General equations and fixed points for system

Supplementary Material for the manuscript “Bacteriophage diet breadth is impacted by interactions between bacteria.” This *Mathematica* file contains analytical calculations used to derive fixed point concentrations and corresponding eigenvalues. Queries may be address to Ave T. Bisesi at bis-es004@umn.edu.

Defining the four equations:

In[ ]:=
```
Clear[mu1, mu2, b1, b2, a1, a2, k1, k2, z1, z2, R, v1, v2, δ]
xRHS = mu1 * X * ((a1 * Y) / (a1 * Y) + k1) * (R - X - b1 * Y) - z1 * g * X - δ * X;
yRHS = mu2 * Y * ((a2 * X) / (a2 * X) + k2) * (R - Y - b2 * X) - z2 * s * Y - z1 * g * Y - δ * Y;
gRHS = v1 * z1 * g * Y + v1 * z1 * g * X - δ * g;
sRHS = v2 * z2 * s * Y - δ * s;
fps =
  Simplify[Solve[{xRHS == 0, yRHS == 0, gRHS == 0, sRHS == 0}, {X, Y, g, s}], Reals];
fps // Length
```

Out[ ]=
```
10
```

Get all fixed points of the system, of which there are 10.

In[ ]:=
```
fpsXY = FullSimplify[(fps〚1〛 // Normal)];
fpsY = FullSimplify[(fps〚2〛 // Normal)];
fpsYgen = FullSimplify[(fps〚3〛 // Normal)];
fpsYsp = FullSimplify[(fps〚4〛 // Normal)];
fpsXYgen = FullSimplify[(fps〚5〛 // Normal)];
fpsX = FullSimplify[(fps〚6〛 // Normal)];
fpsnone = FullSimplify[(fps〚7〛 // Normal)];
fpsXgen = FullSimplify[(fps〚8〛 // Normal)];
fpsXYsp = FullSimplify[(fps〚9〛 // Normal)];
fpsXYspgen = FullSimplify[(fps〚10〛 // Normal)];
```

We will first define the Jacobian matrix, and then create the Jacobian matrices evaluated on each fixed point.

```
In[ ]:= J = {{D[xRHS, X], D[xRHS, Y], D[xRHS, g], D[xRHS, s]}, {D[yRHS, X], D[yRHS, Y],
           D[yRHS, g], D[yRHS, s]}, {D[gRHS, X], D[gRHS, Y], D[gRHS, g], D[gRHS, s]},
          {D[sRHS, X], D[sRHS, Y], D[sRHS, g], D[sRHS, s]} } // FullSimplify;
        J1 = J /. fpsXY;
        J2 = J /. fpsY;
        J3 = J /. fpsYgen;
        J4 = J /. fpsYsp;
        J5 = J /. fpsXYgen;
        J6 = J /. fpsX;
        J7 = J /. fpsnone;
        J8 = J /. fpsXgen;
        J9 = J /. fpsXYsp;
        J10 = J /. fpsXYspgen;
```

To understand the linear stability of each fixed point, we will solve by setting the determinant of the matrix (Jacobian - λ*I) equal to 0.

```
In[ ]:= eigens1 = Solve[Det[J1 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens2 = Solve[Det[J2 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens3 = Solve[Det[J3 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens4 = Solve[Det[J4 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens5 = Solve[Det[J5 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens6 = Solve[Det[J6 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens7 = Solve[Det[J7 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens8 = Solve[Det[J8 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens9 = Solve[Det[J9 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
        eigens10 = Solve[Det[J10 - λ * IdentityMatrix[4]] == 0, λ] /. Rule → Equal;
```

Cooperative parameters

In[ ]:=
```
Clear[mu1, mu2, b1, b2, a1, a2, k1, k2, z1, z2, R, v1, v2, δ]
mu1 = 0.5; mu2 = 0.5;
a1 = 1; a2 = 1;
k1 = 1;
k2 = 1;
R = 1;
δ = 3 / 100; b1 = 0; b2 = 0; v1 = 20; z1 = 1 / 1000;  z2 = 1 / 1000; v2 = 20;
eigens1
eigens2
eigens3
eigens4
eigens5
eigens6
eigens7
eigens8
eigens9
eigens10
```

