MAT250 Proof 5

Kishan S Patel

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Lemma 1 ($\mathcal{L}_1$)**.** For any linear subspace U, a basis of U constructed from some set S of linearly independent vectors in U must have a cardinality of $\dim U$.

$$\forall n \in \mathbb{N}, U \subseteq \mathbb{R}^n : U = \operatorname{span}(\{\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_{\dim U}\})$$

Lemma 2 ($\mathcal{L}_2$)**.** For any linearly independent set of vectors $\{\, \vec{v}_1, \cdots, \vec{v}_n \,\}$, for all v_i there does not exist a linear combination with all other vectors $(v_i : i \neq j)$ that sums to v_i.

Let $\mathbb{R}^n$ be a Euclidean space, and let $U, V \subseteq \mathbb{R}^n$ be subspaces such that $V \subseteq U$
Let $p = \dim U$ and $q = \dim V$.

1. Prove that $q \geq p$ (P_1).

   Proof. Assuming $\neg P_1 \implies \exists U \subseteq \mathbb{R}^n, V \subseteq U : \dim V > \dim U$.

   Let $\{\, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_q \,\}$ be a basis of V.

   Assuming $q > p$, we can take p vectors from the set $\{\, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p, \cdots, \vec{v}_q \,\}$ to form a basis of U as the set $\{\, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \,\}$.

   Because $V \subseteq U$, all vectors must in both sets must be inside U.

   $$V \subseteq U \implies \forall i \, v_i \in U$$

   Forming a linearly independent set that spans U from $\{\, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p \,\}$, the vector $\vec{v}_q$ is not inside this set but is an element in U. This must mean there exists a linear combination between the set and some vector $\vec{x}$ such that it equals $\vec{v}_q$.

   $$\neg P_1 \implies \exists \vec{x} : \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_p \end{bmatrix} x = \vec{v}_q$$

   This imposes a contradiction to $\mathcal{L}_2$, as the set $\{\, \vec{v}_1, \vec{v}_2, \cdots, \vec{v}_p, \cdots, \vec{v}_q \,\}$ is linearly independent, therefore P_1 must be true. QED

2. If $q = p$ then $V = U$ (P_2).

   Proof. Let $\vec{u} \in U$. Because $V \subseteq U$ and $\dim U = \dim V$, any basis $\{\, \vec{v}_1, \cdots, \vec{v}_q \,\}$ of V is also a basis of U as all elements in the set are elements of U.

   Because any basis of V is a basis of U, $\vec{u}$ can be written as a linear combination of the basis vectors of V, therefore $\vec{u} \in V$.

   $$\begin{aligned} u \in V \land u \in U &\implies U \subseteq V \\ V \subseteq U \land U \subseteq V &\implies U = V \end{aligned}$$

   QED