# MAT250S25 Proof 1

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Let $\mathbb{R}^n$ be any Euclidian Space, and let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$ be vectors such that the set of vectors $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is linearly independent.

**Lemma 1.** $\forall S, b[S \in L_I \land b \subset S \implies b \in L_I]$, where $S$ and $B$ are ordered sets of vectors, and $L_I$ is the set of all linearly independent sets.

Prove that the set of vectors $\{ \vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3 \}$ is linearly independent (Proposition $P$).

*Proof.* Let $\mathcal{F}$ be the set of all ordered sets that have any number of elements $\in \mathbb{R}$, but not any ordered set with all 0 values.

Let $Q$ be the proposition that $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is linearly independent.

Assuming $Q \land \neg P \to \{ \vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3 \}$ is linearly dependent.

$$\neg P \to \exists \{ x_1, x_2, x_3 \} \in \mathcal{F} : x_1\vec{v}_1 + x_2\vec{v}_2 + x_3(\vec{v}_1 + \vec{v}_3) = \vec{0}$$

$$\begin{aligned} \sim (x_1 + x_3)\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 &= \vec{0} \\ (x_1 + x_3)\vec{v}_1 + x_2\vec{v}_2 &= -x_3\vec{v}_3 \\ -\left(\frac{x_1}{x_3} + 1\right)\vec{v}_1 - \frac{x_2}{x_3}\vec{v}_2 &= \vec{v}_3 \end{aligned}$$

$$\text{let } t = -\frac{x_1}{x_3}, s = -\frac{x_2}{x_3}$$

$$\neg P \implies \exists t, v : t\vec{v}_1 + s\vec{v}_2 = \vec{v}_3$$

This brings a contradiction: assuming $\neg P$ implies that there exists two real numbers $t, v$ (who are not both 0) that can be into a linear combination with $\{ \vec{v}_1, \vec{v}_2 \}$ to equal $\vec{v}_3$, therefore $\neg P \implies \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$ is linearly dependent.

As shown by Lemma 1, $\{ \vec{v}_1, \vec{v}_2 \}$ is linearly independent as it is a subset of the linearly independent set $\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \}$.

This shows that $Q \land \neg P \implies \neg Q$, which is a contradiction which must mean that $P$ must be true, and the set $\{ \vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3 \}$ is linearly independent.

QED