MAT258S25 Proof 1

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Definition 1. Any element in the set $\mathbb{Q}$ can be expressed as $\frac{q}{p}$, where $p, q \in \mathbb{Z}$, $q \neq 0$

$$P_{\mathbb{Q}} = \forall (p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{\,0\,\}) \left[\frac{p}{q} \in \mathbb{Q} \right]$$

Assumption 1. *The set $\mathbb{Z}$ is closed under addition.*

$$P^{+} = \forall x, y \in \mathbb{Z}[(x + y) \in \mathbb{Z}]$$

Assumption 2. *The set $\mathbb{Z}$ is closed under multiplication.*

$$P^{\times} = \forall x, y \in \mathbb{Z}[xy \in \mathbb{Z}]$$

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1. For any given $r, s \in \mathbb{Q}$, prove that $(r + s) \in \mathbb{Q}$.

    Proof.

    $$\text{let } r = \frac{p_r}{q_r}, s = \frac{p_s}{q_s}$$
    $$[r, s \in \mathbb{Q}] \implies \{\, p_r, q_r, p_s, q_s \,\} \subset \mathbb{Z}$$
    $$0 \notin \{\, q_r, q_s \,\}$$

    $$r + s = \frac{p_r}{q_r} + \frac{p_s}{q_s}$$
    $$= \frac{p_r q_s}{q_r q_s} + \frac{p_s q_r}{q_s q_r}$$
    $$= \frac{p_r q_s + p_s q_r}{q_r q_s}$$

    $$P^{\times} \implies \{\, p_r q_s, p_s q_r, q_r q_s \,\} \subset \mathbb{Z}$$
    $$P^{\times} \wedge P^{+} \implies (p_r q_s + p_s q_r) \in \mathbb{Z}$$
    $$0 \notin \{\, q_r, q_s \,\} \implies q_r q_s \neq 0$$

    $$P_{\mathbb{Q}}$$
    $$p_r q_s + p_s q_r \in \mathbb{Z}$$
    $$q_r q_s \in (\mathbb{Z} \setminus \{\,0\,\})$$
    $$\overline{\therefore \frac{p_r q_s + p_s q_r}{q_r q_s} \in \mathbb{Q} \rightarrow r + s \in \mathbb{Q}}$$

    $\square$

2. let $r \in \mathbb{Q}$, let $s \in \mathbb{R} \setminus \mathbb{Q}$, prove that $r + s \in \mathbb{R} \setminus \mathbb{Q}$

Proof. Assume the the proposition P_2 to be false, $\forall r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q} \left[r + s \in \mathbb{R} \setminus \mathbb{Q} \right]$

$$\text{let } r = \frac{p}{q}$$

$$\text{let } r + s = \frac{t}{v}$$

$$P_{\mathbb{Q}} \implies p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{\, 0 \,\}$$

$$P_2 \wedge P_{\mathbb{Q}} \implies t \in \mathbb{Z}, v \in \mathbb{Z} \setminus \{\, 0 \,\}$$

$$r + s = \frac{t}{v}$$

$$\frac{p}{q} + s = \frac{t}{v}$$

$$\frac{vp}{q} + vs = t$$

$$vp + vsq = tq$$

$$s = \frac{tq - vp}{vq}$$

$$= \frac{tq + (-1)(vp)}{vq}$$

$$[q, v \in \mathbb{Z} \setminus \{\, 0 \,\}] \wedge P^{\times} \implies vq \in \mathbb{Z} \setminus \{\, 0 \,\}$$

$$[p, q \in \mathbb{Z}] \wedge P^{\times} \implies \{\, tq, vp \,\} \subset \mathbb{Z}$$

$$P^{+} \wedge P^{+} \implies tq + (-1)vp \in \mathbb{Z}$$

$$\overline{\therefore x}$$

$\square$