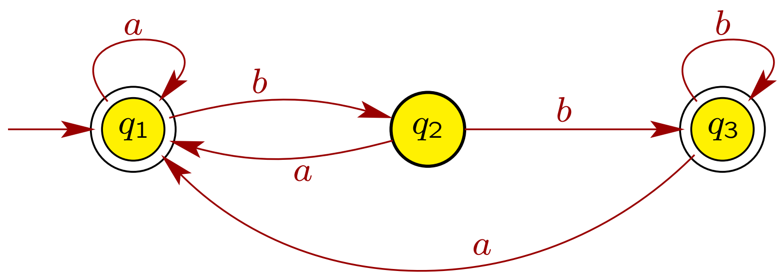
Homework 2 Solutions

1. For the drawing below,



   we can formally express the DFA as $M = (Q, \Sigma, \delta, q_1, F)$, where

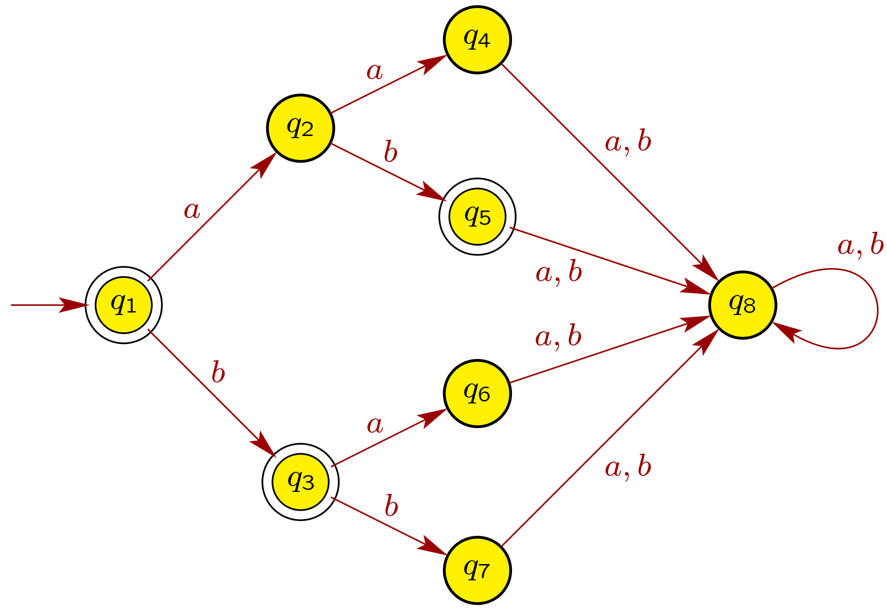
   - $Q = \{q_1, q_2, q_3\}$
   - $\Sigma = \{a, b\}$
   - transition function δ is given by

	a	b
q_1	q_1	q_2
q_2	q_1	q_3
q_3	q_1	q_3

   - q_1 is the start state
   - $F = \{q_1, q_3\}$ is the set of accept states.

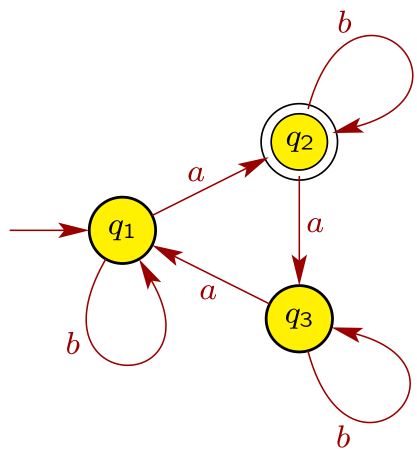
2. There are (infinitely) many correct DFAs for each part below.

   (a) A DFA that recognizes the language $A = \{\varepsilon, b, ab\}$ is

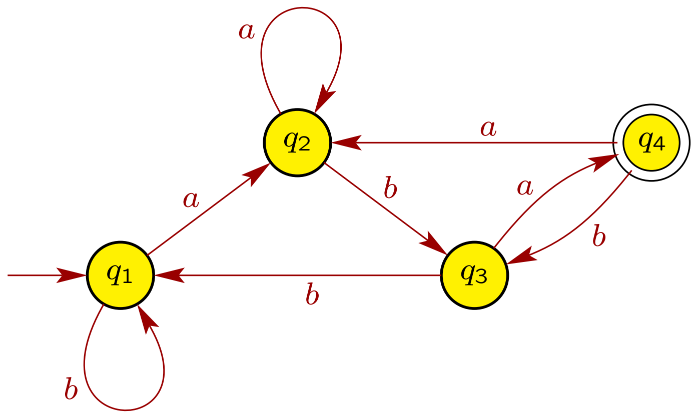


There are simpler DFAs that recognize this language. Can you come up with one with only 4 states?

