MAT250 Proof 3

Kishan S Patel

March 7, 2025

This proof is an altered version of one I submitted for MAT258 this semester (about 3 weeks ago for the 2nd proof assignment).

Let A be an $n \times k$ matrix, and let B be a $k \times m$ matrix. AB is their product $n \times m$ matrix.

1. If AB is one-to-one (injective) as a linear transformation, then so is B (P_1).

   Proof.

$$\begin{aligned} \text{let } & f : Y \to Z \\ & f : \vec{x} \mapsto A\vec{x} \end{aligned}$$

$$\begin{aligned} \text{let } & g : X \to Y \\ & g : \vec{x} \mapsto B\vec{x} \end{aligned}$$

   $f \circ g$ is injective because $f \circ g = AB$.

$$\text{let } P_2 = \nexists x_0, x_1 \in X : (f \circ g)(x_0) = (f \circ g)(x_1) \land x_0 \neq x_1$$

$$\begin{aligned} \neg P_1 &\implies \exists x_0, x_1 \in X : g(x_0) = g(x_1) \land x_0 \neq x_1 \\ &\implies \exists x_0, x_1 \in X : (f \circ g)(x_0) = (f \circ g)(x_1) \land x_0 \neq x_1 \end{aligned}$$

$$P_2 \land \neg P_1 \implies \neg P_2 \therefore P_1 \text{ is true.}$$

   QED

2. If AB is surjective as a transformation, then so is A.

   Proof.

$$\begin{aligned} \forall z \in Z, \exists x \in X : (f \circ g)(x) &= z \\ f(g(x)) &= z \\ \text{let } y \in Y &= g(x) \\ f(y) = f(g(x)) &= z \end{aligned}$$

$$\therefore f \text{ is surjective.}$$

   QED