MAT250S25 Proof 1

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Any set B is Linearly Independent if it is a subset of a linearly independent set.

$$B \in L_I \iff (B \subset A) \wedge A \in L_I$$

Let $\mathbb{R}^n$ be any Euclidian Space, and let $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^n$ be vectors such that the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Prove that the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3\}$ is linearly independent (Proposition P).

Proof. Let Q be the proposition that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

Assume that $Q \wedge \neg P \rightarrow \{\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3\}$ is linearly dependent.

Let $\mathcal{F}$ be the set of all sets that have elements $\in \mathbb{R}$, but not any set with all 0 values.

$$\neg P \rightarrow \exists \{x_1, x_2, x_3\} \in \mathcal{F} : x_1\vec{v}_1 + x_2\vec{v}_2 + x_3(\vec{v}_1 + \vec{v}_3) = \vec{0}$$

$$\sim (x_1 + x_3)\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$
$$(x_1 + x_3)\vec{v}_1 + x_2\vec{v}_2 = -x_3\vec{v}_3$$
$$-\left(\frac{x_1}{x_3} + 1\right)\vec{v}_1 - \frac{x_2}{x_3}\vec{v}_2 = \vec{v}_3$$

$$\text{let } t = -\frac{x_1}{x_3}, s = -\frac{x_2}{x_3}$$

$$\neg P \implies \exists t, v : t\vec{v}_1 + s\vec{v}_2 = \vec{v}_3$$

Assuming $\neg P$ implies that there exists two real numbers $\{t, v\} \in \mathcal{F}$ that can be made into a linear combination with $\{\vec{v}_1, \vec{v}_2\}$ to construct $\vec{v}_3$. Therefore the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

As shown by thereom , $\{\vec{v}_1, \vec{v}_2\}$ must be linearly independent as it is a subset of the defined linearly independent set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

This shows that $Q \wedge \neg P \implies \neg Q$, which is a contradiction, meaning that P must be true ($\{\vec{v}_1, \vec{v}_2, \vec{v}_1 + \vec{v}_3\}$ is linearly independent).

QED