50. Find the probability that the driver is female given that the car is yellow.

Soln

If A is the event 'the car is yellow' and B the event 'the driver is female', then we are given that P(A) = 0.03, P(B) = 0.2, and P(A and B) = 0.02.

$$P(B|A)=\frac{P(A \text{ and } B)}{P(A)} \qquad P(B|A)=\frac{0.02}{0.03}=0.667$$

Conditional probability …

Exercise 2

Suppose that an office has 100 calculating machines. Some of these machines are electric (E) while others are manual (M). And some of the machines are new (N) while others are used (U). The following table gives the number of machines in each category. A person enters the office, picks a machine at random and discovers that it is new. What is the probability that it is electric?

	E	M	
N	40	30	70
U	20	10	30
	60	40	100

Soln

$$P(E/N) = \frac{P(E \cap N)}{P(N)} = \frac{40/100}{70/100} = \frac{4}{7}$$

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4.2 Multiplication, Bayes' and total probability theorems

Multiplication theorem of probability

Definition: If A and B are two events in S and P(A) is not zero, then the probability that A and B will both occur is the product of the probability of A and the conditional probability of B given A.

That is,

$$P(A \cap B) = P(A)P(B/A).$$

Similarly,

$$P(A \cap B) = P(B)P(A/B)$$

The *multiplication rules* can be used to find the probability of two or more **events that occur in sequence**.

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Multiplication …

Multiplication Rule 1

- When two events are **independent**, the probability of both occurring is

P(A and B)= P(A)·P(B)

Examples of independent events:

- Rolling a die and tossing a coin; and getting a tail on the coin and a 5 on the die
- Drawing a card twice from a deck **with replacement and getting a 10**
- Tossing a coin twice and getting two heads

Multiplication Rule 2

- When two events are **dependent**, the probability of both occurring is

P(A and B)= P(A)·P(B|A)

Examples of dependent events:

- Drawing a card from a deck, **not replacing it**, and then drawing a second card.
- Scoring excellent grades and winning a scholarship award.
- Cheating on exams and obtaining nil.

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Multiplication…

The above multiplication rule can be generalized to more than two events as shown below.

Let A1, A2, …, An be a sequence of events and assume

$$\bigcap_{i=1}^{n} A_i \neq 0$$

then,

$$P\left(\bigcap_{I=1}^{n} A_i\right) = P(A_1)P(A_2/A_1)P(A_3/A_1\,, A_2)\ldots P(A_n/A_1\,, A_2\ \ldots,, A_{n-1})$$

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Multiplication ….

Example 1

Suppose a card is drawn from a deck and ***not* replaced**, and then a second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

Soln:

Let

A= Selecting an ace

B= Selecting a king

Are the two evets dependent or independent?

The event of getting a king on the second draw *given* that an ace was drawn the first time is called a ***conditional probability***.

$$P(A \cap B) = P(A) \times P(A/B)$$

$$4/52 \times 4/51 = 16/2652 = 4/663$$

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Multiplication …

Example

Three cards are drawn from an ordinary deck and **not replaced**. Find the probability of the following events:

a) Getting 3 jacks
b) Getting an ace, a king and a queen in order
c) Getting a diamond, a spade and a heart in order
d) Getting 3 hearts

Soln : (dependent or independent?)

a) $$P\ (3\ jacks) = \frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} = \frac{24}{132{,}600} = \frac{1}{5525}$$

b) $$\text{P(ace and king and queen)} = \frac{4}{52} \times \frac{4}{51} \times \frac{4}{50} = \frac{64}{132{,}600} = \frac{8}{16{,}575}$$

c) $$\text{P(diamond and spead and heart)} = \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} = \frac{2197}{132{,}600} = \frac{169}{10{,}200}$$

d) $$\text{P(3 hearts)} = \frac{13}{52} \times \frac{12}{51} \times \frac{11}{50} = \frac{1{,}716}{132{,}600} = \frac{11}{850}$$

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Multiplication …

Bayes' Theorem

Given two dependent events *A* and *B*, the formulas for conditional probability allow us to find *P*(*A* and *B*), or *P*(*B/A*). Related to these formulas is a rule developed by Thomas Bayes (1702–1761). The rule is known as **Bayes' Theorem.**

Let *A* and *B* be two events with their corresponding probabilities P(A) and P(B), then

$$P(A/B) = \frac{P(A)}{P(B)} P(B/A)$$

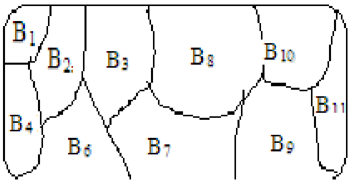
Bayes'Rule is based on the notion of a partition of a sample space. Events are said to partition a sample space S if two conditions exist:

1) $\boldsymbol{B_i \cap B_j = \emptyset}$ **for any pair i and j**

2) $\boldsymbol{B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_k = S}$

Thus the events $\boldsymbol{B_1}$, $\boldsymbol{B_2}$, $\boldsymbol{B_3}$, … , $\boldsymbol{B_k}$ are mutually exclusive and exhaustive.

