MAT250S25 Proof 2

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Definition 1 ($\mathcal{L}_1$, from the **Big Thereom™**)**.** A set of vectors is linearly independent if and only if it is injective.

Definition 2 ($\mathcal{L}_2$, from the **Big Thereom™**)**.** A set of vectors $\{\, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \,\}$ is linearly independent if and only if:

$$\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{bmatrix} \vec{b} = \vec{0} \iff \vec{b} = \vec{0}$$

Given:

- let $\mathbb{R}^n$,$\mathbb{R}^m$ be Euclidean spaces.
- let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation with corresponding $n \times m$ matrix A.
- let $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \in \mathbb{R}^m$ be vectors such that the set of vectors
- $\{\, T(\vec{x}_1), T(\vec{x}_2), \ldots, T(\vec{x}_k) \,\}$ is linearly independent.

Prove that the set of vectors $\{\, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k \,\}$ is linearly independent.

Proof.

$$\forall X \in M_{n\times m}\; f_X : \vec{b} \mapsto X\vec{b}$$

$$\begin{aligned}
\text{let } C &= \begin{bmatrix} T(\vec{x}_1) & T(\vec{x}_2) & \cdots & T(\vec{x}_k) \end{bmatrix} \\
\text{let } B &= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k \end{bmatrix} \\
C &= A \circ B
\end{aligned}$$

$$\begin{aligned}
f_C &= T \circ f_B \\
\mathcal{L}_1 \implies \forall \vec{a}, \vec{b} :\ f_C(\vec{a}) = f_C(\vec{b}) &\iff \vec{a} = \vec{b}
\end{aligned}$$

$$\begin{aligned}
\text{let } \vec{b}, \vec{c} &\in \mathbb{R}^k \\
f_B(\vec{b}) &= f_B(\vec{c}) \\
(T \circ f_B)(\vec{b}) &= (T \circ f_B)(\vec{c}) \\
f_C(\vec{b}) &= f_C(\vec{c}) \\
\vec{b} &= \vec{c}
\end{aligned}$$

$$\begin{aligned}
&\therefore f_B \text{ is injective} \\
&\implies \forall \vec{b}, \vec{c} \in \mathbb{R}^k :\ B\vec{b} = B\vec{c} \iff \vec{b} = \vec{c} \\
&\implies B\vec{b} = \vec{0} \iff \vec{b} = \vec{0} \\
\cdots \wedge \mathcal{L}_2 &\implies \{\, \vec{x}_1, \vec{x}_2, \cdots, \vec{x}_k \,\} \text{ is linearly independent.}
\end{aligned}$$

QED