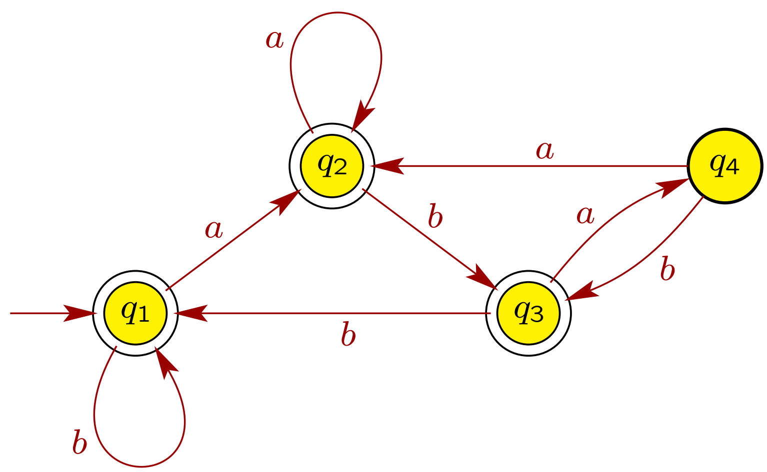
(b) A DFA that recognizes the language $B = \{\, w \in \Sigma^* \mid n_a(w) \mod 3 = 1 \,\}$ is



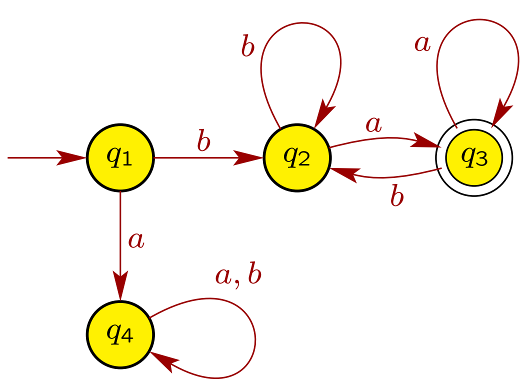
(c) A DFA that recognizes the language $C = \{\, w \in \Sigma^* \mid w = saba$ for some string $s \in \Sigma^* \,\}$ is



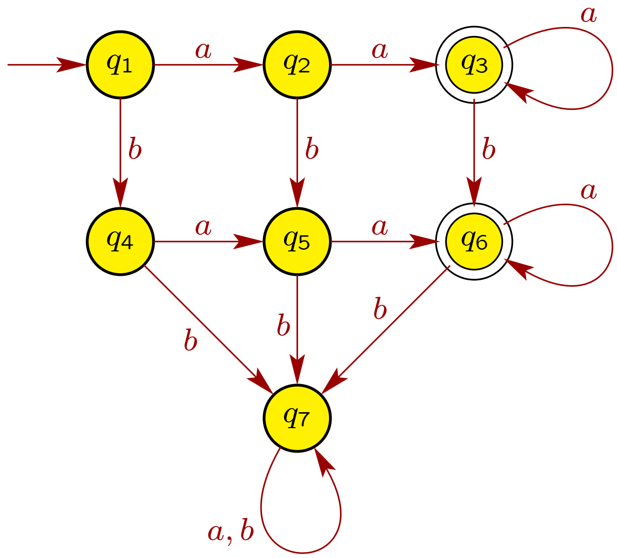
(d) Since $D = \overline{C}$, the complement of C, we can convert the DFA for C into a DFA for D by swapping the accept and non-accept states:



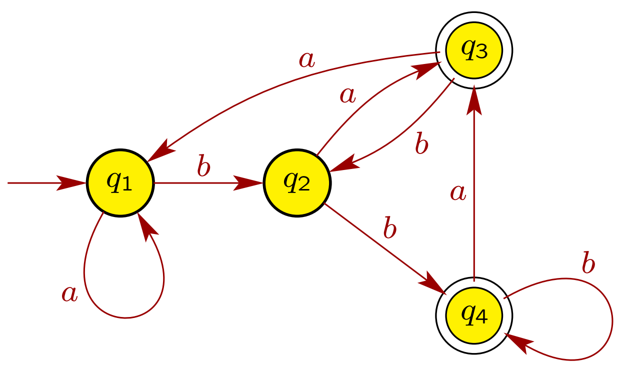
(e) A DFA for $E = \{\, w \in \Sigma^* \mid w$ begins with b and ends with $a \,\}$ is



(f) A DFA for $F = \{\, w \in \Sigma^* \mid n_a(w) \geq 2, n_b(w) \leq 1 \,\}$ is



(g) A DFA for $G = \{ w \in \Sigma^* \mid |w| \geq 2,$ second-to-last symbol of w is $b \}$ is



3. Show that, if M is a DFA that recognizes language B, swapping the accept and non-accept states in M yields a new DFA that recognizes $\overline{B}$, the complement of B. Conclude that the class of regular languages is closed under complement.