This can be graphically represented as



Multiplication …

Let A be some event associated with S and $\boldsymbol{B_1}$, $\boldsymbol{B_2}$, … , $\boldsymbol{B_k}$ be partitions of S. Hence we can write

$$A = (A \cap B_1) \cup (A \cap B_2) \cup \ \ldots \cup (A \cap B_k)$$

- Some of $A \cap B_i$ may be empty. Note that since B's are partitions and have no intersections, all sets $(A \cap B_1)$, $(A \cap B_2)$, … , $(A \cap B_k)$ are pair-wise mutually exclusive. Hence we may apply the addition property for mutually exclusive events and write

$$\text{P(A)} = P(A \cap B_1) + P(A \cap B_2) + \cdots P(A \cap B_k).$$

- We know that by conditional probability, the term $P(A \cap B_i)$ for each i can be expressed as $P(A \,/\, B_i)\ P(B_i)$ and hence we obtain what is called the theorem on **total probability**:

Multiplication …

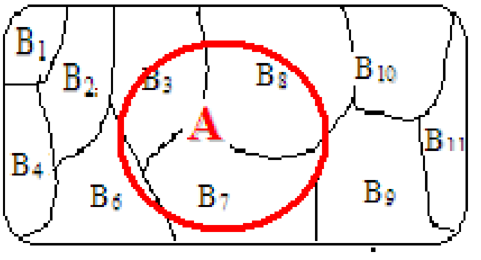
Total Probability Theorem

If we have $\boldsymbol{B_1}$, $\boldsymbol{B_2}$, … , $\boldsymbol{B_k}$ as partitions of **S** for any event A associated with **S**,

$$P(A) = P(A/B_1)P(B_1) + P(A/B_2)P(B_2) + \ \dots + P(A/B_k)P(B_k)$$

$$= \sum_{i=1}^{k} P(A/B_i)P(B_i)$$

This can be represented graphically as follows:



Multiplication …

Example 1

Suppose a statistics class contains 60% girls and 40% boys. Suppose that 30% of the girls and 20% of the boys scored A in the final exam. A student is chosen at random from the class. What is the probability that the chosen student will have an A grade?

To answer this, we let

A be the set of all students who scored A grade
B_1 be the set of girls
B_2 be the set of boys.

Then $\{B_1, B_2\}$ is a partition of the class.

Given: $P(B_1)= 0.6$ $P(B_2) =0.4$
$P(A/B_1)=0.3$ $P(A/B_2)=0.2$

We are interested in P(A). We compute this by the total probability theorem as

$P(A) = P(B_1)P(A|\ B_1) + P(B_2)P(A|\ B_2)$
$= (0.6)(0.3) + (0.4)(0.2) = 0.26,$

so there is a 26% chance that the randomly chosen student has scored an A grade.

Multiplication …

Exercise 1

A company buys chairs from two suppliers, 1 and 2. Supplier 1 has a record of delivering chairs containing 10% defectives, whereas supplier 2 has a defective rate of only 5%. Suppose 40% of the current supply came from supplier 1. If a chair is selected randomly from this supply, what is the probability of the chair to be defective?

Soln

Let,

A be the set of all defective chairs.

B_1 be the set of chairs from supplier 1

B_2 be the set of chairs from supplier 2.

Then $\{B_1, B_2\}$ is a partition of the supplies.

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Multiplication …

Thinking about the previous exercise (Exercise 1), we may want to get another important result. *Suppose one chair is chosen and is found to be defective. What is the probability that it is from Supplier 1?* **The Bayes' Theorem** which has been discussed previously can be used to answer this question.

Bayes' Theorem

Let $B_1, B_2, \ldots, B_k$ be a partition of the sample space S and let A be an event associated with S. Then

$$P(B_i/A) = \frac{P(A/B_i)P(B_i)}{\sum_{J=1}^{k} P(A/B_j)P(B_j)}$$

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Multiplication …

Example 2

Take the two suppliers information in Exercise 1. If a chair is selected randomly from this supply and observed to be defective, find the probability that it came from supplier 1.

Let B_i denote the event that a chair comes from supplier i (i = 1,2), and note that B_1 and B_2 form a partition of the sample space for the experiment of selecting one chair. Let A denote the event that the selected chair is defective. Then

$$P(B_1/A) = \frac{P(B_1)P(A/B_1)}{P(B_1)P(A/B_1) + P(B_2)P(A/B_2)}$$

$$= \frac{0.40\times0.10}{(0.40\times0.10)+(0.60\times0.05)}$$

$$= 0.5714$$

Supplier 1 has a greater probability of being the party supplying the defective chair than does supplier 2.

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Multiplication …

Exercise 2

Consider the statistic class students' information in Example 1. It was observed that a randomly selected student has an A grade. What is the probability that the chosen student is a girl?

Soln

We found out that a randomly selected student has scored A. Our interest is to quantify the chance of occurrence that the selected student is a girl.

Let B_i denote the event that a student belongs to one of the sexes (M,F) in the class, and note that B_1 and B_2 form a partition of the sample space for the experiment of selecting a student. Let A denote the event that the selected student's grade is A. Then ----

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4.3 Independent events

Independence: Two events *A* and *B* are **independent events** if the occurrence of *A* does not affect the probability of *B* occurring.

Example 1: A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Soln : (dependent or independent?)

- P(A and B)= P(A)·P(B).
- P(head and 4)= $\frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$

Using the sample space:

- H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6.
- There is only one way to get the head-4 outcome. The solution is, therefore, $\frac{1}{12}$.

… independence

Example 2: If a coin is tossed twice, what is the probability of getting two heads? (dependent or independent?)

P(HH)= $1/2 \times 1/2 = 1/4$

Using the sample space HH,HT,TH,TT. Then P(HH)= $1/4$

Example 3: A card is drawn from a deck and **replaced**; then a second card is drawn. Find the probability of getting a queen and then an ace. (dependent or independent?)

Solution

The probability of getting a queen is $4/52$,
since the card is **replaced**, the probability of getting an ace is $4/52$

Hence, the probability of getting a queen and an ace is

$$\mathrm{P(queen\ and\ ace)} = \mathrm{P(queen)} \times \mathrm{P(ace)} = \frac{4}{52} \times \frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$$