Out[ ]=
$$\{\{\lambda == 0.0088\}, \{\lambda == -0.0106\}, \{\lambda == -0.97 - 1.49012 \times 10^{-8}\, i\}, \{\lambda == -0.97 + 1.49012 \times 10^{-8}\, i\}\}$$

Out[ ]=
$$\{\{\lambda == -0.97\}, \{\lambda == 0.97\}, \{\lambda == -0.0106\}, \{\lambda == -0.0106\}\}$$

Out[ ]=
$$\{\{\lambda == 1.5\}, \{\lambda == 0\}, \{\lambda == -1.51053\}, \{\lambda == 0.0105261\}\}$$

Out[ ]=
$$\{\{\lambda == 0\}, \{\lambda == 0.97\}, \{\lambda == -1.51053\}, \{\lambda == 0.0105261\}\}$$

Out[ ]=
$$\{\{\lambda == -0.015\}, \{\lambda == -0.75 + 0.\, i\}, \{\lambda == -0.00890575 + 0.\, i\}, \{\lambda == -0.741094 + 0.\, i\}\}$$

Out[ ]=
$$\left\{\left\{\lambda == -\frac{3}{100}\right\}, \{\lambda == -0.97\}, \{\lambda == 0.97\}, \{\lambda == -0.0106\}\right\}$$

Out[ ]=
$$\left\{\{\lambda == 0.97\}, \{\lambda == 0.97\}, \left\{\lambda == -\frac{3}{100}\right\}, \left\{\lambda == -\frac{3}{100}\right\}\right\}$$

Out[ ]=
$$\left\{\left\{\lambda == -\frac{3}{100}\right\}, \{\lambda == 1.5\}, \{\lambda == -1.51053\}, \{\lambda == 0.0105261\}\right\}$$

Out[ ]=
$$\{\{\lambda == 0.0194\}, \{\lambda == -1.51053 - 5.55112 \times 10^{-17}\, i\}, \{\lambda == 0.0105261 + 2.77556 \times 10^{-17}\, i\}, \{\lambda == -0.97 + 5.55112 \times 10^{-17}\, i\}\}$$

Out[ ]=

$$\{\{\lambda == -1.51053 - 6.80124 \times 10^{-20}\, i\}, \{\lambda == -3.44169 \times 10^{-15} + 7.96901 \times 10^{-9}\, i\}, \{\lambda == 2.49453 \times 10^{-15} - 7.96901 \times 10^{-9}\, i\}, \{\lambda == 0.0105261 - 9.75547 \times 10^{-18}\, i\}\}$$

What are the important trends when the parameter values are set to those used when prey are mutualistic?

If neither v2*z2 nor 2*z1*v1 are greater than delta, then the only stable fixed point is the two bacterial species point (eigens1).

If 2*v1*z1 > delta and (z2 < 2*z1 or v2 < 2*v1), then the only stable fixed point is the two bacterial species point with the generalist (eigens5).

If 2*v1*z1 > delta and (2*z1 < z2 < 2.83*z1 or 2*v1 < v2 < 2.83*v1), then the only stable fixed point is the four species fixed point (eigens10).

Outside of these ranges, the only stable fixed point is the two bacterial species point with the specialist (eigens9).

Note that the lower 2*v1*z1 is while still being > delta, the more likely it is there there is no region where the four species fixed point is stable (eigens10), instead switching immediately to eigens9 (bacterial plus specialist).

These trends are largely invariant to growth rates and alpha values; increasing the benefit or growth rate of E simply shifts these observed ranges to the right (see Figure 3C in the manuscript, which visualizes this quite well).

