EXAMPLE PROBLEM :-

Obj. Func. :- Max. $Z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

Subject To. :- $x_1 + x_2 \leq 2$ — (I)

$2x_1 + 3x_2 \leq 12$ — (II)

BOUNDS :- $x_1, x_2 \geq 0$

SOLUTION :-

We will solve this optimization problem using the KKT's four conditions :-

① Stationarity :-

$$\boxed{\nabla f_i(x^*) - \sum \lambda_i \nabla h^i(x^*) = 0}$$

② Complementary slackness :-

$$\boxed{\lambda_i h^i(x^*) = 0}$$

③ Primal feasibility :-

$$\boxed{h^i(x^*) \leq 0}$$

④ Dual feasibility :-

$$\boxed{\lambda^i \geq 0}$$

Where, $f(x)$ is the objective function, λ_i's are the Lagrangian Multipliers, $h^i(x)$ represents both the inequality and equality constraints.

We will solve the above problem using the four KKT's Condition :-

Let $f(x) = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$

$h^1(x) = x_1 + x_2 - 2$

$h^2(x) = 2x_1 + 3x_2 - 12$

From condition - 1 :-

$$\nabla f_j - \sum \lambda_i \nabla h^i = 0$$

Partial Derivative w.r.t x_1 :-

$$\frac{\partial}{\partial x_1}\{-x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2\} - \frac{\partial}{\partial x_1}\{\lambda_1(x_1 + x_2 - 2)\}$$
$$- \frac{\partial}{\partial x_1}\{\lambda_2(2x_1 + 3x_2 - 12)\} = 0$$

$$\Rightarrow \boxed{-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0} \longrightarrow 1a$$

Similarly,

Partial Derivative w.r.t x_2 :-

$$\frac{\partial}{\partial x_2}\{-x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2\} - \frac{\partial}{\partial x_2}\{\lambda_1(x_1 + x_2 - 2)\}$$
$$- \frac{\partial}{\partial x_2}\{\lambda_2(2x_1 + 3x_2 - 12)\} = 0$$

$$\Rightarrow \boxed{-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0} \longrightarrow 1b$$

Similarly,

Partial Derivative w.r.t x_3 :-

$$\frac{\partial}{\partial x_3}\{-x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2\} - \frac{\partial}{\partial x_3}\{\lambda_1(x_1 + x_2 - 2)\}$$
$$- \frac{\partial}{\partial x_3}\{\lambda_2(2x_1 + 3x_2 - 12)\} = 0$$

$$\Rightarrow -2x_3 = 0 \Rightarrow \boxed{x_3 = 0} \longrightarrow 1c$$

Condition 2 :-

$$\lambda_i h^i = 0$$

$$\Rightarrow \boxed{\lambda_1 (x_1 + x_2 - 2) = 0} \longrightarrow 2a$$

$$\boxed{\lambda_2 (2x_1 + 3x_2 - 12) = 0} \longrightarrow 2b$$

Condition 3 :-

$$h^i \leq 0$$

$$\Rightarrow h^1(x) \leq 0 \quad ; \quad h^2(x) \leq 0$$

$$\boxed{x_1 + x_2 - 2 \leq 0} \longrightarrow 3a$$

$$\boxed{2x_1 + 3x_2 - 12 \leq 0} \longrightarrow 3b$$

Condition 4 :-

$$\lambda_i \geq 0$$

$$\Rightarrow \boxed{\lambda_1 \geq 0} \longrightarrow 4a$$

$$\boxed{\lambda_2 \geq 0} \longrightarrow 4b$$

Using the above equations, we have to find the values of x_1, x_2, x_3, λ_1 & λ_2.

CASE 1 :- $\lambda_1 = 0$ & $\lambda_2 = 0$ [Let].

From equation - 1a :-

$$-2x_1 + 4 = 0 \Rightarrow \boxed{x_1 = 2}$$

From equation - 1b :-

$$-2x_2 + 6 = 0 \Rightarrow \boxed{x_2 = 3}$$

NOW,

we have to check whether these values of x_1, x_2 are acceptable —

Putting the values of x_1 & x_2 in (3a) —

$$x_1 + x_2 - 2 \leq 0$$

$$\Rightarrow 2 + 3 - 2 \leq 0$$

$$\Rightarrow 3 \nleq 0$$ [But LHS is not satisfying RHS].

$\therefore$ our assumption that $\lambda_1 = 0$, $\lambda_2 = 0$ is incorrect.

CASE - 2 :- Let $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$.

from eq - (2a) & (2b), since λ_1 & λ_2 are not equal to zero, then,

$$x_1 + x_2 - 2 = 0 \text{ and}$$

$$2x_1 + 3x_2 - 12 = 0$$

Now, solving for x_1 & x_2 —

$$\cancel{2x_1} + 2x_2 - 4 = 0$$

$$\cancel{2x_1} + 3x_2 - 12 = 0$$

$$- \quad - \quad +$$

$$-x_2 + 8 = 0 \Rightarrow \boxed{x_2 = 8}$$

Now,

$$x_1 + x_2 - 2 = 0$$

$$\Rightarrow x_1 + 8 - 2 = 0 \Rightarrow \boxed{x_1 = -6}$$

Putting x_1 & x_2 in eq - (1a) & (1b).