   Answer:
   Suppose language B over alphabet Σ has a DFA

$$M = (\, Q,\ \Sigma,\ \delta,\ q_1,\ F \,).$$

   Then, a DFA for the complementary language $\overline{B}$ is

$$\overline{M} = (\, Q,\ \Sigma,\ \delta,\ q_1,\ Q - F \,).$$

   The reason why $\overline{M}$ recognizes $\overline{B}$ is as follows. First note that M and $\overline{M}$ have the same transition function δ. Thus, since M is deterministic, $\overline{M}$ is also deterministic. Now consider any string $w \in \Sigma^*$. Running M on input string w will result in M ending in some state $r \in Q$. Since M is deterministic, there is only one possible state that M can end in on input w. If we run $\overline{M}$ on the same input w, then $\overline{M}$ will end in the same state r since M and $\overline{M}$ have the same transition function. Also, since $\overline{M}$ is deterministic, there is only one possible ending state that $\overline{M}$ can be in on input w.

Now suppose that $w \in B$. Then M will accept w, which means that the ending state $r \in F$, i.e., r is an accept state of M. But then $r \notin Q - F$, so $\overline{M}$ does not accept w since $\overline{M}$ has $Q - F$ as its set of accept states. Similarly, suppose that $w \notin B$. Then M will not accept w, which means that the ending state $r \notin F$. But then $r \in Q - F$, so $\overline{M}$ accepts w. Therefore, $\overline{M}$ accepts string w if and only M does not accept string w, so $\overline{M}$ recognizes language $\overline{B}$. Hence, the class of regular languages is closed under complement.

4. We say that a DFA M for a language A is *minimal* if there does not exist another DFA M' for A such that M' has strictly fewer states than M. Suppose that $M = (Q, \Sigma, \delta, q_0, F)$ is a minimal DFA for A. Using M, we construct a DFA $\overline{M}$ for the complement $\overline{A}$ as $\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$. Prove that $\overline{M}$ is a minimal DFA for $\overline{A}$.

   Answer:
   We prove this by contradiction. Suppose that $\overline{M}$ is not a minimal DFA for $\overline{A}$. Then there exists another DFA D for $\overline{A}$ such that D has strictly fewer states than $\overline{M}$. Now create another DFA D' by swapping the accepting and non-accepting states of D. Then D' recognizes the complement of $\overline{A}$. But the complement of $\overline{A}$ is just A, so D' recognizes A. Note that D' has the same number of states as D, and $\overline{M}$ has the same number of states as M. Thus, since we assumed that D has strictly fewer states than $\overline{M}$, then D' has strictly fewer states than M. But since D' recognizes A, this contradicts our assumption that M is a minimal DFA for A. Therefore, $\overline{M}$ is a minimal DFA for $\overline{A}$.

5. Suppose A_1 and A_2 are defined over the same alphabet Σ. Suppose DFA M_1 recognizes A_1, where $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$. Suppose DFA M_2 recognizes A_2, where $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$. Define DFA $M_3 = (Q_3, \Sigma, \delta_3, q_3, F_3)$ for $A_1 \cap A_2$ as follows:

   - Set of states of M_3 is
   $$Q_3 = Q_1 \times Q_2 = \{\, (x, y) \mid x \in Q_1,\ y \in Q_2 \,\}.$$
   - The alphabet of M_3 is Σ.
   - M_3 has transition function $\delta_3 : Q_3 \times \Sigma \to Q_3$ such that for $x \in Q_1$, $y \in Q_2$, and $\ell \in \Sigma$,
   $$\delta_3(\,(x, y),\ \ell\,) \;=\; (\,\delta_1(x, \ell),\ \delta_2(y, \ell)\,)\,.$$
   - The initial state of M_3 is $s_3 = (q_1, q_2) \in Q_3$.
   - The set of accept states of M_3 is
   $$F_3 \;=\; \{\,(x, y) \in Q_1 \times Q_2 \mid x \in F_1 \text{ and } y \in F_2\,\} = F_1 \times F_2.$$

   Since $Q_3 = Q_1 \times Q_2$, the number of states in the new DFA M_3 is $|Q_3| = |Q_1| \cdot |Q_2|$. Thus, $|Q_3| < \infty$ since $|Q_1| < \infty$ and $|Q_2| < \infty$.