Competitive parameters

In[∘]:=
```
Clear[mu1, mu2, b1, b2, a1, a2, k1, k2, z1, z2, R, v1, v2, δ]
mu1 = 0.5; mu2 = 0.5;
a1 = 1; a2 = 1;
k1 = 0;
k2 = 0;
R = 2;
δ = 3 / 100; b1 = 0.9; b2 = 0.9; v1 = 20; z1 = 1 / 1000;  z2 = 1 / 1000; v2 = 20;
eigens1
eigens2
eigens3
eigens4
eigens5
eigens6
eigens7
eigens8
eigens9
eigens10
```

Out[∘]=

$\{\{\lambda == 0.0108421\}, \{\lambda == -0.00957895\}, \{\lambda == -0.97\}, \{\lambda == -0.0510526\}\}$

Out[∘]=

$\{\{\lambda == -0.97\}, \{\lambda == 0.097\}, \{\lambda == 0.0088\}, \{\lambda == 0.0088\}\}$

Out[∘]=

$\{\{\lambda == 0.075\}, \{\lambda == 0\}, \{\lambda == -0.741094\}, \{\lambda == -0.00890575\}\}$

Out[∘]=

$\{\{\lambda == 0\}, \{\lambda == 0.295\}, \{\lambda == -0.741094\}, \{\lambda == -0.00890575\}\}$

Out[∘]=

$\{\{\lambda == -0.015\}, \{\lambda == -0.701488 - 1.48249 \times 10^{-14}\, i\}, \{\lambda == -0.0110123 - 3.71619 \times 10^{-13}\, i\}, \{\lambda == -0.0375 + 3.86441 \times 10^{-13}\, i\}\}$

Out[∘]=

$\{\{\lambda == -\frac{3}{100}\}, \{\lambda == -0.97\}, \{\lambda == 0.097\}, \{\lambda == 0.0088\}\}$

Out[∘]=

$\{\{\lambda == 0.97\}, \{\lambda == 0.97\}, \{\lambda == -\frac{3}{100}\}, \{\lambda == -\frac{3}{100}\}\}$

Out[∘]=

$\{\{\lambda == -\frac{3}{100}\}, \{\lambda == 0.075\}, \{\lambda == -0.741094\}, \{\lambda == -0.00890575\}\}$

Out[∘]=

$\{\{\lambda == 0.0118\}, \{\lambda == -1.00409 - 3.46945 \times 10^{-18}\, i\}, \{\lambda == 0.00817143 - 5.55112 \times 10^{-17}\, i\}, \{\lambda == -0.0490775 + 5.55112 \times 10^{-17}\, i\}\}$

Out[ ]=

$$\{\{\lambda == -0.741094\}, \{\lambda == -0.00890575\}, \{\lambda == 3.4972\times10^{-15} - 8.33\times10^{-9}\, i\}, \{\lambda == 3.4972\times10^{-15} + 8.33\times10^{-9}\, i\}\}$$

What are the important trends when the parameter values are set to those used when prey compete?

If neither v2*z2 nor 2*z1*v1 are greater than delta, then the only stable fixed point is the two bacterial species point (eigens1).

If 2*v1*z1 >= delta and (z2 < 2*z1 or v2 < 2*v1), then the only stable fixed point is the two bacterial species point with the generalist (eigens5).

If 2*v1*z1 >=delta and (z2 = 2*z1 or v2 = 2*v1), then there are two stable fixed points: the bacterial species point with the generalist (eigens5) and the four species point (eigens10). This will be the only case in which we would expect the final state to be dependent on initial conditions.

If 2*v1*z1 >=delta and z2 > 2*z1 or v2 > 2*v1, then the only stable fixed point is the four species fixed point (eigens10).

Both bacterial species plus the specialist (eigens5) is stable assuming that the v1 * z1 * < delta.

Changing the growth rates and the competition coefficients can shift the only stable fixed point to be the fixed point with both the generalist and E (eigens8), but otherwise the stable fixed point, if there is one, is the four species fixed point (eigens10). The specialist with the bacterial species is not stable under any domain tested here.

Notably, reducing the value of R to 1 changes these outcomes, such that the only two points that are stable over any relevant domains are the two bacterial point (eigens1) and the point with both bacterial and the specialist (eigens9). In this case, eigens1 is stable in domains where neither v2*z2 nor 2*z1*v1 are greater than 2*delta. It is also stable where mu1 is greater than mu2 or b2 is greater than or equal to b1 (i.e., it is stable where E is the better competitor). Eigens9 is stable if v2*z2 > 2*delta and S is the better or equal competitor (mu2 >= mu1 or b1 >= b2).