$$-2x_1 + 4 - \lambda_1 - 2\lambda_2 = 0$$

$$\Rightarrow 12 + 4 - \lambda_1 - 2\lambda_2 = 0$$

$$\Rightarrow \lambda_1 + 2\lambda_2 = 16$$

$$-2x_2 + 6 - \lambda_1 - 3\lambda_2 = 0$$

$$\Rightarrow -16 + 6 - \lambda_1 - 3\lambda_2 = 0$$

$$\Rightarrow \lambda_1 + 3\lambda_2 = -10$$

Solving for λ_1 & λ_2 —

$$\cancel{\lambda_1} + 2\lambda_2 = 16$$
$$\cancel{\lambda_1} + 3\lambda_2 = -10$$
$$- \quad - \quad +$$
$$-\lambda_2 = 26 \Rightarrow \lambda_2 = -26$$

But from condition $\rightarrow$ (4b), $\lambda_2 \geq 0$.

$\therefore$ our assumption that $\lambda_1 \neq 0$ & $\lambda_2 \neq 0$ is incorrect

CASE - 3 :- Let $\lambda_1 = 0$ & $\lambda_2 \neq 0$

From eq - (2b),

$$2x_1 + 3x_2 - 12 = 0 \quad \text{— (I)}$$

from eq - (1a),

$$-2x_1 + 4 - 2\lambda_2 = 0 \quad \text{— (II)}$$

From eq - (1b),

$$-2x_2 + 6 - 3\lambda_2 = 0 \quad \text{— (III)}$$

Solving for x_1, x_2 & λ_2.

$$2\lambda_2 = -2x_1 + 4 \qquad 3\lambda_2 = -2x_2 + 6.$$
$$\lambda_2 = -x_1 + 2 \qquad \lambda_2 = -\frac{2}{3}x_2 + 2$$

$\therefore$

$$-x_1 + \cancel{2} = -\frac{2}{3}x_2 + \cancel{2}$$

$$\Rightarrow x_1 = \frac{2}{3}x_2$$

$$2x_1 + 3x_2 - 12 = 0$$

$$\Rightarrow \cancel{4x_1 + 3} \ \frac{4}{3}x_2 + 3x_2 = 12$$

$$\Rightarrow \frac{(4+9)}{3}x_2 = 12$$

$$\Rightarrow x_2 = \frac{36}{13} \qquad x_1 = \frac{2}{\cancel{3}} \times \frac{\cancel{36}^{12}}{13} = \frac{24}{13}$$

So, $\boxed{x_1 = \frac{24}{13}}$, $\boxed{x_2 = \frac{36}{13}}$

Putting these in eq- (3a),

$$x_1 + x_2 - 2 \leq 0$$

$$\Rightarrow \frac{24}{13} + \frac{36}{13} - 2 \leq 0$$

$$\Rightarrow \frac{24+36-26}{13} \leq 0$$

$$\Rightarrow \frac{34}{13} \not\leq 0$$ [∵ Not satisfying].

∴ Our assumption that $\lambda_1 = 0$ & $\lambda_2 \neq 0$ is incorrect.

CASE 4 :- $\lambda_1 \neq 0$, $\lambda_2 = 0$ (Let)

From eq- (2a),

$$x_1 + x_2 - 2 = 0 \quad \text{--- (i)}$$

From eq- (1a),

$$-2x_1 + 4 - \lambda_1 = 0 \quad \text{--- (ii)}$$

From eq- (1b),

$$-2x_2 + 6 - \lambda_1 = 0 \quad \text{--- (iii)}$$

Solving, for x_1, x_2 & λ_1.

$$-2x_1 + 4 = -2x_2 + 6$$

$$\Rightarrow -2x_1 + 2x_2 = 2$$

$$\Rightarrow x_2 - x_1 = 1$$

$$x_1 + x_2 = 2$$

$$2x_2 = 3$$

$$\Rightarrow \boxed{x_2 = 3/2}$$

∴ $x_2 - x_1 = 1$

$\Rightarrow \boxed{x_1 = 1/2}$

$$\lambda_1 = -2x_1 + 4$$
$$= -1 + 4 = 3$$

Scanned with CamScanner

So,

$$\boxed{x_1 = \frac{1}{2}} \qquad \boxed{x_2 = \frac{3}{2}}$$

$$\boxed{\lambda_1 = 3} \qquad \boxed{\lambda_2 = 0}$$

Putting these in eq - 3a & 3b,

$$x_1 + x_2 - 2 \leq 0$$

$$\Rightarrow \frac{1}{2} + \frac{3}{2} - 2 \leq 0$$

$$\Rightarrow 0 \leq 0 \qquad [\text{satisfied}]$$

$$2x_1 + 3x_2 - 12 \leq 0$$

$$\Rightarrow 1 + \frac{9}{2} - 12 \leq 0$$

$$\Rightarrow \frac{2 + 9 - 24}{2} \leq 0$$

$$\Rightarrow \frac{-13}{2} \leq 0 \qquad [\text{satisfied}].$$

Also,

eq - 4a & 4b are also satisfied.

$\therefore$ Our assumption is correct.

$$\therefore f(x) \Rightarrow z = -x_1^2 - x_2^2 - x_3^2 + 4x_1 + 6x_2$$

$$= -\frac{1}{4} - \frac{9}{4} - 0 + 2 + 9$$

$$= \frac{-10}{4} + 11$$

$$= -\frac{5}{2} + 11 = \frac{17}{2}. \text{ Ans.}$$