MAT258S25 Proof 1

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Definition 1. Any element in the set $\mathbb{Q}$ can be expressed as $\frac{q}{p}$, where $p, q \in \mathbb{Z}$, $q \neq 0$

$$P_{\mathbb{Q}} = \forall (p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}) \left[\frac{p}{q} \in \mathbb{Q} \right]$$

Assumption 1. *The set $\mathbb{Z}$ is closed under addition.*

$$P^{+} = \forall x, y \in \mathbb{Z}[(x+y) \in \mathbb{Z}]$$

Assumption 2. *The set $\mathbb{Z}$ is closed under multiplication.*

$$P^{\times} = \forall x, y \in \mathbb{Z}[xy \in \mathbb{Z}]$$

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1. For any given $r, s \in \mathbb{Q}$, prove that $(r+s) \in \mathbb{Q}$.

   Proof.

$$\begin{gathered} \text{let } r = \frac{p_r}{q_r}, s = \frac{p_s}{q_s} \\ [r, s \in \mathbb{Q}] \implies \{p_r, q_r, p_s, q_s\} \subset \mathbb{Z} \\ 0 \notin \{q_r, q_s\} \end{gathered}$$

$$\begin{aligned} r + s &= \frac{p_r}{q_r} + \frac{p_s}{q_s} \\ &= \frac{p_r q_s}{q_r q_s} + \frac{p_s q_r}{q_s q_r} \\ &= \frac{p_r q_s + p_s q_r}{q_r q_s} \end{aligned}$$

$$\begin{aligned} P^{\times} &\implies \{p_r q_s, p_s q_r, q_r q_s\} \subset \mathbb{Z} \\ P^{\times} \wedge P^{+} &\implies (p_r q_s + p_s q_r) \in \mathbb{Z} \\ 0 \notin \{q_r, q_s\} &\implies q_r q_s \neq 0 \end{aligned}$$

$$\begin{array}{r} P_{\mathbb{Q}} \\ p_r q_s + p_s q_r \in \mathbb{Z} \\ q_r q_s \in (\mathbb{Z} \setminus \{0\}) \\ \hline \therefore \dfrac{p_r q_s + p_s q_r}{q_r q_s} \in \mathbb{Q} \to r + s \in \mathbb{Q} \end{array}$$

QED

2. let $r \in \mathbb{Q}$, let $s \in \mathbb{R} \setminus \mathbb{Q}$, prove that $r + s \in \mathbb{R} \setminus \mathbb{Q}$

Proof. Assume the the proposition P_2 to be false, $\forall r \in \mathbb{Q}, s \in \mathbb{R} \setminus \mathbb{Q}\, [r + s \in \mathbb{R} \setminus \mathbb{Q}]$

$$\begin{aligned} \text{let } r &= \frac{p}{q} \\ \text{let } r + s &= \frac{t}{v} \\ P_{\mathbb{Q}} &\implies p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \\ P_2 \wedge P_{\mathbb{Q}} &\implies t \in \mathbb{Z}, v \in \mathbb{Z} \setminus \{0\} \end{aligned}$$

$$\begin{aligned} r + s &= \frac{t}{v} \\ \frac{p}{q} + s &= \frac{t}{v} \\ \frac{vp}{q} + vs &= t \\ vp + vsq &= tq \\ s &= \frac{tq - vp}{vq} \\ &= \frac{tq + (-1)(vp)}{vq} \end{aligned}$$

$$\begin{aligned} [q, v \in \mathbb{Z} \setminus \{0\}] \wedge P^{\times} &\implies vq \in \mathbb{Z} \setminus \{0\} \\ [p, q \in \mathbb{Z}] \wedge P^{\times} &\implies \{tq, vp\} \subset \mathbb{Z} \\ P^{+} \wedge P^{+} &\implies tq + (-1)vp \in \mathbb{Z} \\ & \therefore \frac{tq + (-1)vp}{vq} \in \mathbb{R} \setminus \mathbb{Q} \\ & \to s \in \mathbb{R} \setminus \mathbb{Q} \end{aligned}$$

This provides a contradiction regarding $s \in \mathbb{R} \setminus \mathbb{Q}$, as $\neg P_2 \implies s \in \mathbb{Q}$ QED