MAT258S25 Proof 1

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Definition 1. Any element in the set $\mathbb{Q}$ can be expressed as $\frac{q}{p}$, where $p, q \in \mathbb{Z}$, $q \neq 0$

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{ 0 \} \right\}$$

Assumption 1. *The set $\mathbb{Z}$ is closed under addition.*

$$P^{+} = (\forall x, y \in \mathbb{Z} \,:\, (x + y) \in \mathbb{Z})$$

Assumption 2. *The set $\mathbb{Z}$ is closed under multiplication.*

$$P^{\times} = (\forall x, y \in \mathbb{Z} \,:\, xy \in \mathbb{Z})$$

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1. For any given $r, s \in \mathbb{Q}$, prove that $(r + s) \in \mathbb{Q}$.

Proof.

$$\text{let } r = \frac{a}{b}, s = \frac{c}{d}$$
$$r \in \mathbb{Q} \implies a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{ 0 \}$$
$$s \in \mathbb{Q} \implies c \in \mathbb{Z}, d \in \mathbb{Z} \setminus \{ 0 \}$$

$$\begin{aligned} r + s &= \frac{a}{b} + \frac{c}{d} \\ &= \frac{ad}{bd} + \frac{cb}{db} \\ &= \frac{ad + cb}{bd} \end{aligned}$$

$$\begin{aligned} P^{\times} &\implies ad, cb, bd \in \mathbb{Z} \\ P^{\times} \wedge P^{+} &\implies (ad + cb) \in \mathbb{Z} \\ 0 \notin \{ b, d \} &\implies bd \in \mathbb{Z} \setminus \{ 0 \} \end{aligned}$$

$$\begin{matrix} (ad + cb) \in \mathbb{Z} \\ bd \in (\mathbb{Z} \setminus \{ 0 \}) \end{matrix} \implies \frac{ad + cb}{bd} \in \mathbb{Q} \to r + s \in \mathbb{Q}$$

QED

2. let $r \in \mathbb{Q}$, let $s \in \mathbb{R} \setminus \mathbb{Q}$, prove that $r + s \in \mathbb{R} \setminus \mathbb{Q}$

Proof. Assuming the the proposition P_2 to be false, $\forall [r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}] : [r + s \in \mathbb{R} \setminus \mathbb{Q}]$

$$\begin{aligned}
\text{let } r &= \frac{p}{q} \\
\text{let } r + s &= \frac{t}{v} \\
r \in \mathbb{Q} &\implies p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \\
\neg P_2 \wedge ((r + s) \in \mathbb{Q}) &\implies t \in \mathbb{Z}, v \in \mathbb{Z} \setminus \{0\}
\end{aligned}$$

$$\begin{aligned}
r + s &= \frac{t}{v} \\
\frac{p}{q} + s &= \frac{t}{v} \\
\frac{vp}{q} + vs &= t \\
vp + vsq &= tq \\
s &= \frac{tq - vp}{vq} \\
s &= \frac{tq + (-1)(vp)}{vq}
\end{aligned}$$

$$\begin{aligned}
[p, t \in \mathbb{Z}] \wedge P^{\times} &\implies tq, vp \in \mathbb{Z} \\
[v, q \in \mathbb{Z} \setminus \{0\}] \wedge P^{\times} &\implies vq \in \mathbb{Z} \setminus \{0\} \qquad \therefore \left(\frac{tq + (-1)vp}{vq} \right) \in \mathbb{Q} \implies s \notin \mathbb{R} \setminus \mathbb{Q} \\
P^{+} \wedge P^{+} &\implies tq + (-1)vp \in \mathbb{Z}
\end{aligned}$$

Assuming $\neg P_2$ contradicts the definition of $s \in \mathbb{R} \setminus Q$, therefore the proposition $\neg P_2$ must be false, meaning P_2 is true.

$$\text{let } s \in \mathbb{R} \setminus \mathbb{Q}, r \in \mathbb{R}$$

$$\neg P_2 \implies s \notin \mathbb{R} \setminus \mathbb{Q} \therefore P_1 \text{ is true.}$$

QED